Covering Relations and Non-autonomous Perturbations of ODEs

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Abstract

Covering relations are a topological tool for detecting periodic orbits, symbolic dynamics and chaotic behavior for autonomous ODE. We extend the method of the covering relations onto systems with a time dependent perturbation. As an example we apply the method to non-autonomous perturbations of the Rössler equations to show that for small perturbation they posses symbolic dynamics.

Keywords: covering relations, non-autonomous ODEs, chaotic behavior

1 Introduction

The goal of this paper is to answer the following
QUESTION: *Assume that the equation*

$$x' = v(x) \tag{1.1}$$

has a symbolic dynamics. Consider now small non-autonomous perturbation of (1.1)

$$x'(t) = v(x(t)) + \epsilon(t, x(t)). \tag{1.2}$$

Will equation (1.2) also have the symbolic dynamics if ϵ is sufficiently small?

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Of course to make the above question mathematically meaningful we need to explain what is the precise meaning of the statement "ODE (1.1) or (1.2) has a symbolic dynamics". For the purpose of this introduction it is enough to say that it means that for suitably defined Poincaré maps there exists orbits which could be coded by infinite sequences of finite number of symbols. This is made precise in Section 3, where we consider the non-autonomous perturbations of the Rössler equations [R] and provide an affirmative answer to the above question.

The content of the paper can be described as follows. In Section 2 we recall from paper [ZGi] the notion of covering relations for maps. This is the basic technical tool used in this paper. In Section 2.2 we prove Theorem 2 - the basic theorem about continuation of covering relations for Poincaré maps for the non-autonomous perturbations of ODEs. In the following sections we apply Theorem 2 to answer positively our question in the context of the Rössler equation.

2 Topological theorems

2.1 Covering relations - basic definitions

Definition 1 *[ZGi] An h-set, N, is an object consisting of the following data*

1. *$|N|$ - a compact subset of $\mathbb{R}^k$*
2. *$u(N), s(N) \in \{0, 1, 2, 3, \dots\}$, such that $u(N) + s(N) = k$*
3. *a homeomorphism $c_N : \mathbb{R}^k \to \mathbb{R}^k = \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)}$ such that*

$$c_N(|N|) = \overline{B_{u(N)}}(0,1) \times \overline{B_{s(N)}}(0,1)$$

We set

$$\begin{aligned} N_c &= \overline{B_{u(N)}}(0,1) \times \overline{B_{s(N)}}(0,1), \\ N_c^- &= \partial\overline{B_{u(N)}}(0,1) \times \overline{B_{s(N)}}(0,1), \\ N_c^+ &= \overline{B_{u(N)}}(0,1) \times \partial\overline{B_{s(N)}}(0,1), \\ N^- &= c_N^{-1}(N_c^-), \quad N^+ = c_N^{-1}(N_c^+) \end{aligned}$$

Later we will quite often drop the parallel lines in $|N|$ and write N instead of $|N|$ to indicate the support of an h-set N.

Definition 2 *[ZGi] Assume N,M are h-sets, such that $u(N) = u(M) = u$ and $s(N) = s(M) = s$. Let $f : |N| \to \mathbb{R}^k$ be a continuous map. Let $f_c = c_M \circ f \circ c_N^{-1} : N_c \to \mathbb{R}^u \times \mathbb{R}^s$. We say that*

$$N \overset{f}{\Longrightarrow} M$$

(N f-covers M) if the following conditions are satisfied

1. *There exists a continuous homotopy* $h : [0,1] \times N_c \to \mathbb{R}^u \times \mathbb{R}^s$ *such that the following conditions hold true*

$$\begin{aligned} h_0 &= f_c, \\ h([0,1], N_c^-) \cap M_c &= \emptyset, \\ h([0,1], N_c) \cap M_c^+ &= \emptyset. \end{aligned}$$

2.1. *If* $u > 0$, *then there exists a linear map* $A : \mathbb{R}^u \to \mathbb{R}^u$, *such that*

$$h_1(p,q) = (Ap, 0), \quad \text{where } p \in \mathbb{R}^u \text{ and } q \in \mathbb{R}^s, \tag{2.1}$$

$$A(\partial B_u(0,1)) \subset \mathbb{R}^u \backslash \overline{B_u}(0,1). \tag{2.2}$$

2.2. *If u=0, then*

$$h_1(x) = 0, \quad \text{for } \ x \in N_c.$$

With above definition we have the following theorem (see also [MM, Z0, Z1] for its precursors).

