FIXED POINT RESULTS BASED ON THE WAŻEWSKI METHOD

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1. Introduction

Topological methods are frequently used in proving results on qualitative properties of differential equations, especially in problems of the existence of solutions satisfying some boundary value data. They are usually based on fixed point theorems or on properties of the Brouwer and Leray-Schauder degrees. The most common approach applies those tools to integral operators corresponding to the considered equations in infinite-dimensional spaces of functions satisfying the prescribed boundary data. Another approach, which is restricted to equations representing some evolution in time, applies them to translation operators along solutions. In the case of ordinary differential equations those operators are finite-dimensional; they are infinite-dimensional if one considers parabolic partial differential equations or delay differential equations. Frequently, the existence of a required solution is a consequence of the fact that the translation operator preserves some compact convex subset of the phase space of the equation, hence the Brouwer or Schauder fixed point theorem applies. In particular, such an approach applies to dissipative equations. Using the Lefschetz fixed point theorem, the same idea can be applied to compact subsets being absolute neighborhood retracts. However, for non-dissipative equations usually there are no reasonable compact subsets which are invariant with respect to the translation operator. The aim of this note is to describe a method which sometimes can be applied in that context. It is based on the concept of isolating segment and applies the Lefschetz fixed point theorem, the fixed point index, and the retract method of Ważewski. It provides results on the existence of periodic solutions, and, after suitable arrangements, also on the existence of solutions of some other two-point boundary value problems, homoclinic and multibump solutions, and chaotic dynamics of various kinds. Our aim is to

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present those results and illustrate them using some concrete equations. Finally, we make some comments on a numerical algorithm leading to construction of isolating segments.

We use a standard notation concerning fixed points; in particular, $\mathrm{Fix}(f)$ denotes the set of fixed points of a map $f\colon U \to X$, where $U \subset X$. If U is open and $\mathrm{Fix}(f)$ is compact then $\mathrm{ind}(f)$ denotes the fixed point index (compare [3]). The singular homology functor with coefficients in the field of rational numbers $\mathbb{Q}$ is denoted by H. The n-th iterate $f \circ \cdots \circ$ of f is denoted by f^n.

2. Local semi-flows and the retract method of Ważewski

In 1947, Tadeusz Ważewski published the paper [17] in which he presented a new topological method of proving the existence of solutions remaining in a given set for positive values of time. Below we briefly describe some basic notions concerning the method. Actually, our presentation will use a contemporary approach to the method which slightly differs from the original one.

Let X be a topological space and let D be an open subset of $X \times [0, \infty)$. A continuous map $\phi\colon D \to X$ is called a *local semi-flow* on X if for every $x \in X$ the set $\{t \in [0, \infty)\colon (x, t) \in D\}$ is equal to an interval $[0, \omega_x)$ for some $\omega_x > 0$ or $\omega_x = \infty$,

$$\phi(x, 0) = x, \tag{1}$$

and if $(x, t) \in D$, $(\phi(x, t), s) \in D$ then $(x, t + s) \in D$ and

$$\phi(x, s + t) = \phi(\phi(x, t), s). \tag{2}$$

We write also $\phi_t(x)$ instead of $\phi(x, t)$, hence $\phi_0 = \mathrm{id}$ and $\phi_{s+t} = \phi_t \circ \phi_s$. Obviously, ϕ is called a *semi-flow* if $D = X \times [0, \infty)$. In a natural way a more restrictive notion notions of *local flow* and *flow* are defined – one should extend the above definition symmetrically to negative values of t. In this case one assumes that the set $\{t \in \mathbb{R}\colon (x, t) \in D\}$ is equal to an open interval (α_x, ω_x) with some α_x and ω_x,

$$-\infty \leq \alpha_x < 0 < \omega_x \leq \infty,$$

and the equations (1) and (2) hold. Ordinary differential equations deliver the most natural examples of local flows: a smooth vector-field $v\colon M \to TM$ on a manifold M determines a local flow ϕ such that the orbit $t \mapsto \phi_t(x_0)$ of $x_0 \in M$ is the unique solution of the initial value problem

$$\dot{x} = v(x), \quad x(0) = x_0.$$

Let ϕ be a local semi-flow on X; in this case X is called the *phase space* of ϕ. Usually, the t parameter is interpreted as the time.

Let $x \in X$. The map $t \mapsto \phi_t(x)$ is called an *orbit* of x and its image

$$\phi^+(x) := \{\phi_t(x)\colon t \in [0, \omega_x)\}$$

is called the *positive semi-trajectory* of x. In the case of a local flow, one defines the *trajectory* as

$$\phi(x) := \{\phi_t(x)\colon t \in (\alpha_x, \omega_x)\}.$$

If the orbit of x constant then it is called *stationary* and x is called a *stationary point*; if it is a periodic map then it is called a *periodic orbit* and x is a *periodic point*. In the case of a local flow generated by a vector-field v a point x_0 is stationary if and only if $v(x_0) = 0$.

It is a natural question to ask for the existence of periodic or stationary points of a given local semi-flow. That question is closely connected to problems in the fixed point theory: a periodic point of period $T > 0$ is a fixed point of the map ϕ_T and a stationary point is a fixed point of ϕ_t with each $t \geq 0$. In order to apply results of the topological fixed point theory one usually assumes that the whole phase space is compact or the points of interest are located in some its compact subset. We distinguish a class of subsets (not invariant, in general) which are particularly convenient to deal with the problem. To this aim we recall some facts related to the Ważewski retract method.

Let $W \subset X$. Define the *exit set* of W as

$$W^- := \{x \in W : \phi(x, [0,t]) \not\subset W \,\forall t \in (0, \omega_x)\}.$$

We call W a *Ważewski set* for ϕ if it is closed and its exit set W^- is closed as well. (That notion was introduced by Charles Conley, compare [1]. Actually, the original Conley's definition is more general.). A compact Ważewski set is called here a *block*; an example of a block for a local flow generated by some planar vector-field is drawn in Figure 1. In the case ϕ is a local flow we say that a block W is *isolating* if the boundary of W is equal to the union of W^- and the *entry set* W^+ defined as the exit set of W with respect to the local flow obtained from ϕ by the reversal of time $t \to -t$. Obviously, the block in Figure 1 is isolating.

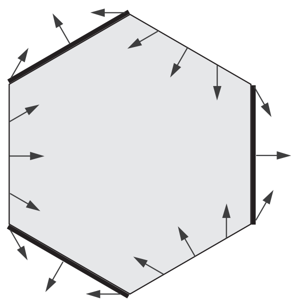

Figure 1. A block for a planar vector-field. The exit set consists of three thickened sides of the hexagon.

Let another subset of W (called the *asymptotic part* of W) be defined as

$$W^* := \{x \in W : \exists t \in (0, \omega_x) : \phi_t \notin W\},$$

The main property of the notion of Ważewski set is given in the following lemma.

Lemma 2.1. *If W is a Ważewski set then the mapping*

$$\sigma\colon W^* \ni x \to \sup\{t \in [0, \omega_x) : \phi(x, [0,t]) \subset W\} \in [0, \infty)$$

is continuous.

The mapping σ in the lemma is called the *escape-time map*. As a consequence of the continuity of σ one instantly gets the following

Corollary 2.1. *If W is a Ważewski set then W^- is a strong deformation retract of W^*.*

That corollary can be reformulated into a version of the Ważewski retract theorem:

Theorem 2.1. *If W^- is not a strong deformation retract of a Ważewski set W then there exists an $x \in W$ such that $\phi^+(x) \subset W$.*

Actually, if ϕ is a local flow and W is a block then x can be chosen such that the whole trajectory $\phi(x)$ is contained in W. As we see below, in some cases one can get a stationary trajectory x; the block in Figure 1 represents such a case.

The Ważewski retract method consists in applications of Lemma 2.1 and its consequences to problems in differential equations. In particular, simple results on asymptotic behavior of solutions can be derived from Theorem 2.1. Other applications, like results on the existence of solutions of two-point boundary value problems, require more advanced theorems.

3. Stationary points in blocks

Results on the existence of periodic or stationary points in Ważewski sets which we present here are based on the Lefschetz fixed point theorem. We consider compact Ważewski sets (i.e. blocks) only.

Theorem 3.1 (compare [11]). *Let W be a block and let $T > 0$. Then the set*

$$U := \{x \in W : \phi_t(x) \in W \setminus W^- \ \forall t \in [0, T]\}$$

is an open subset of W and the set of fixed points $\mathrm{Fix}(\phi_T|_U)$ of the restriction

$$\phi_T|_U : U \to W$$

is compact. Moreover, if W and W^- are ANRs then

$$\operatorname{ind}(\phi_T|_U) = \chi(W) - \chi(W^-). \tag{3}$$

In particular, if

$$\chi(W) - \chi(W^-) \neq 0 \tag{4}$$

then ϕ_T has a fixed point in W.

We do not provide a proof of the above result here since it is a corollary of a more general Theorem 5.1. It follows by Theorem 3.1 that the fixed point index does not distinguish the essential periodic orbits in W from the stationary ones:

Corollary 3.1. *If W is a block, W and W^- are ANRs, and* (4) *holds then there exists a stationary point in W.*

Let a local flow on an n-dimensional manifold M be generated by a smooth vector-field v. In this case Corollary 3.1 states that v has a zero in W provided (4) holds. Moreover, using Theorem 3.1 one can prove that that if $M = \mathbb{R}^n$ and v has no zeros on the boundary of W then the Brouwer degree of v in the interior of W is given by

$$\deg(0, v, \operatorname{int} W) = (-1)^n (\chi(W) - \chi(W^-)) \tag{5}$$

(compare [10]). If the block W is a smooth n-dimensional submanifold of M with boundary and W^- is an $n-1$ dimensional submanifold of ∂W with boundary, then the formulas (3) and (5) are particular cases of the generalized Poincaré-Bendixson formula (see [5] for the history and references related it). For extension of that formula to the non-smooth case we refer to [4].

Example 3.1. Let a planar local flow ϕ has a block W represented in Figure 1. By above results, there exists a stationary point of ϕ inside the block, since

$$\chi(W) = 1, \quad \chi(W^-) = 3.$$

An example of equation which generates such a local flow is given by

$$\dot{z} = \bar{z}^2 + f(z)$$

(written in the complex-number notation) where $f : \mathbb{C} \to \mathbb{C}$ is a smooth such that $f(z)/|z|^2 \to 0$ if $|z| \to \infty$ (compare [11]; see also Example 5.1 below).

4. Local processes and segments

By a *local semi-process* on a topological space X we mean a continuous map $\Phi\colon D \to X$, where D is an open subset of $\mathbb{R} \times X \times [0, \infty)$, such that the map

$$\phi\colon D \ni ((\sigma, x), t) \to (\sigma + t, \Phi(\sigma, x, t)) \in \mathbb{R} \times X$$

is a local semi-flow on $\mathbb{R} \times X$. (In that case ϕ is called a local semi-flow *generated* by Φ.) In a similar way we define a *local process* Φ if the corresponding map ϕ is a local flow. In particular, for a local semi-process (or a local process) Φ one has

$$\Phi(\sigma, x, 0) = x,$$
$$\Phi(\sigma, x, s + t) = \Phi(\sigma + s, \Phi(\sigma, x, s), t)$$

whenever it is defined (compare [7]). In the sequel we write $\Phi_{(\sigma,t)}(x)$ instead of $\Phi(\sigma, x, t)$; in that notation

$$\Phi_{(\sigma,0)} = \mathrm{id}, \quad \Phi_{(\sigma,s+t)} = \Phi_{(\sigma+s,t)} \circ \Phi_{(\sigma,s)}.$$

The space $\mathbb{R}\times X$ is called the *extended phase space* of Φ. The notion of local process is motivated by properties of solutions of non-autonomous differential equations: if $v\colon \mathbb{R} \times M \to TM$ is a smooth time-dependent vector-field on a manifold M then the system of equations

$$\dot{x} = v(t, x), \quad \dot{t} = 1$$

generates a local flow on $\mathbb{R} \times M$, hence a local process Φ on M such that for $t_0 \in \mathbb{R}$ and $x_0 \in M$ the map

$$\tau \mapsto \Phi_{(t_0,\tau-t_0)}(x_0)$$

is the solution of the initial value problem

$$\dot{x} = v(t, x), \quad x(t_0) = x_0.$$

Let $T > 0$. A local semi-process Φ is called *T-periodic* if

$$\Phi_{(\sigma,t)} = \Phi_{(\sigma+T,t+T)}$$

for each σ and t. In that case the map $\Phi_{(0,T)}$, called the *Poincaré map*, satisfies

$$\Phi_{(0,nT)} = \Phi^n_{(0,T)}.$$

Observe that if v is a smooth time-dependent vector-field which is T-periodic with respect to t then the local process Φ is generated by v is T-periodic. Moreover, in this case fixed points of the Poincaré map correspond to initial points of T-periodic solutions of the equation $\dot{x} = v(x, t)$.

