Topological entropy for multidimensional perturbations of snap-back repellers and one-dimensional maps

Ming-Chia Li*, Ming-Jiea Lyu†, and Piotr Zgliczyński‡

Abstract

We consider a one-parameter family of maps F_λ on $\mathbb{R}^m \times \mathbb{R}^n$ with the singular map F_0 having one of the two forms (i) $F_0(x, y) = (f(x), g(x))$, where $f : \mathbb{R}^m \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^n$ are continuous; and (ii) $F_0(x, y) = (f(x), g(x, y))$, where $f : \mathbb{R}^m \to \mathbb{R}^m$ and $g : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous and g is locally trapping along the second variable y. We show that if f is one-dimensional and has a positive topological entropy, or if f is high-dimensional and has a snap-back repeller, then F_λ has a positive topological entropy for all λ close enough to 0.

1 Introduction

In this paper, we consider multidimensional perturbations from a continuous map f on a low-dimensional phase space, say $\mathbb{R}^m$, to a continuous family of maps F_λ on a high-dimensional space, say $\mathbb{R}^m \times \mathbb{R}^n$, where $\lambda \in \mathbb{R}^\ell$ is a parameter, such that at $\lambda = 0$, the singular map F_0 is one of the following forms:

(i) $F_0(x, y) = (f(x), g(x)) \in \mathbb{R}^m \times \mathbb{R}^n$;

(ii) $F_0(x, y) = (f(x), g(x, y)) \in \mathbb{R}^m \times \mathbb{R}^n$ and $g(\mathbb{R}^m \times S) \subset int(S)$ for some compact set $S \subset \mathbb{R}^n$ homeomorphic to the closed unit ball in $\mathbb{R}^n$; here $int(S)$ denotes the interior of S.

Let $h_{\text{top}}(\varphi)$ denote the supremum of topological entropies of a map φ restricted to compact invariant sets. The basic question we study here is the following:

(#) *If $h_{\text{top}}(f) > 0$, will $h_{\text{top}}(F_\lambda) > 0$ for λ near 0?*

*Department of Applied Mathematics, National Chiao Tung University, 1001 Ta Hsueh Road, Hsinchu 300, TAIWAN, Tel: +866-3-5712121 ext. 56463, Fax: +866-3-5131223, E-mail: mcli@math.nctu.edu.tw

†Department of Applied Mathematics, National Chiao Tung University, 1001 Ta Hsueh Road, Hsinchu 300, TAIWAN

‡Institute of Computer Science, Jagiellonian University, Nawojki 11, 30-072 Krakow, POLAND

In the present paper, we establish two kind of results addressing question (#). First we show that if f is one-dimensional (without any other additional assumption) then $\liminf_{\lambda\to 0} h_{\text{top}}(F_\lambda) \geq h_{\text{top}}(f)$ (see Theorems 1 and 2). Second, we allow f to be possibly high-dimensional and show that if f has a snap-back repeller (for a discussion of its definition see Definition 3 and remarks in the next section) then $h_{\text{top}}(F_\lambda) > 0$ for all λ near enough 0 (see Theorems 4 and 5).

Our methodology is based on the concept of covering relations (see Section 3 for the definition and basic properties), which was introduced by Zgliczyński in [11, 12]. It allows to prove the existence of periodic points, the symbolic dynamics, and the positive topological entropy without using hyperbolicity. As a by-product of using such a method, we give a new proof of Blanco Garcia's result in [1] that the existence of a snap-back repeller implies the positive topological entropy (see Proposition 15). It is also possible that the notion of snap-back repeller can be changed by other structure, such as a hyperbolic horseshoe, in order to obtain similar results.

Let us compare our results to the existing literature. Assuming that f is one-dimensional (i.e., $m = 1$) and some additional conditions are satisfied, affirmative answers to question (#) have been given in literature. For the case when f is an interval map and $g = 0$, Misiurewicz and Zgliczyński in [8] proved that $\liminf_{\lambda\to 0} h_{\text{top}}(F_\lambda) \geq h_{\text{top}}(f)$. They used the covering relation approach in the same way as we use in the present paper.

For the planar case (i.e., $m = n = 1$), Marotto in [6] restricted perturbations to the two types: one is that $F_\lambda(x, y) = (\varphi(x, \lambda y), x)$ and $\lambda \in \mathbb{R}$, and the one that is $F_\lambda(x, y) = (\varphi(x, \lambda_1 y), g(\lambda_2 x, y))$, $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$, and the map $y \mapsto g(0, y)$ has a stable fixed point. Assuming the map $x \mapsto \varphi(x, 0)$ is $C^1$and has a snap-back repeller (for a discussion of its definition see Definition 3 and remarks in the next section), he showed that for all λ near 0, the map F_λ has a transverse homoclinic point. His method relies heavily on the planar structure of the map F_0 and the Birkhoff-Smale transverse homoclinic point theorem.

The results from [2] and [4] about difference equations can be applied to question (#), but these are in fact perturbations of one-dimensional maps.

Our results are applicable to a high-dimensional version of the Hénon-like maps. Define a family of maps $H_b(x, y)$ on $\mathbb{R}^m \times \mathbb{R}^n$, with parameter $b \in \mathbb{R}^\ell$, by its components, for $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_n)$,

$$\begin{cases} \bar{x}_i = a_i - x_i^2 + o_i(b)\varphi_i(x, y), & 1 \leq i \leq m, \\ \bar{y}_j = g_j(x, y), & 1 \leq j \leq n, \end{cases}$$

where each a_i is a constant, o_i, φ_i, g_j are real-valued continuous functions, and $\lim_{b\to 0} o_i(b)/|b| = 0$. If $m = n = 1$, one can reduce H_b to the original Hénon map $(x, y) \mapsto (a - x^2 + by, x)$ and apply results from the present paper as well as from [2, 4, 6]. For the general case when $m \geq 1$ and $n \geq 1$, we assume that each g_j is either dependent only on x or bounded (hence, the conditions in form (i) or (ii) are satisfied, respectively). At the singular value $b = 0$, the first m components of H_0, i.e., $\bar{x}_i = a_i - x_i^2$ for $1 \leq i \leq m$, form a decoupled map from $\mathbb{R}^m$ into itself, and such a map has a positive topological entropy or a snap-back repeller by suitably choosing a_i's. By applying results presented in this paper, we get that $h_{\text{top}}(H_b) > 0$ for all b sufficiently near 0. Nevertheless, if $m > 1$ (the high-dimensional case), we can not apply to H_b the results in [2, 4, 6, 8]. Even when $m = 1$, we can not apply those results neither for many situations: more precisely, in [8] if one of g_j's is not the zero function, in [6] if one of g_j's depends on the variable y, and in [2, 4] if each coordinate of the full orbits of H_b is not reduced to solutions of a difference equation.