Theorem 1 *[ZGi] Let* N_i, $i = 0, \ldots, n$ *be an h-set and* $N_n = N_0$. *Assume that for each* $i = 1, \ldots, n$ *we have*

$$N_{i-1} \overset{f_i}{\Longrightarrow} N_i \tag{2.3}$$

then there exists a point $x \in int|N_0|$, *such that*

$$\begin{aligned} f_i \circ f_{i-1} \circ \ldots \circ f_1(x) &\in int|N_i|, \quad i = 1, \ldots, n \\ f_i \circ f_{i-1} \circ \ldots \circ f_1(x) &= x. \end{aligned}$$

2.2 Continuation of covering relations for Poincaré maps for non-autonomous perturbations

Let $v : \mathbb{R}^k \to \mathbb{R}^k$ be a C^1 function. Let us consider an autonomous differential equation

$$x' = v(x) \tag{2.4}$$

Let $V_0, V_1, \ldots, V_{n-1}, V_n$ be Poincaré sections of the system generated by the equation (we do not require that they are different). Let $1 \leq i \leq n$, and let x be the solution of the problem

$$\begin{aligned} x' &= v(x) \\ x(0) &= x_j \end{aligned}$$

where $x_j \in V_j$ for some $j \in \{0, \ldots, n\}$.

For $i = 0, \ldots, n$, by $\sigma_i(x_j)$ we will denote the first time for which the solution x reaches the section V_i,

$$\sigma_i(x_j) := \inf\{t > 0 : x(t) \in V_i\}$$

Let $A_{ji} \subset V_j$ denote the set of all the points x_j for which the function $\sigma_i(x_j)$ is defined. When it will be evident from the context which sections we wish to consider, we will sometimes omit the index i.

We will also define functions (Poincaré maps)

$$\begin{aligned} f_{ji} &: A_{ji} \to V_i \\ f_{ji}(x_j) &:= x(\sigma_i(x_j)). \end{aligned}$$

Now in the context of the question asked in the introduction assume for example that we have the following covering relations $N_i \overset{f_{ij}}{\Longrightarrow} N_j$ for $i, j = 0, 1$ (compare with topological horseshoes in [Z3]). Then from Theorem 1 it follows that we have a semiconjugacy onto the Bernoulli shift on two symbols.

Let us now consider the equation (2.4) with a time dependent perturbation

$$x'(t) = v(x(t)) + \epsilon(t, x(t)) \tag{2.5}$$

where $\epsilon(t, x)$ is an *admissible* function, which means that it is continuous with respect to (t, x) and is locally Lipschitz with respect to the x variable.

We will try to show a similar result (for example for topological horseshoe) for this perturbed equation. It seems very likely that for small perturbation the above result should hold. Let us start with the fact that for small perturbations of the equation the covering relations (2.3) for the solution still hold. Let us clarify what we will exactly understand by the functions f_{ji} in the setting of the perturbed equation (2.5). Let us consider the equation (2.5) with the following initial conditions

$$\begin{aligned} x' &= v(x) + \epsilon(t, x) \\ x(T) &= x_j \\ x_j &\in N_j \end{aligned} \tag{2.6}$$

Let x be the solution of problem (2.6). We will define functions f_{ji}^T which will be analogous to the functions f_{ji}. As before

$$\begin{aligned} f_{ji}^T &: V_j \to V_i \\ f_{ji}^T(x_j) &: = x(\sigma_i(x_j, T)) \\ \sigma_i(x_j, T) &: = \inf\{t > T : x(t) \in V_i\} \end{aligned} \tag{2.7}$$

Let us note that for $|\epsilon|$ sufficiently small the functions above are well defined [ZGi]. What is more if there exists a covering relation $N_j \overset{f_{ji}}{\Longrightarrow} N_i$ then for $|\epsilon| < \delta_1$ there exists $\beta_1 > \beta_2 > 0$ such that the term

$$\beta_1 > \sigma_i(x_j, T) - T > \beta_2 \quad \text{is bounded for all } x_j \in N_j \text{ and } |\epsilon| < \delta_1. \tag{2.8}$$

The goal of this section is to establish the following

Theorem 2 *Let $v : \mathbb{R}^k \to \mathbb{R}^k$ be C^1-function, let $V_1, \dots, V_n$ be the Poincaré sections for the equation*

$$x' = v(x) \tag{2.9}$$

Let $N_i \subset V_i$, $i = 1, \dots, n$ be h-sets, we denote this family by $\mathcal{H}$

Assume that we have a set Γ of covering relations $N_i \overset{f_{ij}}{\Longrightarrow} N_j$ for some $N_i, N_j \in \mathcal{H}$, where f_{ji} are Poincaré maps for (2.9).

