Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

Website: http://AIMsciences.org

pp. **X–XX**

CONE CONDITIONS AND COVERING RELATIONS FOR NORMALLY HYPERBOLIC INVARIANT MANIFOLDS

Maciej J. Capiński

AGH University of Science and Technology, Faculty of Applied Mathematics
al. Mickiewicza 30, 30-059 Kraków, Poland

Piotr Zgliczyński

Jagiellonian University, Institute of Computer Science,
Łojasiewicza 6, 30–348 Kraków, Poland

Abstract. We present a new topological proof of the existence of normally hyperbolic invariant manifold for maps. The proof is conducted in the phase space of the system. In our approach we do not require that the map is a perturbation of some other map for which we already have an invariant manifold. We provide conditions which imply the existence of the manifold within an investigated region of the phase space. The required assumptions are formulated in a way which allows for rigorous computer assisted verification. We apply our method to obtain a normally hyperbolic invariant manifold within an explicit range of parameters for the rotating Hénon map.

1. **Introduction.** The goal of our paper is to present a topological method of the proof of the existence of the normally hyperbolic invariant manifold for a map in a vicinity of an approximate invariant manifold. In our opinion there are two main advantages of our approach: 1) we do not assume that the given map is a perturbation of some other map for which we have a normally hyperbolic invariant manifold, 2) the assumptions could be rigorously checked with computer assistance if our approximation of the invariant manifold is good enough.

Our results about normally invariant manifolds are weaker than the standard ones [1, 3, 7, 10]. In fact we do not prove any smoothness result and the existence of fibration of the stable and unstable manifolds of normally hyperbolic invariant manifold by points on that manifold. We believe though that topological assumptions considered by us should be sufficient to prove normal hyperbolicity in the usual sense. This paper should be considered as a first step towards this end. In a subsequent publication we intend to improve our results in this direction.

In the standard approach to the proof of various invariant manifold theorems all consideration are done in suitable function spaces or sequences spaces, moreover the

2000 *Mathematics Subject Classification.* Primary: 34D10, 34D35; Secondary: 37C25.

Key words and phrases. Normally hyperbolic manifolds, covering relations, cone conditions, Brouwer degree.

The first author would like to thank the African Institute For Mathematical Sciences, Muizenberg, Cape Town, South Africa, where the work was partially developed.

The second author is supported by the Polish State Ministry of Science and Information Technology grant N201 024 31/2163.

Both authors are supported by the Polish State Ministry of Science and Information Technology grant N201 543238.

existence of the invariant manifold for nearby map (or ODE) is always assumed, see for example [3, 7, 10] and the references given there. Usually these proofs do not give any computable bounds for the size of perturbation for which the invariant manifold exists, with only exception known to us being the result of Bates, Lu and Zeng [1].

In contrast to the above mentioned standard approach, in our method the whole proof in made in the phase space. This method of proof of the existence of normally hyperbolic invariant manifolds is not entirely new, see for example the proof of Jones [8] in the context of slow-fast system of ODEs. But still he considered the perturbation of some normally hyperbolic invariant manifold. In [2] a somewhat similar approach has been applied to obtain a topologically normally hyperbolic invariant set. The result relied only on covering relations without the use of cone conditions. There it has not been shown that the invariant set is a manifold, hence the result was much weaker than the one presented in this paper.

The work is organised as follows. In Section 2 we introduce notations and and give preliminaries on vector bundles. In Section 3 we introduce central-hyperbolic sets (ch-sets), covering relations and cones. A ch-set will play the role of a region in which we suspect to find a normally hyperbolic invariant manifold. Covering relations will ensure the existence of an invariant set within a ch-set. In Section 4 we introduce cone conditions for maps and main results. Cone conditions combined with covering relations will give the existence of a normally hyperbolic invariant manifold inside of a ch-set. In Section 5 we show how cone conditions and covering relations can be verified in practice. In Section 6 we discuss how our result relates so far to the classical normally hyperbolic invariant manifold theorem. To demonstrate clearly the strength of our approach, in Section 7 we prove that for the rotating Hénon map considered in [5] for an explicit range of parameters there exists a normally hyperbolic invariant manifold.

2. **Preliminaries.**

2.1. **Notation.** By Id $: X \to X$ we will denote the identity map.

Let $\mathbb{R}^n$ be equipped with some norm $\|\cdot\|$. For $x_0 \in \mathbb{R}^n$ and $r > 0$ we define $B_n(x_0, r) = \{x \in \mathbb{R}^n \mid \|x - x_0\| < r\}$. Since most of the time we will be using unit balls centered at 0, we set $B_n = B_n(0, 1)$. When considering a linear map on $A : \mathbb{R}^n\mathbb{R}^k$ the symbol $\|A\|$ will always denote the standard operator norm of A, i.e. $\|A\| = \sup_{\|x\|=1} \|Ax\|$. We will also use $\|A\|_m = \inf_{\|x\|=1} \|Ax\|$.

Let $\Pi X = X_1 \times X_2 \times \cdots \times X_n$. For $i = 1, \ldots, n$ we define projection $\pi_i : \Pi X \to X_i$ by $\pi_i(x_1, \ldots, x_n) = x_i$. Sometimes when the variables in the cartesian product have names, for example (x, y, z), we will use names of variables as indices for projections, hence $\pi_y(x, y, z) = y$.

Let $\mathcal{U} = \{U_i\}_{i\in I}$ and $\mathcal{V} = \{V_j\}_{j\in J}$ be coverings of a set X (i.e. $X = \bigcup \mathcal{U} = \bigcup \mathcal{V}$), we say that covering $\mathcal{V}$ is inscribed in $\mathcal{U}$, denoted by $\mathcal{V} \prec \mathcal{U}$, iff for any $j \in J$ there exists $i \in I$, such that $V_j \subset U_i$. If X is a topological space, then we say that covering $\mathcal{U}$ is open iff it consists from open sets.

Let U be a topological space. We say that U is contractible, when there exists a deformation retraction of U onto single point space, i.e. there exists continuous map $h : [0, 1] \to U$ and $x_0 \in U$, such that $h(0, q) = q$ and $h(1, q) = x_0$ for all for $q \in U$.

For $x, y \in \mathbb{R}^n$ we set $[x, y] = \{z = tx + (1 - t)y,\ t \in [0, 1]\}$.

2.2. **Vector bundles.** We start with recalling the definition of the vector bundle [6].

Definition 2.1. Let B, E be topological spaces. Let $p : E \to B$ be a continuous map. A *vector bundle chart* on (p, E, B) with *domain* U and *dimension* n is a homeomorphism $\varphi : p^{-1}(U) \to U \times \mathbb{R}^n$, where $U \subset B$ is open and such that

$$\pi_1 \circ \varphi(z) = p(z), \qquad \text{for } z \in p^{-1}(U). \tag{1}$$

We will denote such bundle chart by a pair (φ, U).

For each $x \in U$ we define the homeomorphism φ_x to be the composition

$$\varphi_x : p^{-1}(x) \xrightarrow{\varphi} \{x\} \times \mathbb{R}^n \to \mathbb{R}^n. \tag{2}$$

A *vector bundle atlas* Φ on (p, E, B) is a family of vector bundle charts on (p, E, B) with the values in the same $\mathbb{R}^n$, whose domains cover B and such that whenever (φ, U) and (ψ, V) are in Φ and $x \in U \cap V$, the homeomorphism $\psi_x \varphi_x^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is linear. The map

$$U \cap V \ni x \mapsto \psi_x \varphi_x^{-1} \in GL(n) \tag{3}$$

is continuous for all pairs of charts in Φ.

A maximal vector bundle atlas Φ is a *vector bundle structure* on (p, E, B).

We then call $\xi = (p, E, B, \Phi)$ a *vector bundle* having *(fibre) dimension* n, *projection* p, *total space* E and *base space* B. Often Φ will not be explicitly mentioned. In fact we may denote ξ by E. Sometimes it is convenient to put $E = E\xi$, $B = B\xi$, etc.

The *fibre* over $x \in B$ is the space $p^{-1}(x) = \xi_x = E_x$. ξ_x has the vector space structure.

If the E, B are C^r manifolds and all maps appearing in the above definition are C^r, then we will say that the bundle (p, E, B, Φ) is a C^r-bundle.

One can introduce the notion of subbundles, morphisms etc (see [6] and references given there). The fibers can have a structure: for example a scalar product, a norm, which depend continuously on the base point.

Definition 2.2. We say that the vector bundle ξ is a Banach vector bundle with fiber being the Banach space $(F, \|\cdot\|)$, if for each $x \in B\xi$ the fiber ξ_x is a Banach space with norm $\|\cdot\|_x$ such that for each bundle chart (φ, U) the map $\varphi_x : E_x \to F$ is an isometry ($\|\varphi(v)\| = \|v\|_x$).

For vector bundles ξ_1, ξ_2 over the same base space one can define $\xi = \xi_1 \oplus \xi_2$ by setting $\xi_x = \xi_{1,x} \oplus \xi_{2,x}$. In the following, points in $\xi_1 \oplus \xi_2$ will be denoted by a triple (z, x_1, x_2), where $z \in B\xi_1 = B\xi_2$, $x_1 \in \xi_{1,z}$ and $x_2 \in \xi_{2,z}$. If ξ_1 and ξ_2 are both Banach bundles, then $\xi_i \oplus \xi_2$ is also a Banach vector bundle with the norm on $\xi_{1,z} \oplus \xi_{2,z}$ defined by $\|(x_1, x_2)\|_z = \max(\|x_1\|_z, \|x_2\|_z)$. We will always use this norm on $\xi_1 \oplus \xi_2$. We will also always assume that the atlas on bundle $\xi_1 \oplus \xi_2$ respects this structure, namely if (η, U) is a bundle chart for $\xi_1 \oplus \xi_2$, then its restriction (obtained through projection) to ξ_i is also a bundle chart for ξ_i for $i = 1, 2$.

3. **Central hyperbolic sets, covering relations and cones.** In this section we introduce the setup for our approach. First we introduce the concept of a central hyperbolic set (ch-set). This set will play the role of a compact region in which we expect to find a normally hyperbolic invariant manifold. We will consider maps defined on ch-sets which have certain contraction and expansion properties. On a ch-set we will have three directions: the direction of topological expansion, the

direction of topological contraction, and a third, the central direction, in which the dynamics will be weaker than in the first two directions. The contraction and expansion properties will be expressed in terms of covering relations. The ch-sets will be equipped with cones. The cones will help us to narrow down and specify in more detail the directions of expansion and contraction.

3.1. **Central hyperbolic sets.** Before we introduce the definition of a central hyperbolic set we will need the following definition.

Definition 3.1. Let ξ_u and ξ_s be Banach vector bundles, with the base space Λ. Let $Z \subset \Lambda$. For $r > 0$ we define set $N(Z,r), N^{\pm}(Z,r) \subset \xi_u \oplus \xi_s$ by

$$\begin{aligned} N(Z,r) &= \{(z,z_u,z_s) \in \xi_u \oplus \xi_s \mid z \in Z, \quad \|z_u\| \le r, \quad \|z_s\| \le r\}, \\ N^{+}(Z,r) &= \{(z,z_u,z_s) \in \xi_u \oplus \xi_s \mid z \in Z, \quad \|z_u\| \le r, \quad \|z_s\| = r\}, \\ N^{-}(Z,r) &= \{(z,z_u,z_s) \in \xi_u \oplus \xi_s \mid z \in Z, \quad \|z_u\| = r, \quad \|z_s\| \le r\} \end{aligned}$$

If $r = 1$, then we will write $N(Z), N^{\pm}(Z)$ instead of $N(Z,1), N^{\pm}(Z,1)$ respectively.

Definition 3.2. Assume that, we have Banach vector bundles ξ_u, ξ_s, $\xi = \xi_u \oplus \xi_s$. Let u and s be the fiber dimension of ξ_u and ξ_s, respectively. Let the base space for ξ, denoted by Λ, be a compact manifold without boundary of dimension c.

A *central-hyperbolic set* D (a ch-set), is an object consisting of the following data

1. $|D|$ a compact subset of $E\xi$.
2. a homeomorphism $\phi : \xi_u \oplus \xi_s \to \xi_u \oplus \xi_s$ such that

$$p_\xi(\phi(z)) = p_\xi(z) \quad \text{for } z \in E\xi, \tag{4}$$
$$\phi(|D|) = N(\Lambda). \tag{5}$$

We will usually drop vertical bars in the symbol $|D|$ and write D instead.

The notation u, s and c for the dimensions in Definition 3.2, stand for the unstable, stable and central directions respectively. Their roles will become apparent once we introduce the definitions of covering relations and cone conditions for maps.

For a central-hyperbolic set D we define

$$\begin{aligned} D_\phi &= \phi(|D|) = N(\Lambda), \\ D_\phi^- &= \{(x,x_u,x_s) \in D_\phi, x \in \Lambda, x_u \in \xi_{u,x}, x_s \in \xi_{s,x} \mid \|x_u\| = 1\}, \\ D_\phi^+ &= \{(x,x_u,x_s) \in D_\phi, x \in \Lambda, x_u \in \xi_{u,x}, x_s \in \xi_{s,x} \mid \|x_s\| = 1\}, \\ D^- &= \phi^{-1}(D_\phi^-), \quad D^+ = \phi^{-1}(D_\phi^+). \end{aligned}$$

The sets D^+ and D^- will later on be the entrance and exit sets for a map defined on D, respectively.

From now on we assume that we work in the Banach vector bundle $\xi_u \oplus \xi_s$ with the base space given by a compact manifold without boundary Λ. We will now define an atlas on Λ, which will later be used for the definitions of covering relation and cone conditions.

Definition 3.3. For our vector bundle we define the *good atlas* as follows:

Assume that we have an atlas $\{(\eta_i, U_i)\}_{i\in I}$ for Λ, where I is a finite set and

$$\eta_i : U_i \to \eta_i(U_i) \subset \mathbb{R}^c, \quad \text{where } U_i \subset \Lambda \text{ is open.} \tag{6}$$

Moreover, we assume that for all $i \in I$ there exists $(\varphi, W) \in \Phi\xi$, such that $U_i \subset W$. We fix one such φ for each i. Therefore on $p^{-1}(U_i)$ we have a chart map

$$\tilde{\eta}_i : p^{-1}(U_i) \ni z \mapsto (\eta_i(p(z)), \pi_2\varphi(z)) \in \eta_i(U_i) \times (\mathbb{R}^u \oplus \mathbb{R}^s)\,. \tag{7}$$

Therefore the atlas is good if the domains of the chart maps on Λ are also domains for some bundle chart maps. From now we will always implicitly assume that we work with a good atlas.