This paper is organized as follows. In the next section, we give precise statement of our main results along with a definition of snap-back repellers. In Section 3, we present background information about covering relations, mainly from the work of Zgliczyński and Gidea in [13]. In Section 4, we prove our results concerning a one-dimensional map with a positive topological entropy (Theorems 1 and 2). Then, in Section 5, we show that the existence of a snap-back repeller implies the existence of two closed loops of covering relations, as well as a positive topological entropy (Proposition 15). Finally, in Section 6, we prove our results concerning a high-dimensional map with a snap-back repeller (Theorems 4 and 5).

2 Definitions and statement of main results

In this section, we state our main results and define snap-back repellers. First, we consider multidimensional perturbations of a one-dimensional map f. If the singular map F_0 depends only on the phase variable of f (refer to form (i) in Section 1), we have the following result.

Theorem 1. *Let F_λ be a one-parameter family of continuous maps on $\mathbb{R} \times \mathbb{R}^n$ such that $F_\lambda(x, y)$ is continuous as a function jointly of $\lambda \in \mathbb{R}^\ell$ and $(x, y) \in \mathbb{R} \times \mathbb{R}^n$. Assume that $F_0(x, y) = (f(x), g(x))$ for all $(x, y) \in \mathbb{R} \times \mathbb{R}^n$, where $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}^n$. Then $\liminf_{\lambda\to 0} h_{top}(F_\lambda) \geq h_{top}(f)$.*

For the case when the singular map is locally trapping along the normal direction (refer to form (ii) in Section 1), we have the following.

Theorem 2. *Let F_λ be a one-parameter family of continuous maps on $\mathbb{R} \times \mathbb{R}^n$ such that $F_\lambda(x, y)$ is continuous as a function jointly of $\lambda \in \mathbb{R}^\ell$ and $(x, y) \in \mathbb{R} \times \mathbb{R}^n$. Assume that $F_0(x, y) = (f(x), g(x, y))$ for all $(x, y) \in \mathbb{R} \times \mathbb{R}^n$, where $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, and $g(\mathbb{R} \times S) \subset int(S)$ for some compact set $S \subset \mathbb{R}^n$ homeomorphic to the closed unit ball in $\mathbb{R}^n$. Then $\liminf_{\lambda\to 0} h_{top}(F_\lambda) \geq h_{top}(f)$.*

Next, we consider multidimensional perturbations of a map on a space of dimension possibly bigger than one. Recently, Marotto [7] redefined snap-back repellers and stated that his earlier result in [5] that the existence of a snap-back repeller implies Li-Yorke chaos is still correct. Both definitions of snap-back repellers in [5] and [7] depend on the norms of the phase space. In the following, we give a slightly different definition so that it is independent of norms defined on the phase space.

Definition 3. *Let $f : \mathbb{R}^m \to \mathbb{R}^m$ be a C^1 function. A fixed point x_0 for f is called a* snap-back repeller *if (i) all eigenvalues of the derivative $df(x_0)$ are greater than one in absolute value and (ii) there exists a sequence $\{x_{-i}\}_{i\in\mathbb{N}}$ such that $x_{-1} \neq x_0$, $\lim_{i\to\infty} x_{-i} = x_0$, and for all $i \in \mathbb{N}$, $f(x_{-i}) = x_{-i+1}$ and $\det(df(x_{-i})) \neq 0$.*

Roughly speaking, a snap-back repeller of a map is a repelling fixed point associated with a transverse homoclinic orbit. Notice that if there exists a norm $\|\cdot\|_*$ on $\mathbb{R}^m$ such that for some constants $r > 0$ and $\rho > 1$, one has that $\|f(x) - f(y)\|_* > \rho\|x - y\|_*$ for all $x, y \in B(x_0, r)$, where $B(x_0, r) = \{x \in \mathbb{R}^m : \|x - x_0\|_* < r\}$, then f is one-to-one on $B(x_0, r)$ and $f(B(x_0, r)) \supset B(x_0, r)$; hence item (ii) of the above definition is satisfied provided that there is a point $q \in B(x_0, r)$ such that $f^k(q) = x_0$ and $\det(df^k(q)) \neq 0$ for some positive integer k. In fact, item (i) implies that such a norm must exist (refer to Theorem V.6.1 of Robinson [10]). Furthermore, if all eigenvalues of $(df(x_0))^T df(x_0)$ are greater than 1, then such a norm can be chosen to be the Euclidean norm on $\mathbb{R}^m$ (see Lemma 5 of Li and Chen [3]).

If the singular map depends only on the phase variable of a snap-back repeller, we have the following result.

Theorem 4. *Let F_λ be a one-parameter family of continuous maps on $\mathbb{R}^m \times \mathbb{R}^n$ such that $F_\lambda(x, y)$ is continuous as a function jointly of $\lambda \in \mathbb{R}^\ell$ and $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$. Assume that $F_0(x, y) = (f(x), g(x))$ for all $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$, where $f : \mathbb{R}^m \to \mathbb{R}^m$ is C^1 and has a snap-back repeller and $g : \mathbb{R}^m \to \mathbb{R}^n$. Then F_λ has a positive topological entropy for all λ sufficiently close to* 0.

When the singular map is locally trapping along the normal direction, we have the following.

Theorem 5. *Let F_λ be a one-parameter family of continuous maps on $\mathbb{R}^m \times \mathbb{R}^n$ such that $F_\lambda(z)$ is continuous as a function jointly of $\lambda \in \mathbb{R}^\ell$ and $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$. Assume that $F_0(x, y) = (f(x), g(x, y))$ for all $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$, where $f : \mathbb{R}^m \to \mathbb{R}^m$ is C^1 and has a snap-back repeller, $g : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$, and $g(\mathbb{R}^m \times S) \subset int(S)$ for some compact set $S \subset \mathbb{R}^n$ homeomorphic to the closed unit ball in $\mathbb{R}^n$. Then F_λ has a positive topological entropy for all λ sufficiently close to* 0.