Then there exits $\delta = \delta(\Gamma)$, such that for all admissible functions ϵ such that $|\epsilon| < \delta$ we have:

For any $t_0 \in \mathbb{R}$ and for any infinite chain of covering relations from Γ

$$N_0 \overset{f_{01}}{\Longrightarrow} N_1 \overset{f_{12}}{\Longrightarrow} N_2 \overset{.}{\Longrightarrow} ..$$

where $N_i \in \mathcal{H}$ and $\left(N_i \overset{f_{i,i+1}}{\Longrightarrow} N_{i+1}\right) \in \Gamma$ for $i = 0, 1, \dots,$

there exists a point $x_0 \in N_0$ and a sequence $\{t_i\}_{i=0}^{\infty}$, $t_0 < t_1 < \ldots < t_m < \ldots$, such that for the solution x of the equation

$$\begin{aligned} x' &= v(x) + \epsilon(t, x) \\ x(t_0) &= x_0 \end{aligned} \tag{2.10}$$

we have

$$x(t_i) \in intN_i, \qquad i = 1, 2, \dots$$

Before we move on to the proof of this theorem we shall need some preliminary results.

Lemma 1 *Assume that f_i is a Poincaré map for (2.9). If*

$$N_{i-1} \overset{f_i}{\Longrightarrow} N_i$$

then there exists a $\delta > 0$ such that for all admissible ϵ such that $|\epsilon| < \delta$, for all $T \in \mathbb{R}$

$$N_{i-1} \overset{f_i^T}{\Longrightarrow} N_i.$$

Furthermore for all i there exists a homotopy $H^i : [0, 1] \times \mathbb{R} \times N_{i-1,c} \to \mathbb{R}^u \times \mathbb{R}^s$,

$$\begin{aligned} H^i(0, T, x) &= f_{i,c}^T(x), & (2.11)\\ H^i(1, T, (p, q)) &= (A_i p, 0), & (2.12)\\ H^i([0, 1], T, N_{i-1,c}^-) \cap N_{i,c} &= \emptyset, & (2.13)\\ H^i([0, 1], T, N_{i-1,c}) \cap N_{i,c}^+ &= \emptyset. & (2.14) \end{aligned}$$

where $x = (p, q)$ and A_i is the linear map from the definition of the covering relation for the covering $N_{i-1} \overset{f_i}{\Longrightarrow} N_i$.

This Lemma states that for all the Poincaré maps f_i^T , for any T, there exists a homotopy $H_T^i = H^i(\cdot, T, \cdot)$ which transports the function $f_{i,c}^T$ into the linear function $(A_i\,, 0)$, which is independent from T. What is more, the family of functions H_T^i is continuous with respect to T.

Proof: The first part of the lemma regarding the fact that

$$N_{i-1} \overset{f_i^T}{\Longrightarrow} N_i$$

is a consequence of the Theorem 13 from [ZGi]. To prove the second part of the lemma, let us consider the following differential equation

$$x' = v(x) + (\frac{1}{2} - \lambda)\epsilon(t + T, x)$$

where $\lambda \in [0, \frac{1}{2}]$. We can define Poincaré maps $f_i^{\lambda,T}$ in the same manner as we have defined the functions f_i^T in (2.7). From the first part of the lemma we know that

$$N_{i-1} \overset{f_i^{\lambda,T}}{\Longrightarrow} N_i.$$

Let us note that $f_i^{\frac{1}{2},T} = f_i$. Since

$$N_{i-1} \overset{f_i}{\Longrightarrow} N_i$$

we know that there exists a homotopy $h^i : [0, 1] \times N_{i-1,c} \to \mathbb{R}^u \times \mathbb{R}^s$, which satisfies the conditions 1, 2.1 and 2.2, from the definition of the covering relation.

We can now define our homotopy as

$$H^i(\lambda, T, x) := \begin{cases} c_{N_i} \circ f_i^{\lambda,T}(x) \circ c_{N_{i-1}}^{-1} & \text{for } \lambda \in [0, \frac{1}{2}] \\ h^i(2\lambda - 1, x) & \text{for } \lambda \in (\frac{1}{2}, 1] \end{cases}$$

We need to show that this homotopy satisfies the conditions (2.11), (2.12), (2.13), (2.14). The first two conditions are evident from the definition of H. From the fact that

$$N_{i-1} \overset{f_i^{\lambda,T}}{\Longrightarrow} N_i$$

we know that

$$\begin{aligned} H^i([0, \frac{1}{2}], T, N_{i-1,c}^-) \cap N_{i,c} &= \emptyset, \\ H^i([0, \frac{1}{2}], T, N_{i-1,c}) \cap N_{i,c}^+ &= \emptyset. \end{aligned}$$

The fact that

$$\begin{aligned} H^i((\frac{1}{2}, 1], T, N_{i-1,c}^-) \cap N_{i,c} &= \emptyset, \\ H^i((\frac{1}{2}, 1], T, N_{i-1,c}) \cap N_{i,c}^+ &= \emptyset, \end{aligned}$$

follows from the conditions 2.1 and 2.2 for the covering

$$N_{i-1} \overset{f_i}{\Longrightarrow} N_i.$$

Hence all the conditions (2.11), (2.12), (2.13), (2.14) hold. ∎

The following lemma will be the main tool for the proof of the Theorem 2.