Definition 3.4. Let $Z \subset U_i$, where (η_i, U_i) is a chart from some good atlas. For $r > 0$ we define $N_{\eta_i}(Z,r)$, the normal neighborhood, and $N^{\pm}_{\eta_i}(Z,r)$ normal exit and entry sets, as follows

$$N_{\eta_i}(Z,r) = \eta_i(Z) \times \overline{B}_u(0,r) \times \overline{B}_s(0,r) \subset \mathbb{R}^c \times \mathbb{R}^u \times \mathbb{R}^s, \tag{8}$$

$$N^{-}_{\eta_i}(Z,r) = \eta_i(Z) \times \partial B_u(0,r) \times \overline{B}_s(0,r) \subset N_{\eta_i}(Z,r), \tag{9}$$

$$N^{+}_{\eta_i}(Z,r) = \eta_i(Z) \times \overline{B}_u(0,r) \times \partial B_s(0,r) \subset N_{\eta_i}(Z,r). \tag{10}$$

If $r = 1$ then we will often drop it and write $N^{\pm}_{\eta_i}(Z)$ instead.

Obviously, for $Z \subset U_i$ we have

$$N(Z,r) = \tilde{\eta}_i^{-1}\left(N_{\eta_i}(Z,r)\right), \quad N^{\pm}(Z,r) = \tilde{\eta}_i^{-1}\left(N^{\pm}_{\eta_i}(Z,r)\right) \quad .$$

Let D be ch-set and let

$$f : |D| \to \xi_u \oplus \xi_s$$

be a continuous map. We define

$$f_\phi := \phi \circ f \circ \phi^{-1} : N(\Lambda) \to \xi_u \oplus \xi_s.$$

When using local coordinates around z and $f(z)$, given by charts (η_j, U_j) and (η_i, U_i) respectively, we will consider functions f_{ij} given by

$$f_{ij} := \tilde{\eta}_i \circ f_\phi \circ \tilde{\eta}_j^{-1}. \tag{11}$$

The following commutative diagram illustrates the meaning and mutual relations between maps f, f_ϕ and f_{ij}.

$$\begin{array}{ccc} D & \xrightarrow{f} & \xi_u \oplus \xi_s \\ \downarrow \phi & & \downarrow \phi \\ N(\Lambda) & \xrightarrow{f_\phi} & \xi_u \oplus \xi_s \\ \uparrow \tilde{\eta}_j^{-1} & & \uparrow \tilde{\eta}_i^{-1} \\ \eta_j(U_j) \times \overline{B}_u \times \overline{B}_s & \xrightarrow{f_{ij}} & \eta_i(U_i) \times \mathbb{R}^{u+s} \end{array}$$

3.2. **Covering relations for ch-sets.** In this section we give the definition of a covering relation for a ch-set. Covering relations will be used later on in the proof of our main result to ensure that we have an invariant set in the interior of a ch-set.

Definition 3.5. Assume that we have two Banach vector bundles ξ_u and ξ_s with the same base space Λ, a compact manifold without boundary of dimension c.

Let $D \subset \xi_u \oplus \xi_s$ be a ch-set and a map $f : D \to \xi_u \oplus \xi_s$ be continuous.

Assume that $\{(\eta_i, U_i)\}_{i \in I}$ is a good atlas on Λ, where I is some finite set, and $\{V_j\}_{j \in J}$ (J a finite set) is an open covering of Λ, such that for every $j \in J$ there exists $i_0, i_1 \in I$ (not necessarily unique) such that

$$V_j \subset U_{i_0} \tag{12}$$

$$f(p^{-1}(V_j) \cap |D|) \subset p^{-1}(U_{i_1}). \tag{13}$$

We will say that ch-set D *f-covers itself*, which we will denote by

$$D \overset{f}{\Longrightarrow} D,$$

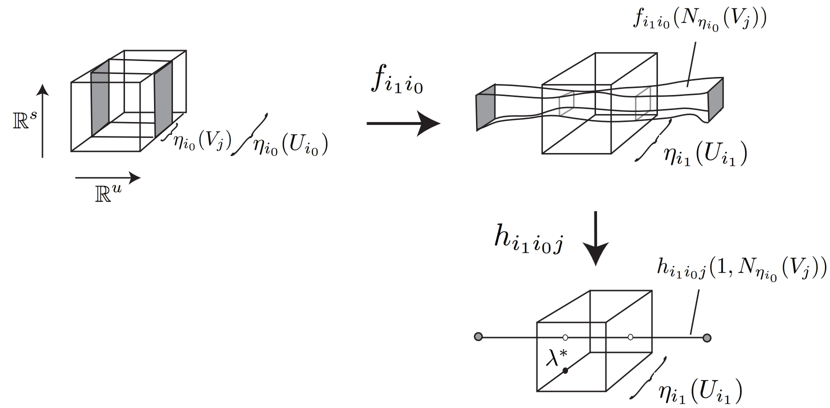

FIGURE 1. Covering relation for a ch-set.

if the following conditions are satisfied for every V_j.

Let U_{i_0} and U_{i_1} by any sets such that conditions (12) and (13) hold. The conditions are:

1. There exists a continuous homotopy

$$h_{i_1i_0j} : [0,1] \times N_{\eta_{i_0}}(V_j) \to \eta_{i_1}(U_{i_1}) \times \mathbb{R}^u \times \mathbb{R}^s$$

such that the following conditions hold true

$$h_{i_1i_0j}(0,z) = f_{i_1i_0}(z) \qquad \text{for } z \in N_{\eta_{i_0}}(V_j), \tag{14}$$

$$h_{i_1i_0j}\left([0,1], N^-_{\eta_{i0}}(V_j)\right) \cap N_{\eta_{i_1}}(U_{i_1}) = \emptyset, \tag{15}$$

$$h_{i_1i_0j}\left([0,1], N_{\eta_{i_0}}(V_j)\right) \cap N^+_{\eta_{i_1}}(U_{i_1}) = \emptyset. \tag{16}$$

2. If $u > 0$, then there exists a linear map $A_{i_1i_0j} : \mathbb{R}^u \to \mathbb{R}^u$ and $\lambda^* \in \eta_{i_1}(U_{i_1})$ such that

$$h_{i_1i_0j}(1,(\lambda,x,y)) = (\lambda^*, A_{i_1i_0j}x, 0),$$
$$A_{i_1i_0j}(\partial B_u(0,1)) \subset \mathbb{R}^u \backslash \overline{B_u}(0,1). \tag{17}$$

3. If $u = 0$, then there exists $\lambda^* \in \eta_{i_1}(U_{i_1})$ such that

$$h_{i_1i_0j}(1,(\lambda,y)) = (\lambda^*, 0).$$

The idea behind the definition can be summarized as follows. In local coordinates the map topologically expands the ch-set in the unstable direction and contracts it in the stable direction. In the central direction we do not require any contraction or expansion properties. We just require that the local maps are properly defined on sets $N_{\eta_{i_0}}(V_j)$ (see Figure 1).

Remark 1. In fact it is enough to require that for a given j there exists a required homotopy for just one pair of U_{i_0} and U_{i_1} satisfying conditions (12) and (13). For the other pairs the homotopy can be obtained using the transition maps from the atlas.

3.3. **Cones, horizontal and vertical disks.** In this section we introduce horizontal and vertical discs. These will be later on used as building blocks to construct both the normally hyperbolic invariant manifold and its stable and unstable manifolds in Section 4. Equipping the ch-set with cones will allow us to consider horizontal discs which are appropriately flat and vertical discs which are appropriately steep.

In this section we assume that we have a fixed Banach vector bundle $\xi_u \oplus \xi_s$ with a base space Λ, which is a compact manifold without boundary of dimension c.

Definition 3.6. Let $Q : \mathbb{R}^n \to \mathbb{R}$ be a quadratic form. We define positive and negative cone by

$$C^+(Q) = \{x \in \mathbb{R}^n \mid Q(x) > 0\},$$
$$C^-(Q) = \{x \in \mathbb{R}^n \mid Q(x) < 0\}.$$

If $z \in \mathbb{R}^n$ then we define positive and negative cones centered at z by

$$C^+(Q, z) = \{x \in \mathbb{R}^n \mid Q(z - x) > 0\} = z + C^+(Q),$$
$$C^-(Q, z) = \{x \in \mathbb{R}^n \mid Q(z - x) < 0\} = z + C^-(Q).$$

In the sequel we will work with cones given by locally defined quadratic forms on $\xi_u \oplus \xi_s$ and the sets of interests will be the intersection of positive or negative cones centered at z with $N(\Lambda)$. Let $Q : \mathbb{R}^c \times \mathbb{R}^u \times \mathbb{R}^s \to \mathbb{R}$ be a quadratic form. Let $z \in \mathbb{R}^c \times \mathbb{R}^u \times \mathbb{R}^s$, we set

$$C_r^{\pm}(Q, z) = C^{\pm}(Q, z) \cap \{(\theta, x, y) \in \mathbb{R}^c \times \mathbb{R}^u \times \mathbb{R}^s \mid \|x\| \le r,\ \|y\| \le r\}.$$

Definition 3.7. Let $D \subset \xi_u \oplus \xi_s$ be a ch-set.

Let $\{(\eta_i, U_i)\}_{i\in I}$ be a good atlas on Λ and assume that for all $i \in I$ we have quadratic forms $Q_{i,h}, Q_{i,v} : \mathbb{R}^c \times \mathbb{R}^u \times \mathbb{R}^s \to \mathbb{R}$ given by

$$Q_{i,h}(\theta, x, y) = \alpha_i(x) - \beta_i(y) - \gamma_i(\theta), \tag{18}$$
$$Q_{i,v}(\theta, x, y) = \alpha_i(x) - \beta_i(y) + \gamma_i(\theta), \tag{19}$$

where $\alpha_i : \mathbb{R}^u \to \mathbb{R}$, $\beta_i : \mathbb{R}^s \to \mathbb{R}$ and $\gamma_i : \mathbb{R}^c \to \mathbb{R}$ are positive definite quadratic forms. We assume that these forms are uniformly bounded in the following sense: there exist $M_+, M_- \in \mathbb{R}_+$, such that for every $i \in I$ hold

$$\begin{aligned} M_-\|x\|^2 \le \alpha_i(x) \le M_+\|x\|^2 &\qquad \text{for } x \in \mathbb{R}^u,\\ M_-\|y\|^2 \le \beta_i(y) \le M_+\|y\|^2 &\qquad \text{for } y \in \mathbb{R}^s,\\ M_-\|\theta\|^2 \le \gamma_i(\theta) \le M_+\|\theta\|^2 &\qquad \text{for } \theta \in \mathbb{R}^c. \end{aligned} \tag{20}$$

Assume that there exists an open covering $\{V_j\}_{j\in J}$ of Λ inscribed in $\{U_i\}_{i\in I}$, such that V_j is contractible for each j and the following two conditions are satisfied

1. for any point $q \in |D|$ there exists $j \in J$ and $i \in I$ such that $p(q) \in V_j$, $V_j \subset U_i$ and
$$\overline{C_1^+(Q_{i,h}, \tilde{\eta}_i(\phi(q)))} \subset N_{\eta_i}(V_j), \tag{21}$$
$$\overline{C_1^-(Q_{i,v}, \tilde{\eta}_i(\phi(q)))} \subset N_{\eta_i}(V_j). \tag{22}$$
2. for any $j \in J$ there exists an $i \in I$, such that for all $q \in |D| \cap p^{-1}(V_j)$ there exists a $V_{j(q)} \subset U_i$ such that
$$\overline{C_1^+(Q_{i,h}, \tilde{\eta}_i(\phi(q)))} \subset N_{\eta_i}(V_{j(q)}) \subset N_{\eta_i}(U_i), \tag{23}$$
$$\overline{C_1^-(Q_{i,v}, \tilde{\eta}_i(\phi(q)))} \subset N_{\eta_i}(V_{j(q)}) \subset N_{\eta_i}(U_i). \tag{24}$$

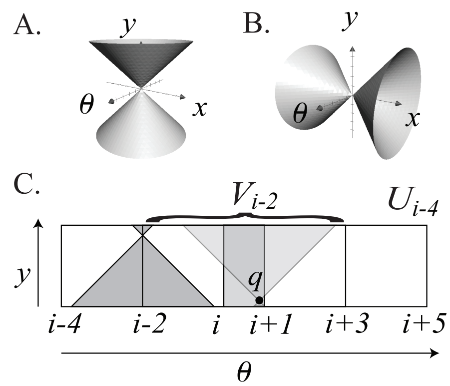

FIGURE 2. A vertical cone (A) and a horizontal cone (B), together with a projection of the cones chart pair (V_{i-2}, U_{i-4}) and two vertical cones (one centered at q and the other at a point from $|D| \cap p^{-1}(V_{i-2})$) onto the θ, y coordinates (C).

The structure consisting of $(D, \{\eta_i, U_i, Q_{i,h}, Q_{i,v}\}_{i\in I}, \{V_j\}_{j\in J})$ will be called a *ch set with cones*. Usually we will refer to this structure by D.

Remark 2. In fact we may allow to have different γ_i in $Q_{i,h}$ and $Q_{i,h}$, but α_i and β_i must coincide.

Definition 3.8. Assume that D is a ch-set with cones.

Let $q \in |D|$. We will say that (V_j, U_i) is a *cone enclosing pair for* q, when V_j and U_i are as in point 1. of Def. 3.7.

We will say that a pair (V_j, U_i) is a *cones chart pair* iff condition 2 in Def. 3.7. is satisfied.

Example 1. *Let us choose some $k \in \mathbb{N}$ such that $k \geq 9$. Consider a vector bundle $(\mathbb{R}/k) \times \mathbb{R}^u \times \mathbb{R}^s = S^1 \times \mathbb{R}^u \times \mathbb{R}^s$. Let $D = N(\mathbb{R}/k)$ and let $\alpha(x) = x^2$, $\beta(y) = y^2$, $\gamma(\theta) = \theta^2$, $Q_h = \alpha - \beta - \gamma$ and $Q_v = \alpha - \beta + \gamma$. If for $i, j \in \{0, \ldots, k\}$ we define the sets*

$$
\begin{aligned}
V_j &= j + (0,5) \ \mathit{mod}\ k,\\
U_i &= i + (0,9) \ \mathit{mod}\ k,
\end{aligned}
$$

then for any point q for which $\pi_\theta(q) \in [i, i+1]$ the pair (V_{i-2}, U_{i-4}) is both a cones chart pair, as well as a cone enclosing pair for q (see Figure 2).