In order to establish results on fixed points which refer to local semi-processes (hence also to non-autonomous equations) we introduce a special class of blocks, called *segments* in the extended phase space. At first we introduce the following notation: we denote by π_1 and π_2 the projections of $\mathbb{R}\times X$ onto $\mathbb{R}$ and, respectively, X, and if Z is a subset of $\mathbb{R} \times X$ and $t \in \mathbb{R}$, $[t_1, t_2] \subset \mathbb{R}$ then we put

$$Z_t := \{z \in X \colon (t, z) \in Z\}, \quad Z_{[t_1,t_2]} = Z \cap ([t_1, t_2] \times X).$$

Let Φ be a local semi-process on X and let ϕ be the corresponding local semi-flow on $\mathbb{R} \times X$. Assume that $a < b$. A set $W \subset [a, b] \times X$ is called a *segment* over $[a, b]$ if it is a block with respect ϕ such that the following conditions hold:

(a) there exists a compact subset W^{--} of W^- (called the *essential exit set*) such that

$$W^- = W^{--} \cup (\{b\} \times W_b), \quad W^- \cap ([a, b) \times X) \subset W^{--},$$

(b) there exists a homeomorphism $h\colon [a, b] \times W_a \to W$ such that $\pi_1 \circ h = \pi_1$ and

$$h([a, b] \times W_a^{--}) = W^{--}.$$

If Φ is a local process then a segment is called *isolating* if it is an isolating block for ϕ.

The notion of segment is explained in a simple case in Figure 2. Intuitively, W

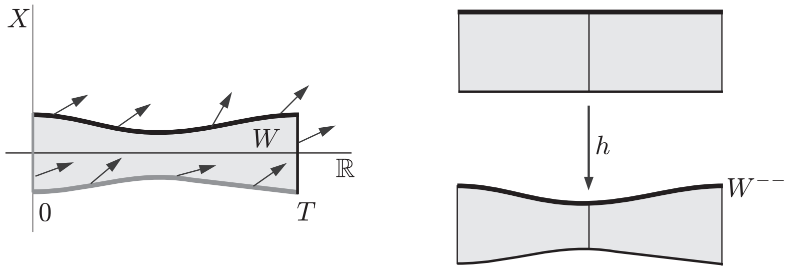

FIGURE 2. A segment W over $[0, T]$ and a monodromy homeomorphism h.

consists of the left-hand side $\{a\} \times W_a$, the right-hand side $\{b\} \times W_b$, and the main part located over the open interval (a, b). The condition (b) means that (W, W^{--}) is a pair of trivial bundles over $[a, b]$ with the fibre (W_a, W_a^{--}). Because of the specific behavior of ϕ (it moves along the time-axis with speed 1), it is clear that that the right-hand side must belong to the exit set.

Let a homeomorphism h satisfies (b). We define the corresponding *monodromy map*

$$m\colon (W_a, W_a^{--}) \to (W_b, W_b^{--})$$

by

$$m(x) = \pi_2 h(b, \pi_2 h^{-1}(a, x)).$$

The monodromy map is actually a homoeomorphism. It can be proved that a different choice of the homeomorphism satisfying (b) provides the monodromy map homotopic to m. It follows, in particular, that the isomorphism in homologies

$$\mu_W := H(m)\colon H(W_a, W_a^{--}) \to H(W_b, W_b^{--})$$

is an invariant of the segment W.

Segments can be glued in a natural way: if W is a segment over $[a, b]$, Z is a segment over $[b, c]$ and

$$(W_b, W_b^{--}) = (Z_b, Z_b^{--}) \tag{6}$$

then their union $W \cup Z$ is a segment over $[a, c]$ and its monodromy map is a composition of monodromy maps of W and Z.

Remark 4.1. Even if the condition (6) is not satisfied, the union of W and Z is a useful object (with respect to problems considered in this note) provided $W \cup Z$ is a block with respect to the local semi-process generated by Φ. In this case W and Z are called *contiguous* and $W \cup Z$ is called a *chain*. More generally, a chain is a union of a finite number of segments $U(1), \ldots, U(r)$ such that $U(i)$ is contiguous to $U(i+1)$ for $i = 1, \ldots, r-1$ (see [15].

5. DETECTION OF PERIODIC SOLUTIONS

Let W be a segment over $[a, b]$. The segment W is called *periodic* if

$$(W_a, W_a^{--}) = (W_b, W_b^{--}).$$

In that case, if $H(W_a, W_a^{--})$ is of finite type then the Lefschetz number $\Lambda(\mu_W)$ is correctly defined. It is called the *Lefschetz number of the periodic segment* W.

Our results on the existence of periodic solutions of non-autonomous differential equations are based on the following theorem:

Theorem 5.1 (compare [11]). *Let W be a periodic segment over $[a,b]$. Then the set*

$$U =: U_W := \{x \in W_a \colon \Phi_{(a,t-a)}(x) \in W_t \setminus W_t^{--}\ \forall t \in [a,b]\}$$

is open in W_a and the set of fixed points of the restriction

$$\Phi_{(a,b-a)}|_U \colon U \to W_a$$

is compact. Moreover, if W and W^{--} are ANRs then

$$\operatorname{ind}(\Phi_{(a,b-a)}|_U) = \Lambda(\mu_W).$$

In particular, if

$$\Lambda(\mu_W) \neq 0 \tag{7}$$

then $\Phi_{(a,b-a)}$ has a fixed point in W_a.

If Φ is the local process generated by a time-dependent vector-field v then x_0 is a fixed point of $\Phi_{(a,b-a)}$ if and only if $\tau \mapsto \Phi_{(a,\tau-a)}(x_0)$ is a solution of the periodic problem

$$\dot{x} = v(t,x), \quad x(a) = x(b), \tag{8}$$

hence Theorem 5.1 immediately implies:

Corollary 5.1. *Let W be a periodic segment and let W and $W^{-}-$ be ANRs. If* (7) *holds then the periodic problem* (8) *has a solution which passes through W_a at time a. In particular, if W is a segment over $[0,T]$,* (7) *holds, and the vector-field v is T-periodic in t then the equation $\dot{x} = v(t,x)$ has a T-periodic solution passing through W_0 at time* 0. □

Remark 5.1. Using the notion of chain (see Remark 4.1), a direct generalization of Theorem 5.1 is presented in [15].

Proof of Theorem 5.1. By the definition of segment, W is an ANR if and only if W_a is an ANR and the same holds for W^{--} and W_a^{--}. We define maps $m_s \colon W_s \to W_a$ ($s \in [a,b]$), by

$$m_s(x) = \pi_2 h(b, \pi_2 h^{-1}(s,x)).$$

In particular $m_a = m$ and $m_b = \mathrm{id}$. Let σ be the escape-time map for W (see Lemma 2.1; here obviously $W = W^*$). Consider a homotopy $H \colon W_a \times [0,1] \to W_a$, $H_t := H(\cdot,t)$, given by

$$H_t(x) := \begin{cases} m_{a+\sigma(a,x)}(\Phi_{(a,\sigma(a,x))}(x)), & \text{if } \sigma(a,x) \leq (1-t)(b-a), \\ m_{a+(1-t)(b-a)}(\Phi_{(a,(1-t)(b-a))}(x)), & \text{if } \sigma(a,x) \geq (1-t)(b-a). \end{cases}$$

In particular, $H_1 = m$. Moreover, it is easy to check that

$$H_t(x) = m(x), \quad \text{if } t \in [0,1] \text{ and } x \in W_a^{--},$$

hence

$$H_t(W_a^{--}) = W_a^{--}, \quad \text{if } t \in [0,1].$$

By the homotopy property of the Lefschetz number we get

$$\Lambda(m) = \Lambda(H_1) = \Lambda(H_0). \tag{9}$$

Since

$$H_0(x) = m_{a+\sigma(a,x)}(\Phi_{(a,\sigma(a,x))}(x)), \tag{10}$$

for every $x \in W_a$, one has $H_0(x) = \Phi_{(a,b-a)}(x)$ if $\sigma(a,x) = b-a$, hence

$$H_0|_U = \Phi_{(a,b-a)}|_U. \tag{11}$$

Let us observe that

$$U = \{x \in W_a \colon \sigma(a,x) = b-a,\ \Phi_{(a,b-a)}(x) \in W_a \setminus W_a^{--}\}, \tag{12}$$

so by (10),

$$U = (H_0)^{-1}(W_a \setminus W_a^{--})$$

and consequently U is open in W_a. If $x \in W_a \setminus W_a^{--}$ and $H_0(x) = x$ then necessarily $\sigma(a,x) = b-a$ (since in the other case $H_0(x) \in W_a^{--}$), hence $x \in U$ by (12), and thus

$$\mathrm{Fix}(\Phi_{(a,b-a)}|_U) = \mathrm{Fix}(H_0) \cap \{x \in W_a : \sigma(a,x) = b-a\}.$$

In particular $\mathrm{Fix}(\Phi_{(a,b-a)}|_U)$ is compact. Put

$$V := \{x \in W_a \colon \sigma(a,x) < b-a\}.$$

It follows that V is open in W_a, $W_a^{--} \subset V$, and $H_0(V) = W_a^{--}$. One can easy check that

$$\mathrm{Fix}(H_0) = \mathrm{Fix}(H_0|_U) \cup \mathrm{Fix}(H_0|_V) = \mathrm{Fix}(\Phi_{(a,b-a)}|_U) \cup \mathrm{Fix}(m|_{W_a^{--}}).$$

Since both the sets $\mathrm{Fix}(\Phi_{(a,b-a)}|_U)$ and $\mathrm{Fix}(m|_{W_a^{--}})$ are compact and disjoint, by the Lefschetz fixed point theorem and the additivity of the fixed point index we get

$$\Lambda(H_0) = \mathrm{ind}(H_0|_U) + \mathrm{ind}(H_0|_V), \tag{13}$$

By the commutativity of the fixed point index and the Lefschetz fixed point theorem we obtain

$$\mathrm{ind}(H_0|_V) = \mathrm{ind}(H_0|_{W_a^{--}}) = \Lambda(H_0|_{W_a^{--}}) = \Lambda(m|_{W_a^{--}}).$$

Combining (9), (11), and (13) we get

$$\mathrm{ind}(\Phi_{(a,b-a)}|_U) = \Lambda(m) - \Lambda(m|_{W_a^{--}}) = \Lambda(\mu_W),$$

so the proof is complete. □

As it was already pointed out, Theorem 3.1 is a particular case of Theorem 5.1. Indeed, a local flow ϕ on X generates a local process Φ given by

$$\Phi_{(a,t)} := \phi_t$$

for each $a \in \mathbb{R}$. If B is a block for ϕ then $[a,b] \times B$ is a segment for Φ and its proper exit set is equal to $[a,b] \times B^-$; since the identity is a monodromy, one has

$$\Lambda(\mu_W) = \chi(B) - \chi(B^-).$$

In the following examples (taken from [11]) we provide some natural applications of the obtained results.

Example 5.1. Consider a planar non-autonomous equation

$$\dot{z} = \bar{z}^n + f(t,z), \tag{14}$$

where n is an integer, $n \geq 1$, $z \in \mathbb{C}$ and $f \colon \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ is a smooth function T-periodic with respect to t for some $T > 0$. Assume that

$$\frac{f(t,z)}{|z|^n} \to 0, \quad \text{as } |z| \to \infty \text{ uniformly in } t. \tag{15}$$

Then the equation (14) has a T-periodic solution.

Indeed, by (15) the term $\bar{z}^n$ becomes dominating as $|z| \to \infty$, hence the behavior of solutions of (14) near infinity resembles the phase portrait of the autonomous equation

$$\dot{z} = \bar{z}^n. \tag{16}$$

For the local flow generated by the latter equation there exists a family of isolating blocks $\{B_r\}_{r>0}$, where B_r is an equilateral $2(n+1)$-gon centered at zero with the diameter equal to $2r$ and the exit set B_r^- consists of $n+1$ disjoint sides of B_r, one of which intersects perpendicularly the positive real semi-axis (compare Figure 1 in the case $n = 2$). It follows that for r sufficiently large the prism $[0, T] \times B_r$ is an isolating segment for (14). It is depicted in Figure 3 for $n = 2$. Its essential exit

FIGURE 3. An isolating segment over $[0, T]$ for the equation (14) with $n = 2$.

set $[0, T] \times B_r^-$ consists of $n + 1$ faces of the prism (in the picture they are marked in gray). Since the identity is a monodromy map for the segment, one concludes that its Lefschetz number is equal $-n$, hence there exists a T-periodic solution of the equation (14) by Corollary 5.1.

Example 5.2. Let us modify the previous example by multiplying the leading term of the right-hand side of the equation by e^{it}:

$$\dot{z} = e^{it}\bar{z}^n + f(t, z). \tag{17}$$

Here we assume that $n \geq 2$ and $f\colon \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ is smooth and 2π-periodic in t. As before we assume that (15) holds. It follows by results in [11] that for r sufficiently large the set

$$W := \{(t, z) \in [0, 2\pi] \times \mathbb{C}\colon e^{-it/(n+1)}z \in B_r\}$$

is an isolating segment over $[0, 2\pi]$ for (17) with the essential exit set

$$W^{--} = \{(t, z) \in [0, 2\pi] \times \mathbb{C}\colon e^{-it/(n+1)}z \in B_r^-\},$$

where B_r and B_r^- are defined in Example 5.1. It means that W is a twisted prism with a $2(n+1)$-gon base centered at the origin and its time sections W_t are obtained by rotating the base with the angular velocity $\frac{1}{n+1}$ over the time interval $[0, 2\pi]$. The set W^{--} consists of $n + 1$ disjoint ribbons winding around the prism, as is shown in Figure 4 in the case $n = 2$. One can choose the rotation by the angle $2\pi/3$

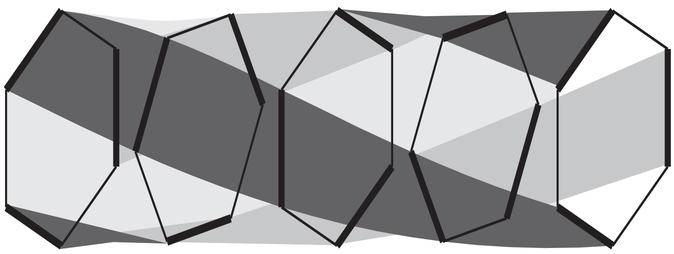

FIGURE 4. An isolating segment over $[0, 2\pi]$ for the equation (17) with $n = 2$.

as a monodromy map of the segment, hence the Lefschetz number of the segment W is equal to 1. It follows by Corollary 5.1 that (17) has a 2π-periodic solution.