3 Covering relations

In this section, we give the background information about covering relations. First of all, we introduce some notations. Suppose that $\mathbb{R}^k$ has a norm $\|\cdot\|$. For $x \in \mathbb{R}^k$ and $r > 0$, we denote $B_k(x, r) = \{z \in \mathbb{R}^k : \|z - x\| < r\}$, that is, the open ball of radius r centered at the origin 0 in $\mathbb{R}^k$; in short, we write $B_k = B_k(0, 1)$, the open unit ball in $\mathbb{R}^k$. Moreover, for a subset S of $\mathbb{R}^k$, let $\overline{S}$, $int(S)$ and ∂S denote the closure, the interior and the boundary of S, respectively. It will be always clear from the context, which norm is used.

We briefly recall some definitions and results in [13].

Definition 6. *[13, Definition 1] A* h-set *in $\mathbb{R}^k$ is a quadruple consisting of the following data:*

- *a compact subset N of $\mathbb{R}^k$;*
- *a pair of numbers $u(N), s(N) \in \{0, 1, ..., n\}$ with $u(N) + s(N) = k$;*

- *a homeomorphism* $c_N : \mathbb{R}^k \to \mathbb{R}^k = \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)}$ *such that*

$$c_N(N) = \overline{B_{u(N)}} \times \overline{B_{s(N)}}.$$

For simplicity, we will denote such a quadruple by N. Furthermore, we set

$$N_c = \overline{B_{u(N)}} \times \overline{B_{s(N)}}, \quad N_c^- = \partial B_{u(N)} \times \overline{B_{s(N)}}, \quad N_c^+ = \overline{B_{u(N)}} \times \partial B_{s(N)},$$

and

$$N^- = c_N^{-1}(N_c^-), \quad N^+ = c_N^{-1}(N_c^+).$$

A covering relation between two h-sets are defined as follows.

Definition 7. *[13, Definition 6] Let N, M be h-sets in $\mathbb{R}^k$ with $u(N) = u(M) = u$ and $s(N) = s(M) = s$, $f : N \to \mathbb{R}^u \times \mathbb{R}^s$ be a continuous function, $f_c = c_M \circ f \circ c_N^{-1} : N_c \to \mathbb{R}^u \times \mathbb{R}^s$, and w be a nonzero integer. We say N f-*cover *M with degree w, denoted by*

$$N \overset{f,w}{\Longrightarrow} M,$$

if the following conditions are satisfied:

1. *There exists a homotopy $h : [0,1] \times N_c \to \mathbb{R}^u \times \mathbb{R}^s$ such that*

$$h(0, x) = f_c(x) \text{ for } x \in N_c, \tag{1}$$
$$h([0,1], N_c^-) \cap M_c = \emptyset, \tag{2}$$
$$h([0,1], N_c) \cap M_c^+ = \emptyset. \tag{3}$$

2. *There exists a map $A : \mathbb{R}^u \to \mathbb{R}^u$ such that*

$$h(1, p, q) = (A(p), 0) \text{ for } p \in \overline{B_u} \text{ and } q \in \overline{B_s},$$
$$A(\partial B_u) \subset \mathbb{R}^u \setminus \overline{B_u}.$$

3. *The local Brouwer degree of A at 0 in B_u is w; refer to [13, Appendix] for its properties.*

Usually, we will be not interested in the values of w among covering relations and we just write $N \overset{f}{\Longrightarrow} M$ if there exists $w \neq 0$ such that $N \overset{f,w}{\Longrightarrow} M$.

We will need the following two theorems proved by Zgliczyński and Gidea in [13]. The first one says that a closed loop of covering relations implies the existence of a periodic point.

Theorem 8. *[13, Theorem 9] Let N_i for $0 \leq i \leq m$ be h-sets in $\mathbb{R}^k$ such that $N_m = N_0$ and let f_i for $1 \leq i \leq m$ be continuous maps on $\mathbb{R}^k$ such that the covering relations $N_{i-1} \overset{f_i,w_i}{\Longrightarrow} N_i$ with $w_i \neq 0$ for all $1 \leq i \leq m$. Then there exists a point $x \in int(N_0)$ such that*

$$f_i \circ f_{i-1} \circ \cdots \circ f_1(x) \in int(N_i) \text{ for } 1 \leq i \leq m,$$
$$f_m \circ f_{m-1} \circ \cdots \circ f_1(x) = x.$$

The following one shows that a covering relation is persistent under C^0 small perturbations.

Theorem 9. *[13, Theorem 14] Let N, M be h-sets in $\mathbb{R}^k$ such that $u(N) = u(M)$ and $s(N) = s(M)$. Let $f, g : N \to \mathbb{R}^k$ be continuous maps. Assume that $N \overset{f,w}{\Longrightarrow} M$ and that the map c_M satisfies a Lipschitz condition. Then there exists $\varepsilon > 0$ such that if $\|f(x) - g(x)\| < \varepsilon$ for all $x \in N$, then $N \overset{g,w}{\Longrightarrow} M$.*

4 Proofs of Theorems 1 and 2

In this section, we will prove the first two of our main results. To this end, we need the following lemma, which can be easily derived from [9]; see also Theorem 3.1 of Misiurewicz and Zgliczyński in [8]. It says that for continuous interval maps, the positive topological entropy is realized by horseshoes.

Lemma 10. *Let I be a closed interval in $\mathbb{R}$ and $f : I \to I$ be a continuous map with a positive topological entropy, i.e., $h_{top}(f) > 0$. Then there exist sequences $\{s_k\}_{k=1}^{\infty}$ and $\{t_k\}_{k=1}^{\infty}$ of positive integers such that for each $k \in \mathbb{N}$ there exist s_k disjoint closed intervals, $N_1, \cdots, N_{s_k}$, which are h-sets in $\mathbb{R}$ and satisfy the covering relations $N_i \overset{f^{t_k}, w_{i,j}}{\Longrightarrow} N_j$ with $w_{i,j} \in \{-1, 1\}$ for all $1 \le i, j \le s_k$; moreover, one has $\lim_{k\to\infty}(\log(s_k)/t_k) = h_{top}(f)$.*

Now, we are ready to prove the first main result.