We now move on to the definitions of horizontal and vertical discs.

Definition 3.9. Let D be a ch-set. We say that $b : \overline{B}_u \to D$ is a *horizontal disc* in D if there exists U_i and a homotopy $h : [0,1] \times \overline{B}_u \to N_{\eta_i}(U_i)$ such that

$$
\begin{aligned}
h(0,x) &= \tilde{\eta}_i(b_\phi(x)), \quad x \in \overline{B}_u,\\
h(1,x) &= (\lambda^*, x, 0), \quad \text{for all } x \in \overline{B}_u \text{ and some } \lambda^* \in \eta_i(U_i),\\
h(t,x) &\in N^-_{\eta_i}(U_i), \quad \text{for all } t \in [0,1] \text{ and } x \in \partial\overline{B}_u, \qquad (25)
\end{aligned}
$$

where $b_\phi = \phi \circ b$. Let us set $|b| = b(\overline{B}_u)$.

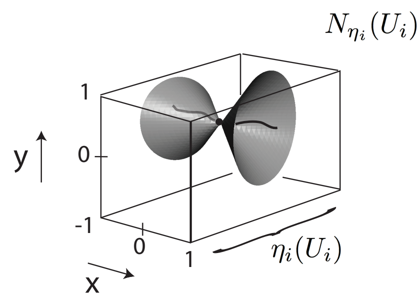

FIGURE 3. A horizontal disc which satisfies cone conditions (here $u = s = 1$).

If additionally, $|b| \subset Z \subset U_i$ and $h([0,1] \times \overline{B}_u) \subset N_{\eta_i}(Z)$, then we will say that b is a horizontal disk for the pair (Z, U_i).

Definition 3.10. Let D be a ch-set with cones, $Z \subset U_i$ and let $b : \overline{B}_u \to D$ be a horizontal disk for (Z, U_i). We say that b is a *horizontal disc satisfying the cone condition* for (Z, U_i) if the following condition holds

$$Q_{i,h}(\tilde{\eta}_i(b_\phi(x_0)) - \tilde{\eta}_i(b_\phi(x_1))) > 0, \quad \text{for all } x_0, x_1 \in \overline{B}_u. \tag{26}$$

If condition (26) holds for all U_j, such that $|b| \subset U_j$, then we say that b is a horizontal disk in D.

The idea behind Definition 3.10 is that when we consider a horizontal disc in local coordinates and attach a horizontal cone to any of its points, then the entire disc will be contained in the cone (see Figure 3).

Analogously we define a vertical disk as follows.

Definition 3.11. Let D be a ch-set. We say that $b : \overline{B}_s \to D$ is a *vertical disc* in D if there exists U_i and a homotopy $h : [0,1] \times \overline{B}_s \to N_{\eta_i}(U_i)$ such that

$$\begin{aligned} h(0,y) &= \tilde{\eta}_i(b_\phi(y)), \quad y \in \overline{B}_s, \\ h(1,y) &= (\lambda^*, 0, y), \quad \text{for all } y \in \overline{B}_s \text{ and some } \lambda^* \in \eta_i(U_i), \\ h(t,y) &\in N^+_{\eta_i}(U_i), \quad \text{for all } t \in [0,1] \text{ and } y \in \partial\overline{B}_s, \end{aligned} \tag{27}$$

where $b_\phi = \phi \circ b$. Let us set $|b| = b(\overline{B}_s)$.

If additionally, $|b| \subset Z \subset U_i$ and $h([0,1] \times \overline{B}_s) \subset N_{\eta_i}(Z)$, then we will say that b is a vertical disk for (Z, U_i).

Definition 3.12. Let D be a ch-set with cones, $Z \subset U_i$ and let $b : \overline{B}_s \to D$ be a vertical disk for (Z, U_i). We say that b is a *vertical disc satisfying the cone condition* for (Z, U_i) if the following condition holds

$$Q_{i,v}(\tilde{\eta}_i(b_\phi(y_0)) - \tilde{\eta}_i(b_\phi(y_1))) < 0, \quad \text{for all } y_0, y_1 \in \overline{B}_s. \tag{28}$$

If condition (28) holds for all U_j, such that $|b| \subset U_j$, then we say that b is a vertical disk in D.

The following lemma was proved in a slightly different setting in [11, Lemma 5]

Lemma 3.13. *Assume that $b : \overline{B}_u \to D$ is a horizontal disk satisfying cone conditions in U_i.*

Then b can be represented as the graph of the function in the following sense: there exists continuous functions $d_c : \overline{B}_u \to U_i$, $d_y : \overline{B}_u \to \overline{B}_s$ such that for any $x_1 \in \overline{B}_u$ there exists a uniquely defined $x \in \overline{B}_u$, such that

$$\tilde{\eta}_i(b_\phi(x_1)) = (d_c(x), x, d_y(x)). \tag{29}$$

An analogous lemma is valid for vertical disks satisfying cone conditions.

The lemma below shows that for a graph of a function over "unstable coordinate" with its range in $p^{-1}(V_j)$ for some V_j is a horizontal disk in V_j.

Lemma 3.14. *Assume that $V_j \subset \Lambda$ is contractible and that we have continuous functions $d_c : \overline{B}_u \to V_j$, $d_y : \overline{B}_u \to \overline{B}_s$. Let $d : \overline{B}_u \to N_{\eta_i}(V_j)$ be given by $d(x) = (d_c(x), x, d_y(x))$.*

Then there exists homotopy $h : [0,1] \times \overline{B}_u \to N_{\eta_i}(V_j)$ such that

$$\begin{aligned} h(0,x) &= d(x), \quad x \in \overline{B}_u, \\ h(1,x) &= (\lambda^*, x, 0), \quad \text{for all } x \in \overline{B}_u \text{ and some } \lambda^* \in \eta_i(V_j), \\ h(t,x) &\in N^-_{\eta_i}(V_j), \quad \text{for all } t \in [0,1] \text{ and } x \in \partial \overline{B}_u, \end{aligned} \tag{30}$$

which means that $\phi^{-1} \circ \tilde{\eta}_i^{-1} \circ d : \overline{B}_u \to D$ is horizontal disc in V_j.

Proof. From the contractibility of V_j it follows that there exists $\lambda^* \in V_j$ and a continuous homotopy $g : [0,1] \times V_j \to V_j$ such that

$$g(0,\lambda) = \lambda, \quad g(1,\lambda) = \lambda^*, \qquad \text{for all } \lambda \in V_j.$$

We define homotopy h by setting

$$h(t,x) = (g(t, d_c(x)), x, (1-t)d_y(x)). \tag{31}$$

□

An analogous lemma is valid for vertical disks.

4. **Cone-conditions for maps and main theorems.** Our main result is given in Theorem 4.7. There we will show that for a map which satisfies cone conditions (see Def. 4.1) there exists an invariant manifold together with its stable and unstable manifolds inside of the ch-set. For the proof of Theorem 4.7 we will first show that if a map satisfies cone conditions then an image of a horizontal disc which satisfies cone conditions is also a horizontal disc which satisfies cone conditions. This fact will be used to construct vertical discs of points whose forward iterations remain inside of the ch-set. Using these discs we will construct our invariant manifold together with its stable and unstable manifolds.

We start by defining cone conditions for a map.

Definition 4.1. Assume that D is a central-hyperbolic set with cones. Let $f : D \to \xi_u \oplus \xi_s$ be continuous and assume that $D \overset{f}{\Longrightarrow} D$. Moreover, we assume that the inscribed coverings $\{V_j\}_j \in J$ used in covering relation $D \overset{f}{\Longrightarrow} D$ (Def. 3.5) and cones in D (Def. 3.7) coincide.

We say that f *satisfies (forward) cone conditions on D* if there exists an $m > 1$ such that:

For any $x \in D$ and any (V_{j_0}, U_{i_0}) which is a cone enclosing pair for x, if $f(x) \in D$ then there exist $i_1 \in I$, $j_1 \in J$ such that (V_{j_1}, U_{i_1}) is a cone enclosing pair for $f(x)$, and $f(p^{-1}(V_{j_0}) \cap |D|) \subset p^{-1}(U_{i_1})$; what is more, for any $x_1, x_2 \in N_{\eta_{i_0}}(V_{j_0})$ such that

$$Q_{i_0,h}(x_1 - x_2) \geq 0 \tag{32}$$

we have

$$Q_{i_1,h}(f_{i_1 i_0}(x_1) - f_{i_1 i_0}(x_2)) > m Q_{i_0,h}(x_1 - x_2). \tag{33}$$

We will now define backward cone conditions for the inverse map. In order to do so we will need the following notations. Let $Q_{i,h}, Q_{i,v} : \mathbb{R}^c \times \mathbb{R}^u \times \mathbb{R}^s \to \mathbb{R}$ be defined as in (18) and (19). We define $Q^-_{i,h}, Q^-_{i,v} : \mathbb{R}^c \times \mathbb{R}^u \times \mathbb{R}^s \to \mathbb{R}$ as

$$Q^{inv}_{i,h}(\theta, x, y) := -\alpha_i(x) + \beta_i(y) - \gamma_i(\theta), \tag{34}$$

$$Q^{inv}_{i,v}(\theta, x, y) := -\alpha_i(x) + \beta_i(y) + \gamma_i(\theta). \tag{35}$$

Definition 4.2. Assume that $(D, \{\eta_i, U_i, Q_{i,h}, Q_{i,v}\}_{i\in I}, \{V_j\}_{j\in J})$ is a ch-set with cones. Let $f : \xi_u \oplus \xi_s \supset \operatorname{dom}(f) \to \xi_u \oplus \xi_s$ be a homeomorphism, such that $D \subset \operatorname{dom}(f^{-1})$. Assume that $D \overset{f^{-1}}{\Longrightarrow} D$, with reversed roles of the coordinates u and s. We say that f *satisfies backward cone conditions* if f^{-1} satisfies forward cone conditions on $(D, \{\eta_i, U_i, Q^{-inv}_{i,h}, Q^{inv}_{i,v}\}_{i\in I}, \{V_j\}_{j\in J})$, with reversed roles of the coordinates u and s.

We will now show that an intersection of the ch-set with an image of a horizontal disc which satisfies cone conditions is a horizontal disc which satisfies cone conditions.

Lemma 4.3. *Assume that*

$$D \overset{f}{\Longrightarrow} D,$$

and that f satisfies (forward) cone conditions. Let $b : \overline{B}_u \to D$. Assume that for any $x \in B^u$ we have a $(V_{j(x)}, U_{i(x)})$ which is a cone enclosing pair for $b(x)$, and that b is a horizontal disc which satisfies cone condition for $(V_{j(x)}, U_{i(x)})$. Then

$$\beta = \{q \in D | q = f(b(x)) \text{ for some } x \in B^u\}$$

is nonempty and for any $q \in \beta$ there exists a $(V_{j(q)}, U_{i(q)})$ which is a cone enclosing pair for q, such that β is a horizontal disc which satisfies cone condition for $(V_{j(q)}, U_{i(q)})$.

Proof. Without any loss of generality we can assume that $\phi = \mathrm{Id}$ and therefore $D_\phi = D = N(\Lambda)$, $b_\phi = b$ and $f_\phi = f$.

Take any V_j, U_{i_0}, U_{i_1} such that $b(\overline{B}_u) \subset p^{-1}(V_j)$, $V_j \subset U_{i_0}$ and

$$f(p^{-1}(V_j) \cap |D|) \subset p^{-1}(U_{i_1}). \tag{36}$$

The existence of such sets is a direct consequence of Def. 3.5. From Def. 3.10 it follows that there exists a homotopy $h : [0,1] \times \overline{B}_u \to N_{\eta_{i_0}}(V_j)$ and $\lambda^* \in \eta_{i_0}(V_j)$ satisfying

$$h(0, x) = \eta_{i_0}(b(x)), \quad x \in \overline{B}_u, \tag{37}$$

$$h(1, x) = (\lambda^*, x, 0), \quad x \in \overline{B}_u, \tag{38}$$

$$h(t, \partial B_u) \subset N^-_{\eta_{i_0}}(V_j), \quad t \in [0,1]. \tag{39}$$

Let us denote by g the homotopy $g = h_{i_1 i_0 j}$ appearing in Def. 3.5 and also let $A = A_{i_1 i_0 j}$. To study the set $f(b(\overline{B}_u))$ and its behavior under the homotopy h and g it is enough to use charts (η_{i_1}, U_{i_1}) in the range and (η_{i_0}, U_{i_0}) in the domain.

We will show now that for any $z \in \overline{B}_u$ there exits $x \in B_u$, such that

$$\pi_x f_{i_1 i_0}(\tilde{\eta}_{i_0}(b(x))) = z. \tag{40}$$

Later we will prove the uniqueness of such x. This will allow us to define map $z \mapsto f(b(x(z)))$, which we will show to be the horizontal disc we are looking for.

To study equation (40) we consider a parameter dependent map $H : [0,1] \times \overline{B}_u \to \mathbb{R}^u$ given by

$$H(t,x) = \pi_x g(t, h(t,x)) - (1-t)z. \tag{41}$$

Using the local Brouwer degree we will study the parameter dependent equation

$$H_t(x) = 0. \tag{42}$$

Any solution of (42) with $t = 0$ is a solution of problem (40) and vice versa.

Observe first that the local Brouwer degree (see Section 8 for the properties of the degree) of H_t on B_u at 0, denoted by $\deg(H_t, B_u, 0)$, is defined because from Def. 3.5 it follows that

$$H_t(\partial B_u) \subset \pi_x g(t, \partial B_u) \subset \mathbb{R}^u \setminus \overline{B}_u. \tag{43}$$

Hence

$$0 \notin H_t(\partial B_u). \tag{44}$$

Therefore from the homotopy property of local degree it follows that

$$\deg(H_t, B_u, 0) = \deg(H_1, B_u, 0) \quad t \in [0,1]. \tag{45}$$

For $t = 1$ we have

$$H_1(x) = Ax, \tag{46}$$

and since A is a nonsingular linear map, therefore

$$\deg(H_t, B_u, 0) = \det A = \pm 1 \quad t \in [0,1], \tag{47}$$

hence $\deg(H_t, B_u, 0) \neq 0$ and by the solution property of the local degree, there exists an x in B_u solving (42). Therefore there exists a solution for (40).