Example 5.3. Now we consider a special case of (17) (recall that $n \geq 2$):

$$\dot{z} = e^{it}\bar{z}^n + \bar{z}. \tag{18}$$

The zero solution is 2π-periodic, hence one should look for a nontrivial one. By the previous example, there is a large segment W for the equation such that

$$\Lambda(\mu_W) = 1.$$

Since the term $\bar{z}$ on the right-hand side of (18) dominates as $|z| \to 0$, it can be proved that there is another segment Z for that equation: it is a prism having a sufficiently small square centered at the origin as a base (see Figure 5). Moreover,

FIGURE 5. The isolating segment Z for the equation (18).

$Z \subset W$ and

$$\Lambda(\mu_Z) = -1$$

If there is no 2π-periodic solution of (18) then 0 is the only fixed point of the Poincaré map $\Phi_{(0,2\pi)}$ for the equation, hence by Theorem 5.1,

$$\Lambda(\mu_W) = \operatorname{ind}(\Phi_{(0,2\pi)}|_{\operatorname{int} W_0}) = \operatorname{ind}(\Phi_{(0,2\pi)}|_{\operatorname{int} Z_0}) = \Lambda(\mu_Z)$$

which is a contradiction. Thus (18) has a nonzero 2π-periodic solution.

Example 5.4. Finally, we consider another special case of (17):

$$\bar{z} = e^{it}\bar{z}^n + z. \tag{19}$$

As before, we have a large segment W over $[0, 2\pi]$ which is a twisted prism having an equilateral $(2n+1)$-polygon as a base. There exist also a smaller segment $Y \subset W$ being a cylinder over a disc centered at 0 such that Y^{--} is equal to the whole boundary, hence its Lefschetz number is also equal to 1 and the argument used in Example 5.3 fails here. However, if we glue three copies of W along the interval $[0, 6\pi]$ we obtain a new segment $\widetilde{W}$ such that its monodromy map comes from the full 2π-rotation, hence it is equal to the identity and thus

$$\Lambda(\mu_{\widetilde{W}}) = -n.$$

A similar gluing of three copies of Y provides the segment $\widetilde{Y}$ for which again

$$\Lambda(\mu_{\widetilde{Y}}) = 1.$$

Thus, by the argument in the previous example we conclude that there exists a nonzero 6π-periodic solution of the equation (19).

Results on the existence of periodic solutions of planar non-autonomous equations extending those presented in the above examples can be found in [12].

6. Detection of chaotic dynamics

In order to formulate results on chaotic dynamics we use the notion of *shift on r symbols*, where r is some positive integer. It is a pair (Σ_r, σ), where Σ_r, called the *shift space*, is defined as

$$\Sigma_r := \{0, \dots, r-1\}^{\mathbb{Z}},$$

i.e. the set of bi-infinite sequences of r symbols, and the *shift map* σ is given by

$$\sigma\colon \Sigma_r \ni (\dots s_{-1}.s_0 s_1 \dots) \to (\dots s_0.s_1 s_2 \dots) \in \Sigma_r.$$

In the above notation the dot . marks the 0th term of a sequence, hence σ moves the sequence by one position to the left. The shift on r symbols is a model example of complicated dynamics; in particular it satisfies all three conditions from the classic definition of chaos: sensitive dependence on initial conditions, topological transitivity, and the density of periodic orbits (compare [2]). The term *chaotic dynamics* for a non-autonomous T-periodic equation (and, more generally, for a T-periodic local process Φ) is used by us if the following two conditions are satisfied. The first condition is the existence of a semi-conjugacy between the Poincaré map $\Phi_{(0,T)}$ restricted to some compact subset I of the phase space of the equation and the shift map, i.e. there exists a continuous surjective map $g\colon I \to \Sigma_r$ such that

$$\sigma \circ g = g \circ \Phi_{(0,T)} \tag{20}$$

holds (that condition is often called *symbolic dynamics*). The second condition asserts that for infinitely many of periodic sequences $c \in \Sigma_r$ the counter-image $g^{-1}(c)$ contains an initial point of a periodic solution of the equation.

Below we present results on the existence of chaotic dynamics based on a proper configuration of segments. We consider a local process Φ on a topological space X and we assume that it is T-periodic for some $T > 0$. Our first result is a simple consequence of Theorem 5.1.

Theorem 6.1 (compare [13])**.** *Let $W(0), \dots, W(r)$ be periodic segments over $[0,T]$. Assume that $W(0)_0$ and $W(0)_0^{--}$ are ANRs and*

(A1) $(W(0)_0, W(0)_0^{--}) = \dots = (W(r)_0, W(r)_0^{--})$,
(A2) *there exists $s \in (0,T)$ such that $W(i)_s \cap W(j)_s = \emptyset$ for every $i, j = 0, \dots, r$,*
(A3) *there exists $n \in \mathbb{N}$ such that*

$$H_n(W(0)_0, W(0)_0^{--}) = \mathbb{Q}, \quad H_k(W(0)_0, W(0)_0^{--}) = 0\, \forall k \neq n.$$

Then there are a compact set $I \subset X$, invariant for the Poincaré map $\Phi_{(0,T)}$, and a continuous surjective map $g : I \to \Sigma_{r+1}$ such that (20) holds and for every k-periodic sequence $c \in \Sigma_{r+1}$ there exists $x \in g^{-1}(c)$ such that

$$\Phi^k_{(0,T)}(x) = x.$$

An example of segments satisfying the assumptions of the above theorem for $r = 1$ and $n = 1$ is shown in Figure 6. Before we give a proof the above theorem we define an operation of gluing of periodic segments. If W and Z are periodic segments over $[0,T]$ having the same cross-sections at 0, i.e.

$$(W_0, W_0^{--}) = (Z_0, Z_0^{--})$$

holds, put

$$WZ := \{(t,x) \in [0,2T] \times X \colon x \in W_t \text{ if } t \in [0,T],\ x \in Z_{t-T} \text{ if } t \in [T,2T]\}.$$

(see Figure 7). It is a periodic segment over $[0,2T]$. If $Z(1), \dots, Z(r)$ are periodic segments over $[0,T]$ having the same cross-sections at 0 then we define recurrently another periodic segment

$$Z(1)\dots Z(r) := (Z(1)\dots Z(r-1))Z(r).$$

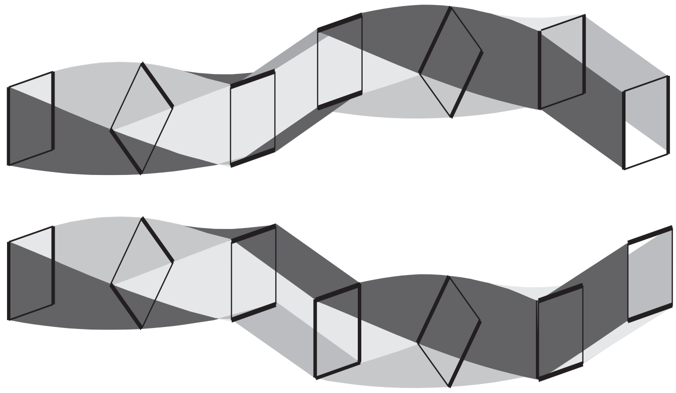

FIGURE 6. Two periodic segments satisfying (A1), (A2), and (A3).

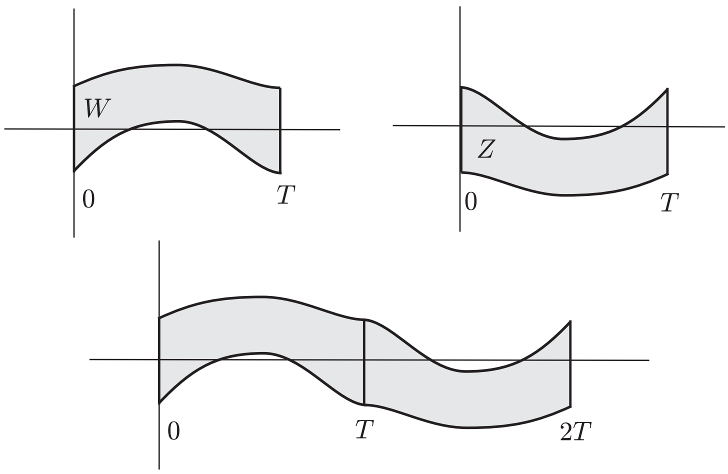

FIGURE 7. Periodic segments W and Z satisfying (A2), and the segment WZ.

If $Z(i) = W$ for each $i = 1, \dots, r$ then w put

$$W^r := Z(1) \dots Z(r).$$

Proof of Theorem 6.1. The required set I is defined as

$$I := \{x \in W(0)_0 \colon \forall k \in \mathbb{Z}\, \exists i = 0, \dots, r \colon \Phi_{(0,kT+t)} \in W(i)_t\, \forall t \in [0, T]\}.$$

By (A2), the map g given by

$$g(x) = c \quad \text{if and only if } \Phi_{(0,kT+s)} \in W(c_k)_s\, \forall k \in \mathbb{Z}$$

is continuous and provides the required semi-conjugacy (20) because the considered local process is T-periodic. Since the set of periodic sequences in Σ_{r+1} is dense, in order to prove the surjectivity of g (and also the remaining claim of the theorem) it suffices to prove that for each periodic sequence $c \in \Sigma_{r+1}$ there exists a corresponding periodic point of the Poincaré map. Let c be such a k-periodic sequence; it is uniquely determined by a sequence $(c_0, \dots, c_{k-1})$ in the set $\{0, \dots, r\}^{\{0,\dots,k-1\}}$. Define a periodic segment

$$W := W(c_0) \dots W(c_{k-1})$$

over $[0, kT]$. Since the homologies of (W_0, W_0^-) are one-dimensional by (A3), and μ_W is an automorphism (since each monodromy map is a homeomorphism), $\Lambda(\mu_W) \neq 0$. Thus, by Theorem 5.1, there exists an $x \in W_0$ such that

$$\Phi^k_{(0,T)}(x) = \Phi_{(0,kT)}(x) = x.$$

It follows by the T-periodicity of Φ that $x \in I$ and $g(x) = c$, hence the result follows. $\square$

Example 6.1. One can verify the existence of two isolating segments satisfying the assumptions of Theorem 6.1 for the planar equation

$$\dot{z} = \frac{1}{2} e^{-i\kappa t} z \left(\frac{1}{2} i\kappa (z+1) + e^{i\kappa t}(\overline{z}+1) \right) \left(\frac{1}{2} i\kappa (z-1) + e^{i\kappa t}(\overline{z}-1) \right)$$

provided $\kappa > 0$ is small enough. They are similar to those in Figure 6; for an explanation we refer to [13].

The other results stated in this note assert the existence of chaotic dynamics in presence of two periodic segments, one of which contains the other.

Theorem 6.2 (compare [16, 18])**.** *Let Z and W be periodic segments over $[0, T]$ which satisfy*

$$Z \subset W, \quad (Z_0, Z_0^{--}) = (W_0, W_0^{--}). \tag{21}$$

Assume that Z_0 and Z_0^{--} are ANRs. Assume moreover that there exists an $n_0 \in \mathbb{N} \setminus \{1\}$ such that

- (B1) $\mu_Z = \mu_W^{n_0} = \mathrm{id}_{H(Z_0, Z_0^{--})}$,
- (B2) $\Lambda(\mu_W) = \Lambda(\mu_W^i)$ *for* $i \in \{1, \ldots, n_0 - 1\}$,
- (B3) $\Lambda(\mu_W) \neq \chi(W_0, W_0^{--})$ *and* $\chi(W_0, W_0^{--}) \neq 0$,

Then there are a compact set $I \subset X$, invariant for the Poincaré map $\Phi_{(0,T)}$, and a continuous surjective map $g \colon I \to \Sigma_2$ such that the equation (20) *holds and*

- $(*)$ *if n_0 is even then for each n-periodic sequence $c \in \Sigma_2$ there exists $x \in g^{-1}(c)$ such that $\Phi^n_{(0,T)}(x) = x$,*
- $(**)$ *if n_0 is odd then for each n-periodic sequence $c \in \Sigma_2$ such that the symbol 1 appears k times in $(c_0, \ldots, c_{n-1})$ and k is not an odd multiplicity of n_0, there exists $x \in g^{-1}(c)$ such that $\Phi^n_{(0,T)}(x) = x$.*

The theorem appeared first in [16] in the case $n_0 = 2$ and then in [18] in a full generality. In order to present a sketch of its proof we introduce the following convenient notation for the segments W and Z: For a finite sequence

$$c = (c_0, \ldots, c_{n-1}) \in \{0, 1\}^{\{0, \ldots, n-1\}}$$

we write $W^n(c)$ for the segment $W(0) \ldots W(n-1)$, where $W(i) = W$ if $c_i = 1$ and $W(i) = Z$ if $c_i = 0$ (see Figure 8). In particular, if $c_i = 1$ for all $i = 0, \ldots, n-1$ then $W^n(c) = W^n$.