Proof of Theorem 1. We only need to consider the case when f has a positive topological entropy. Let δ be an arbitrary number such that $0 < \delta < h_{\text{top}}(f)$. From Lemma 10, there exist $k, p \in \mathbb{N}$ such that f^k has p disjoint closed intervals, denoted by $N_i' = [a_{2i}, a_{2i+1}]$ for $0 \le i \le p-1$ with $a_0 < \cdots < a_{2p-1}$, which are h-sets satisfying

$$N_i' \overset{f^k, w_{i,j}}{\Longrightarrow} N_j' \text{ for } 0 \le i \le p-1 \text{ and } 0 \le j \le p-1.$$

where $w_{i,j} = 1$ or -1, and $\log(p)/k > \delta$.

Set $N' = \cup_{i=0}^{p-1} N_i'$. Since $g \circ f^{k-1}$ is continuous and N' is compact, there exists $r > 0$ such that $g \circ f^{k-1}(N') \subset B_n(0, r)$. Set $N_i = N_i' \times \overline{B_n(0, r)}$ for $0 \le i \le p-1$ and $N = \cup_{i=0}^{p-1} N_i$. Then every N_i is an h-set for $0 \le i \le p-1$ and N is compact in $\mathbb{R} \times \mathbb{R}^n$. For $\lambda = 0$, we have $F_0^k(x, y) = (f^k(x), g \circ f^{k-1}(x))$. Hence there are covering relations:

$$N_i \overset{F_0^k, w_{i,j}}{\Longrightarrow} N_j \text{ for } 0 \le i \le p-1 \text{ and } 0 \le j \le p-1.$$

Since $F_\lambda^k(z)$ is uniformly continuous on a compact set, say $[-1, 1] \times N$, as a function jointly of λ and z, by using Theorem 9 for p^2 times while each c_{N_j} is linear and satisfies the Lipschitz condition, there exists $\lambda_0 > 0$ such that if $|\lambda| < \lambda_0$ then we have

$$N_i \overset{F_\lambda^k, w_{i,j}}{\Longrightarrow} N_j \text{ for } 0 \le i \le p-1 \text{ and } 0 \le j \le p-1.$$

Let m be a positive integer and $|\lambda| < \lambda_0$. Consider any closed loop

$$N_{\alpha_0} \overset{F_\lambda^k}{\Longrightarrow} N_{\alpha_1} \overset{F_\lambda^k}{\Longrightarrow} \cdots \overset{F_\lambda^k}{\Longrightarrow} N_{\alpha_m},$$

where every $\alpha_i \in \{0, 1, \ldots p-1\}$ and $\alpha_m = \alpha_0$. By using Theorem 8, F_λ^k has a periodic point $x = x(\lambda) \in int(N_{\alpha_0})$ such that $F_\lambda^{km}(x) = x$. Since there are p^m choices of such closed loops, F_λ^k has at least p^m periodic points in N. These periodic points provide a (m, ε)-separated set for F_λ^k as long as ε is a positive number less than gaps of N_i''s, i.e., $0 < \varepsilon < \min\{a_{2i} - a_{2(i-1)+1} : 1 \le i \le p-1\}$. Since m is arbitrarily chosen, we have $h_{\text{top}}(F_\lambda^k) \ge \log(p)$ and so $h_{\text{top}}(F_\lambda) \ge \log(p)/k > \delta$. Therefore, $\liminf_{\lambda \to 0} h_{\text{top}}(F_\lambda) \ge h_{\text{top}}(f)$. □

The proof of the second main result is the following.

Proof of Theorem 2. Define $G_\lambda = (id, c) \circ F_\lambda \circ (id, c)^{-1}$, where id denotes the identity map on $\mathbb{R}$ and c is a homeomorphism from S to $\overline{B_n}$. Then the topological entropies of G_λ and F_λ are equal. By applying the above argument to the family G_λ while the corresponding c_M of a covering relation $N \overset{G_\lambda, w}{\Longrightarrow} M$ is the identity now, we have the desired result. □

5 Snap-back repeller and closed loops of covering relations

Throughout this section, we assume that $f : \mathbb{R}^m \to \mathbb{R}^m$ is a C^1 map having a snap-back repeller x_0 associated with a transverse homoclinic orbit. We shall construct two closed loops of covering relations for f: the first one is from the snap-back repeller to a homoclinic point then back to the repeller, and the second one consists of just one relation $N_r \overset{f}{\Longrightarrow} N_r$, where N_r is one of the h-sets in the first closed loop. Then, we use the covering relations approach to prove that f has a positive topological entropy.

Let L be a linearization of f at x_0, that is, $L(z) = x_0 + df(x_0)(z - x_0)$ for $z \in \mathbb{R}^m$. Since all eigenvalues of $df(x_0)$ are greater than one in absolute value, there exist a norm $\|\cdot\|$ on $\mathbb{R}^m$ and a constant $\rho > 1$ such that

$$\|df(x_0)z\| \ge \rho \|z\| \text{ for } z \in \mathbb{R}^m. \tag{4}$$

¿From now on, we keep this norm fixed.

For any $r > 0$ and $x \in \mathbb{R}^m$, we denote the closed ball with the center x and radius r by

$$N(x, r) = \{x\} + \overline{B_m(0, r)}.$$

For any $r > 0$ we define an h-set $N_{x,r}$ in $\mathbb{R}^m$ as follows: we set $N_{x,r} = N(x, r)$, $c_{N_{x,r}}(z) = (z - x)/r$, $u(N_{x,r}) = m$, and $s(N_{x,r}) = 0$. Since the point x_0 is a fixed point for f and will play a distinguished role in the following, we will write N_r instead of $N_{x_0,r}$. Next, we define a homotopy from the map f to L, its linearization at x_0, as follows:

$$f_\mu(z) = (1 - \mu) f(z) + \mu L(z) \text{ for } \mu \in [0, 1] \text{ and } z \in \mathbb{R}^m. \tag{5}$$

It is easy to see that $f_0(z) = f(z)$, $f_1(z) = L(z)$ and $df_\mu(z) = (1-\mu)df(z)+\mu df(x_0)$ for all μ and z. This homotopy will be later used in covering relations in the vicinity of the snap-back repeller.

First, we show the size of the repulsion set for snap-back repeller x_0 can be chosen uniformly for all f_μ for $\mu \in [0,1]$.