Lemma 2 *Let $v : \mathbb{R}^k \to \mathbb{R}^k$ be C^1-function, let $V_1, \ldots, V_n$ be the Poincaré sections for the equation*

$$x' = v(x) \tag{2.15}$$

Let $N_i \subset V_i$, $i = 1, \ldots, n$ be h-sets. Let f_i be Poincaré maps for (2.15). Let us also assume that we have the following chain of covering relations

$$N_0 \overset{f_1}{\Longrightarrow} N_1 \overset{f_2}{\Longrightarrow} \ldots \overset{f_n}{\Longrightarrow} N_n. \tag{2.16}$$

Then there exists a $\delta > 0$ which depends only on the set of covering relations in the chain (2.16) and not on the length of the chain, such that for any admissible function ϵ, $|\epsilon| < \delta$ and for all $T \in \mathbb{R}$ we have

$$N_{i-1} \overset{f_i^T}{\Longrightarrow} N_i \quad \textit{for } i = 1, \ldots, n$$

and for any $t_0 \in \mathbb{R}$ there exists a point $x_0 \in N_0$ and a sequence $t_0 < t_1 < \ldots < t_n$, such that for the solution x of the equation

$$\begin{aligned} x' &= v(x) + \epsilon(t, x) \\ x(t_0) &= x_0 \end{aligned} \tag{2.17}$$

we have

$$x(t_i) \in intN_i \quad \textit{for } i = 1, \ldots, n.$$

Remark 1 *Let us note that in the above Lemma a strong emphasis should be put on the fact that δ is not dependent on the length of the chain. This fact when used in the proof of Theorem 2 will allow us to extend the chain of relations forward to infinity without affecting our δ.*

Let us also note that one could be tempted into proving the Lemma using an argument that the existence of the point x_0 and the times $t_0 < t_1 < \ldots < t_n$ should follow from the fact that Poincaré maps are continuous with respect to both initial conditions and perturbations. Using this argument we would indeed obtain our x_0 and the times $t_0 < t_1 < \ldots < t_n$ for all $|\epsilon| < \delta$, but since the perturbation ϵ is time dependent the δ would strongly depend on the length of the chain.

Proof of Lemma 2: From Lemma 1 we know that the first part of the lemma is true.

Without any loss of generality we will give the proof for $t_0 = 0$. We will also assume that

$$\begin{array}{rcl} c_{N_i} & = & \mathrm{Id} \quad \text{for } i = 0, \ldots, n \\ f_i & = & f_{i,c} \quad \text{for } i = 1, \ldots, n \\ |N_i| & = & N_{i,c} \quad N_i^{\pm} = N_{i,c}^{\pm}. \end{array}$$

Let us define a function

$$\begin{array}{rcl} g & : & N_n \to V_0 \\ g := & & (A_{n+1}, 0) \end{array}$$

where $A_{n+1} : R^u \to R^u$ is any linear map such that $A_{n+1}(\partial B_u(0,1)) \subset R^u \backslash \overline{B_u}(0,1)$. Clearly we have

$$N_n \overset{g}{\Longrightarrow} N_0$$

This artificial function will be needed to close the loop of covering relations (compare Thm. 1), so that it is possible to define the function (2.20) later on.

Let us define functions

$$\begin{array}{rcl} F_i & : & N_{i-1} \times \mathbb{R} \to V_i \times \mathbb{R} \quad \text{for } i = 1, \ldots, n \\ F_i(x, T) := & & (f_i^T(x), \sigma_i(x, T)). \end{array}$$

If we start from the set N_0, then from (2.8) we know that there exists $s_1, s_2, \ldots s_n$ and $r_1, r_2, \ldots, r_n$ such that

$$\begin{array}{rcl} F_1(N_0, 0) & \subset & V_1 \times \mathrm{int} I_1 \\ F_2(N_1, I_1) & \subset & V_2 \times \mathrm{int} I_2 \\ & & \ldots \\ F_n(N_{n-1}, I_{n-1}) & \subset & V_n \times \mathrm{int} I_n \end{array} \tag{2.18}$$

where

$$I_j = [s_j - r_j, s_j + r_j]. \tag{2.19}$$

Let us define

$$XN := (N_0 \times [-1, 1]) \times (N_1 \times I_1) \times \ldots \times (N_n \times I_n)$$

and

$$F : XN \to (\mathbb{R}^k \times \mathbb{R})^{n+1} \tag{2.20}$$

$$\begin{array}{rcl} F((x_0, t_0), \ldots, (x_{n-1}, t_{n-1})) = ((x_0 - g(x_n) & , & t_0), \\ (x_1 - f_1^{t_0}(x_0) & , & t_1 - \sigma_1(x_0, t_0)), \\ & & \ldots, \\ (x_n - f_n^{t_{n-1}}(x_{n-1}) & , & t_n - \sigma_n(x_{n-1,} t_{n-1}))) \end{array}$$