Observe that this solution is unique. We can take any $x_0, x_1 \in \overline{B}_u$, $x_0 \neq x_1$, such that $f(b(x_0)) \in D$ and $f(b(x_1)) \in D$. We will use the notation $q = f(b(x_0))$. By Def. 4.1 we can choose $i(q), j(q)$ such that $(V_{j(q)}, U_{i(q)})$ is a cone enclosing pair for q and

$$f(p^{-1}(V_{j(x_0)}) \cap |D|) \subset p^{-1}(U_{i(q)}). \tag{48}$$

From the cone condition for b and map f it follows, that

$$\begin{aligned} &Q_{i(q),h}(f_{i(q)i(x_0)}(\tilde{\eta}_{i(x_0)}(b(x_0))) - f_{i(q)i(x_0)}(\tilde{\eta}_{i(x_0)}(b(x_1)))) > \\ &\qquad mQ_{i(x_0),h}(\tilde{\eta}_{i(x_0)}(b(x_0)) - \tilde{\eta}_{i(x_0)}(b(x_1))) > 0. \end{aligned} \tag{49}$$

This immediately implies that

$$\pi_x f_{i(q)i(x_0)}(\tilde{\eta}_{i(x_0)}(b(x_0)) \neq \pi_x f_{i(q)i(x_0)}(\tilde{\eta}_{i(x_0)}(b(x_1)).$$

Therefore, we have a well defined map $v : \overline{B}_u \to B_u$, given as the solution of implicit equation $\pi_x f_{i(q)i(x_0)}(\tilde{\eta}_{i(x_0)}(b(v(x))) = x$. We define a map $d : \overline{B}_u \to N_{\eta_{i(q)}}(U_{i(q)})$ by $d(x) = f(b(v(x)))$.

Take now any x_0 such that $q = f(b(x_0)) \in |D|$, and take a cone enclosing pair $(V_{j(q)}, U_{i(q)})$ for q which satisfies (48–49). We will show that d is a horizontal disc which satisfies cone condition for $(V_{j(q)}, U_{i(q)})$. From (49)) it follows that for any $x_1, x_2 \in \overline{B}_u$, $x_1 \neq x_2$ we have

$$Q_{i(q),h}(\eta_{i(q)}(d(x_1)) - \eta_{i(q)}(d(x_2))) > 0. \tag{50}$$

Observe that this implies that the map $\tilde{d} = \tilde{\eta}_{i(q)} \circ d$ is Lipschitz. Namely, for any $x_1, x_2 \in \overline{B}_u$ from (50) we have

$$\beta_{i(q)}(\pi_3(\tilde{d}(x_1) - \tilde{d}(x_2))) + \gamma_{i(q)}(\pi_1(\tilde{d}(x_1) - \tilde{d}(x_2))) \leq \alpha_{i(q)}(\pi_2(\tilde{d}(x_1) - \tilde{d}(x_2)), \quad (51)$$

therefore we obtain

$$\begin{aligned} M_- \|\pi_3(\tilde{d}(x_1) - \tilde{d}(x_2))\|^2 + M_- \|\pi_1(\tilde{d}(x_1) - \tilde{d}(x_2))\|^2 \\ \leq M_+ \|\pi_2(\tilde{d}(x_1) - \tilde{d}(x_2))\|^2 = M_+ \|x_1 - x_2\|^2, \end{aligned}$$

which ensures continuity of d. To finish the argument that d is a horizontal disk for $(V_{j(q)}, U_{i(q)})$ we need the homotopy. This homotopy is obtained by applying Lemmas 3.13 and 3.14. □

From Lemma 4.3 we obtain by induction the following lemma.

Lemma 4.4. *Let $n \geq 1$. Assume that*

$$D \overset{f}{\Longrightarrow} D,$$

and that f satisfies (forward) cone conditions. Let $b : \overline{B}_u \to D$. Assume that for any $x \in B^u$ we have a $(V_{j(x)}, U_{i(x)})$ which is a cone enclosing pair for $b(x)$, and that b is a horizontal disc which satisfies cone condition for $(V_{j(x)}, U_{i(x)})$. Then

$$\beta = \{y : y = f^n(b(x)) \text{ and } f^i(b(x)) \in D \text{ for } i = 1, \ldots, n \text{ for some } x \in B^u\}$$

is nonempty and for any $y \in \beta$ there exists a $(V_{j(y)}, U_{i(y)})$ which is a cone enclosing pair for y, such that β is a horizontal disc which satisfies cone condition for $(V_{j(y)}, U_{i(y)})$.

Using Lemma 4.4 we will now show that on a horizontal disc which satisfies cone conditions we have a unique point whose forward iterations remain inside of the ch-set.

Lemma 4.5. *Assume that*

$$D \overset{f}{\Longrightarrow} D,$$

and that f satisfies (forward) cone conditions. Let $b : \overline{B}_u \to D$. Assume that for any $x \in B_u$ we have a $(V_{j(x)}, U_{i(x)})$ which is a cone enclosing pair for $b(x)$, and that b is a horizontal disc which satisfies cone condition for $(V_{j(x)}, U_{i(x)})$. Then there exists a unique point $x^* \in \overline{B}_u$, such that

$$f^n(b(x^*)) \in \operatorname{int} D, \qquad n = 0, 1, 2, \ldots. \quad (52)$$

Proof. Without any loss of generality we can assume that $\phi = \mathrm{Id}$ and therefore $D_\phi = D = N(\Lambda)$, $b_\phi = b$ and $f_\phi = f$.

From Lemma 4.4 it follows that for any $n \in \mathbb{N}$ there exists a point $x_n \in \overline{B}_u$ such that

$$f^k(b(x_n)) \in D \quad \text{for } k = 0, 1, \ldots, n.$$

Because $\overline{B}_u$ is compact and D is closed, we can pass to a convergent subsequence and obtain a point $x^* \in \overline{B}_u$, such that

$$f^k(b(x^*)) \in D \quad \text{for } k = 0, 1, \ldots. \quad (53)$$

In fact we have

$$f^k(b(x^*)) \in \operatorname{int} D \quad \text{for } k = 0, 1, \ldots \quad (54)$$

because from the definition of covering relation it follows that points from D^+ are not in $f(D)$ and points from D^- are mapped out of D.

Let us show that such a point x^* is unique. Let us assume that we have two points $x_0, x_1 \in \overline{B}_u$, $x_0 \neq x_1$ satisfying (52). From Lemma 4.4 it follows that for any $n = 0, 1, 2, \ldots$ points $q_{l,n}$ given by

$$q_{l,n} := f^n(b(x_l)) \in \operatorname{int} D, \quad l = 0, 1 \tag{55}$$

belong to b_n, a horizontal disc satisfying cone conditions for a cone enclosing pair (V_{j_n}, U_{i_n}) for $q_{0,n}$.

From cone conditions for f and b_n we have

$$Q_{i_n,h}(\tilde{\eta}_{i_n}(q_{0,n}) - \tilde{\eta}_{i_n}(q_{1,n})) > mQ_{i_{n-1},h}(\tilde{\eta}_{i_{n-1}}(q_{0,n-1}) - \tilde{\eta}_{i_{n-1}}(q_{1,n-1})). \tag{56}$$

Hence

$$Q_{i_n,h}(\tilde{\eta}_n(q_{0,n}) - \tilde{\eta}_n(q_{1,n})) > m^n Q_{0,h}(\tilde{\eta}_{i_0}(q_{0,0}) - \tilde{\eta}_{i_0}(q_{1,0})). \tag{57}$$

From (57) it follows that

$$\begin{aligned} 4M_+ \geq \alpha(\pi_x\tilde{\eta}_{i_n}(q_{0,n}) - \pi_x\tilde{\eta}_{i_n}(q_{1,n})) \geq \\ Q_{i_n,h}(\tilde{\eta}_{i_n}(q_{0,n}) - \tilde{\eta}_{i_n}(q_{1,n})) > m^n Q_{i_0,h}(\tilde{\eta}_{i_0}(q_{0,0}) - \tilde{\eta}_{i_0}(q_{1,0})) \\ = m^n\alpha(x_0 - x_1) \geq m^n M_- \|x_0 - x_1\|^2. \end{aligned}$$

This for $\|x_0 - x_1\| > 0$, since $m > 1$, can not be true for all $n = 0, 1, 2, \ldots$. Hence $x_0 = x_1$. This finishes the proof. □

We will now show that points whose forward iterations remain inside of the ch-set form vertical discs which satisfy cone conditions.

Theorem 4.6. *Let $n \geq 1$. Assume that*

$$D \overset{f}{\Longrightarrow} D,$$

and that f satisfies cone conditions. Then for any $\lambda_0 \in \Lambda$ there exists a $b : \overline{B}_s \to D$, such that

$$p(b(\overline{B}_s)) = \{\lambda_0\},$$

and for any $y \in \overline{B}_s$

$$f^n(b(y)) \in \operatorname{int} D \quad \textit{for } n = 0, 1, 2, \ldots. \tag{58}$$

In addition, b is a vertical disc which satisfies cone conditions for any cones chart pair (V_j, U_i) such that $\lambda_0 \in V_j$.

What is more, if $p(q) = \lambda_0$ and $f^n(q) \in D$ for all $n \in \mathbb{N}$, then $q \in b(\overline{B}_s)$.

Proof. Without any loss of generality we can assume that $\phi = \mathrm{Id}$, hence $b_\phi = b$ for all vertical or horizontal discs and $f_\phi = f$.

Let us choose $\lambda_0 \in \Lambda$ and some $q \in p^{-1}(\{\lambda_0\}) \cap |D|$. Let (V_j, U_i) be a cones chart pair and a cone enclosing pair for q. Let us fix a $y \in \overline{B}_s$ and let $b_0 : \overline{B}_u \to D$ be a horizontal disc defined by

$$b_0(x) = \tilde{\eta}_i^{-1}(\lambda_0, x, y). \tag{59}$$

Since (V_j, U_i) is a cones chart pair, for any $x \in \overline{B}_u$ we have a $(V_{j(x)}, U_{i(x)})$ which is a cone enclosing pair for $b_0(x)$. This means that we can apply Lemma 4.5. Hence there exists a uniquely defined $x^*(\lambda_0, y) \in \overline{B}_u$, such that

$$f^n(\tilde{\eta}_i^{-1}(\lambda_0, x^*(\lambda_0, y), y) \in \operatorname{int} D, \quad n = 0, 1, 2, \ldots. \tag{60}$$

We define a map $b : \overline{B}_s \to D$

$$b(y) = \tilde{\eta}_i(\lambda_0, x^*(\lambda_0, y), y). \tag{61}$$

We will show that the map b is a disc satisfying the assertion of our theorem.

We need to show that b is a vertical disc which satisfies cone conditions for (V_j, U_i). We will first show that for any two points $y_1, y_2 \in \overline{B}_s$ such that $y_1 \neq y_2$ we have

$$Q_{i,v}\left(\tilde{\eta}_i(b(y_1)) - \tilde{\eta}_i(b(y_2))\right) < 0. \tag{62}$$

Let us assume that (62) does not hold. Then since $\pi_\lambda(b(y_1)) = \pi_\lambda(b(y_2)) = \lambda_0$ we have

$$Q_{i,h}\left(\tilde{\eta}_i(b(y_1)) - \tilde{\eta}_i(b(y_2))\right) = Q_{i,v}\left(\tilde{\eta}_i(b(y_1)) - \tilde{\eta}_i(b(y_2))\right) \geq 0. \tag{63}$$

Let (V_{j_1}, U_{i_1}) be a cone enclosing pair for $f(b(y_1))$ such that $f(p^{-1}(V_j) \cap |D|) \subset p^{-1}(U_{i_1})$. From the fact that f satisfies cone conditions we have

$$Q_{i_1,h}(\tilde{\eta}_{i_1}(f(b(y_1))) - \tilde{\eta}_{i_1}(f(b(y_2)))) > mQ_{i,h}\left(\tilde{\eta}_i(b(y_1)) - \tilde{\eta}_i(b(y_2))\right) \geq 0. \tag{64}$$

Let us take any horizontal disc satisfying the cone condition for (V_{j_1}, U_{i_1}) passing through $f(b(y_1))$ and $f(b(y_2))$. From Lemma 4.5 it follows that since $f^n(b(y_i)) \in \operatorname{int} D$ for $i = 1, 2$ and all $n = 0, 1, 2, \ldots$ that $f(b(y_1)) = f(b(y_2))$, but this is in a contradiction with (64). Hence (62) holds.

From (62) we have

$$\begin{aligned}
0 > Q_{i,v}\left(\tilde{\eta}_i(b(y_1)) - \tilde{\eta}_i(b(y_2))\right) = \\
\alpha_i(\pi_x(\tilde{\eta}_i(b(y_1))) - \pi_x(\tilde{\eta}_i(b(y_2)))) - \beta_i(\pi_y(\tilde{\eta}_i(b(y_1))) - \pi_y(\tilde{\eta}_i(b(y_2)))) \\
= \alpha(x^*(\lambda_0, y_1) - x^*(\lambda_0, y_2)) - \beta(y_1 - y_2).
\end{aligned} \tag{65}$$

From (65) and (20) we have

$$\|x^*(\lambda_0, y_1) - x^*(\lambda_0, y_2)\|^2 \leq \frac{M_+}{M_-}\|y_1 - y_2\|^2, \tag{66}$$

which together with (61) b guarantees its continuity. This by Lemma 3.14 means that b is a vertical disc. Condition (62) now ensures that it satisfies cone conditions. □

We now come to our main result. Theorem 4.7 will give us the existence of a normally hyperbolic invariant manifold together with its stable an unstable manifolds. At this stage, the assumptions of the theorem might seem somewhat abstract. In Section 5 we will show how these assumptions can be verified in practice, and in Section 6 we will highlight how the theorem relates to the classical version of the normally hyperbolic invariant manifold theorem [7].

Theorem 4.7. *Let $(D, \{\eta_i, U_i, Q_{i,h}, Q_{i,v}\}_{i\in I}, \{V_j\}_{j\in J})$ be a ch-set with cones. Let $f : \xi_u \oplus \xi_s \to \xi_u \oplus \xi$ be a homeomorphism. If f satisfies forward and backward cone conditions, then:*

1. *There exists a continuous function*
$$\chi : \Lambda \to \operatorname{int} D,$$
such that $p(\chi(\lambda)) = \lambda$ for all $\lambda \in \Lambda$, and
$$\chi(\Lambda) = \operatorname{inv}(f, D) := \{z \in D | f^n(z) \in D \text{ for all } n \in \mathbb{Z}\}.$$
2. *There exist C^0 submanifolds W^u and W^s such that*
$$W^u \cap W^s = \chi(\Lambda),$$
W^u consists of all points whose backward iterations converge to $\chi(\Lambda)$, and W^s consists of all points whose forward iterations converge to $\chi(\Lambda)$.