Sketch of a proof of Theorem 6.2. The following idea of the proof comes from [18] (compare also [16]). Let

$$I := \bigcap_{n=-\infty}^{\infty} \{x \in W_0 \colon \Phi_{(0,t+nT)}(x) \in W_t \ \forall t \in [0, T]\}$$

be the set of all points in W_0 whose full trajectories are contained in the bigger segment W. It follows that I is compact. Let σ_Z be the escape-time map for the smaller segment Z (see Lemma 2.1). It follows by (21) that $\sigma(0, x)$ is defined for every $x \in W_0$ and if $x \in I$ then either

$$\sigma_Z(0, x) < T, \tag{22}$$

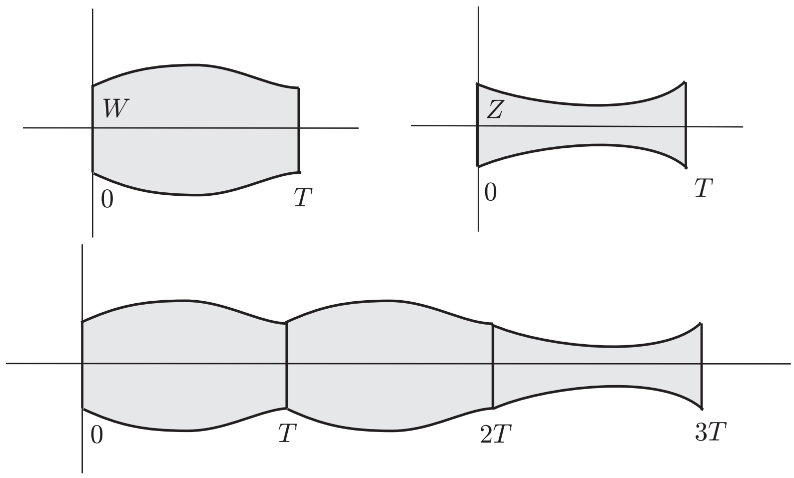

FIGURE 8. Periodic segments W and Z satisfying (21), and the segment $W^3((1,1,0))$.

or

$$\sigma_Z(0,x) = T \quad \text{and} \quad \Phi_{(0,T)}(x) \in W_0 \setminus W_0^{--}. \tag{23}$$

For $x \in I$ we define $g(x) \in \Sigma_2$ by the following rule:

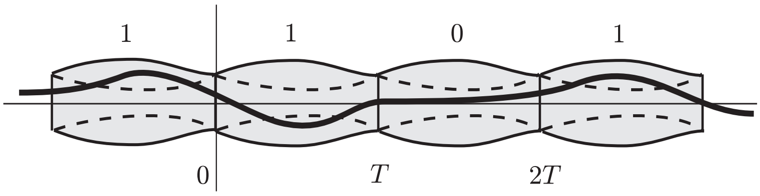

FIGURE 9. Coding of the trajectory of an $x \in I$.

- if on the time interval $[iT, (i+1)T]$ the trajectory of x is contained in Z, then $g(x)_i = 0$,
- if $\Phi^i_{(0,T)}(x)$ leaves Z in time less then T, then $g(x)_i = 1$.

It follows by (22) and (23) that the map $g\colon I \to \Sigma_2$ is continuous and satisfies (20). By compactness of I and density of the set of periodic sequences in the shift space Σ_2 it is sufficient to show that $(*)$ and $(**)$ hold.

Let $c = (c_0, \dots, c_{n-1}) \in \{0,1\}^{\{0,\dots,n-1\}}$. According to the notation in the statement of Theorem 5.1, we have

$$U_{W^n(c)} := \{x \in W_0 \colon \Phi_{(0,t)}(x) \in W^n(c)_t \setminus W^n(c)_t^{--} \ \forall t \in [0, nT]\}.$$

We define $U_{W^n(c),c} \subset U_{W^n(c)}$ as follows: $x \in U_{W^n(c)}$ belongs to the set $U_{W^n(c),c}$ if and only if for each $i \in \{0,\dots,n-1\}$ such that $c_i = 1$ there exists $t \in (0,T)$ such that

$$\Phi_{(0,t_iT+t)}(x) \in W_t \setminus Z_t,$$

i.e. $\Phi_{(0,t_iT)}(x) \in Z_0$ leaves Z in less time then T. It is easy to check that $U_{W^n(c),c}$ is open in W_0 and the sets $U_{W^n(c),c}$ over all n-element sequences c from $\{0,1\}^{\{0,\dots,n-1\}}$ form open and disjoint covering of U_{W^n}. We define

$$F_c := \{x \in g^{-1}(c) \colon \Phi^n_{(0,T)}(x) = x\} \subset I.$$

The set F_c consists of all fixed points of the nth iterate of the Poincaré map $\Phi^n_{(0,T)}$ whose trajectories are coded by the sequence c. It is easy to check that F_c is compact and it is equal to $\mathrm{Fix}(\Phi^n_{(0,T)}|_{U_{W^n(c),c}})$, so $\mathrm{ind}(\Phi^n_{(0,T)}|_{U_{W^n(c),c}})$ is defined. It can be proved that

$$\mathrm{ind}(\Phi^n_{(0,T)}|_{U_{W^n(c),c}}) = \begin{cases} \left(\sum_{s\colon\, n_0|s}(-1)^{k-s}\binom{k}{s}\right)(\chi(Z_0,Z_0^{--})-\Lambda(\mu_W)), & \text{if } k\geq 1,\\ \chi(Z_0,Z_0^{--}), & \text{if } k=0,\end{cases} \tag{24}$$

where the symbol 1 appears exactly k-times in the sequence c. The equation (24) is a consequence of Theorem 5.1, elementary properties of the fixed point index, and some combinatorial calculations. We skip its proof here referring the reader to [18]. One can check that

$$\sum_{s\colon\, n_0|s}(-1)^{k-s}\binom{k}{s}=0$$

if and only if n_0 is odd and k is an odd multiplicity of n_0, hence the proof of Theorem 6.2 is finished. $\square$

Example 6.2. Consider the following planar non-autonomous equation

$$\dot z = (1+e^{i\kappa t}|z|^2)\bar z^n, \tag{25}$$

where $\kappa>0$ is a real parameter and $n\geq 1$ is an integer. The right-hand side of the equation (25) is $2\pi/\kappa$ periodic. It was proved in [18] (and in [16] in the case $n=1$; see also [23]) that for sufficiently small κ there are two periodic segments $Z(n)$ and $W(n)$ over $[0,2\pi/\kappa]$ which satisfy all assumptions of Theorem 6.2 with $n_0=n+1$. We describe briefly how the segments look like. For a small $|z|$ the dynamics generated by (25) is close to the one of the autonomous equation (16), hence by a similar argument then the one in Example 5.1 (but now we are near the origin, not infinity) we conclude the existence of a periodic isolating segment

$$Y(n) := \left[0,\frac{2\pi}{\kappa}\right]\times B_r.$$

for (25), independent of the choice of $\kappa>0$, with $r>0$ sufficiently small. Recall that B_r is an equilateral $2(n+1)$-gon centered at the origin with the diameter equal to $2r$. The essential exit set is given by

$$Y(n)^{--} = \left[0,\frac{2\pi}{\kappa}\right]\times B_r^-,$$

where B_r^- consists of $n+1$ disjoint sides of B_r. If $|z|$ is large then the term $e^{i\kappa t}|z|^2\bar z^n$ dominates in (25) and it follows by results in [11] that for R sufficiently large and each $\kappa>0$ the set

$$W(n) := \left\{(t,z)\in\left[0,\frac{2\pi}{\kappa}\right]\times\mathbb{C}\colon e^{-\frac{it\kappa}{n+1}}z\in B_R\right\}$$

is an isolating segment over $[0,2\pi]$ for (25) with the essential exit set

$$W(n)^{--} = \left\{(t,z)\in\left[0,\frac{2\pi}{\kappa}\right]\times\mathbb{C}\colon e^{-\frac{it\kappa}{n+1}}z\in B_R^-\right\}.$$

It is obvious that if $R>\sqrt{2}r$ then $Y(n)\subset W(n)$ for every $n\in\mathbb{N}$ and $\kappa>0$, but in this case the zero-sections $Y(n)_0$ and $W(n)_0$ are not equal each to the other, hence the condition (21) is not satisfied. In order to get (21) we modify the smaller segment $Y(n)$. It can be done for $0<\kappa<\kappa_0$, where κ_0 is sufficiently small and as result we obtain a new segment, which we denote by $Z(n)$. Its construction can be described as follows. Like for $Y(n)$, the time t-section $Z(n)_t$ is a regular $2(n+1)$-gon based prism centered at the origin and the essential exit set $Z(n)^{--}$ consists of $n+1$ disjoint parts. However, contrary to $Y(n)$, the diameter of $Z(n)_t$ decreases linearly from $2R$ to $2r$ as t passes through the interval $[0,\Delta]$ to some $\Delta < \pi/\kappa$, then stays constant in $[\Delta, 2\pi/\kappa - \Delta]$, and then increases linearly from $2r$ to $2R$ in $[2\pi/\kappa - \Delta, 2\pi/\kappa]$.

In particular, the segments $Z(2)$ and $W(2)$ are similar to the ones shown in Figure 10. It follows that

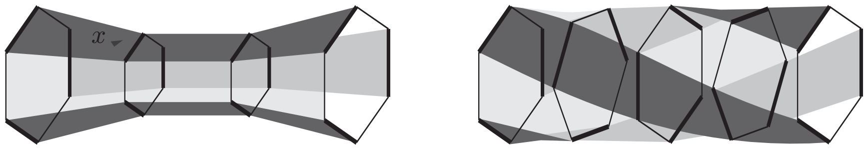

FIGURE 10. Isolating segments $Z(2) \subset W(2)$ for the equation (25) with $n = 2$. The shaded faces are the exit sets Z^{--} and W^{--}.

$$\Lambda(\mu_{W(2)}) = \Lambda(\mu^2_{W(2)}) = 1, \quad \chi(Z(2)_0, Z(2)_0^{--}) = -2$$

and

$$\mu^3_{W(2)} = \mathrm{id}_{H(Z(2)_0, Z(2)_0^{--})}.$$

By generalizing those equations to the case of arbitrary n one can get the following conclusion:

Theorem 6.3 (compare [18])**.** *For every $n \in \mathbb{N}$ there exists $\kappa_0 > 0$ such that for each $0 < \kappa < \kappa_0$ the local process generated by the equation* (25) *satisfies the assumptions of Theorem 6.2 with $n_0 = n+1$.*

Assume that Φ is a T-periodic local process on $\mathbb{R}^n$ generated by a time-dependent vector-field f such that

$$f(t, -x) = -f(t, x).$$

In this case we present a modified version of the previous theorem.

Theorem 6.4 (compare [19])**.** *Let W and Z be two periodic segments over $[0, T]$ such that the condition* (21) *holds. Assume that Z_0 and Z_0^{--} are ANRs and*

- (C1) $(Z_0, Z_0^{--}) = (-Z_0, -Z_0^{--})$ *i.e. the pair (Z_0, Z_0^{--}) is symmetric with respect to the origin,*
- (C2) $\mu_W = H(-\mathrm{id}_{(Z_0, Z_0^{--})})\colon H(Z_0, Z_0^{--}) \to H(Z_0, Z_0^{--})$,
- (C3) $\mu_Z = \mathrm{id}_{H(Z_0, Z_0^{--})}$,
- (C4) $\Lambda(\mu_W) \neq \chi(Z_0, Z_0^{--})$, $\quad \chi(Z_0, Z_0^{--}) \neq 0$.