We will show that there exists an $x = ((x_0, t_0), \ldots, (x_n, t_n)) \in \operatorname{int} XN$ such that $F(x) = 0$. Once we find the x, we will have our $x_0, t_0, \ldots, t_n$, because from the definition we know that for $i = 1, \ldots, n$

$$f_i^T(x_0) = x(\sigma_i(x_0, T))$$

and from the fact that $F(x) = 0$ we shall have

$$\begin{aligned} x_i - x(t_i) &= x_i - x(\sigma_i(x_{i-1}, t_{i-1})) \\ &= x_i - f_i^{t_{i-1}}(x_{i-1}) \\ &= 0 \end{aligned}$$

which will mean that

$$x(t_i) \in \operatorname{int} N_i \quad \text{for } i = 1, \ldots, n$$

What is more, from our construction and the fact that $F(x) = 0$ we will know that $t_0 = 0$ and that

$$t_i - \sigma(x_{i-1}, t_{i-1}) = 0 \tag{2.21}$$

which means that

$$0 = t_0 < t_1 < \ldots < t_n.$$

Our goal is therefore to find the $x \in \operatorname{int} XN$ for which $F(x) = 0$. Let us define a homotopy

$$H : [0,1] \times XN \to (\mathbb{R}^k \times \mathbb{R})^{n+1}$$

$$\begin{aligned} H(\lambda, (x_0, t_0), \ldots, (x_{n-1}, t_{n-1})) \;=\;& ((x_0 - G(\lambda, t_n, x_n), t_0), \\ & (x_1 - H^1(\lambda, t_0, x_0), t_1 - \lambda s_1 - (1-\lambda)\sigma(x_0, t_0)), \\ & , \ldots, \\ & (x_n - H^n(\lambda, t_{n-1}, x_{n-1}), t_n - \lambda s_n \\ & -(1-\lambda)\sigma(x_{n-1}, t_{n-1}))) \end{aligned}$$

where for $i = 1, \ldots, n$, H^i is the homotopy from the Lemma 1 and $G(\lambda, \cdot, \cdot) = (A_{n+1}, 0)$. Let us note that $H(0, x) = F(x)$ and that

$$H(1, x) = B(x - ((0,0), (0, s_1), \ldots, (0, s_n))) \tag{2.22}$$

where

$$\begin{aligned} B((x_0, t_0), \ldots, (x_{n-1}, t_{n-1})) \;=\;& (((p_0, q_0) - (A_{n+1} p_n, 0), t_0), \\ & ((p_1, q_1) - (A_1\, p_0, 0), t_1), \\ & \ldots, \\ & ((p_n, q_n) - (A_n\, p_{n-1}, 0), t_n)) \end{aligned}$$

$$x_i = (p_i, q_i) \quad \text{for } i = 0, \ldots, n$$

Let us assume that we have the following two lemmas which we will prove after completing this proof

Lemma 3

$$\textit{deg}(H(1,\cdot), \textit{int}XN, 0) = \pm 1$$

Lemma 4 *For all $\lambda \in [0,1]$ the local Brouwer degree deg($H(\lambda,\cdot)$,int$XN,0$) is defined, constant and independent from λ.*

Let us now complete our proof using the two lemmas. From Lemmas 3 and 4 we know that

$$\deg(F, \mathrm{int}XN, 0) = \deg(H(0,\cdot), \mathrm{int}XN, 0) = \deg(H(1,\cdot), \mathrm{int}XN, 0) = \pm 1$$

which means that there exists an $x \in \mathrm{int}XN$ such that $F(x) = 0$,
$x = ((x_0, t_0), \ldots, (x_n, t_n))$ hence we have found our $x_i \in \mathrm{int}\, N_i$ and t_i. ∎

Now to finish of the argument, let us prove the Lemmas 3 and 4.
Proof of Lemma 3: From (2.22) we know that

$$H(1,x) = B(x - ((0,0),(0,s_1),\ldots,(0,s_n)))$$

where B is linear. From the degree for affine maps (4.2) we have

$$\deg(H(1,\cdot), \mathrm{int}XN, 0) = \mathrm{sgn}(\det B)$$

which means that to prove the lemma it is sufficient to show that B is an isomorphism. Let us recall the definition of B.

$$\begin{aligned} B((x_0,t_0),\ldots,(x_{n-1},t_{n-1})) &= (((p_0,q_0) - (A_{n+1}p_n, 0), t_0), \\ &\quad ((p_1,q_1) - (A_1\, p_0, 0), t_1), \\ &\quad \ldots, \\ &\quad ((p_n,q_n) - (A_n\, p_{n-1}, 0), t_n)) \end{aligned}$$

$$x_i = (p_i, q_i) \quad \text{for } i = 0,\ldots,n-1$$

We have to show that $B(x) = 0$ implies $x = 0$. If $B(x) = 0$ then

$$\begin{aligned} t_0 &= t_1 = \ldots = t_n = 0 \\ q_0 &= q_1 = \ldots = q_n = 0. \end{aligned}$$