Remark 3. Let us note that the stable and unstable manifolds W^u and W^s from Theorem 4.7 are different from the vector bundles ξ_u and ξ_s. The bundles ξ_u and ξ_s provide only approximate coordinates in which we look for W^u and W^s, in practice W^u will be close to ξ_u and W^s close to ξ_s, but they need not precisely match.

Proof of Theorem 4.7. Observe first that vertical cones for the inverse map coincide with horizontal cones for the forward map (see (18), (35))

$$C^-(Q^-_{i,v}) = C^+(Q_{i,h}). \tag{67}$$

Take λ_0 from Λ. From the fact that f satisfies cone conditions, by Theorem 4.6 we have a vertical disc $b : B_s \to D \cap p^{-1}(\lambda_0)$ of forward invariant points (b is vertical for the ch-set $(D, \{\eta_i, U_i, Q_{i,h}, Q_{i,v}\}_{i\in I}, \{V_j\}_{j\in J})$). Since f^{-1} satisfies cone conditions, by Theorem 4.6 we also have a "vertical" disc $b^- : B_u \to D \cap p^{-1}(\lambda_0)$ of backward invariant points (b^- is vertical with respect to the ch-set $(D, \{\eta_i, U_i, Q^-_{i,h}, Q^-_{i,v}\}_{i\in I}$, $\{V_j\}_{j\in J})$ with reversed roles of u and s). We can define

$$\chi(\lambda_0) := b(B_s) \cap b^-(B_u).$$

The function is χ properly defined since by (67) b^- is a horizontal disc for the ch-set $(D, \{\eta_i, U_i, Q_{i,h}, Q_{i,v}\}_{i\in I}, \{V_j\}_{j\in J})$. A horizontal disc and a vertical disc which satisfy cone conditions and are contained in $p^{-1}(\lambda_0)$ intersect at a single point.

We now have to show that χ is continuous. Take any sequence $\lambda_n \to \lambda_0$. By the fact that D is compact we can take a convergent subsequence $\chi(\lambda_{v_n}) \to z$. We need to show that $z = \chi(\lambda_0)$. For any $k \in \mathbb{N}$, by continuity of functions f, f^{-1} and closeness of D, we know that $\lim_{n\to\infty} f^k(\lambda_{v_n}), \lim_{n\to\infty} f^{-k}(\lambda_{v_n}) \in D$, hence $z \in \mathrm{inv}(f, D)$. The fact that $z = \chi(\lambda_0)$ follows from the uniqueness of the choice of $\chi(\lambda_0)$.

Now we will construct the manifold W^s. For any λ from Λ, by Theorem 4.6 we have a vertical disc $b_\lambda : B_s \to D \cap p^{-1}(\lambda)$ of forward invariant points. We can define

$$W^s = \bigcup_{\lambda\in\Lambda} b_\lambda(B_s). \tag{68}$$

We need to show that W^s is C^0. We take any Cauchy sequence x_n from W^s. From the compactness of D we know that x_n converges to some $x_0 \in D$. We will show that $x_0 \in W^s$. Since for any $k, n > 0$ we have $f^k(x_n) \in D$, by continuity of f we have that $f^k(x_0) \in D$. Letting $\lambda_0 = \pi_\lambda(\phi(x_0))$, we can see that $x_0 \in p^{-1}(\lambda_0)$, which by Theorem 4.6 means that x_0 lies on the unique vertical disc b_{λ_0} of forward invariant points in $p^{-1}(\lambda_0)$, hence by (68) we have $x_0 \in W^s$.

The construction of W^u is analogous.

We will now show that for any $x_0 \in W^s$ $f^n(x_0)$ converges to $\chi(\Lambda)$ as n goes to infinity. Let us consider the limit set of the point x_0

$$\omega(f, x_0) = \{q | \lim_{k\to\infty} f^{n_k}(x_0) = q \text{ for some } n_k \to \infty\}.$$

If we can show that $\omega(f, x_0)$ is contained in $W^u \cap W^s = \chi(\Lambda)$, then this will conclude our proof. We take any $q = \lim_{k\to\infty} f^{n_k}(x_0)$ from $\omega(f, x_0)$. We need to show that $q \in W^u \cap W^s$. By continuity of W^s we know that $q \in W^s$. Suppose now that $q \notin W^u$. This would mean that there exists an $n > 0$ for which $f^{-n}(q) \notin D$. Since

$$\lim_{k\to\infty} f^{n_k - n}(x_0) = f^{-n}(q),$$

we have that $f^{-n}(q) \in \omega(f, x_0)$, but this contradicts the fact that $\omega(f, x_0) \subset D$.

Showing that all backward iterations of points in W^u converge to $\chi(\Lambda)$ is analogous. □

5. **Rigorous numerical verification of covering and cone conditions.** In this section we will show how assumptions of Theorem 4.7 can be verified. We set up our conditions so that they are verifiable using rigorous numerics. To verify them it is enough to obtain estimates of the derivatives of the local maps on the ch-set. We will show that the assumptions of Theorem 4.7 follow from explicit algebraic conditions on these estimates. An example of how this works in practice will be shown in Section 7.

We start with the definition of an interval enclosure of a derivative.

Definition 5.1. Let $U \subset \mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ be a C^1 function. We define the *interval enclosure* of *df* on the set U as

$$[df(U)] = \left\{ A \in \mathbb{R}^{n\times n} | A_{ij} \in \left[\inf_{x\in U} \frac{df_i}{dx_j}(x), \sup_{x\in U} \frac{df_i}{dx_j}(x) \right] \text{ for all } i,j = 1,\ldots,n \right\}.$$

5.1. **Verification of the covering condition.** In this section we show how covering relations on ch-sets follow from algebraic conditions on the derivatives of local maps.

Theorem 5.2. *Assume that $(D, \{\eta_i, U_i, Q_{i,h}, Q_{i,v}\}_{i\in I}, \{V_j\}_{j\in J})$ is a ch-set with cones, with convex $\eta_i(V_j)$ for all $V_j \subset U_i$. Assume that $f : D \to \xi_u \oplus \xi_s$ is such that for any $V_j \subset U_{i_0}$ there exists i_1 for which*

$$f(p^{-1}(V_j) \cap |D|) \subset p^{-1}(U_{i_1}).$$

Assume that for any such j, i_0, i_1 the function $f_{i_1 i_0}$ is differentiable, and for any $\theta \in \eta_{i_0}(V_j)$ we have

$$f_{i_1 i_0}(\theta, 0, 0) \in \eta_{i_1}(U_{i_1}) \times \overline{B_u}(0, \varepsilon_u) \times \overline{B_s}(0, \varepsilon_s) \tag{69}$$

for some $1 > \varepsilon_u, \varepsilon_s > 0$ ($\varepsilon_u, \varepsilon_s$ can be dependent on the choice of j, i_0, i_1). If for any matrix $A \in [Df_{i_1 i_0}\left(N_{\eta_{i_0}}(V_j)\right)]$, the following conditions hold

$$\inf\{\|(\pi_x \circ A)(0, x, y)\| : ||x|| = 1, ||y|| \le 1\} > 1 + \varepsilon_u, \tag{70}$$

$$\sup\{\|(\pi_y \circ A)(0, x, y)\| : ||x|| \le 1, ||y|| \le 1\} < 1 - \varepsilon_s, \tag{71}$$

then $D \overset{f}{\Longrightarrow} D$.

Proof. For j, i_0, i_1 such that

$$f(p^{-1}(V_j) \cap |D|) \subset p^{-1}(U_{i_1}),$$

we need to define the homotopy $h = h_{i_1 i_0, j}$ from Definition 3.5. We will use the notations $p = (x, y)$ and $f_c := \pi_\theta \circ f_{i_1 i_0}$, $f_u := \pi_x \circ f_{i_1 i_0}$, $f_s := \pi_y \circ f_{i_1 i_0}$. We take any $\theta^* \in \eta_{i_0}(V_j)$ and define h as

$$\begin{aligned} h_\alpha(\theta, p) = {} & (f_c((1-\alpha)\theta + \alpha\theta^*, (1-\alpha)p), (1-\alpha)f_u(\theta, 0), (1-\alpha)f_s(\theta, 0)) \\ & + \left(0, \pi_x \left(\int_0^1 Df_{i_1 i_0}((1-\alpha)\theta + \alpha\theta^*, (1-\alpha)tp)\, dt \cdot (0, x, (1-\alpha)y) \right)\right. \\ & \left. , \pi_y \left((1-\alpha) \int_0^1 Df_{i_1 i_0}(\theta, tp) dt \cdot (0, p) \right) \right). \end{aligned}$$

Since
$$f_{i_1 i_0}(\theta,p) - f_{i_1 i_0}(\theta,0) = \int_0^1 Df_{i_1 i_0}(\theta,tp)\,dt\cdot(0,p),$$
we have $h_0(\theta,p) = f_{i_1 i_0}(\theta,p)$. For $\alpha=1$ we have
$$h_1(\theta,p) = (\lambda^*, Ax, 0)\,,$$
with
$$\lambda^* = f_c(\theta^*,0),$$
$$Ax = \pi_x\left(Df_{i_1 i_0}(\theta^*,0,0)\cdot(0,x,0)\right).$$

For any α from $[0,1]$, from the fact that
$$A^\alpha := \int_0^1 Df_{i_1 i_0}((1-\alpha)\theta+\alpha\theta^*,(1-\alpha)tp)dt \in \left[Df_{i_1 i_0}(N_{\eta_{i_0}}(V_j))\right],$$
for any $(\theta,p)\in N^-_{\eta_{i_0}}(V_j)$ using (69) and (70) we have
$$\begin{aligned}
&|\pi_x(h_\alpha(\theta,p))| \\
&= \bigg|(1-\alpha)f_u(\theta,0) \\
&+\pi_x\left(\int_0^1 Df_{i_1 i_0}((1-\alpha)\theta+\alpha\theta^*,(1-\alpha)tp)\,dt\cdot(0,x,(1-\alpha)y)\right)\bigg| \\
&\geq \left|\pi_x\left(\int_0^1 Df_{i_1 i_0}((1-\alpha)\theta+\alpha\theta^*,(1-\alpha)tp)dt\cdot(0,x,(1-\alpha)y)\right)\right| - \varepsilon_u \\
&= \left|\pi_x\left(A^\alpha\left(0,x,(1-\alpha)y\right)\right)\right| - \varepsilon_u \\
&> 1.
\end{aligned} \tag{72}$$

This proves that for any $\alpha\in[0,1]$ we have $h_\alpha(N^-_{\eta_{i_0}}(V_j))\cap N^-_{\eta_{i_1}}(U_{i_1})=\emptyset$. Also for $\alpha=1$ since $A(x)=A^1(0,x,0)=\pi_x(h_1(\theta,x,0))$ from (72) we have $A\left(\partial B_u(0,1)\right)\cap\overline{B_u}(0,1)=\emptyset$.

For $(\theta,p)\in N^-_{\eta_{i_0}}(V_j)$, using (69) and (71) we have
$$\begin{aligned}
|\pi_y\left(h_\alpha(\theta,p)\right)| &= \left|(1-\alpha)f_s(\theta,0)+\pi_y\left((1-\alpha)\int_0^1 Df_{i_1 i_0}(\theta,tp)dt\cdot(0,p)\right)\right| \\
&\leq \left|(1-\alpha)\pi_y\left(\int_0^1 Df_{i_1 i_0}(\theta,tp)dt\cdot(0,p)\right)\right| + \varepsilon_s \\
&= (1-\alpha)|\pi_y(A^0(0,p))| + \varepsilon_s \\
&< 1,
\end{aligned}$$
which means that $h_\alpha(N_{\eta_{i_0}}(V_j))\cap N^+_{\eta_{i_1}}(U_{i_1})=\emptyset$. This finishes our proof. □

5.2. **Verification of cone conditions.** In this section we will show how to verify cone conditions using interval enclosures of derivatives of local maps.

We start with a technical lemma.

Lemma 5.3. *Let* $p=(p_1,p_2,p_3)\in\mathbb{R}^c\times\mathbb{R}^u\times\mathbb{R}^s$ *and*
$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \tag{73}$$

be an $(c+u+s)\times(c+u+s)$ *matrix. If* $Q_h(p_1,p_2,p_2) = -\|p_1\|^2 + \|p_2\|^2 - \|p_3\|^2$, *then*

$$Q_h\left(Ap\right) \geq -\mathbf{a}\left\|p_1\right\|^2 + \mathbf{b}\left\|p_2\right\|^2 - \mathbf{c}\left\|p_3\right\|^2,$$

where

$$\begin{aligned}
\mathbf{a} &= \left\|A_{11}\right\|^2 - \left\|A_{21}\right\|_m^2 + \left\|A_{31}\right\|^2 + \sum_{i=1}^{3}\left\|A_{i1}\right\|\left(\left\|A_{i2}\right\| + \left\|A_{i3}\right\|\right),\\
\mathbf{b} &= -\left\|A_{12}\right\|^2 + \left\|A_{22}\right\|_m^2 - \left\|A_{32}\right\|^2 - \sum_{i=1}^{3}\left\|A_{i2}\right\|\left(\left\|A_{i1}\right\| + \left\|A_{i3}\right\|\right),\\
\mathbf{c} &= \left\|A_{13}\right\|^2 - \left\|A_{23}\right\|_m^2 + \left\|A_{33}\right\|^2 + \sum_{i=1}^{3}\left\|A_{i3}\right\|\left(\left\|A_{i1}\right\| + \left\|A_{i2}\right\|\right).
\end{aligned} \tag{74}$$

Proof. Using the fact that

$$\pm 2\left\langle A_{ki}p_i, A_{kj}p_j\right\rangle \geq -||A_{ki}||\cdot||A_{kj}||\left(||p_i||^2 + ||p_j||^2\right),$$

we can compute

$$\begin{aligned}
&Q(Ap) =\\
&= -||\sum_{i=1}^{3}A_{1i}p_i||^2 + ||\sum_{i=1}^{3}A_{2i}p_i||^2 - ||\sum_{i=1}^{3}A_{3i}p_i||^2\\
&= -\sum_{i,j=1}^{3}\left\langle A_{1i}p_i, A_{1j}p_j\right\rangle + \sum_{i,j=1}^{3}\left\langle A_{2i}p_i, A_{2j}p_j\right\rangle - \sum_{i,j=1}^{3}\left\langle A_{3i}p_i, A_{3j}p_j\right\rangle\\
&= -\sum_{i=1}^{3}||A_{1i}p_i||^2 + \sum_{i=1}^{3}||A_{2i}p_i||^2 - \sum_{i=1}^{3}||A_{3i}p_i||^2 - 2\sum_{k=1}^{3}\sum_{i<j}\pm\left\langle A_{ki}p_i, A_{kj}p_j\right\rangle\\
&\geq \left\|p_1\right\|^2\left(-||A_{11}||^2 + ||A_{21}||_m^2 - ||A_{31}||^2\right)\\
&\quad + \left\|p_2\right\|^2\left(-||A_{12}||^2 + ||A_{22}||_m^2 - ||A_{32}||^2\right)\\
&\quad + \left\|p_3\right\|^2\left(-||A_{13}||^2 + ||A_{23}||_m^2 - ||A_{33}||^2\right)\\
&\quad - \sum_{k=1}^{3}\sum_{i<j}||A_{ki}||||A_{kj}||\left(||p_i||^2 + ||p_j||^2\right)\\
&= -\mathbf{a}\left\|p_1\right\|^2 + \mathbf{b}\left\|p_2\right\|^2 - \mathbf{c}\left\|p_3\right\|^2.
\end{aligned}$$

□

The theorem below gives conditions which imply cone conditions.