Then there are a compact set I invariant with respect to the Poincaré map $\Phi_{(0,T)}$ and a continuous surjective map $g : I \to \Sigma_2$ such that the equation (20) *holds and*

- $(*)$ *for each k-periodic sequence $c \in \Sigma_2$ there exists a fixed point $x \in g^{-1}(c)$ of $\Phi^k_{(0,T)}$,*
- $(**)$ *for each k-periodic sequence $c \in \Sigma_2$ there exists a fixed point $x \in g^{-1}(c)$ of $\Phi^{2k}_{(0,T)}$ such that $\Phi^k_{(0,T)}(x) = -x$.*

An idea of a proof. By (C2) and (C3), we get

$$\mu_W \circ \mu_W = \mu_Z = \mathrm{id}_{H(Z_0, Z_0^{--})},$$

hence the existence of g and $(*)$ follow by Theorem 6.2 with $n_0 = 2$. The proof of $(**)$ is based on the version of Theorem 5.1 concerning the existence of antiperiodic solutions inside of a periodic segment given in [14]. We skip it here referring to [19]. □

Example 6.3. As an example of applications of Theorem 6.4 we consider the equation (25) with $n = 1$, i.e. the equation

$$\dot{z} = (1 + e^{i\kappa t}|z|^2)\bar{z}. \tag{26}$$

In this case we present a more detailed description of the periodic isolating segments Z and W over $[0, 2\pi/\kappa]$ which appear Example 6.2. They are shown in Figure 11. The larger segment and its essential exit set are of the form

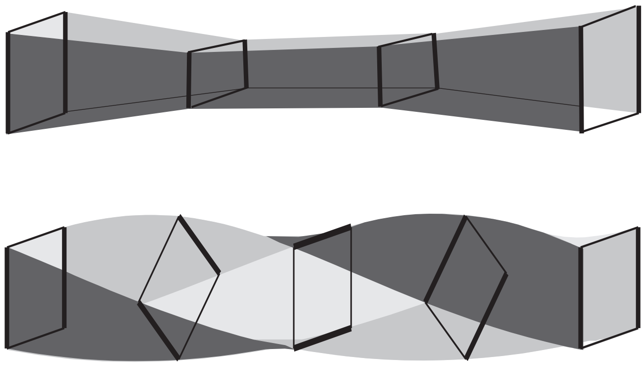

Figure 11. Isolating segments for (26): Z at the top and W at the bottom.

$$W = \{(t,z) \in [0, 2\pi/\kappa] \times \mathbb{C} \colon |\Re(e^{-it\kappa/2}z)| \leq R, |\Im(e^{-it\kappa/2}z)| \leq R\},$$
$$W^{--} = \{(t,z) \in W \colon |\Re(e^{-it\kappa/2}z)| = R\}.$$

In order to construct the smaller segment, we set

$$\omega = \frac{R - r}{\Delta} \tag{27}$$

Let $s : \mathbb{R} \to \mathbb{R}$ be a $2\pi/\kappa$-periodic function such that

$$s(t) := \begin{cases} R - \omega t, & \text{if } t \in [0, \Delta], \\ r, & \text{if } t \in [\Delta, \frac{2\pi}{\kappa} - \Delta], \\ R - \omega(\frac{2\pi}{\kappa} - t), & \text{if } t \in [\frac{2\pi}{\kappa} - \Delta, \frac{2\pi}{\kappa}]. \end{cases}$$

Then smaller segment and its essential exit set are given by

$$Z = \{(t,z) \in [0, 2\pi/\kappa] \times \mathbb{C} \colon |\Re z| \leq s(t),\ |\Im z| \leq s(t)\},$$
$$Z^{--} = \{(t,z) \in Z \colon |\Re z| = s(t)\}.$$

For the proof of the following result we refer the reader to Lemma 19 in [21] (some more restrictive estimates were given earlier in [16]).

Lemma 6.1 (compare [21]). *Assume $\kappa \in (0, 0.495]$, then for $R = 1.15$, $r = 0.5946$, and $\Delta = 0.935$, the above sets W and Z are periodic isolating segments over $[0, 2\pi/\kappa]$ for* (26) *which satisfy the assumptions of Theorem 6.4.*

As an conclusion we get the following precise information on the range of values of the parameter κ for which a chaotic dynamics occur:

Corollary 6.1 (compare [16, 21]). *The local process generated by the equation* (26) *satisfies the conclusion of Theorem 6.4 if* $0 < \kappa \leq 0.495$. □

Remark 6.1. Results on the existence of chaotic dynamics using isolating chains (see Remark 4.1), similar to Theorem 6.2, are presented in [8]. Examples in that paper are based on equations different from the ones considered above and to which theorems given here cannot be directly applied.

7. Detection of homoclinic and multibump solutions

Results on isolating segments can be applied in proofs of other properties of time-periodic non-autonomous equations, like the existence of solutions asymptotic to zero at $\pm\infty$ or approaching zero in some intervals, as we indicate using the previously considered equation (26). In the following result we gather several properties of that equation, including the ones stated in Corollary 6.1.

Theorem 7.1 (compare [20]). *Let Φ be the local process generated by* (26). *Put*

$$T := \frac{2\pi}{\kappa}. \tag{28}$$

If $0 < \kappa < 0.495$ then there exists a compact set I such that $\Phi_{(0,T)}(I) = I$ and a continuous map $g\colon I \to \Sigma_2$ with the following properties:

(D1) $\sigma \circ g = g \circ \Phi_{(0,T)}$,
(D2) $g(I) = \Sigma_2$,
(D3) *if $c \in \Sigma_2$ is n-periodic sequence, then $g^{-1}(c)$ contains a point x such that $\Phi^n_{(0,T)}(x) = x$,*
(D4) *if $c \in \Sigma_2$ is n-periodic sequence, then $g^{-1}(c)$ contains a point x such that $\Phi^n_{(0,T)}(x) = -x$ and $\Phi^{2n}_{(0,T)}(x) = x$,*
(D5) *for each $c \in \Sigma_2$ such that $c_i = 0$ for $i \geq i_0$, $g^{-1}(c)$ contains a point x such that $\lim_{t\to\infty} \Phi_{(0,t)}(x) = 0$,*
(D6) *for each $c \in \Sigma_2$ such that $c = 0$ for $i \leq i_0$, $g^{-1}(c)$ contains a point x such that $\lim_{t\to-\infty} \Phi_{(0,t)}(x) = 0$,*
(D7) *for each $c \in \Sigma_2$ such that $c_i = 0$ for $|i| \geq i_0$, $g^{-1}(c)$ contains a point x such that $\lim_{t\to\pm\infty} \Phi_{(0,t)}(x) = 0$,*
(D8) *for each $t_1 < t_2$ and $\epsilon > 0$ there is infinitely many geometrically distinct* subharmonic *(i.e. kT-periodic for some $k \in \mathbb{N}$) solutions z of* (26) *such that $|z(t)| < \epsilon$ for $t \in [t_1, t_2]$.*

Solutions satisfying (D7) are called *homoclinic* to the zero solution, while solutions satisfying (D8) belong to the class of *multibump* solutions.

As we pointed out above, the properties (D1)–(D4) are already proved. For a proof of the other properties we extend a notation used in the proof of Theorem 6.2 in Section 6. For a moment we consider T arbitrary, i.e. (28) is not necessarily satisfied. Assume that W and Z are periodic segments over $[0,T]$ satisfying (21). Let V be a periodic isolating segment over $[0, lT]$ (where $l \in \mathbb{N}$) for which there are integers

$$0 \leq k_0 < k_1 < \ldots < k_{n-1} \leq l - 1$$

such that

$$V_{t+k_iT} = W_t, \quad \text{for } i \in \{0,\ldots,n-1\},\ t \in [0,T],$$

hence $V_{[k_iT,(k_i+1)T]}$ is equal to the segment W translated to the interval $[k_iT, (k_i + 1)T]$. For a finite sequence $c = (c_0, \ldots, c_{n-1}) \in \{0,1\}^{\{0,\ldots,n-1\}}$ we define $V(c)$ as the periodic segment over $[0, lT]$ obtained from V by replacing $V_{[k_iT,(k_i+1)T]}$ by the translated copy of Z for each $i \in \{0, \ldots n-1\}$ such that $c_i = 0$, i.e.

$$V(c)_t := \begin{cases} Z_{t \bmod T}, & \text{if } t \in [k_iT, (k_i + 1)T] \text{ and } c_i = 0, \\ V_t, & \text{otherwise} \end{cases}$$

(see Figure 12). In particular, if $V = W^n$ then that notation coincides with the one given in Section 6. Using the notation in the statement of Theorem 5.1, we put

$$U_{V(c)} := \{x \in V_0 \colon \Phi_{(0,t)}(x) \in V(c)_t \setminus V(c)^{--}_t \ \forall t \in [0, lT]\}.$$

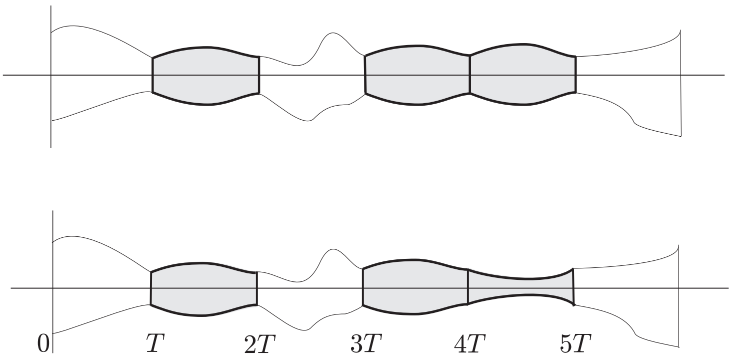

FIGURE 12. A segment V (top) and $V((1,1,0))$ (bottom) with $k_0 = 1$, $k_1 = 3$, and $k_2 = 4$.

Similarly like in the proof of Theorem 6.2, w define $U_{V(c),c} \subset U_{V(c)}$ by the following rule: $x \in U_{V(c)}$ belongs to the set $U_{V(c),c}$ if and only if for each $i \in \{0, \dots, n-1\}$ such that $c_i = 1$ there exists $t \in (0, T)$ such that

$$\Phi_{(0,k_iT+t)}(x) \in W_t \setminus Z_t.$$

Lemma 7.1. *Let V be the segment given above. Assume that Z_0 and Z_0^{--} are ANRs, and*

(E1) $\mu_W \circ \mu_W = \mu_Z = \mathrm{id}_{H(Z_0, Z_0^{--})}$,
(E2) $\Lambda(\mu_W) \neq \chi(Z_0, Z_0^{--})$ *and* $\chi(Z_0, Z_0^{--}) \neq 0$.
(E3) *for each* $c = (c_0, \dots, c_{n-1}) \in \{0,1\}^{\{0,\dots,n-1\}}$ *such that* 1 *appears exactly* k *times in* c,

$$\Lambda(\mu_{V(c)}) = \Lambda(\mu_W^k).$$

Then the set $U_{V(c),c}$ is open in V_0, the set of fixed points of the restriction

$$\Phi_{(0,lT)}|_{U_{V(c),c}} \colon U_{V(c),c} \mapsto V_0$$

is compact, and

$$\mathrm{ind}(\Phi_{(0,lT)}|_{U_{V(c),c}}) = \begin{cases} (-2)^{k-1}(\Lambda(\mu_W) - \chi(Z_0, Z_0^{--})), & \textit{if } k \geq 1, \\ \chi(Z_0, Z_0^{--}), & \textit{if } k = 0. \end{cases}$$

An idea of a proof. One can use a similar argument as the one in the proof of (24). For details we refer the reader to the proof of Lemma 1 in [16]. □

Now we return to the local process generated by (26) and from now we assume that κ, R, r, and Δ are the numbers given in Lemma 6.1, and W and Z are the corresponding isolating segments. We built some other segments related to the equation. Let $r_1 > 0$. Put

$$P(r_1) := \{(t,z) \in \mathbb{R} \times \mathbb{C} \colon |\Re z| \leq r_1,\ |\Im z| \leq r_1\}$$

By Lemma 3 in [16] we have

Lemma 7.2. *Let $\kappa > 0$ be arbitrary and let $r_1 \leq 1/3$. For all $a < b$ the set $P(r_1)_{[a,b]}$ is a periodic isolating segment for (26) and its essential exit set is given by*

$$P(r_1)^{--}_{[a,b]} = \{(t,z) \in P(r_1)_{[a,b]} \colon |\Re z| = r_1\}.$$

Through reminder of this section we assume (28), i.e. $T = 2\pi/\kappa$. For ω defined by (27), $\gamma > 0$, and $t \geq 0$ set

$$s_V(t) := \begin{cases} R - \omega t, & \text{if } t \in [0, \Delta], \\ r, & \text{if } t \in [\Delta, T], \\ re^{-\gamma(t-T)}, & \text{if } t \geq T. \end{cases}$$

and extend the definition to the whole real line by

$$s_V(t) := s_V(-t) \quad \text{for } t < 0.$$

Using s_V we define a set

$$V := \{(t, z) \in \mathbb{R} \times \mathbb{C} \colon |\Re z| \leq s_V(t),\ |\Im z| \leq s_V(t)\},$$

In particular, $V_{[0,T]} \subset Z$. The following result essentially appeared in [22] as Lemma 9:

Lemma 7.3. *Let $\gamma = 0.25$. Then for all $a < b$ the set $V_{[a,b]}$ is an isolating segment for* (26) *and*

$$V_{[a,b]}^{--} = \{(t, z) \in V_{[a,b]} \colon |\Re z| = s_V(t)\}.$$

The next lemma is helpful in a proof of the property (D8).

Lemma 7.4. *Let $t_1 < t_2$ and $\epsilon > 0$ be fixed. Then there exists a periodic segment N for* (26) *over the interval $[\mu T, \nu T]$ for some integers $\mu < \nu$ such that*

(F1) $\mu T < t_1 < t_2 < \nu T$,
(F2) $N_{\mu T} = Z_0$, $N_{\mu T}^{--} = Z_0^{--}$,
(F3) $|z| < \epsilon$ *for each* $t \in [t_1, t_2]$ *and* $z \in N_t$.

The required segment is schematically shown in Figure 13.

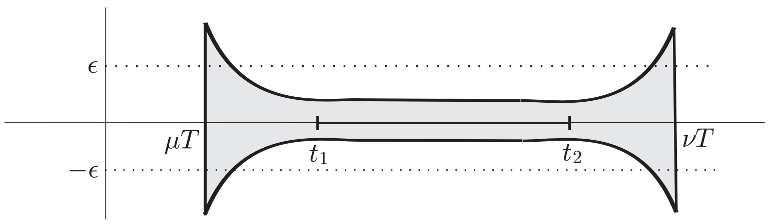

FIGURE 13. Segment N.