Lemma 11. *Let $\beta = (\rho+1)/2$. Then there exists $r_0 > 0$ such that for any $\mu \in [0,1]$, $0 < r \le r_0$, $z \in N_r$ with $\|z - x_0\| = r$, the following holds:*

$$\|f_\mu(z) - x_0\| > \beta r.$$

Proof. By using Taylor's theorem with an integral remainder, we have

$$f_\mu(z) - x_0 = f_\mu(z) - f_\mu(x_0) = C(z - x_0),$$

where

$$C = C(\mu, z, x_0) = \int_0^1 df_\mu(x_0 + t(z - x_0))dt.$$

By (5), we get that

$$\begin{aligned} C - df_\mu(x_0) &= \int_0^1 (1-\mu)df(x_0 + t(z - x_0)) + \mu df(x_0)dt - df_\mu(x_0) \\ &= \int_0^1 (1-\mu)[df(x_0 + t(z - x_0)) - df(x_0)]dt. \end{aligned} \tag{6}$$

Since df is continuous at x_0 and $\rho > 1$, there exists $r_0 > 0$ such that if $\|y - x_0\| \le r_0$ then $\|df(y) - df(x_0)\| < (\rho - 1)/2$. Hence, from (6), we have that for any $\mu \in [0,1]$ and $z \in \overline{B_m(x_0, r_0)}$,

$$\begin{aligned} \|C - df_\mu(x_0)\| &\le \int_0^1 (1-\mu) \|df(x_0 + t(z - x_0)) - df(x_0)\| \, dt \\ &< \int_0^1 (1-\mu)\frac{\rho - 1}{2} dt \le \frac{\rho - 1}{2}. \end{aligned}$$

Therefore, by using (4), we have that for any $\mu \in [0,1]$, $0 < r \le r_0$, $z \in N_r$ with $\|z - x_0\| = r$,

$$\begin{aligned} \|f_\mu(z) - x_0\| &= \|C(z - x_0)\| = \|(C - df_\mu(x_0) + df_\mu(x_0))(z - x_0)\| \\ &\ge \|df(x_0)(z - x_0)\| - \|(C - df_\mu(x_0))(z - x_0)\| \\ &> \rho r - \frac{\rho - 1}{2} r = \beta r. \end{aligned}$$

□

Throughout the rest of this section, we fix the two constants β and r_0 as given in Lemma 11. In the following, we establish a covering relation between two h-sets around the snap-back repeller.

Proposition 12. *Let r and r_1 be two numbers satisfying $0 < r \le r_0$ and $0 < r_1 \le \beta r$. Then the following covering relation holds:*

$$N_r \overset{f}{\Longrightarrow} N_{r_1}$$

Proof. Define $h(\mu, z) = c_{N_{r_1}}(f_\mu(c_{N_r}^{-1}(z))$. We need to check whether all conditions for the covering relation $N_r \overset{f}{\Longrightarrow} N_{r_1}$ are satisfied.

First we deal with the conditions in the first item of Definition 7. Condition (1) is implied by $f_0 = f$, (2) follows from Lemma 11, and since $N_{r_1}^+ = \emptyset$, (3) is also satisfied.

Next, we define a map A on $\mathbb{R}^m$ by $A(z) = (r/r_1)df(x_0)z$. Then for $z \in \overline{B_m}$, we have

$$h(1, z) = \frac{L(rz + x_0) - x_0}{r_1} = \frac{df(x_0)(rz)}{r_1} = A(z).$$

Moreover, from (4) it follows that for $z \in \overline{B_m}$ with $\|z\| = 1$,

$$\|A(z)\| \ge \frac{\rho r}{r_1} \ge \frac{\rho r}{\beta r} > 1.$$

Since A is linear, from the above equation we have that $\deg(A, B_m, 0) = \pm \det(A) \neq 0$. □

Next, we give a covering relation from the snap-back repeller x_0 to points near x_0, which will be homoclinic points near x_0 as the result is used later.

Lemma 13. *Let $r > 0$, $r_1 > 0$, and $z_1 \in \mathbb{R}^m$ near x_0 satisfy that $(\|z_1 - x_0\| + r_1)/\beta < r < r_0$. Then*

$$N_r \overset{f}{\Longrightarrow} N_{z_1, r_1}$$

Proof. As in the proof of Proposition 12, we set $h(\mu, z) = c_{N_{z_1, r_1}}(f_\mu(c_{N_r}^{-1}(z))$. Again, we need to check all conditions for the covering relation $N_r \overset{f}{\Longrightarrow} N_{z_1, r_1}$.

Condition (1) is implied by $f_0 = f$, and since $N_{z_1, r_1}^+ = \emptyset$, (3) is also satisfied.

To verify condition (2), observe that it is equivalent to the following one

$$f_\mu(N_r^-) \cap N_{z_1, r_1} = \emptyset \text{ for } \mu \in [0, 1]. \tag{7}$$

¿From Lemma 11, it follows that for any $z \in N_r^-$ (hence $\|z - x_0\| = r$),

$$\begin{aligned} \|f_\mu(z) - z_1\| &= \|f_\mu(z) - x_0 + x_0 - z_1\| \ge \|f_\mu(z) - x_0\| - \|x_0 - z_1\| \\ &\ge \beta r - \|x_0 - z_1\| > \|x_0 - z_1\| + r_1 - \|x_0 - z_1\| = r_1. \end{aligned}$$

This proves (7).

It remains to investigate $h(1, z)$. Define a map A on $\mathbb{R}^m$ by $A(z) = (rdf(x_0)z + x_0 - z_1)/r_1$. Then A is affine and for $z \in \overline{B_m}$,

$$h(1, z) = \frac{L(rz + x_0) - z_1}{r_1} = \frac{x_0 + df(x_0)(rz) - z_1}{r_1} = A(z).$$

To prove that $\deg(A, B_m, 0) = \det(df(x_0)) = \pm 1$, it is sufficient to show that the unique solution $\hat{z} = (1/r)df(x_0)^{-1}(z_1 - x_0)$ of the equation $A(z) = 0$ is in B_m. To this end, observe that from (4), we have $\|df(x_0)^{-1}\| \leq \rho^{-1}$ and hence

$$\|\hat{z}\| \leq \frac{1}{r}\|df(x_0)^{-1}\| \cdot \|z_1 - x_0\| \leq \frac{\|z_1 - x_0\|}{\rho r} < \frac{\|z_1 - x_0\| + r_1}{\beta r} < 1.$$

□

The following lemma gives a covering relation from a homoclinic point to the snap-back repeller.