Theorem 5.4. *Assume that* $(D, \{\eta_i, U_i, Q_{i,h}, Q_{i,v}\}_{i\in I}, \{V_j\}_{j\in J})$ *is a ch-set with cones, with convex* $\eta_i(V_j)$ *for all* $V_j \subset U_i$. *Assume that the maps used for* $Q_{i,h}, Q_{i,v}$ *(see (18, 19)) are*

$$\alpha(x) = \|x\|^2, \quad \beta(y) = \|y\|^2, \quad \gamma(\theta) = \|\theta\|^2. \tag{75}$$

Assume that for any $x \in D$ *and any* (V_{j_0}, U_{i_0}) *which is a cone enclosing pair for* x, *if* $f(x) \in D$ *then there exist* $i_1 \in I$, $j_1 \in J$ *such that* (V_{j_1}, U_{i_1}) *is a cone enclosing pair for* $f(x)$ *and* $f(p^{-1}(V_{j_0}) \cap |D|) \subset p^{-1}(U_{i_1})$. *What is more for any*

$$\begin{aligned}
&A \in [df_{i_1 i_0}(N_{\eta_{i_0}}(V_j))],\\
&A = (A_{ik})_{i,k=1,\ldots,3},
\end{aligned}$$

we assume that we have

$$\begin{array}{lll} \|A_{11}\| \leq C, & \|A_{12}\| \leq \varepsilon_c, & \|A_{13}\| \leq \varepsilon_c, \\ \mu \leq \|A_{21}\|_m \leq \|A_{21}\| \leq M, & \alpha \leq \|A_{22}\|_m \leq \|A_{22}\| \leq \mathcal{A}, & \|A_{23}\| \leq \varepsilon_u, \\ \|A_{31}\| \leq M, & \|A_{32}\| \leq \varepsilon_s, & \|A_{33}\| \leq \beta \end{array} \tag{76}$$

(The $C, M, \mu, \mathcal{A}, \alpha, \beta, \varepsilon_u, \varepsilon_s$ and ε_c can depend on j_0, i_0, j_1 and i_1). If there exists an $m > 1$ such that

$$\begin{aligned} C^2 - \mu^2 + M^2 + 2C\varepsilon_c + M(\mathcal{A} + \varepsilon_u + \varepsilon_s + \beta) &< m, \\ -\varepsilon_c^2 + \alpha^2 - \varepsilon_s^2 - \varepsilon_c(C + \varepsilon_c) - \mathcal{A}(M + \varepsilon_u) - \varepsilon_s(M + \beta) &> m, \\ \varepsilon_c^2 + \beta^2 + \varepsilon_c(C + \varepsilon_c) + \varepsilon_u(M + \mathcal{A}) + \beta(M + \varepsilon_s) &< m, \end{aligned} \tag{77}$$

then f satisfies cone conditions.

Proof. We only need to prove condition (33). Take any $x_1, x_2 \in N_{\eta_{i_0}}(V_j)$ for which $Q_{i_0,h}(x_1 - x_2) \geq 0$. We have

$$f_{i_1 i_0}(x_1) - f_{i_1 i_0}(x_2) = \int_0^1 df_{i_1 i_0}(x_2 + t(x_1 - x_2))dt \cdot (x_1 - x_2).$$

Let us define a matrix $A = \int_0^1 df_{i_1 i_0}(x_2 + t(x_1 - x_2))dt \cdot (x_1 - x_2)$. Clearly $A \in [df_{i_1 i_0}(N_{\eta_{i_0}}(V_j))]$. By Lemma 5.3 we have

$$\begin{aligned} Q_{i_1,h}(f_{i_1 i_0}(x_1) - f_{i_1 i_0}(x_2)) = &Q_{i_1,h}(A(x_1 - x_2)) \\ \geq &-\mathbf{a}\|\pi_1(x_1 - x_2)\|^2 + \mathbf{b}\|\pi_2(x_1 - x_2)\|^2 \\ &- \mathbf{c}|\pi_3(x_1 - x_2)\|^2, \end{aligned} \tag{78}$$

with $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ defined as in (74). By using the estimates (76) in the formulas (74) for the coefficients $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$, and applying (77), from (78) we obtain

$$Q_{i_1,h}(f_{i_1 i_0}(x_1) - f_{i_1 i_0}(x_2)) > mQ_{i_0,h}(x_1 - x_2).$$

This finishes our proof. □

Remark 4. It might turn out that the conditions (77) will not hold due to the fact that the bounds

$$\left\| \frac{d(\pi_x f_{i_1 i_0})}{d\theta}(N_{\eta_{i_0}}(V_j)) \right\| \leq M,$$
$$\left\| \frac{d(\pi_y f_{i_1 i_0})}{d\theta}(N_{\eta_{i_0}}(V_j)) \right\| \leq M,$$

will produce a large coefficient M. In such case, a change of local coordinates

$$(\theta, x, y) \to (v\theta, x, y),$$

will result in conditions

$$\begin{aligned} C^2 - \left(\frac{\mu}{v}\right)^2 + \left(\frac{M}{v}\right)^2 + 2Cv\varepsilon_c + \frac{M}{v}(\mathcal{A} + \varepsilon_u + \varepsilon_s + \beta) &< m, \\ -(v\varepsilon_c)^2 + \alpha^2 - \varepsilon_s^2 - v\varepsilon_c(C + v\varepsilon_c) - \mathcal{A}\left(\frac{M}{v} + \varepsilon_u\right) - \varepsilon_s\left(\frac{M}{v} + \beta\right) &> m, \\ (v\varepsilon_c)^2 + \beta^2 + v\varepsilon_c(C + v\varepsilon_c) + \varepsilon_u\left(\frac{M}{v} + \mathcal{A}\right) + \beta\left(\frac{M}{v} + \varepsilon_s\right) &< m, \\ 1 &< m. \end{aligned} \tag{79}$$

The conditions (79) with appropriately large v will hold more readily than (77), since in practice we usually have the bound ε_c small in comparison with M.

Remark 5. Let us note that if our choice of the local coordinates in the stable and unstable direction results in having $\varepsilon_u = \varepsilon_s = \varepsilon_c = 0$, then the condition

$$\beta \leq C < \alpha,$$

implies conditions (79) for any $m \in (\max\{C^2, 1\}, \alpha^2)$ and sufficiently large v.

6. **Comparison of the results with the classical normally hyperbolic invariant manifold theorem.** In the classical version of the normally hyperbolic invariant manifold theorem (see [7]) we have the following setting. We have a smooth Riemann manifold M and a diffeomorphism $f : M \to M$ with an invariant submanifold $\Lambda \subset M$

$$f(\Lambda) = \Lambda.$$

We say that f is *r-normally hyperbolic* at Λ if the tangent bundle of M splits into invariant by the tangent of f subbundles

$$T_V M = \xi^u \oplus T\Lambda \oplus \xi^s,$$

such that

1. Tf expands ξ^u more sharply than Tf^r expands $T\Lambda$,
2. Tf contracts ξ^s more sharply than Tf^r contracts $T\Lambda$.

Theorem 6.1. [7] *Let f be r-normally hyperbolic at Λ. Through Λ pass stable and unstable manifolds invariant by f and tangent at Λ to $T\Lambda \oplus \xi^s$, $\xi^u \oplus T\Lambda$. They are of class C^r. The stable manifold is invariantly fibred by C^r submanifolds tangent at Λ to the subspaces ξ^s. Similarly for the unstable manifold and ξ^u. These structures are unique, and permanent under small perturbations of f.*

We will now highlight how our result obtained in Theorem 4.7 (see also Theorems 5.2, 5.4 and Remark 5) relates to Theorem 6.1.

In our approach we do not need to start with a normally hyperbolic invariant manifold. We start with a region (ch-set) D in which we suspect the manifold to be contained. Theorem 4.7 gives us both the existence the normally hyperbolic manifold, and of the stable and unstable manifolds inside of D.

Using Theorem 6.1 it is not straightforward to obtain an estimate on the size of the perturbation under which the structures persist. This is due to the fact that the proof is conducted by the use of the implicit function theorem on infinite dimensional functional spaces. In contrast, our proof is designed so that this estimate is explicitly given (see Theorems 5.2, 5.4, and Section 7 for an example how this is done in the case of the rotating Hénon map). This was possible due to the fact that the proof relies only on topological arguments, and is conducted in the phase space of the system.

Theorem 6.1 is in many ways stronger than our result. Due to the fact that our proof relies only on topological arguments we have lost the C^r regularity results and our manifolds are only C^0. As of yet the method also does not give us the fibration of the stable and unstable manifolds. This issue will be addressed in forthcoming work. We also point out that in the method of verification of cone conditions given by Theorem 5.4 (see also Remark 5) we assume that we have strong expansion properties for the first iterate of the map. This is not required for classical normal hyperbolicity, where it is enough that one has strong expansion and contraction properties for some higher iteration of the map (for more details see [7]). This means that we can apply our results in the classical setting, but to do so we need to consider a higher iterate of the map, and take the higher iterate to be the function considered in Theorem 4.7.

7. Rotating Hénon map.

In this Section we will apply Theorem 4.7 to obtain a normally hyperbolic invariant manifold for the rotating Hénon map. We will obtain an explicit estimate of the region in which the manifold is contained. The size of this region depends on the size of the perturbation. The smaller the perturbation is the more exact our estimate.

7.1. Statement of the problem.

We will consider the rotating Hénon map

$$\begin{aligned} \bar{\theta} &= \theta + \omega \pmod 1, \\ \bar{x} &= 1 + y - ax^2 + \varepsilon\cos(2\pi\theta), \\ \bar{y} &= bx. \end{aligned} \tag{80}$$

The dynamics of (80) with $a = 0.68$ and $b = 0.1$ has been investigated by Haro and de la Llave in [5], for a demonstration of a numerical algorithm for finding invariant manifolds and their whiskers in quasi periodically forced systems.

In this section we will prove that for the the same parameters a and b for all $\varepsilon \leq \frac{1}{2}$, there exists an invariant C^0 manifold of (80) which is homeomorphic to $\mathbb{T}^1$ and is contained in a set

$$U_\varepsilon = \mathbb{T}^1 \times [x_- - 1.1\varepsilon, x_- + 1.1\varepsilon] \times [y_- - 0.12\varepsilon, y_- + 0.12\varepsilon], \tag{81}$$

where (x_-, y_-) is a fixed point for the (standard) Hénon map,

$$\begin{aligned} x_- &= \frac{-(1-b) - \sqrt{(1-b)^2 + 4a}}{2a} \approx -2.043\,3, \\ y_- &= bx_- \approx -0.204\,33. \end{aligned}$$

7.2. The unperturbed map.

We start by investigating the case of $\varepsilon = 0$. We will ignore the coordinate θ and concentrate on a map

$$F(x, y) = (1 + y - ax^2, bx).$$

The point (x_-, y_-) is one of the two fixed points $(x_\pm, y_\pm)$ of the map F

$$x_\pm = \frac{-(1-b) \pm \sqrt{(1-b)^2 + 4a}}{2a}, \quad y_\pm = bx_\pm.$$

We have

$$DF(x, y) = \begin{pmatrix} -2ax & 1 \\ b & 0 \end{pmatrix},$$

with two eigenvalues $\lambda_1 = -ax + \sqrt{b + a^2x^2}$, $\lambda_2 = -ax - \sqrt{b + a^2x^2}$. For (x_-, y_-) the eigenvalues are

$$\lambda_1 \approx 2.814\,4, \quad \lambda_2 \approx -3.553\,1 \times 10^{-2}.$$

We will consider the following Jordan forms of the matrix $DF(x_-, y_-)$

$$DF(x_-, y_-) = \Phi_\varepsilon^{-1} J \Phi_\varepsilon,$$

$$\Phi_\varepsilon^{-1} = \varepsilon\kappa \begin{pmatrix} \tau & \eta \\ -\tau\lambda_2 & -\lambda_1\eta \end{pmatrix}, \quad J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \Phi_\varepsilon = \frac{1}{\varepsilon} \begin{pmatrix} -\frac{1}{\tau}\lambda_1 & -\frac{1}{\tau} \\ \frac{1}{\eta}\lambda_2 & \frac{1}{\eta} \end{pmatrix},$$

where $\kappa = 1/(\lambda_2 - \lambda_1)$. The constants τ, η serve the purpose of an appropriate rescaling of the stable and unstable directions in the local coordinates, and will be chosen later on. When we will consider the perturbed Hénon map in Section 7.4, for a given $\varepsilon > 0$ we will use the maps Φ_ε^{-1} and Φ_ε.