Proof. Without loss of generality we can assume that $t_1 = i_1 T$ and $t_2 = i_2 T$ for some integers $i_1 < i_2$, and $\epsilon < 1/3$. Let an integer $p \geq 2$ be such that

$$r_1 := re^{-\gamma(p-1)T} < \epsilon/4,$$

where γ is given in Lemma 7.3. For a real number s define the time-s translation along the time axis in $\mathbb{R} \times \mathbb{C}$ as

$$\tau_s(t, z) := (t + s, z). \tag{29}$$

Then we can put $\mu := i_1 - p$, $\nu := i_2 + p$, and

$$N := \tau_{\mu T}(V_{[0,pT]}) \cup P(r_1)_{[i_1 T, i_2 T]} \cup \tau_{\nu T}(V_{[-pT,0]}),$$

where $P(r_1)$ is taken from Lemma 7.2. $\square$

Proof of Theorem 7.1. Recall that by Corollary 6.1 it suffices to prove (D5)–(D8). In order to deal with solutions that are asymptotic to the trivial solution we will use the set V defined above. We describe only the main idea of the proof of assertion (D7). Proofs of (D5) and (D6) are similar. Let i_0 be a positive integer and let c be a sequence of symbols 0 and 1 such that $c_i = 0$ if $|i| \geq i_0$. Denote by

$$d = (d_0, \dots, d_{2i-1})$$

the shifted fragment of c given by $d_i = c_{i-i_0}$. For $k \in \mathbb{N}$ define a periodic segment $Y(k)$ over $[0, 2(i_0+k)T]$ as

$$Y(k) := \tau_{kT}(V_{[-kT,0]}) \cup \tau_{kT}(W^{2i_0}(d)) \cup \tau_{(2i_0+k)T}(V_{[0,kT]}),$$

where the translations τ are given by (29) and V satisfies Lemma 7.3. By an application of Lemma 7.1 to the segment $Y(k)$, for every k we get a point $y_k \in Y(k)_0$ such that

$$\Phi_{(0,T)}^{2(k+i_0)}(y_k) = y_k$$

and $\Phi_{(0,t)}(y_k) \in Y(k)_t$ for $t \in [0, 2(i_0+k)T]$, and if $k \leq s \leq k + 2i_0 - 1$ and $Y(k)_{[sT,(s+1)T]} = W$ then $\Phi_{(0,t)}(y_n) \notin Z_t$ for some time $t \in (sT, (s+1)T)$. Put

$$x_k := \Phi_{(0,(k+i_0)T)}(y_k).$$

An accumulation point x of the sequence $\{x_k\}$ has the required property (see Figure 14). The proof of (D8) is based on the same idea as the proof of (D5)–(D7).

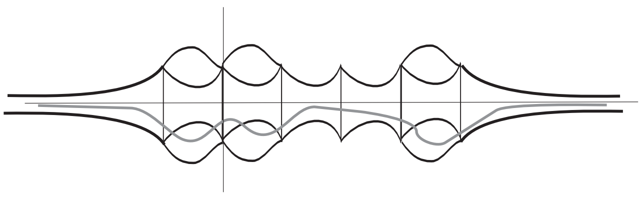

FIGURE 14. A trajectory homoclinic to the trivial solution coded by the sequence $\dots 0001.1001000 \dots$.

In order to deal with multibump solutions described in (D8), for fixed $t_1 < t_2$ and $\epsilon > 0$ we consider the auxiliary segment N like in Lemma 7.4. The result follows by application of Lemma 7.1 to the segments being the union of N followed by the translated copy of $W^n(c)$, where $c \in \{0,1\}^{\{0,\dots,n-1\}}$ is a finite sequence. □

8. Continuation Theorem

We assume that X is a metric space with a distance function ρ. By the same letter ρ we denote also a corresponding distance on $\mathbb{R} \times X$. By $B(D, \delta)$ we denote an open ball of the radius δ around the set D contained either in X or in $\mathbb{R} \times X$.

Let Φ be a local semi-process on X generating the local semi-flow ϕ on $\mathbb{R} \times X$. Let $T > 0$ and W and Z be two subsets of $\mathbb{R} \times X$. We consider the following conditions:

(G1) W and Z are T-periodic segments for Φ which satisfy (21), and Z_0 and Z_0^{--} are ANRs,

(G2) there exists $\eta > 0$ such that for every $w \in W^{--}$ and $z \in Z^{--}$) there exists $t > 0$ such that for $0 < \tau < t$ holds $\phi_\tau(w) \notin W$, $\rho(\phi_t(w), W) > \eta$, $\phi_\tau(z) \notin Z$, and $\rho(\phi_t(z), Z) > \eta$).

Let K be a positive integer and let $E(1), \dots, E(K)$ be disjoint closed subsets of the essential exit set Z^{--} which are T-periodic, i.e. $E(l)_0 = E(l)_T$, and such that

$$Z^{--} = \bigcup_{l=1}^{K} E(l).$$

(In applications we will use the decomposition of Z^{--} into connected components). For $n \in \mathbb{N}$ and every finite sequence $c = (c_0, \dots, c_{n-1}) \in \{0, 1, \dots, K\}^{\{0,\dots,n-1\}}$ and $D \subset W_0$, we define D_c as a set of points x satisfying the following conditions:

(H1) $\Phi_{(0,lT)}(x) \in D$ for $l \in \{0, \dots, n\}$,
(H2) $\Phi_{(0,t+lT)}(x) \in W_t \setminus W_t^{--}$ for $t \in [0, T]$ and $l \in \{0, \dots, n-1\}$,
(H3) for each $l = 0, 1, \dots, n-1$, if $c_l = 0$, then $\Phi_{(0,lT+t)}(x) \in Z_t \setminus Z_t^{--}$ for $t \in (0, T)$,
(H4) for each $l = 0, 1, \dots, n-1$, if $c_l > 0$, then $\Phi_{(0,lT)}(x)$ leaves Z in time less than T through $E(c_l)$.

Now let

$$[0, 1] \times \mathbb{R} \times X \times [0, \infty) \ni (\lambda, \sigma, x, t) \to \Phi^{\lambda}_{(\sigma,t)}(x) \in X$$

be a continuous family of T-periodic semi-processes on X. Let ϕ^{λ} denotes the local semi-flow on $\mathbb{R} \times X$ generated by the semi-process Φ^{λ}. We say that the conditions (G1) and (G2) are satisfied *uniformly* (with respect to λ) if they are satisfied with Φ replaced by Φ^{λ} and the same η in (G2) is valid for all $\lambda \in [0, 1]$.

We write D_c^{λ} for the set defined by the conditions (H1)–(H4) for the semi-process Φ^{λ}.

Lemma 8.1. *If D is open in W_0, then D_c^{λ} is also open in W_0.* □

The main result of this section is the following:

Theorem 8.1 (Continuation Theorem, compare [21])**.** *Let Φ^{λ} be a continuous family of T-periodic semi-processes such that (G1) and (G2) hold uniformly. Then for every $n > 0$ and every finite sequence $c = (c_0, \dots, c_{n-1}) \in \{0, \dots, K\}^{\{0,\dots,n-1\}}$ the fixed point indices $\operatorname{ind}(\Phi^{\lambda}_{(0,nT)}|_{(W_0 \setminus W_0^{--})^{\lambda}_c})$ are correctly defined and equal each to the other (i.e. do not depend on $\lambda \in [0, 1]$).*

Sketch of a proof. We follow the argument in [21]. Let $0 < \beta < \eta$ be such that

$$\beta < \rho(E(l), E(j)) \quad \text{for } l \neq j.$$

One can check that there exists a $\delta > 0$ such that for each $\lambda \in [0, 1]$ the following condition hold: every point x from $\overline{B(W_0^{--}, \delta)} \cap W_0$ leaves W in time $\tau < T$, i.e.

$$\sigma^{\lambda}_W(0, x) < T \quad \text{for } x \in \overline{B(W_0^{--}, \delta)} \cap W_0 \tag{30}$$

(where σ^{λ}_W is the escape-time function for ϕ^{λ}). By decreasing δ we can assume that

$$\beta > 2\delta.$$

We define two open subsets of W_0 by

$$D := W_0 \setminus \overline{B(W_0^{--}, \delta)}, \quad C := W_0 \setminus \overline{B(W_0^{--}, \delta/2)}.$$

It follows that

$$W_0 \cap B(D, \delta/2) \subset C, \quad W_0 \cap B(C, \delta/2) \subset W_0 \setminus W_0^{--}.$$

Let us fix $\lambda_0 \in [0, 1]$. There exists a set Λ open in $[0, 1]$, $\lambda_0 \in \Lambda$, such that for every λ_1, $\lambda_2 \in \Lambda$,

$$\rho(\Phi^{\lambda_1}_{(0,t)}(x), \Phi^{\lambda_2}_{(0,t)}(x)) \leq \frac{\delta}{2}$$

for $0 \le t \le nT$ and $x \in W_0$. One can check using (G2) that

$$D_c^\lambda \subset C_c^{\lambda_0} \subset (W_0 \setminus W_0^{--})_c^\lambda, \quad \text{for} \quad \lambda \in \Lambda. \tag{31}$$

From the choice of δ, the definition of the set D, and (30) one can conclude that

$$\Phi_{(0,nT)}^\lambda(x) \neq x, \quad \text{for} \quad x \in (W_0 \setminus W_0^{--})_c^\lambda \setminus D_c^\lambda. \tag{32}$$

From (31) we deduce that for λ, $\lambda_0 \in \Lambda$ all sets ∂D_c^λ, ∂C_c^λ, and $\partial C_c^{\lambda_0}$ are contained in $(W_0 \setminus W_0^{--})_c^\lambda \setminus D_c^\lambda$, and, in consequence, by (32) the fixed point index for the map $\Phi_{(0,nT)}^\lambda$ relative to those sets is correctly defined. By (31), (32), and the excision property of the fixed point index we obtain that for $\lambda \in \Lambda$,

$$\operatorname{ind}(\Phi_{(0,nT)}^\lambda|_{D_c^\lambda}) = \operatorname{ind}(\Phi_{(0,nT)}^\lambda|_{C_c^{\lambda_0}}) = \operatorname{ind}(\Phi_{(0,nT)}^\lambda|_{(W_0\setminus W_0^{--})_c^\lambda}). \tag{33}$$

In particular, for $\lambda = \lambda_0$ we can assert that

$$\operatorname{ind}(\Phi_{(0,nT)}^{\lambda_0}|_{D_c^{\lambda_0}}) = \operatorname{ind}(\Phi_{(0,nT)}^{\lambda_0}|_{C_c^{\lambda_0}}). \tag{34}$$

Combining (31) with (32) we see that for $\lambda \in \Lambda$ and $x \in \partial C_c^{\lambda_0}$,

$$\Phi_{(0,nT)}^\lambda(x) \neq x,$$

hence by the homotopy property of the fixed point index we get

$$\operatorname{ind}(\Phi_{(0,nT)}^\lambda|_{C_c^{\lambda_0}}) = \operatorname{ind}(\Phi_{(0,nT)}^{\lambda_0}|_{C_c^{\lambda_0}}), \quad \lambda \in \Lambda \tag{35}$$

and finally from (33), (34), and (35) we conclude

$$\operatorname{ind}(\Phi_{(0,nT)}^\lambda|_{D_c^\lambda}) = \operatorname{ind}(\Phi_{(0,nT)}^{\lambda_0}|_{D_c^{\lambda_0}}), \quad \lambda \in \Lambda. \tag{36}$$

By (36), the fixed point index $\operatorname{ind}(\Phi_{(0,nT)}^\lambda|_{D_c^\lambda})$ is locally constant with respect to λ, and consequently

$$\operatorname{ind}(\Phi_{(0,nT)}^0|_{D_c^0}) = \operatorname{ind}(\Phi_{(0,nT)}^1|_{D_c^1}).$$

Therefore

$$\operatorname{ind}(\Phi_{(0,nT)}^0|_{(W_0\setminus W_0^{--})_c^0}) = \operatorname{ind}(\Phi_{(0,nT)}^1|_{(W_0\setminus W_0^{--})_c^1})$$

by (33), and the proof is complete. □

9. Model semi-processes and applications of Continuation Theorem

In this section we show that a more complete description of the dynamics generated by the equations of the form (25) (where $n = 1, 2$) can be obtained by adapting the continuation method to the context of previous sections. The use of model semi-processes as the terminal objects of continuation for those equations enables us to dig deeper into the structure of the set of periodic solutions than does the method based on the Lefschetz Fixed Point Theorem alone. As in Section 7, in the sequel we put $T := 2\pi/\kappa$.

