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STEADY STATE BIFURCATIONS FOR THE KURAMOTO-SIVASHINSKY EQUATION - A COMPUTER ASSISTED PROOF

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Abstract. We apply the method of self-consistent bounds to prove the existence of multiple steady state bifurcations for Kuramoto-Sivashinski PDE on the line with odd and periodic boundary conditions.

1. **Introduction.** The aim of this paper is to present all necessary tools, which with computer assistance allow to produce a rigorous steady-state bifurcation diagram for dissipative PDEs. For the purpose of this introduction by a dissipative PDE we mean an infinite-dimensional system, which is well approximated by its Galerkin projections.

Our approach is based on the concept of the self-consistent a priori bounds developed in [23, 24, 26], which when applied to steady states can be thought as an improvement of the Cesari method [3] (see [23] for more details). Informally speaking, the self-consistent a priori bounds are used to control rigorously the difference between a PDE and its Galerkin projections, then we apply finite dimensional tools, mainly the local degree and the implicit function theorem to infer information about the steady states and their bifurcations. In an application of the proposed method to a concrete dissipative PDE one needs to check many inequalities (up to several hundred for the Kuramoto-Sivashinsky equation in one spatial dimension). This task, while practically impossible for a human, can be *rigorously* accomplished by a computer program using the interval arithmetic [18, 19].

In this paper we apply our method to Kuramoto-Sivashinsky equation (we will use a shortcut *the KS equation* in sequel), which was introduced in [16, 21] in the context of wave propagation. The KS equation is given by

$$u_t = -\nu u_{xxxx} - u_{xx} + (u^2)_x \qquad (t,x) \in [0,\infty) \times \mathbb{R}, \quad \nu > 0. \tag{1}$$

We assume odd and periodic boundary conditions

$$u(t,x) = u(t,x+2\pi), \quad u(t,x) = -u(t,-x). \tag{2}$$

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The KS-equation is well studied in literature [5, 6, 9, 10, 11, 12, 13, 14, 15, 22]. It serves as one of the model PDE examples for which it was shown rigorously that the dynamics is finite dimensional [9, 13], but there are virtually no rigorous results about the details of the dynamics away from the origin. On the other side a lot is known from the numerical point of view (see [15, 12, 5] and the literature cited there).

As a test for our method we would like to prove that the non-rigorous steady-state bifurcation diagram presented in [12, Fig 1a] and [15, Fig. 3.2, 3.4, 3.5] is correct. In this paper we did not get this far. Our main results about the steady state bifurcations for the KS equation can be formulated as follows (see [15, 12, 22] (also Section 6.1) for the explanation of the names of steady-states branches)

Theorem 1.1. *There are pitchfork symmetry breaking bifurcations for problem (1–2) for the following values of* ν

- $\nu \in 0.247833+2\cdot 10^{-6}\cdot(-1,1)$*, both unimodal branches collide with the negative bimodal branch.*
- $\nu \in 0.177336 + 2 \cdot 10^{-7} \cdot (-1,1)$*, a creation of the bi-tri branch off the positive bimodal branch.*
- *for* $\nu \in 0.075627151 + 5 \cdot 10^{-9} \cdot (-1,1)$*, a creation of the giant branch off the negative bimodal branch.*

For the following values of ν *there are the bifurcations consisting of intersections of two branches of steady states*

- $\nu \in 0.11039383+5\cdot 10^{-8}\cdot(-1,1)$ *an intersection of the trimodal branch with the bi-tri branch. This happens for both positive and negative trimodal branches.*
- $\nu \in 0.078570271 + 5 \cdot 10^{-9} \cdot (-1,1)$ *an intersection of the trimodal branch with the tri-quadratic branch near* R_3t_3*. This happens for both positive and negative trimodal branches.*

Summing up, we were able to establish the existence and determine the (un)stability of most main steady state bifurcations involving the zero solution branch and the unimodal, bimodal and trimodal branches.

The paper is organized in two main parts: an abstract part, where we state and prove abstract theorems and a detailed part, where we provide all necessary formulas, which allow (with computer assistance) to verify the assumptions of the abstract theorems for the KS equation with odd and periodic boundary conditions.

To construct a rigorous steady-state bifurcation diagram for a dissipative PDE one needs to solve the following problems

1.: How to establish the local uniqueness for regular steady states ?
2.: How to obtain the regularity and compute the derivatives of steady states with respect to the parameters?
3.: How to handle the bifurcation point in an infinite dimensional situation, when the bifurcation point is not given explicitly and the spectral data are hard to obtain?

It is apparent that problem **1** is much easier than the other ones. In Section 3 we present the method which allow us to establish the local uniqueness for regular steady states. It works for all the fixed points for the KS equation, whose existence was established in [23]. The main result in this section is Theorem 3.7. In Section 4 we discuss the criteria to establish the stability (asymptotic) and the instability of the fixed point. The main result is Theorem 4.2. Again we were able to rigorously establish the stability/unstability of all fixed points for the KS-equation from [23].

The problem **2** is harder than **1**, but its solution is required also in the solution of **3**. In Section 5 it is proven that, whenever we can prove for the KS-equation the existence and the uniqueness for a parameter range using the self-consistent a priori bounds, then the dependence on the parameters is C^∞. Sections 5 and 10 contain, respectively, the description of an effective algorithm and all necessary formulas for the KS-equation, which allow us to compute rigorously the derivatives of the steady states with respect to the parameters and the derivatives of the functions appearing in the Liapunov-Schmidt method (see [4]), which will be essential in the solution of problem **3**.