We introduce local coordinates of hyperbolic expansion and contraction around the point (x_-, y_-) as

$$(\tilde{x}, \tilde{y}) = \Phi_\varepsilon (x - x_-, y - y_-). \tag{82}$$

The map F in the local coordinates is

$$\tilde{F}(\tilde{x}, \tilde{y}) = \Phi_\varepsilon \left(F\left(\Phi_\varepsilon^{-1}(\tilde{x}, \tilde{y}) + (x_-, y_-) \right) - (x_-, y_-) \right),$$

and its derivative $d\tilde{F}$ is equal to

$$\begin{aligned}
d\tilde{F}(\tilde{x}, \tilde{y}) &= \Phi_\varepsilon \circ dF(\Phi_\varepsilon^{-1}(\tilde{x}, \tilde{y}) + (x_-, y_-)) \circ \Phi_\varepsilon^{-1} \\
&= \Phi_\varepsilon \circ dF \begin{pmatrix} \varepsilon\kappa\,(\tau\tilde{x} + \eta\tilde{y}) + x_- \\ -\varepsilon\kappa\,(\tau\lambda_2\tilde{x} + \eta\lambda_1\tilde{y}) + y_- \end{pmatrix} \circ \Phi_\varepsilon^{-1} \\
&= \Phi_\varepsilon \circ \begin{pmatrix} -2a\,(\varepsilon\kappa\,(\tau\tilde{x} + \eta\tilde{y}) + x_-) & 1 \\ b & 0 \end{pmatrix} \circ \Phi_\varepsilon^{-1} \\
&= \Phi_\varepsilon \circ \left(\begin{pmatrix} -2ax_- & 1 \\ b & 0 \end{pmatrix} + \begin{pmatrix} -2a\varepsilon\kappa\,(\tau\tilde{x} + \eta\tilde{y}) & 0 \\ 0 & 0 \end{pmatrix} \right) \circ \Phi_\varepsilon^{-1} \\
&= J + R_\varepsilon,
\end{aligned} \tag{83}$$

where

$$R_\varepsilon = -2a\varepsilon\kappa^2\,(\tau\tilde{x} + \eta\tilde{y}) \begin{pmatrix} -\lambda_1 & -\frac{\eta}{\tau}\lambda_1 \\ \frac{\tau}{\eta}\lambda_2 & \lambda_2 \end{pmatrix}.$$

For any $\tilde{x}, \tilde{y} \in [-1, 1]$ we have the following estimates, which will be used later on for the verification of the covering and cone conditions

$$\begin{aligned}
&[d\tilde{F}(B(0,1) \times B(0,1))] \\
&\subset \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} + \varepsilon\,(\tau + \eta) \begin{pmatrix} [-\frac{1}{2}, \frac{1}{2}] & [-\frac{1}{2}\frac{\eta}{\tau}, \frac{1}{2}\frac{\eta}{\tau}] \\ [-\frac{6}{1000}\frac{\tau}{\eta}, \frac{6}{1000}\frac{\tau}{\eta}] & [-\frac{6}{1000}, \frac{6}{1000}] \end{pmatrix}.
\end{aligned} \tag{84}$$

Now we turn to the inverse map. The inverse map to F is

$$F^{-1}(x, y) = \left(\frac{1}{b}y, -1 + x + \frac{a}{b^2}y^2 \right),$$

and has a derivative

$$dF^{-1}(x, y) = \begin{pmatrix} 0 & \frac{1}{b} \\ 1 & \frac{2a}{b^2}y \end{pmatrix}.$$

In the local coordinates (82) the inverse map is

$$\tilde{F}^{-1}(\tilde{x}, \tilde{y}) = \Phi_\varepsilon \left(F^{-1}\left(\Phi_\varepsilon^{-1}(\tilde{x}, \tilde{y}) + (x_-, y_-) \right) - (x_-, y_-) \right),$$

and its derivative $d\tilde{F}^{-1}$ is equal to

$$\begin{aligned}
d\tilde{F}^{-1}(\tilde{x},\tilde{y}) &= \Phi_\varepsilon \circ dF^{-1}(\Phi_\varepsilon^{-1}(\tilde{x},\tilde{y}) + (x_-,y_-)) \circ \Phi_\varepsilon^{-1} \\
&= \Phi_\varepsilon \circ dF^{-1} \begin{pmatrix} \varepsilon\kappa\left(\tau\tilde{x}+\eta\tilde{y}\right)+x_- \\ -\varepsilon\kappa\left(\tau\lambda_2\tilde{x}+\eta\lambda_1\tilde{y}\right)+y_- \end{pmatrix} \circ \Phi_\varepsilon^{-1} \\
&= \Phi_\varepsilon \circ \begin{pmatrix} 0 & \frac{1}{b} \\ 1 & \frac{2a}{b^2}\left(-\varepsilon\kappa\left(\tau\lambda_2\tilde{x}+\eta\lambda_1\tilde{y}\right)+y_-\right) \end{pmatrix} \circ \Phi_\varepsilon^{-1} \\
&= \Phi_\varepsilon \circ \left(\begin{pmatrix} 0 & \frac{1}{b} \\ 1 & \frac{2a}{b^2}y_- \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -\frac{2a}{b^2}\varepsilon\kappa\left(\tau\lambda_2\tilde{x}+\eta\lambda_1\tilde{y}\right) \end{pmatrix} \right) \circ \Phi_\varepsilon^{-1} \\
&= J^{-1} + R'_\varepsilon, \qquad (85)
\end{aligned}$$

where

$$R'_\varepsilon = \frac{2a}{b^2}\varepsilon\kappa^2\left(\tau\lambda_2\tilde{x}+\eta\lambda_1\tilde{y}\right)\begin{pmatrix} -\lambda_2 & -\frac{\eta}{\tau}\lambda_1 \\ \frac{\tau}{\eta}\lambda_2 & \lambda_1 \end{pmatrix}.$$

For $\tilde{x},\tilde{y}\in[-1,1]$ this gives us the following estimates

$$\begin{aligned}
&[d\tilde{F}^{-1}(B(0,1)\times B(0,1))] \qquad (86) \\
&\subset \begin{pmatrix} \frac{1}{\lambda_1} & 0 \\ 0 & \frac{1}{\lambda_2} \end{pmatrix} + \varepsilon\left(\tau\left|\lambda_2\right|+\eta|\lambda_1|\right)\begin{pmatrix} [-\frac{6}{10},\frac{6}{10}] & [-50\frac{\eta}{\tau},50\frac{\eta}{\tau}] \\ [-\frac{\tau}{\eta}\frac{6}{10},\frac{\tau}{\eta}\frac{6}{10}] & [-50,50] \end{pmatrix}.
\end{aligned}$$

7.3. **The atlas.** Let us start by introducing a notation $F_\varepsilon:\mathbb{T}^1\times\mathbb{R}^2\to\mathbb{T}^1\times\mathbb{R}^2$ for the perturbed Henon map (80)

$$F_\varepsilon\left(\theta,x,y\right) = \left(\theta+\omega, 1+y-ax^2+\varepsilon\cos(2\pi\theta), bx\right).$$

For computational reasons it will be convenient for us to choose $\Lambda=\mathbb{R}/v$ for some large (later specified) number $v\in\mathbb{N}$, $v\geq 9$. The v will play the role of the rescaling parameter from Remark 4. We choose $p:\mathbb{T}^1\times\mathbb{R}^2\to\mathbb{R}/v$ as $p(\theta,x,y):=v\theta$. For $i,j\in\{0,1,\ldots,v\}$ we define $V_j,U_i\subset\Lambda$ as

$$\begin{aligned}
V_j &:= j+(0,5) \mod v, \qquad (87)\\
U_i &:= i+(0,9) \mod v,
\end{aligned}$$

and define maps $\eta_i:U_i\to\mathbb{R}$ as

$$\eta_i(i+x \text{ mod } v) := i+x. \qquad (88)$$

For all i we define quadratic forms $Q_{i,h}$, $Q_{i,\nu}$ as

$$\begin{aligned}
Q_{i,h}(\theta,x,y) &:= x^2-y^2-\theta^2, \\
Q_{i,\nu}(\theta,x,y) &:= x^2-y^2+\theta^2.
\end{aligned}$$

For any $q=(\theta,x,y)\in N(\Lambda)=\Lambda\times B(0,1)\times B(0,1)$, for which $\theta\in[i,(i+1)]$ mod v, the pair (V_{i-2},U_{i-4}) is both a cones chart pair, as well as a cone enclosing pair for q (see Example 1).

We define $\phi:\mathbb{T}^1\times\mathbb{R}^2\to\Lambda\times\mathbb{R}^2$ as

$$\phi(\theta,x,y) := (v\theta, \Phi_\varepsilon\left(x-x_-, y-y_-\right)).$$

Clearly

$$\phi^{-1}(\theta,x,y) = (\frac{1}{v}\theta, \Phi_\varepsilon^{-1}\left(x,y\right)+(x_-,y_-)).$$

Finally we define our set $|D|\subset\mathbb{T}^1\times\mathbb{R}^2$ as

$$|D| := \phi^{-1}(N(\Lambda)).$$

Let us note that for different ε we will have different sets $|D|$.

With the above notations we can see that $(D, \{\eta_i, U_i, V_i, Q_{i,h}, Q_{i,\nu}\})$ is a ch-set with cones.

7.4. **Verification of the covering conditions.** We will show that

$$D \overset{F_\varepsilon}{\Longrightarrow} D, \tag{89}$$

$$D \overset{F_\varepsilon^{-1}}{\Longrightarrow} D, \tag{90}$$

(with the roles of the stable and unstable coordinates reversed for the inverse map).

We will first apply Theorem 5.2 to establish (89). From the fact that F_ε is a rotation on $\mathbb{T}^1$ and from the definition of the sets V_j, U_i (see (87)), for any $V_j \subset U_{i_0}$ there exists i_1 such that $F_\varepsilon(p^{-1}(V_j) \cap |D|) \subset p^{-1}(U_{i_1})$. We will choose the parameters $\varepsilon_u, \varepsilon_s$ for the estimates (70) and (71) to be independent from the choice of j, i_0, i_1. To simplify notations we will obtain these estimates using the map $(F_\varepsilon)_\phi$ rather than $(F_\varepsilon)_{i_1 i_0}$. We can do so since η_{i_0}, η_{i_1} are identity maps (see (88)).

From the fact that $F_0(\theta, x_-, y_-) = (\theta + \omega, x_-, y_-)$, for any $\theta \in \Lambda$ we have

$$\begin{aligned}
(F_\varepsilon)_\phi (\theta, 0, 0) &= \phi \circ F_\varepsilon \circ \phi^{-1} (\theta, 0, 0) \\
&= \phi \circ F_\varepsilon \left(\frac{1}{v}\theta, x_-, y_- \right) \\
&= \phi \left(F_0 \left(\frac{1}{v}\theta, x_-, y_- \right) + \left(0, \varepsilon \cos 2\pi \frac{1}{v}\theta, 0 \right) \right) \\
&= \left(\theta + v\omega, \Phi_\varepsilon \left((0,0) + \left(\varepsilon \cos 2\pi \frac{1}{v}\theta, 0 \right) \right) \right) \\
&= \left(\theta + v\omega, -\frac{1}{\tau}\lambda_1 \cos 2\pi \frac{1}{v}\theta, \frac{1}{\eta}\lambda_2 \cos 2\pi \frac{1}{v}\theta \right),
\end{aligned}$$

which gives us

$$(F_\varepsilon)_\phi (\Lambda, 0, 0) \subset \Lambda \times \overline{B_u}\left(0, \frac{1}{\tau}|\lambda_1|\right) \times \overline{B_s}\left(0, \frac{1}{\eta}|\lambda_2|\right). \tag{91}$$

Let $g(\theta, x, y) = (0, \cos(2\pi\theta), 0)$, we then have $F_\varepsilon = F_0 + \varepsilon g$ and

$$\begin{aligned}
d(F_\varepsilon)_\phi &= d(F_0)_\phi + \varepsilon \mathrm{diag}(v\mathrm{id}, \Phi_\varepsilon) dg(\phi^{-1}(\cdot)) \mathrm{diag}\left(\frac{1}{v}\mathrm{id}, \Phi_\varepsilon^{-1} \right) \qquad (92) \\
&= \begin{pmatrix} 1 & 0 \\ 0 & d\tilde{F} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ \varepsilon \frac{2\pi\kappa}{v\tau}\lambda_1 \sin 2\pi \frac{1}{v}\theta & 0 & 0 \\ -\varepsilon \frac{2\pi\kappa}{v\eta}\lambda_2 \sin 2\pi \frac{1}{v}\theta & 0 & 0 \end{pmatrix}.
\end{aligned}$$

From (84) we have that for any $A \in [d(F_\varepsilon)_\phi (N(\Lambda))]$

$$\begin{aligned}
\inf\{|A_u(0,x,y)| : |x| = 1, |y| \leq 1\} &\geq |\lambda_1| - \varepsilon(\tau + \eta)\frac{1}{2}\left(1 + \frac{\eta}{\tau}\right), \qquad (93) \\
\sup\{|A_s(0,x,y)| : |x| \leq 1, |y| \leq 1|\} &\leq |\lambda_2| + \varepsilon(\tau + \eta)\frac{6}{1000}\left(1 + \frac{\tau}{\eta}\right).
\end{aligned}$$

From (91) and (93), by Theorem 5.2 (in our case $\varepsilon_u = |\lambda_1|/\tau$, $\varepsilon_s = |\lambda_2|/\eta$) if we have

$$|\lambda_1| - \varepsilon(\tau+\eta)\frac{1}{2}\left(1+\frac{\eta}{\tau}\right) > 1 + \frac{1}{\tau}|\lambda_1|, \tag{94}$$

$$|\lambda_2| + \varepsilon(\tau+\eta)\frac{6}{1000}\left(1+\frac{\tau}{\eta}\right) < 1 - \frac{1}{\eta}|\lambda_2|, \tag{95}$$

then we have established (89). The conditions (94) and (95) hold for all $\varepsilon \leq \frac{1}{2}$ with $\tau = 3$, $\eta = \frac{3}{40}$.

To establish (90) we first compute

$$F_\varepsilon^{-1}(\theta,x,y) = \left(\theta-\omega, \frac{1}{b}y, x' - 1 + \frac{a}{b^2}y^2 - \varepsilon\cos\left(2\pi(\theta-\omega)\right)\right). \tag{96}$$

From the fact that F_ε^{-1} is a rotation on $\mathbb{T}^1$ and from the definition of the sets V_j, U_i (see (87)) we have that for any $V_j \subset U_{i_0}$ there exists i_1 such that $F_\varepsilon^{-1}(p^{-1}(V_j) \cap |D|) \subset p^{-1}(U_{i_1})$.