Let Φ be the local process generated by (26) in $\mathbb{R}^2$. Let W and Z be the T-periodic isolating segments for Φ described in Lemma 6.1. In particular, for $R = 1.15$,

$$Z_0 = W_0 = [-R, R] \times [-R, R].$$

For $0 < c < a < b < R$ we put

$$J_{-1} = [-b, -a], \quad J_0 = [-c, c], \quad J_1 = [a, b].$$

Consider a function

$$f\colon J_{-1} \cup J_0 \cup J_1 \to [-R, R]$$

having the graph shown in Figure 15. We stress that $f(-x) = -f(x)$ and $R = f(c) = f(a) = f(-b)$. Let Z^{+1}, Z^{-1} be two connected components of Z^{--}, the right one ($x > 0$) and the left one ($x < 0$), respectively. Let us observe that after a

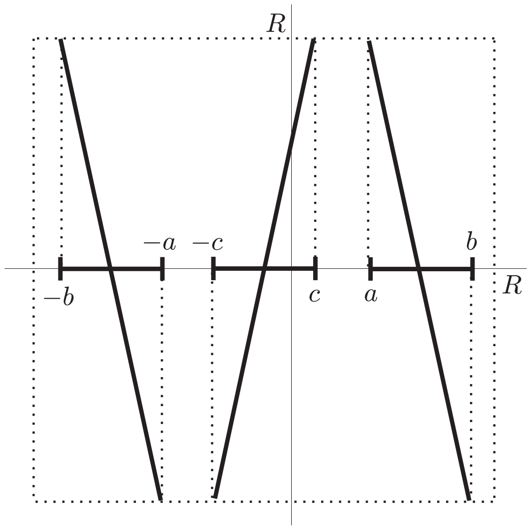

FIGURE 15. Function f in the model for (26).

suitable modifications outside of some large ball we can assume that Φ is a (global) process.

Lemma 9.1. *There exists a semi-process Φ^M on $\mathbb{R}^2$ such that*

(I1) $\{J_{-1} \cup J_0 \cup J_1\} \times [-R, R] = \{z \in W_0 \colon \Phi^M_{(0,t)}(z) \in W_t \ \forall t \in [0, T]\}$,

(I2) $J_0 \times [-R, R] = \{z \in W_0 \colon \Phi^M_{(0,t)}(z) \in Z_t \ \forall t \in [0, T]\}$,

(I3) *for* $l = +1, -1$,

$$J_l \times [-R, R] = \{z \in W_0 \colon z \text{ leaves } Z \text{ through } Z^l \text{ in time} \leq T\},$$

(I4) *for* $z = (x, y) \in \{J_{-1} \cup J_0 \cup J_1\} \times [-R, R]$ *the Poincaré map is given by*

$$\Phi^M_{(0,T)}(x, y) = (f(x), 0),$$

(I5) *Z and W are periodic isolating segments over $[0, T]$ for a family of T-periodic semi-processes Φ^λ such that (G1) and (G2) hold uniformly, and*

$$\Phi^0 = \Phi, \quad \Phi^1 = \Phi^M,$$

(I6) *for every* $c \in \{-1, 0, 1\}^{\{0,\ldots,n-1\}}$,

$$\operatorname{ind}(\Phi^M_{(0,T)}|_{(W_0 \setminus W_0^{--})^1_c}) \neq 0.$$

Φ^M is called a *model semi-process.* We do not provide its construction (actually, intuitive but complicated a little) here, referring the reader to the proof of Theorem 20 in [21]. As a corollary we get the following improvement of a part of Theorem 7.1, in which the shift on two symbols is replaced by the one on three symbols:

Theorem 9.1 (compare [21])**.** *Let Φ be a local process generated by the equation (26) with $0 < \kappa \leq 0.495$. Then there are a compact set $I \subset \mathbb{C}$, invariant with respect to $\Phi_{(0,T)}$, and a continuous surjective map $g \colon I \to \Sigma_3$ such that*

(J1) $\sigma \circ g = g \circ \Phi_{(0,T)}$,

(J2) *if $c \in \Sigma_3$ is n-periodic then $g^{-1}(c)$ contains an n-periodic point of $\Phi_{(0,T)}$.*

Proof. For

$$I := \{x \in W_0 \colon \Phi_{(0,t+kT)}(x) \in W_t \text{ for } t \in [0, T],\ k \in \mathbb{Z}\},$$

we define a semi-conjugacy $g\colon I \to \Sigma_3$ by

$$g(x)_l := \begin{cases} 0, & \text{if } \Phi_{(0,t+lT)}(x) \in Z_t \text{ for all } t \in (0,T), \\ 1, & \text{if } \Phi_{(0,lT)}(x) \text{ leaves } Z \text{ in time less than } T \text{ through } Z^{-1}, \\ 2, & \text{if } \Phi_{(0,lT)}(x) \text{ leaves } Z \text{ in time less than } T \text{ through } Z^{+1}. \end{cases}$$

By Theorem 8.1 and Lemma 9.1, all fixed point indices for periodic sequences of symbols are nontrivial, hence $g(I)$ contains all periodic sequences from Σ_3. But the set of all periodic sequences is dense in Σ_3, so $g(I) = \overline{g(I)} = \Sigma_3$. □

Now we study the dynamics generated by the equation (25) with $n = 2$, i.e.

$$\dot{z} = (1 + e^{i\kappa t}|z|^2)\bar{z}^2. \tag{37}$$

The segments $Z := Z(2)$ i $W := W(2)$ for (37) are shown in Figure 10. It follows that $Z_0 = W_0$ is a hexagon centered at the origin and the exit set Z^{--} has three components E_1, E_2 i E_3. Let R be equal to the radius of the inscribed circle of Z_0 and let

$$h\colon \mathbb{C} \ni z \to ze^{i\frac{2\pi}{3}} \in \mathbb{C}$$

be the rotation. We define

$$S := [0,R] \cup h([0,R]) \cup h^2([0,R]).$$

For $0 < a < b < c < R$ we put

$$\begin{aligned} J :=& [0,a] \cup h([0,a]) \cup h^2([0,a]), \\ J_k :=& h^{k-1}([b,c]), \quad k \in \{1,2,3\}. \end{aligned}$$

Let

$$f\colon J \cup J_1 \cup J_2 \cup J_3 \to S$$

be a continuous function, symmetric with respect to h, with the graph shown in Figure 16.

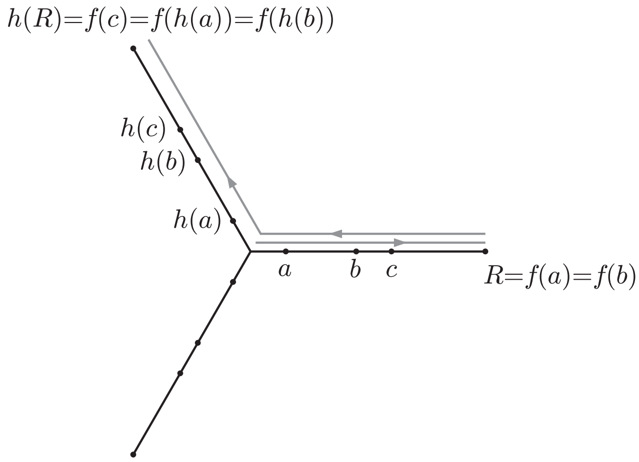

FIGURE 16. Map f in the model for (37). The gray lines above the segment $[0,R]$ represents the image of $[0,a]$ (the lower line) and of $[b,c]$ (the upper line). Actually, the latter image is contained in $[0,R] \cup [0,h(R)]$. The arrows indicate the orientation of the image of f when going from 0 to a on the lower line and from b to c on the upper line.

Let us observe that

(K1) $f(0) = 0$,
(K2) for $k \in \{0, 1, 2\}$,

$$f(h^k(a)) = f(h^k(b)) = h^k(R), \quad f(h^k(c)) = h^{k+1}(R),$$

(K3) the restrictions of f

$$f\colon [0, h^{k-1}(a)] \to [0, h^{k-1}(R)],$$
$$f\colon J_k \to [h^{k-1}(R), 0] \cup [0, h^k(R)]$$

are homeomorphisms.

Let $r\colon W_0 \to S$ be the retraction schematically shown in Figure 17. We put

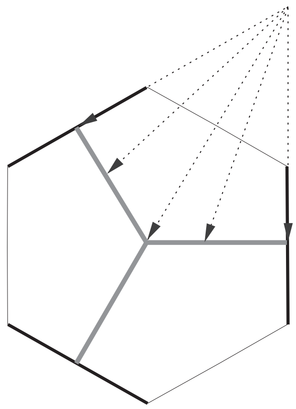

FIGURE 17. The retraction $r\colon W_0 \to S$.

$$K := r^{-1}(J \cup J_1 \cup J_2 \cup J_3).$$

The following result was proved in [23].

Lemma 9.2. *There exists a model semi-process Φ^M for (37) such that*

(L1) *Z and W are periodic isolating segments for a family of T-periodic semi-processes Φ^λ such that (G1) and (G2) hold uniformly and*

$$\Phi^0 = \Phi, \quad \Phi^1 = \Phi^M,$$

(L2) $\{z \in W_0 \colon \Phi^M_{(0,t)}(z) \in W_t \ \forall t \in [0, T]\} = K$,
(L3) $\{z \in W_0 \colon \Phi^M_{(0,t)}(z) \in Z_t \ \forall t \in [0, T]\} = r^{-1}(J)$,
(L4) $\{z \in K \colon z$ *leaves Z through E_k at time* $\leq T\} = r^{-1}(J_k)$,
(L5) *the Poincaré map for Φ^M, denoted by P_M, is of the form*

$$P_M(z) := \Phi^M_{(0,T)}(z) = f(r(z)), \quad z \in K.$$

We want now to investigate the symbolic dynamics on sets J and J_k for the model map. The symbol 0 will correspond to J and the symbols k for $k = 1, 2, 3$ will corresponds to J_k. To simplify the notation we set $J_4 := J_1$. Observe that for $k = 1, 2, 3$, $P_M(J_k)$ covers J_k, J_{k+1} and part of J. The image of $P_M(J)$ covers J and all J_k's. But if we want to see where we can go from J_k and J under P_M^2 we need to consider the parts of J, what they cover and which part is covered by J_k.

We define a set $\Pi \subset \Sigma_4 = \{0, 1, 2, 3\}^{\mathbb{Z}}$ as follows: a sequence c belongs to Π if the following conditions hold:

(M1) if $c_i = k$ for some $k \in \{1, 2, 3\}$, then $c_{i+1} = 0$ or $c_{i+1} = k$ or $c_{i+1} = k \pmod 3 + 1$,
(M2) if $c_p = 0$ for $p \leq i$, then $c_{i+1} \in \{0, 1, 2, 3\}$,

(M3) if $c_i = 0$ and $p < i$ is such that $c_p = k \neq 0$ and for $p < s \leq i$ $c_s = 0$, then $c_{i+1} = 0$ or $c_{i+1} = k$ or $c_{i+1} = (k \bmod 3) + 1$.

The condition (M1) says, for example, that the symbol 1 in the sequence c can be followed by the symbols 0, 1, 2. It is a consequence of the fact that $f(J_1)$ covers J, J_1 i J_2 in a proper way. Hence, if the trajectory coded by the sequence $c \in \Pi$ leaves the segment Z in the time interval $[kT, (k+1)T]$ through the component Z_1^{--}, then, in the time interval $[(k+1)T, (k+2)T]$, it can stay in Z or leaves Z through Z_1^{--} or Z_2^{--}. By the continuation theorem we get:

Theorem 9.2 (compare [23])**.** *Let Φ be a local process generated by* (37)*. There exists a $\kappa_0 > 0$ such that for $0 < \kappa \leq \kappa_0$ there are a compact set $I \subset \mathbb{C}$ invariant under the Poincaré map $\Phi_{(0,T)}$ and a continuous map $g\colon I \to \Sigma_4$ such that*

(N1) $\sigma \circ g = g \circ \Phi_{(0,T)}$,
(N2) $\Pi \subset g(I)$,
(N3) *if $c \in \Pi$ is n-periodic, then $g^{-1}(c)$ contains an n-periodic point for $\Phi_{(0,T)}$.*

Proof. The set I is defined like in the proof of Theorem 9.1. We define a continuous map $g\colon I \to \Sigma_4$ by

$$g(x)_l := \begin{cases} 0, & \text{if } \Phi_{(0,t+lT)}(x) \in Z_t \text{ for all } t \in (0,T), \\ k, & \text{if } \Phi_{(0,lT)}(x) \text{ leaves } Z \text{ in time less than } T \text{ through } E_k. \end{cases}$$

It is easy to check that the condition (N1) is satisfied and (N2) follows by (N3), hence it is enough to prove (N3). Let Π_l denotes the projection of Π onto the coordinates $0, \ldots, l-1$. This means that if $\alpha = (\alpha_0, \ldots, \alpha_{l-1}) \in \{0,1,2,3\}^l$ then $\alpha \in \Pi_l$ if and only if there exists $c \in \Pi$ such that $\alpha_i = c_i$ for $i = 0, \ldots, l-1$. For $\alpha \in \Pi_l$ such that $\alpha_0 \neq 0$ we define

$$s(\alpha) := \max\{j\colon \alpha_j \neq 0\}.$$

Let $c \in \Pi$ be a l-periodic sequence. If $c_i = 0$ for all i, then $g(0) = c$. Let us observe that it is enough to consider $c \in \Pi$, such that $c_0 \neq 0$.