Lemma 14. *Assume that $z_0 \in \mathbb{R}^m$ such that $f^k(z_0) = x_0$ for some integer $k > 0$ and $\det(df^k(z_0)) \neq 0$. Then there exists $R > 0$ such that if $0 < r < R$ then there is $v \equiv v(r)$ with $0 < v < r_0$ such that for any $0 < r_2 \leq v$, we have*

$$N_{z_0,r} \overset{f^k}{\Longrightarrow} N_{r_2}. \tag{8}$$

Proof. By continuity of f, there is $R_1 > 0$ such that

$$f^k(\overline{B_m(z_0, R_1)}) \subset B_m(x_0, r_0).$$

Define a homotopy as follows: for $\mu \in [0, 1]$ and $z \in \overline{B_m(z_0, R_1)}$,

$$g_\mu(z) = (1 - \mu)f^k(z) + \mu(df^k(z_0)(z - z_0) + x_0). \tag{9}$$

Then $g_\mu(z_0) = x_0$ and $dg_\mu(z) = (1 - \mu)df^k(z) + \mu df^k(z_0)$ for all μ and z. Since $df^k(z_0)$ is nonsingular, there is a constant $\alpha > 0$ such that for any $z \in \mathbb{R}^m$,

$$\|df^k(z_0)z\| \geq \alpha\|z\|. \tag{10}$$

Next, we show that there exists a positive number $R < \min\{R_1, 2r_0/\alpha\}$ such that for all $\|z - z_0\| < R$ and $\mu \in [0, 1]$, one has

$$\|g_\mu(z) - x_0\| > \frac{\alpha}{2}\|z - z_0\|. \tag{11}$$

To this end, we have to modify a bit the proof of Lemma 11. By using Taylor's theorem with integral remainder, we have

$$g_\mu(z) - x_0 = g_\mu(z) - g_\mu(z_0) = C(z - z_0),$$

where

$$C = C(\mu, z, z_0) = \int_0^1 dg_\mu(z_0 + t(z - z_0))dt.$$

By (9), we get that

$$\begin{aligned} C - dg_\mu(z_0) &= \int_0^1 (1 - \mu)df^k(z_0 + t(z - z_0)) + \mu df^k(z_0)dt - dg_\mu(z_0) \\ &= \int_0^1 (1 - \mu)[df^k(z_0 + t(z - z_0)) - df^k(z_0)]dt \end{aligned} \tag{12}$$

Since df^k is continuous at z_0, there exists $R > 0$ such that if $\|y - z_0\| < R$ then

$$\left\|df^k(y) - df^k(z_0)\right\| < \alpha/2.$$

Hence, from (12), we have that for any $\mu \in [0,1]$ and $z \in B_m(z_0, R)$,

$$\begin{aligned} \|C - dg_\mu(x_0)\| &\leq \int_0^1 (1-\mu) \left\|df^k(z_0 + t(z - z_0)) - df^k(z_0)\right\| dt \\ &< \int_0^1 (1-\mu)\frac{\alpha}{2} dt \leq \frac{\alpha}{2}. \end{aligned}$$

Therefore, by using (10), we obtain that for any $\mu \in [0,1]$ and $z \in B_m(z_0, R)$,

$$\begin{aligned} \|g_\mu(z) - x_0\| &= \|C(z - z_0)\| = \|(C - dg_\mu(z_0) + dg_\mu(z_0))(z - z_0)\| \\ &\geq \left\|df^k(z_0)(z - z_0)\right\| - \|(C - dg_\mu(z_0))(z - z_0)\| \\ &> (\alpha - \frac{\alpha}{2}) \|(z - z_0)\| = \frac{\alpha}{2} \|(z - z_0)\|. \end{aligned}$$

Now we are ready to prove the desired covering relation (8). Let r a number with $0 < r < R$ and let $v = \alpha r/2$. Let r_2 be a number with $0 < r_2 \leq v$. Since $\alpha > 0$ and $R < 2r_0/\alpha$, we have $0 < v < r_0$. We define a homotopy h_μ by

$$h_\mu(z) = c_{N_{r_2}}(g_\mu(c^{-1}_{N_{z_0,r}}(z))) \text{ for } \mu \in [0,1] \text{ and } z \in \overline{B_m}.$$

The conditions from Definition 7 requiring the proof are only (2) and $\deg(h_1, B_m, 0) \neq 0$ while the rest ones are clear. To verify condition (2), notice that it is equivalent to the following one

$$g_\mu(N^-_{z_0,r}) \cap N_{r_2} = \emptyset \text{ for } \mu \in [0,1]. \tag{13}$$

¿From (11), it follows that for any $z \in N^-_{z_0,r}$ (hence $\|z - z_0\| = r$), one has

$$\|g_\mu(z) - x_0\| > \frac{\alpha}{2}\|z - z_0\| > r_2.$$

This proves (13). Finally, since

$$h_1(z) = \frac{r}{r_2} df^k(z_0)z,$$

We obtain that h_1 is a linear isomorphism; therefore $\deg(h_1, B_m, 0) = \det(df^k(z_0)) \neq 0$. □

The next proposition shows that existence of a snap-back repeller defined in Definition 3 implies a positive topological entropy. In [1], Blanco Garcia gave the same result based on Marotto's definition of a snap-back repeller and results in [5]. Here, we give a new proof by using covering relations.