We also know that

$$\begin{aligned} p_0 &= A_{n+1}p_n \\ p_1 &= A_1 p_0 \\ &\ldots \\ p_n &= A_n p_{n-1} \end{aligned} \tag{2.23}$$

which means that

$$p_0 = A_{n+1} \circ \ldots \circ A_1 p_0$$

The condition (2.2) implies that $\|A_i p\| > \|p\|$ for $i = 1, \ldots n+1$ and $p \neq 0$, which gives us $p_0 = 0$. The fact that $p_1 = \ldots = p_n = 0$ follows from (2.23). ∎

Proof of Lemma 4: From the homotopy property, it is sufficient to show that

$$H(\lambda, x) \neq 0, \quad \text{for all } x \in \partial XN \text{ and } \lambda \in [0,1]. \tag{2.24}$$

We will consider an x from the boundary of XN
$x = ((x_0, t_0), \ldots, (x_n, t_n))$. If $x \in \partial XN$ then there exists an i such that one of the following conditions holds

$$x_i \in N_i^+ \tag{2.25}$$
$$x_i \in N_i^- \tag{2.26}$$
$$t_i \in \{s_i - r_i, s_i + r_i\} \tag{2.27}$$

First let us consider the case (2.25). For $i = 1, \ldots n$ if $x_i \in N_i^+$ and $H(\lambda, x) = 0$ then in particular

$$x_i - H^i(\lambda, t_{i-1}, x_{i-1}) = 0 \tag{2.28}$$

From the statement of Lemma 1, condition (2.14) we know that

$$H^i([0,1], t_{i-1}, N_{i-1}) \cap N_i^+ = \emptyset.$$

This and the fact that $x_{i-1} \in N_{i-1}$ contradicts (2.28). We therefore know that (2.25) does not hold for $i = 1, \ldots, n$. For $i = 0$ if $x_0 \in N_0^+$ and $H(\lambda, x) = 0$ then

$$\begin{aligned} x_0 - G(\lambda, t_n, x_n) &= 0 \\ (p_0, q_0) - (A_{n+1}, 0) &= 0 \end{aligned}$$

which means that $q_0 = 0$ which contradicts the fact that $x_0 = (p_0, q_0) \in N_0^+ = \overline{B_u}(0,1) \times \partial \overline{B_s}(0,1)$.

Let us now consider the case (2.26). For $i = 0, \ldots, n-1$ if $x_i \in N_i^-$ and $H(\lambda, x) = 0$ then

$$x_{i+1} - H^{i+1}(\lambda, t_i, x_i) = 0 \tag{2.29}$$

From Lemma 1, condition (2.13) we have

$$H^{i+1}([0,1], t_i, N_i^-) \cap N_{i+1} = \emptyset,$$

which contradicts (2.29). Condition (2.26) cannot hold for $i = 0, \ldots, n-1$. For $i = n$ if $x_n \in N_n^-$ and $H(\lambda, x) = 0$ then

$$\begin{aligned} x_0 - G(\lambda, t_n, x_n) &= 0 \\ (p_0, q_0) - (A_{n+1} p_n, 0) &= 0 \end{aligned}$$

The fact that $x_n \in N_n^-$ means that $p_n \in \partial\overline{B_u}(0,1)$. We know that $p_0 \in \overline{B_u}(0,1)$ and $p_0 = A_{n+1}p_n$, which contradicts the fact that $A_{n+1}(\partial\overline{B_u}(0,1)) \subset R^u\backslash\overline{B_u}(0,1)$.

We are now left with the case (2.27). For $i = 1,\ldots,n$ if (2.27) holds and $H(\lambda, x) = 0$ then in particular

$$t_i - \lambda s_i - (1-\lambda)\sigma_i(x_{i-1}, t_{i-1}) = 0. \tag{2.30}$$

Our construction of XN (2.18) which guarantees that

$$F_i(N_{i-1}, I_{i-1}) \subset V_i \times \mathrm{int} I_i.$$

gives us

$$\begin{array}{rcl} \sigma_i(x_{i-1}, t_{i-1}) & \in & (s_i - r_i, s_i + r_i) \\ \lambda s_i + (1-\lambda)\sigma_i(x_{i-1}, t_{i-1}) & \in & (s_i - r_i, s_i + r_i) \end{array}$$

and therefore from (2.27)

$$t_i - \lambda s_i - (1-\lambda)\sigma_i(x_{i-1}, t_{i-1}) \neq 0$$

This clearly contradicts (2.30). For $i = 0$ from the definition of $H(\lambda, \cdot)$ and the fact that $H(\lambda, x) = 0$ we get straight away the fact that $t_0 = 0$ which means that it is not possible for $t_0 \in \{-1, 1\}$.