Let l be the principal period of c and let $\alpha = (c_0, \ldots, c_{n-1}) \in \Pi_l$. One can easily check that there exists a closed interval $A \subset J_{\alpha_0}$, such that for the semi-process Φ^M we get

$$\begin{aligned} &(W_0 \setminus W_0^{--})_\alpha = r^{-1}(\operatorname{int} A), \\ &f^l(A) = [0, h^{s(\alpha)-1}(R)] \cup [0, h^{s(\alpha) \bmod 3}(R)], \end{aligned}$$

and $f^l|_A$ is a homeomorphism. Observe that

$$\text{either } c_0 = s(\alpha) \text{ or } c_0 = (s(\alpha) \bmod 3) + 1.$$

It follows the set $r^{-1}(A)$ is topologically a product of a segment A and another interval B, the map $f^l(r(x))$ maps $A \times B$ onto $[0, h^{s(\alpha)-1}(R)] \cup [0, h^{s(\alpha) \bmod 3}(R)]$ containing A in its interior, hence it is easy to see that

$$\operatorname{ind}(P_M^l|_{(W_0 \setminus W_0^{--})_\alpha}) = \operatorname{ind}(f^l|_{\operatorname{int} A}) = \pm 1 \neq 0.$$

By Theorem 8.1 we have

$$\operatorname{ind}(\Phi_{(0,T)}^l|_{(W_0 \setminus W_0^{--})_\alpha}) \neq 0,$$

hence there exists an $x \in (W_0 \setminus W_0))_\alpha$, such that $\Phi_{(0,T)}^l(x) = x$. Observe that $g(x) = c$ and the proof is finished. $\square$

We will finish this section with the following example.

Example 9.1. Consider a time-dependent Hamiltonian system

$$\dot{x} = -\frac{\partial H}{\partial y}, \quad \dot{y} = \frac{\partial H}{\partial x} \tag{38}$$

where

$$H(x, y, t) = x^3 y + xy^3 + H_1(x, y, t),$$
$$H_1(x, y, t) = -\frac{1}{2}y^2 \sin(\kappa t) - xy\cos(\kappa t) + \frac{1}{2}x^2 \sin(\kappa t).$$

One can prove that for $0 < \kappa < \kappa_0$ there are two periodic segments Z and W for (38) that look like the ones shown in Figure 18. We see that when compared to the

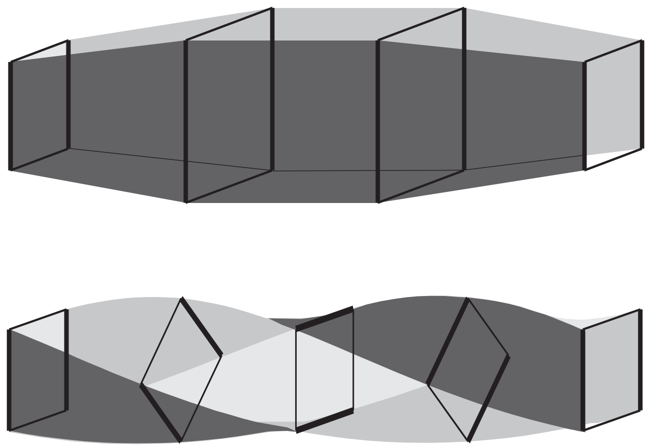

FIGURE 18. Isolating segments for (38): Z at the top and W at the bottom.

previous examples we have a slightly different geometry here. Previously the bigger segment was rotating, while here the smaller one rotates. Both these situation are manifestly homeomorphic. An application of the continuation theorem give us, like in the case of the equation (26), the following result:

Theorem 9.3 (compare [23]). *Let Φ be a local process generated by* (38). *Then there exists a $\kappa > 0$ such that for $0 < \kappa < \kappa_0$ there are a compact set $I \subset \mathbb{R}^2$, invariant with respect to the Poincaré map $\Phi_{(0,T)}$, and a surjective continuous map $g\colon I \to \Sigma_3$ such that $\Phi_{(0,T)}$ is semi-conjugated to the shift map on three symbols by the map g. Moreover, for each n-periodic sequence $c \in \Sigma_3$, $g^{-1}(c)$ contains an n-periodic point of the Poincaré map.*

10. Rigorous numerical shadowing using isolating segments

The aim of this section is to describe a potential application of isolating segments to rigorous numerical shadowing for non-autonomous equations. A minor modifications of the presented argument can provide similar application in the autonomous case. The question of reliability of numerical simulations of chaotic dynamical systems is usually addressed by shadowing algorithms, see for example [6, 9]. When applied to ordinary differential equations, these algorithms require rigorous estimates for Poincaré maps. Here we outline a possible shadowing algorithm based on the concept of isolating segment, without such requirement.

Let us consider an non-autonomous equation

$$\dot{x} = f(t, x) \tag{39}$$

in $\mathbb{R}^d$, where f is Lipschitz with respect to x. Our point of departure is a pseudoorbit (an approximate numerical trajectory)

$$\omega = \{\omega_0, \omega_1, \dots, \omega_N\}$$

for (39) obtained for $t_0 < t_1 < \dots < t_N$ (if our aim is to obtain the periodic orbit then we have $\omega_0 = \omega_N$, $t_0 = 0$, $t_N = kT$, where $k \in \mathbb{N}$, and f is T-periodic in t). Our goal is to show that there is true orbit nearby ω. We construct the following auxiliary objects:

(O1) local sections $\Pi_0, \Pi_1, \dots, \Pi_N$ for (39) defined by $\Pi_i = \{t = t_i\}$, such that $\omega_i \in \Pi_i$. For periodic orbit for T-periodic problem (39) we require that $t_N = kT$ for some $k\mathbb{N}$.

(O2) d linearly independent and approximately invariant vector fields $X_1, \dots, X_d$ along the pseudoorbit ω. By this we understand that the following conditions are satisfied for $l = 1, \dots, d$

$$\begin{aligned} &X_{l,i} \in \mathbb{R}^d \\ &dP_{i\to i+1}X_{l,i} \approx \lambda_{l,i\to i+1}X_{l,i+1} \end{aligned}$$

where $P_{i\to j}\colon \Pi_i \to \Pi_j$ denote the Poincaré map between sections Π_i and Π_j for $i < j$. To obtain the vectors $X_{l,i}$ for a periodic trajectory we perform the diagonalization of the return map and then we evolve forward the unstable eigenvectors and backward the stable ones.

(An effective procedure of construction of $X_{l,i}$ for a very long (not periodic) pseudoorbit is described in [9] for the case of planar periodically forced ODE.)

For any $i < j$ we set

$$\lambda_{l,i\to j} = \lambda_{l,i\to i+1} \cdot \lambda_{l,i+1\to i+2} \cdot \dots \cdot \lambda_{l,j-1\to j}$$

The l-th direction is called *unstable* if $|\lambda_{l,0\to N}| > 1$ and is called *stable* if $|\lambda_{l,0\to N}| < 1$. We assume that each direction is either stable or unstable. Let U be the set of unstable directions. We would like to construct segment $W = \bigcup W_{[t_i,t_{i+1}]}$ isolating a trajectory close to ω, so that W_{t_i} is a parallelogram

$$\begin{aligned} W_{t_i} &:= \{x \in \Pi_i \colon x = \omega_i + \Sigma_{l=1}^d[-1,1]s_{l,i}X_{l,i}\} \\ W_{[t_i,t_{i+1}]} &:= \{(t,x)\colon t = (1-\alpha)t_i + \alpha t_{i+1}, \\ &\qquad x = (1-\alpha)\left(\omega_i + \Sigma_{l=1}c_l s_{l,i}X_{l,i}\right) + \alpha\left(\omega_{i+1} + \Sigma_{l=1}c_l s_{l,i+1}X_{l,i+1}\right) \\ &\qquad c_l \in [-1,1], \alpha \in [0,1]\}, \end{aligned}$$

where $s_{l,i}$ are some positive numbers. We introduce the candidate for the monodromy homeomorphism for $W_{[t_i,t_{i+1}]}$,

$$h_i\colon [0,1]\times[-1,1]^d \to W_{[t_i,t_{i+1}]} \subset [t_i,t_{i+1}]\times\mathbb{R}^d$$

by

$$\begin{aligned} h_i(\alpha, c_1, \dots, c_d) := ((1-\alpha)t_i + \alpha t_{i+1}, \\ (1-\alpha)(\omega_i + \Sigma_{l=1}c_l s_{l,i}X_{l,i}) + \alpha(\omega_{i+1} + \Sigma_{l=1}c_l s_{l,i+1}X_{l,i+1})). \end{aligned}$$

We expect h_i to be a homeomorphism. This happens when a small time step used in the construction of the pseudorbit ω. If this is the case, then also $X_{l,i}$ and $X_{l,i+1}$ are almost collinear and the set of vectors spanning $h_i([0,1]\times[-1,1]^d)$, given approximately by $(1,0), (0,X_{1,i}), \dots, (0,X_{d,i})$, is linearly independent.

Before we describe $W^{--}_{[t_i,t_{i+1}]}$, we define the faces of $W_{[t_i,t_{i+1}]}$ and W_{t_i} for $l = 1,\dots,d$ by

$$\begin{aligned}
W^{l+}_{t_i} &:= \{x = h(0,c_1,\dots,c_l)\colon c_l = 1,\ c_i \in [-1,1] \text{ for } i \neq l\}\\
W^{l-}_{t_i} &:= \{x = h(0,c_1,\dots,c_l)\colon c_l = -1,\ c_i \in [-1,1] \text{ for } i \neq l\}\\
W^{l+}_{[t_i,t_{i+1}]} &:= \{x = h(\alpha,c_1,\dots,c_l)\colon c_l = 1,\ c_i \in [-1,1] \text{ for } i \neq l, \alpha \in [0,1]\}\\
W^{l-}_{[t_i,t_{i+1}]} &:= \{x = h(\alpha,c_1,\dots,c_l)\colon c_l = -1,\ c_i \in [-1,1] \text{ for } i \neq l, \alpha \in [0,1]\}.
\end{aligned}$$

Now we put

$$\begin{aligned}
W^{--}_{t_i} &:= \bigcup_{l\in U} \left(W^{l+}_{t_i} \cup W^{l-}_{t_i}\right)\\
W^{--}_{[t_i,t_{i+1}]} &:= \bigcup_{l\in U} \left(W^{l+}_{[t_i,t_{i+1}]} \cup W^{l-}_{[t_i,t_{i+1}]}\right)\\
W^{++}_{t_i} &:= \bigcup_{l\notin U} \left(W^{l+}_{t_i} \cup W^{l-}_{t_i}\right)\\
W^{++}_{[t_i,t_{i+1}]} &:= \bigcup_{l\notin U} \left(W^{l+}_{[t_i,t_{i+1}]} \cup W^{l-}_{[t_i,t_{i+1}]}\right).
\end{aligned}$$

An algorithm which successively builds the segment can be described as follows:

0. Input values $s_{l,i}$, $l = 1,\dots,d$; Output values $s_{l,i+1}$.
1. For each l we choose $s_{l,i+1}$,
   1.1. $s_{l,i+1} < |\lambda_{l,i\to i+1}|$, for unstable directions,
   1.2. $s_{l,i+1} > |\lambda_{l,i\to i+1}|$ for stable directions.
   Intuitively, this means that in the face unstable face is tilted towards the pseudoorbit and the stable face is tilted away from the pseudoorbit. This should give is exit and entry points on W^{--} and W^{++}, respectively.) To assure that the size of W_{t_i} not change to much and to make sure that in the periodic case $W_{t_N} = W_{t_0}$ it is desirable to choose $s_{l,i+1}$, such that
   $$s_{l,1} \cdot s_{l,2} \cdot \ldots \cdot s_{l,i+1} \approx 1.$$
2. Verification of the algorithm—the following conditions should be checked:
   2.1. the map h_i is a homeomorphism. For this purpose it is enough to check that a certain set of $d+1$ vectors is linearly independent,
   2.2. for $l = 1,\dots,d$ we check that for each $x \in W^{--}_{[t_i,t_{i+1}]}$ the vector $(1, f(x))$ is pointing inside $W_{[t_i,t_{i+1}]}$ and for each $x \in W^{++}_{[t_i,t_{i+1}]}$ the vector $(1, f(x))$ is pointing inside $W_{[t_i,t_{i+1}]}$. For this purpose for each face in $W_{[t_i,t_{i+1}]}$ we have to evaluate the product of $(1, f(x))$ and the normal vector to the face. In interval arithmetic this might be doable in one step for the whole face at once.

We consider the construction successful if we are able to perform N steps (in the case of periodic orbit we also want to have $W_{t_N} = W_{t_0}$). In view of the theory developed in previous sections it is easy that if N steps of the algorithm has been successfully completed, then there exists $x_0 \in W_0$ and a time $T > 0$, such that $\phi([0,T],(0,x_0)) \in \bigcup_i W_{[t_i,t_{i+1}]}$ and $\phi(T,(0,x_0)) \in W_{t_N}$, where ϕ is the local flow induced by (39). Moreover, if the starting pseudoorbit was periodic and $W_{t_0} = W_{t_N}$, then $\phi(T,(0,x_0)) = (0,x_0)$, which means that x_0 is periodic.

The following converse statement is quite obvious: *if the periodic pseudorbit ω is in fact a periodic hyperbolic orbit (with all real eigenvalues) for* (39) *and if the time step is small enough then the algorithm described above yields an isolating segment.*

Of course to make the above statement into the theorem one needs to specify how to choose $s_{l,i}$, but it is quite obvious that it is enough to take $s_{l,i+1} = (1\pm\epsilon_l)\lambda_{l,i\to i+1}$

for some small $\epsilon_l > 0$, with the plus sign for unstable directions and minus sign for stable ones.

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