Proposition 15. *The topological entropy of f is positive.*

Proof. Let β and r_0 as given in Lemma 11. Since x_0 is a snap-back repeller for f, there exists a sequence $\{x_{-i}\}_{i\in\mathbb{N}}$ such that $x_{-1} \neq x_0$, $\lim_{i\to\infty} x_{-i} = x_0$, and for all $i \in \mathbb{N}$, $f(x_{-i}) = x_{-i+1}$ and $\det(df(x_{-i})) \neq 0$. Thus, there is an integer $k > 0$ such that $x_{-k} \in B(x_0, r_0)$. By the chain rule, we have $\det(df^k(x_{-k})) \neq 0$. Furthermore, from Lemma 14, there exist positive constants r_k and r_b such that $r_b < r_0$ and

$$\overline{B(x_{-k}, r_k)} \subset B(x_0, r_0), \tag{14}$$

$$N_{x_{-k},r_k} \cap N_{r_b} = \emptyset, \tag{15}$$

$$N_{x_{-k},r_k} \overset{f^k}{\Longrightarrow} N_{r_b}. \tag{16}$$

Since $\beta > 1$, there exists the minimal positive integer a such that $\beta^a r_b > \|x_{-k} - x_0\| + r_k$. By the minimum of a and equation (14), we have $\beta^{a-1} r_b \leq r\|x_{-k} - x_0\| + r_k < r_0$. From Proposition 12 and Lemma 13, it follows that we have the following chain of covering relations:

$$N_{r_b} \overset{f}{\Longrightarrow} N_{\beta r_b} \overset{f}{\Longrightarrow} \cdots \overset{f}{\Longrightarrow} N_{\beta^{a-1} r_b} \overset{f}{\Longrightarrow} N_{x_{-k},r_k}. \tag{17}$$

Moreover, from Proposition 12, it follows also that

$$N_{r_b} \overset{f}{\Longrightarrow} N_{r_b}. \tag{18}$$

These covering relations are enough to produce symbolic dynamics and a positive topological entropy as follows. Let $w = \max(a, k)$. It is sufficient to construct an f^{2w}-invariant set on which f^{2w} can be semi-conjugated onto the shift map $\sigma : \Sigma_2^+ \to \Sigma_2^+$, where $\Sigma_2^+ = \{0, 1\}^{\mathbb{N}}$, the one-sided shift space on two symbols with the standard Tichonov (product) topology. By using equations (16)-(18), one can consider the following chains of covering relations, each one of length $2w$ (which is counted by the number of iterates of f):

$$
\begin{aligned}
&N_{r_b} \overset{f}{\Longrightarrow} N_{r_b} \overset{f}{\Longrightarrow} N_{r_b} \overset{f}{\Longrightarrow} \cdots \overset{f}{\Longrightarrow} N_{r_b},\\
&N_{r_b} \overset{f}{\Longrightarrow} N_{r_b} \overset{f}{\Longrightarrow} \cdots \overset{f}{\Longrightarrow} N_{r_b} \overset{f}{\Longrightarrow} N_{\beta r_b} \overset{f}{\Longrightarrow} \cdots \overset{f}{\Longrightarrow} N_{\beta^{a-1} r_b} \overset{f}{\Longrightarrow} N_{x_{-k},r_k},\\
&N_{x_{-k},r_k} \overset{f^k}{\Longrightarrow} N_{r_b} \overset{f}{\Longrightarrow} N_{r_b} \overset{f}{\Longrightarrow} \cdots \overset{f}{\Longrightarrow} N_{r_b},\\
&N_{x_{-k},r_k} \overset{f^k}{\Longrightarrow} N_{r_b} \overset{f}{\Longrightarrow} \cdots \overset{f}{\Longrightarrow} N_{r_b} \overset{f}{\Longrightarrow} N_{\beta r_b} \overset{f}{\Longrightarrow} \cdots \overset{f}{\Longrightarrow} N_{\beta^{a-1} r_b} \overset{f}{\Longrightarrow} N_{x_{-k},r_k}.
\end{aligned}
$$

Let us denote $N_0 = N_{r_b}$ and $N_1 = N_{x_{-k},r_k}$. Then N_0 and N_1 are disjoint due to (15). Define Z to be the set of points whose forward orbits under f^{2w} stays in $N_0 \cup N_1$, that is,

$$Z = \{z \in N_0 \cup N_1 \mid f^{2iw}(z) \in N_0 \cup N_1 \text{ for all } i \in \mathbb{N}\}.$$

Then Z is compact. On Z we define a projection $\pi : Z \to \Sigma_2^+$ by

$$\pi(z)_i = j \text{ if and only if } f^{2iw}(z) \in N_j.$$

It is obvious that the map π is continuous and we have a semiconjugacy: $\pi \circ f^{2w} = \sigma \circ \pi$.

Finally, we shall show that π is onto. This gives us that the topological entropy of f^{2w} on Z is greater than or equal to $\log 2$. Let $\alpha = (\alpha_0, \ldots, \alpha_{l-1}) \in \{0,1\}^l$ for some positive integer l. By a suitable concatenation of the above listed chains of covering relations and from Theorem 8, it follows that there exists a point $x_\alpha \in N_{\alpha_0}$ such that

$$f^{2iw}(x_\alpha) \in N_{\alpha_i} \text{ for } 0 \le i \le l-1,$$
$$f^{2lw}(x_\alpha) = x_\alpha.$$

It is clear that $x_\alpha \in Z$ and $\pi(x_\alpha) = (\alpha, \alpha, \ldots) \in \Sigma_2^+$. Since α is arbitrarily chosen, the set $\pi(Z)$ contains all repeating sequences. From the density of repeating sequences in Σ_2^+, it follows that $\pi(Z) = \Sigma_2^+$. □

6 Proofs of Theorems 4 and 5

In this section, we combine all the material in the previous section to prove the last two of our main results. First, we assume that all the hypotheses of Theorem 4 are satisfied. We continue using the notations of the previous section. From the proof of Proposition 15, we have a positive integer a such that the following closed loop of covering relations holds:

$$N_{r_b} \overset{f}{\Longrightarrow} N_{r_b} \overset{f}{\Longrightarrow} N_{\beta r_b} \overset{f}{\Longrightarrow} \cdots \overset{f}{\Longrightarrow} N_{\beta^{a-1} r_b} \overset{f}{\Longrightarrow} N_{x_{-k}, r_k} \overset{f^k}{\Longrightarrow} N_{r_b}.$$