We have shown that for any $\lambda \in [0,1]$ and $x \in \partial XN$, $H(\lambda, x) \neq 0$, this fact and the homotopy property of the index concludes our proof. ∎

Proof of Theorem 2: Let us consider the sequence of covering relations from Γ

$$N_0 \overset{f_{01}}{\Longrightarrow} N_1 \overset{f_{12}}{\Longrightarrow} N_2 \overset{f_{23}}{\Longrightarrow} \ldots \tag{2.31}$$

Let us consider a finite subsequence of the sequence (2.31)

$$N_0 \overset{f_{01}}{\Longrightarrow} N_1 \overset{f_{12}}{\Longrightarrow} \ldots \overset{f_{m-1\,m}}{\Longrightarrow} N_m$$

From Lemma 2 we know that for $|\epsilon| < \delta$ there exists $x_m \in N_0$ and a sequence

$t_0 = t_0^m < t_1^m < \ldots < t_m^m$, such that for the solution x of the equation (2.10) we have

$$x(t_i) \in \mathrm{int} N_i \quad \text{for } i = 1,\ldots,m$$

and that δ depends only on the family Γ and not on the length of the sequence. We therefore have a sequence $\{x_m\}_{m=1}^{\infty} \subset N_0$. Since N_0 is compact there exists a subsequence x_{m_k} which converges to a certain $x_0 \in N_0$. In the course of the proof of Lemma 2 we have shown that (2.21)

$$t_i^m - \sigma_i(x(t_{i-1}^m), t_{i-1}^m) = 0$$

which together with the fact from (2.8), that $\sigma_i(x(t^m_{i-1}), t^m_{i-1}) - t^m_{i-1}$ is bounded, means that $t^m_i - t^m_{i-1} = \sigma_i(x(t^m_{i-1}), t^m_{i-1}) - t^m_{i-1}$ is bounded. From this fact and from the continuity of the solution of the problem

$$x'(t) = v(x(t)) + \epsilon(t, x)$$

with respect to the initial conditions, it follows that the solution $x(t)$, of the problem

$$\begin{aligned} x' &= v(x) + \epsilon(t, x) \\ x(0) &= x_0 \end{aligned} \tag{2.32}$$

passes through the sets $N_0, N_1, \ldots$ and therefore there exists a sequence $t_0 < t_1 < \ldots$ such that

$$x(t_i) \in \mathrm{int} N_i \quad \text{for } i = 1, 2, \ldots$$

∎

3 Application to Rössler equations.

In this section we combine Theorem 2 and results from [Z1] to show that small non-autonomous perturbations of Rössler [R] posses symbolic dynamics.

First we need to recall some definitions.

Let k be a positive integer. Let $\Sigma_k := \{0, 1, \ldots, k-1\}^{\mathbb{Z}}$, $\Sigma^+_k := \{0, 1, \ldots, k-1\}^{\mathbb{N}}$. Σ_k, Σ^+_k are topological spaces with the Tichonov topology. On Σ_k, Σ^+_k we have the shift map σ given by

$$(\sigma(c))_i = c_{i+1}$$

Let $A = [\alpha_{ij}]$ be a $k \times k$-matrix,
$\alpha_{ij} \in \mathbb{R}_+ \cup \{0\}$, $i, j = 0, 1, \ldots, k-1$. We define $\Sigma_A \subset \Sigma_k$ and $\Sigma^+_A \subset \Sigma^+_k$ by

$$\Sigma_A := \{c = (c_i)_{i \in \mathbb{Z}} \quad | \quad \alpha_{c_i c_{i+1}} > 0\} \tag{3.1}$$

$$\Sigma^+_A := \{c = (c_i)_{i \in \mathbb{N}} \quad | \quad \alpha_{c_i c_{i+1}} > 0\} \tag{3.2}$$

Obviously Σ^+_A, Σ_A are invariant under σ.

Let $F : X \to X$ be any continuous map and $N \subset X$. By $F_{|N}$ we will denote the map obtained by restricting the domain of F to the set N. The maximal invariant part of N (with respect to F) is defined by

$$\mathrm{Inv}(N, F) = \bigcap_{i \in \mathbb{Z}} F^{-i}_{|N}(N).$$

The Rössler equations are given by [R]

$$\begin{aligned} \dot{x} &= -(y + z) \\ \dot{y} &= x + by \\ \dot{z} &= b + z(x - a) \end{aligned} \tag{3.3}$$

where $a = 5.7, b = 0.2$. These are parameters values originally considered by Rössler. The flow generated by Eq. (3.3) exhibits a so-called strange attractor.

We will investigate the Poincaré map P generated by (3.3) on the section $\Theta := \{(x,y,z) | \quad x = 0, y < 0, \dot{x} > 0\}$.