Once again, to simplify notations, we will consider $\left(F_\varepsilon^{-1}\right)_\phi$ instead of $\left(F_\varepsilon^{-1}\right)_{i_0 i_1}$. Using the fact that $F_0^{-1}(\theta,x_-,y_-) = (\theta-\omega,x_-,y_-)$ we have

$$\begin{aligned}
\left(F_\varepsilon^{-1}\right)_\phi(\theta,0,0) &= \phi\circ F_\varepsilon^{-1}\circ\phi^{-1}(\theta,0,0)\\
&= \phi\left(F_0^{-1}(\theta,x_-,y_-) + \left(0,0,\varepsilon\cos\left(2\pi\left(\frac{1}{v}\theta-\omega\right)\right)\right)\right)\\
&= \left(\theta - v\omega, \left(\Phi_\varepsilon(0,0) + \left(0,\varepsilon\cos\left(2\pi\left(\frac{1}{v}\theta-\omega\right)\right)\right)\right)\right)\\
&= \left(\theta - v\omega, -\frac{1}{\tau}\cos\left(2\pi(\theta-\omega)\right), \frac{1}{\eta}\cos\left(2\pi\left(\frac{1}{v}\theta-\omega\right)\right)\right),
\end{aligned}$$

hence

$$(F_\varepsilon)_\phi^{-1}(\Lambda,0,0) \subset \Lambda\times B\left(0,\frac{1}{\tau}\right)\times B\left(0,\frac{1}{\eta}\right). \tag{97}$$

Let $g^-(\theta,x,y) = (0,0,-\varepsilon\cos(2\pi(\theta-\omega)))$, we then have $F_\varepsilon^{-1} = F_0^{-1} + \varepsilon g^-$ and

$$\begin{aligned}
d\left(F_\varepsilon^{-1}\right)_\phi &= d\left(F_0^{-1}\right)_\phi + \varepsilon\mathrm{diag}(v\mathrm{id},\Phi_\varepsilon)dg^-(\phi^{-1}(\cdot))\mathrm{diag}\left(\frac{1}{v}\mathrm{id},\Phi_\varepsilon^{-1}\right) \qquad (98)\\
&= \begin{pmatrix} 1 & 0\\ 0 & d\tilde{F}^{-1}\end{pmatrix} + \begin{pmatrix} 0 & 0 & 0\\ -\frac{2\pi\varepsilon\kappa}{v\tau}\sin\left(2\pi\left(\frac{1}{v}\theta-\omega\right)\right) & 0 & 0\\ \frac{2\pi\varepsilon\kappa}{v\eta}\sin\left(2\pi\left(\frac{1}{v}\theta-\omega\right)\right) & 0 & 0\end{pmatrix}.
\end{aligned}$$

From (86) we know that for any $A \in [D\left(F_\varepsilon^{-1}\right)_\phi(N(\Lambda))]$ we have (let us note that the roles of the stable and unstable coordinates have been exchanged with respect to the forward map)

$$\begin{aligned}
\inf\left\{|A_u(0,x,y)| : |x| = 1, |y| \leq 1\right\} &\geq \left|\frac{1}{\lambda_2}\right| - \varepsilon\left(\tau|\lambda_2| + \eta|\lambda_1|\right)\left(\frac{\tau}{\eta}\frac{6}{10} + 50\right),\\
\sup\left\{|A_s(0,x,y)| : |x| \leq 1, |y| \leq 1\right\} &\leq \frac{1}{\lambda_1} + \varepsilon\left(\tau|\lambda_2| + \eta|\lambda_1|\right)\left(\frac{6}{10} + 50\frac{\eta}{\tau}\right).
\end{aligned}$$

Hence from (97), by Theorem 5.2 (in our case $\varepsilon_u = 1/\eta$ and $\varepsilon_s = 1/\tau$), if we have

$$\left|\frac{1}{\lambda_2}\right| - \varepsilon\left(\tau\left|\lambda_2\right| + \eta\lambda_1\right)\left(\frac{\tau}{\eta}\frac{6}{10} + 50\right) > 1 + \frac{1}{\eta}, \tag{99}$$

$$\frac{1}{\lambda_1} + \varepsilon\left(\tau\left|\lambda_2\right| + \eta\lambda_1\right)\left(\frac{6}{10} + 50\frac{\eta}{\tau}\right) < 1 - \frac{1}{\tau}, \tag{100}$$

then we have established (90). The conditions (99) and (100) hold for $\varepsilon \leq \frac{1}{2}$ with $\tau = 3$, $\eta = \frac{3}{40}$.

7.5. **Verification of cone conditions.** We will now use Theorem 5.4 to verify cone conditions. For any $x \in |D|$ we can choose a cone enclosing pair (V_{j_0}, U_{i_0}) for x (see (87)). From the fact that F_ε is a rotation on $\mathbb{T}^1$ we have

$$\pi_\theta (F_\varepsilon)_\phi (V_j) \subset V_j + \omega v \bmod v.$$

If $x \in |D|$, $F_\varepsilon(x) \in |D|$, $\pi_\theta(\phi(F_\varepsilon(x))) \in [i, i+1]$ and (V_{j_0}, U_{i_0}) is a cone enclosing pair for x, then we can find a U_{i_1} for which we will both have

$$(V_j + \omega v \bmod v) \subset U_{i_1},$$
$$[i-2, i+3] \bmod v \subset U_{i_1}.$$

Setting $V_{j_1} = [i-2, i+3] \bmod v$ we have found a cone enclosing pair (V_{j_1}, U_{i_1}) for $F_\varepsilon(x)$, for which $F_\varepsilon(p^{-1}(V_j) \cap |D|) \subset p^{-1}(U_{i_1})$. An analogous argument holds for F_ε^{-1}.

Now we will verify (77) for the forward map F_ε. Our estimates will be independent from the choice of j_0, i_0, i_1, hence as in Section 7.4 we will consider the map $(F_\varepsilon)_\phi$ instead of $(F_\varepsilon)_{i_1 i_0}$. From (92) and (83) we have

$$d\left(F_\varepsilon\right)_\phi = \begin{pmatrix} 1 & 0 \\ \begin{matrix} \varepsilon\frac{2\pi\kappa}{v\tau}\lambda_1 \sin 2\pi\frac{1}{v}\theta \\ -\varepsilon\frac{2\pi\kappa}{v\eta}\lambda_2 \sin 2\pi\frac{1}{v}\theta \end{matrix} & J + R_\varepsilon \end{pmatrix}.$$

This by (84) means that our constants from Theorem 5.4 are as follows

$$\begin{array}{lll} C = 1, & \varepsilon_c = 0, & \mu = 0, \\ M = \left|\varepsilon\frac{2\pi\kappa}{v\tau}\lambda_1\right|, & \mathcal{A} = \lambda_1 + \frac{1}{2}\varepsilon\left|\tau + \eta\right|, & \varepsilon_u = \frac{1}{2}\varepsilon\left|\frac{\eta}{\tau}\left(\tau + \eta\right)\right|, \\ \alpha = \lambda_1 - \frac{1}{2}\varepsilon\left|\tau + \eta\right| & \varepsilon_s = \frac{6}{1000}\varepsilon\left|\frac{\tau}{\eta}\left(\tau + \eta\right)\right|, & \beta = \left|\lambda_2\right| + \frac{6}{1000}\varepsilon\left|\tau + \eta\right|. \end{array}$$

By choosing v sufficiently large we can reduce M arbitrarily close to zero. We also note that $\varepsilon_c = 0$. This means that conditions (77) reduce to

$$\begin{aligned} C^2 &< m, \\ \alpha^2 - \varepsilon_s^2 - \varepsilon_u\mathcal{A} - \varepsilon_s\beta &> m, \\ \beta^2 + \varepsilon_u\mathcal{A} + \varepsilon_s\beta &< m. \end{aligned} \tag{101}$$

These conditions hold for $\varepsilon \leq \frac{1}{2}$ with $\tau = 3$, $\eta = \frac{3}{40}$ and $m = 2$.

Now we turn to the conditions for the inverse map. From (98) and (85) we have

$$d\left(F_\varepsilon^{-1}\right)_\phi = \begin{pmatrix} 1 & 0 \\ \begin{matrix} -\varepsilon\frac{2\pi\kappa}{v\tau}\sin\left(2\pi\left(\frac{1}{v}\theta - \omega\right)\right) \\ \varepsilon\frac{2\pi\kappa}{v\eta}\sin\left(2\pi\left(\frac{1}{v}\theta - \omega\right)\right) \end{matrix} & J^{-1} + R'_\varepsilon \end{pmatrix}.$$

This by (86) means that our constants from Theorem 5.4 are as follows (note that for the inverse map the roles of the stable and unstable directions are reversed)

$$
\begin{array}{ll}
C = 1, & \varepsilon_c = 0, \\
M = \left|\varepsilon \frac{2\pi\kappa}{v\tau}\right|, & \mathcal{A} = \frac{1}{\lambda_2} + 50\varepsilon\left(\tau\left|\lambda_2\right| + \eta|\lambda_1|\right), \\
\alpha = \frac{1}{\lambda_2} - 50\varepsilon\left(\tau\left|\lambda_2\right| + \eta|\lambda_1|\right), & \varepsilon_s = \varepsilon\left(\tau\left|\lambda_2\right| + \eta|\lambda_1|\right)\left|50\frac{\eta}{\tau}\right|, \\
\varepsilon_u = \varepsilon\left(\tau\left|\lambda_2\right| + \eta|\lambda_1|\right)\left|\frac{\tau}{\eta}\frac{6}{10}\right|, & \beta = \frac{1}{\lambda_1} + \varepsilon\left(\tau\left|\lambda_2\right| + \eta|\lambda_1|\right)\frac{6}{10}, \\
\mu = 0. &
\end{array}
$$

By choosing v sufficiently large we can once again reduce M arbitrarily close to zero, which means that conditions (77) reduce to conditions same as (101). These conditions hold for $\varepsilon \leq \frac{1}{2}$ with $\tau = 3$, $\eta = \frac{3}{40}$ and $m = 200$.

7.6. **The estimate of the region in which the manifold is contained.** So far we have shown that for $\varepsilon \leq \frac{1}{2}$ our map F_ε satisfies forward and backward cone conditions. This means that we have an invariant manifold inside of

$$|D| = \phi^{-1}\left(N(\Lambda)\right).$$

This gives us the following bounds

$$
\begin{aligned}
D &= \phi^{-1}\left(N(\Lambda)\right) \\
&= \mathbb{T}^1 \times \{(x_-, y_-) + \Phi_\varepsilon^{-1}\left(\overline{B}(0,1) \times \overline{B}(0,1)\right)\} \\
&\subset \mathbb{T}^1 \times \{(x_-, y_-) + [-\varepsilon|\kappa|(\tau+\eta), \varepsilon|\kappa|(\tau+\eta)] \\
&\quad \times [-\varepsilon|\kappa|\left(\tau|\lambda_2| + \eta|\lambda_1|\right), \varepsilon|\kappa|\left(\tau|\lambda_2| + \eta|\lambda_1|\right)]\}.
\end{aligned}
$$

With $\tau = 3$ and $\eta = \frac{3}{40}$ we have $|D| \subset U_\varepsilon$ (where U_ε is given by (81)).

8. Properties of the local Brouwer degree. Solution property. [9]

If $\deg(f, D, c) \neq 0$ then there exists an $x \in D$ with $f(x) = c$.

Homotopy property. [9] Let $H : [0,1] \times D \to R^n$ be continuous. Suppose that

$$\bigcup_{\lambda \in [0,1]} H_\lambda^{-1}(c) \cap D \quad \text{is compact} \tag{102}$$

then

$$\forall \lambda \in [0,1] \quad \deg(H_\lambda, D, c) = \deg(H_0, D, c)$$

If $[0,1] \times \overline{D} \subset \mathrm{dom}(H)$ and $\overline{D}$ is compact, then (102) follows from the condition

$$c \notin H([0,1], \partial D).$$

Degree property for affine maps. [9] Suppose that $f(x) = B(x - x_0) + c$, where B is a linear map and $x_0 \in R^n$. If the equation $B(x) = 0$ has no nontrivial solutions (i.e if $Bx = 0$, then $x = 0$) and $x_0 \in D$, then

$$\deg(f, D, c) = \mathrm{sgn}(\det B). \tag{103}$$

Excision property. [9] Suppose that we have an open set E such that $E \subset D$ and

$$f^{-1}(c) \cap D \subset E,$$

then

$$\deg(f, D, c) = \deg(f, E, c).$$

REFERENCES

[1] P. W. Bates, K. Lu, C. Zeng, Approximately invariant manifolds and global dynamics of spike states. Invent. Math. 174 (2008), no. 2, 355–433.

[2] M. J. Capiński, Covering Relations and the Existence of Topologically Normally Hyperbolic Invariant Sets, Discrete Contin. Dyn. Syst. Ser A. 23 (2009), no. 3, 705 - 725

[3] M. Chaperon, Stable manifolds and the Perron-Irwin method, Ergodic Theory Dynam. Systems 24 (2004), no. 5, 1359–1394.

[4] M. Gidea and P. Zgliczyński, Covering relations for multidimensional dynamical systems, J. of Diff. Equations, 202(2004) 33–58

[5] A. Haro, R. de la Llave, A parameterization method for the computation of invariant tori and their whiskers in quasi-periodic maps: numerical algorithms, Discrete Contin. Dyn. Syst. Ser. B 6 (2006), no. 6, 1261–1300

[6] M. Hirsh, Differential Topology, Graduate Texts in Mathematics, No. 33. Springer-Verlag, New York-Heidelberg, 1976

[7] M. Hirsh, C. Pugh and M. Shub, Invariant Manifolds, Lecture Notes in Mathematics, Vol. 583. Springer-Verlag, Berlin-New York, 1977.

[8] C. K. R. T. Jones, Geometric singular perturbation theory. Dynamical systems (Montecatini Terme, 1994), 44–118, Lecture Notes in Math., 1609, Springer, Berlin, 1995.

[9] N. G. Lloyd, Degree theory, Cambridge Tracts in Math., No. 73, Cambridge Univ. Press, London, 1978

[10] St. Wiggins, Normally hyperbolic invariant manifolds in dynamical systems. Applied Mathematical Sciences, 105. Springer-Verlag, New York, 1994. x+193 pp. ISBN: 0-387-94205-X

[11] P. Zgliczyński, Covering relations, cone conditions and stable manifold theorem. J. Differential Equations 246 (2009), no. 5, 1774–1819.

Received xxxx 20xx; revised xxxx 20xx.

E-mail address: mcapinsk@agh.edu.pl
E-mail address: zgliczyn@ii.uj.edu.pl