By adding the normal direction to the above h-sets and using the persistence of covering relation, we shall construct a closed loop of covering relations for F_λ, similar to the above loop for f. Recall that the singular map F_0 is of the form $F_0(x,y) = (f(x), g(x)) \in \mathbb{R}^m \times \mathbb{R}^n$. Set $N = (\cup_{i=0}^{a-1} N_{\beta^i r_b}) \cup (\cup_{i=0}^{k} f^i(N_{x_{-k}, r_k}))$. Since g is continuous and N is compact, there exists $r > 0$ such that $g(N) \subset B_n(0,r)$. Let us define the corresponding h-sets in $\mathbb{R}^m \times \mathbb{R}^n$ as follows. For $i = 0, 1, \ldots, a-1$, we define h-sets $N'_{\beta^i r_b}$ in $\mathbb{R}^m \times \mathbb{R}^n$ by $N'_{\beta^i r_b} = N_{\beta^i r_b} \times \overline{B_n(0,r)}$, $u(N'_{\beta^i r_b}) = m$, $s(N'_{\beta^i r_b}) = n$, and $c_{N'_{\beta^i r_b}}(x,y) = (c_{N_{\beta^i r_b}}(x), \frac{1}{r} y)$. Moreover, we define an h-set N'_{x_{-k}, r_k} in $\mathbb{R}^m \times \mathbb{R}^n$ by $N'_{x_{-k}, r_k} = N_{x_{-k}, r_k} \times \overline{B_n(0,r)}$, $u(N'_{x_{-k}, r_k}) = m$, $s(N'_{x_{-k}, r_k}) = n$, and $c_{N'_{x_{-k}, r_k}}(x,y) = (c_{N_{x_{-k}, r_k}}(x), \frac{1}{r} y)$.

Observe that we have following closed loop of covering relations for F_0.

Lemma 16. *The following covering relations hold:*

$$N'_{r_b} \overset{F_0}{\Longrightarrow} N'_{r_b} \overset{F_0}{\Longrightarrow} N'_{\beta r_b} \overset{F_0}{\Longrightarrow} \cdots \overset{F_0}{\Longrightarrow} N'_{\beta^{a-1} r_b} \overset{F_0}{\Longrightarrow} N'_{x_{-k}, r_k} \overset{F_0^k}{\Longrightarrow} N'_{r_b}.$$

Proof. For each covering relation under consideration $N' \overset{F_0^j}{\Longrightarrow} M'$ with $j = 1$ or k, we define a homotopy $\hat{h} : [0,1] \times \overline{B_m} \times \overline{B_n} \to \mathbb{R}^{m+n}$ by

$$\hat{h}(\mu, x, y) = (h(\mu, x), \frac{1-\mu}{r} g \circ f^{j-1}(c_N^{-1}(x))).$$

where h is the homotopy from corresponding covering relation $N \overset{f^j}{\Longrightarrow} M$. Then, we have

$$\begin{aligned}\hat{h}(0,x,y) &= (h(0,x), \frac{1}{r} g \circ f^{j-1}(c_N^{-1}(x))) \\ &= (c_M \circ f^j \circ c_N^{-1}(x), \frac{1}{r} g \circ f^{j-1}(c_N^{-1}(x))) = (F_0^j)_c(x,y).\end{aligned}$$

Since $\hat{h}([0,1], N'^{,-}) \subset h([0,1], N^-) \times \mathbb{R}^n$, we get that condition (2) in Definition 7 follows from the analogous condition for h. Condition (3) is satisfied due to

$$\hat{h}([0,1] \times \overline{B_m} \times \overline{B_n}) \subset \mathbb{R}^m \times B_n.$$

Finally, notice that

$$\hat{h}(1,x,y) = (h(1,x), 0).$$

Therefore, the other conditions in Definition 7 are also satisfied. □

¿From Theorem 9, there exists $\lambda_0 > 0$ such that if $|\lambda| < \lambda_0$ then following chain of covering relations holds for F_λ :

$$N'_{r_b} \overset{F_\lambda}{\Longrightarrow} N'_{r_b} \overset{F_\lambda}{\Longrightarrow} N'_{\beta r_b} \overset{F_\lambda}{\Longrightarrow} \cdots \overset{F_\lambda}{\Longrightarrow} N'_{\beta^{a-1} r_b} \overset{F_\lambda}{\Longrightarrow} N'_{x_{-k}, r_k} \overset{F_\lambda^k}{\Longrightarrow} N'_{r_b}. \tag{19}$$

Similar to the proof of Proposition 15, covering relations listed in (19) are sufficient to produce the symbolic dynamics and a positive topological entropy for F_λ with $|\lambda| < \lambda_0$.

This completes the proof of Theorem 4.

For the proof of Theorem 5, define $G_\lambda = (id, c) \circ F_\lambda \circ (id, c)^{-1}$, where id denotes the identity map on $\mathbb{R}^k$ and c is a homeomorphism from S to $\overline{B_n}$. Then the conclusion follows from the above argument applied to G_λ.

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References

[1] Blanco García, C., *Chaos and topological entropy in dimension $n > 1$*, Ergodic Theory Dynam. Systems **6** (1986), 163–165.

[2] Juang, J., Li, M.-C. and Malkin, M., *Chaotic difference equations in two variables and their multidimensional perturbations,* Nonlinearity **21** (2008), 1019-1040.

[3] Li, C. and Chen, G., *An improved version of the Marotto theorem*, Chaos Solitons & Fractals **18** (2003), 69–77.

[4] Li, M.-C. and Malkin, M., *Topological horseshoes for perturbations of singular difference equations*, Nonlinearity, **19** (2006), 795-811.

[5] Marotto, F. R., *Snap-back repellers imply chaos in* $\mathbb{R}^n$, J. Math. Anal. Appl. **63** (1978), 199–223.

[6] Marotto, F. R., *Perturbations of stable and chaotic difference equations.* J. Math. Anal. Appl. 72 (1979), no. 2, 716–729.

[7] Marotto, F. R., *On redefining a snap-back repeller*, Chaos Solitons & Fractals **25** (2005), 25–28.

[8] Misiurewicz, M. and Zgliczyński, P., *Topological entropy for multidimensional perturbations of one-dimensional maps*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **11** (2001), 1443–1446.

[9] Misiurewicz, M. and Szlenk, W., *Entropy of piecewise monotone mappings*, Studia Math. **67** (1980), 45–63.

[10] Robinson, C., *Dynamical systems: Stability, symbolic dynamics, and chaos*, Second edition, CRC Press, Boca Raton, FL, 1999.

[11] Zgliczyński, P., *Fixed point index for iterations, topological horseshoe and chaos*, Topological Methods in Nonlinear Analysis, **8**, 1996, 169–177.

[12] Zgliczyński, P., *Computer assisted proof of chaos in the Rössler equations and in the Hénon map*, Nonlinearity, **10**, (1997), 243–252.

[13] Zgliczyński, P. and Gidea, M., *Covering relations for multidimensional dynamical systems*, J. Differential Equations **202** (2004), 32–58.