The following result was proved in [Z1] (see also [Z3])

Theorem 3 *For all parameter values in sufficiently small neighborhood of* $(a,b) = (5.7, 0.2)$ *there exists Poincaré section* $N \subset \Theta$ *such that the Poincaré map* P *induced by Eq. (3.3) is well defined and continuous.*

There exists continuous map $\pi : Inv(N,P) \to \Sigma_3$*, such that*

$$\pi \circ P = \sigma \circ \pi.$$

$\Sigma_A \subset \pi(Inv(N,P))$*, where*

$$A := \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

The preimage of any periodic sequence from Σ_A *contains periodic points of* P*.*

Above theorem is a consequence of Theorem 1 and the following Lemma, which was established in [Z1] with computer assistance (computer assisted proof)

Lemma 5 *There are h-sets* $N_0, N_1, N_2 \subset \Theta$ *such that for all parameter values in sufficiently small neighborhood of* $(a,b) = (5.7, 0.2)$ $N \subset Dom(P)$ *and the following conditions hold*

$$N_0 \overset{P}{\Longrightarrow} N_2, \quad N_1 \overset{P}{\Longrightarrow} N_0, N_1, \quad N_2 \overset{P}{\Longrightarrow} N_0, N_1 \tag{3.4}$$

Let us denote by $R_{a,b} : \mathbb{R}^3 \to \mathbb{R}^3$ the vector field on the right-hand side of (3.3). By applying Theorem 2 to Lemma 5 we immediately obtain the following

Theorem 4 *Let us fix* $(a,b) = (5.7, 0.2)$*. Let* A *be as in Theorem 3. Consider a non-autonomous perturbation of (3.3)*

$$x' = R_{a,b}(x) + \epsilon(t,x), \tag{3.5}$$

where $\epsilon(t,x)$ *is continuous with respect to* (t,x) *and is locally Lipschitz with respect to the* x *variable.*

There exists $\delta > 0$*, such that for any* $t_0 \in \mathbb{R}$ *and any sequence* $c = (c_i) \in \Sigma_A^+$ *there exists a solution of (3.5),* $x_c : [t_0, \infty) \to \mathbb{R}^3$ *and a sequence* $t_0 < t_1 < t_2 < \cdots < t_n < t_{n+1} < \ldots$*, such that*

$$\begin{array}{rcll} x_c(t) & \in & \Theta, & \text{iff } t = t_i \text{ for some } i \\ x_c(t_i) & \in & |N_{c_i}|. & \end{array}$$

Above theorem says nothing about the size of δ. To obtain a numerical value for δ one can take one of two approaches

analytical from the computer assisted proof in [Z1] one can obtain global bounds $Z \subset \mathbb{R}^3$, Z compact, such that all trajectories linking $|N_i|$ with its Poincaré image are in Z. For ϵ sufficiently small the same will be true for (3.5). Now using bounds for the Poincaré return times on $|N_i|$ we can compute an upper bound of the distance between the solution of (3.3) and (3.5). Then we compute ϵ for which the covering relations listed in Lemma 5 survive.

computational we can replace (3.5) by a differential inclusion

$$x' \in R_{a,b}(x) + [-\delta, \delta]^3. \tag{3.6}$$

Now for various values of δ we can perform an rigorous integration of (3.6) looking for the largest possible δ for which the covering relations listed Lemma 5 are satisfied (for any continuous selector). For an algorithm for rigorous integration of differential inclusions see [Z4].

3.1 Other examples.

There are several other computer assisted proofs of the existence of nontrival symbolic dynamics which are based on Theorem 1. Among these are the results for the following systems

- Lorenz equations [GaZ],
- Chua circuit [G],
- Kuramoto-Shivashinsky ODE [W].

These results would also give rise to theorems analogous to Theorem 4.

4 Appendix. Properties of the local Brouwer degree

Homotopy property. [L] Let $H : [0,1] \times D \to R^n$ be continuous. Suppose that

$$\bigcup_{\lambda\in[0,1]} H_\lambda^{-1}(c) \cap D \quad \text{is compact} \tag{4.1}$$

then

$$\forall \lambda \in [0,1] \quad \deg(H_\lambda, D, c) = \deg(H_0, D, c)$$

If $[0,1] \times \overline{D} \subset \mathrm{dom}(H)$ and $\overline{D}$ is compact, then (4.1) follows from the condition

$$c \notin H([0,1], \partial D)$$

Degree property for affine maps. [L] Suppose that $f(x) = B(x-x_0)+c$, where B is a linear map and $x_0 \in R^n$. If the equation $B(x) = 0$ has no nontrivial solutions (i.e if $Bx = 0$, then $x = 0$) and $x_0 \in D$, then

$$\deg(f, D, C) = \mathrm{sgn}(\det B). \tag{4.2}$$

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