To solve problem **3** we use an approach, which is a combination of the methods presented in [4] and the self-consistent a priori bounds. Section 6 contains the abstract theorems and Section 11 the computational details related to the proof of the existence of the symmetry breaking pitchfork bifurcation off the bimodal branches (most of primary bifurcations are of this type for the KS equation) and intersections of two steady state branches (see Theorem 1.1). Companion file *bifdata.txt* [29] contains the relevant numerical data from the proof of the stability/unstability of two exemplary fixed points and the bifurcations listed in Theorem 1.1.

Sections from 7 to 11 contain the formulas necessary for the application of the theory developed in Sections 3, 4, 5 and 6 to the KS equations. Our intention is to give enough details so that the formulas given here together with the ones in [23, 26] can serve as a documentation of our program.

From the inspection of the bifurcation diagrams in [15, 12] it is quite clear that tools given in this paper should be sufficient for providing a rigorous steady states bifurcation diagram over reasonable range of ν, because apparently all steady state bifurcations appear to be either folds (on regular branches) or the symmetry breaking bifurcations discussed and the intersections discussed in this paper in this paper. Hence linking our method with a continuation approach should be enough for the task. This is the approach taken by Maier-Pappe and coauthors in [17], where using the method of self-consistent a priori bounds the authors were able to continue the branches of steady states for the Cahn-Hillard equation on the square (they did not treat bifurcations).

The first draft version of this paper was prepared in 2003 and later in 2005 Arioli and Koch in [1] using different tools were able to produce the full steady state bifurcation diagram for the KS-equation with odd and periodic boundary conditions. Arioli and Koch method differs considerably from ours, they use the 'integral formulation' of the stationary KS-equation and relay heavily on the abstract PDE theory. In contrast our approach is rather ODE-type as we prove everything relying mainly on the isolation concept originating in the Conley index theory (see references in [23]).

Obviously, while giving some insight, the steady state-bifurcation diagram is only the first step towards a detailed understanding the dynamics of the given dissipative PDE. Next step would be the inclusion of periodic orbits and one is faced with the following problems

- H Creation of periodic orbit from fixed point. How to rigorously show the existence of the Hopf bifurcation?
- E How to prove the existence of an isolated periodic orbit?
- B Bifurcations of periodic orbits.

We believe that the treatment of the Hopf bifurcation is possible within the presented framework i.e. the self-consistent a priori bounds should allow to use the classical bifurcation theory approach, as is done in the present paper for the pitchfork bifurcation.

The problem of proving the existence of an isolated periodic orbit for the KS equations was investigated in [27, 28], where the existence of multiple periodic orbits for the the KS equation for $\nu \in [0.029, 0.127]$ was proved with the computer assistance. In these works using an algorithm, based on the self-consistent a priori bounds, we rigorously integrated a dissipative PDE to obtain topological information about a suitable Poincaré map.

The problem of the bifurcation of periodic orbits, we mean here the rigorous proofs, is hard even for ODEs, as it requires C^r information for $r = 2, 3$ about Poincaré maps. To the best of our knowledge no such algorithm for dissipative PDEs exists in literature.

1.1. **The Kuramoto-Sivashinsky equation in the Fourier domain.** Throughout this paper we look at a dissipative PDE as an infinite ladder of ordinary differential equations. To be more specific the KS-equation with odd and periodic boundary conditions can be reduced (see [23]) to the following infinite system of ordinary differential equations for the coefficients of the Fourier expansion of $u(t,x) = \sum_{k=1}^{\infty} -2a_k(t)\sin(kx)$

$$\dot{a}_k = k^2(1-\nu k^2)a_k - k\sum_{n=1}^{k-1} a_n a_{k-n} + 2k\sum_{n=1}^{\infty} a_n a_{n+k} \quad k = 1, 2, 3, \dots. \tag{3}$$

More abstractly we will write the KS-equation as

$$\dot{a} = F(u), \tag{4}$$

where $F = (F_1, F_2, \dots)$.

2. **The method of the self-consistent a priori bounds.** Consider a Hilbert space H. $\{e_i\}$ an orthogonal basis. $X_k = \mathrm{span}(e_1, \dots, e_k)$.

For $x \in H$, by x_i we will denote the i-th coordinate and for any function $F : \mathrm{dom}\,(F) \to H$ we define $F_i(x)$ the same way.

For any n by P_n denote the projection onto the subspace spanned by $X_n = \{e_1, e_2, \dots, e_n\}$ and by $Y_n = (I - P_n)H$ we will denote the orthogonal complement of X_n in H.

Assume that $F : D(F) \to H$, $D(F) \subset H$. We investigate the equation

$$F(x) = 0. \tag{5}$$

Definition 2.1. Let $m, M \in \mathbb{N}$ with $m \leq M$. A compact set $W \subset X_m$ and a sequence of pairs $\{x_k^{\pm} \in \mathbb{R} \mid x_k^- < x_k^+\}$ form *the self-consistent a priori bounds* for equation (5)

C1: For $k > M$, $x_k^- < 0 < x_k^+$

C2: Let $\hat{x}_k := \max |x_k^{\pm}|$, then $\sum_k \hat{x}_k^2 < \infty$.

C3: The function $x \mapsto F(x)$ is continuous on

$$W \oplus \Pi_{k=m+1}^{\infty}[x_k^-, x_k^+] \subset H.$$

Moreover, if we define $f_k = \max_{x \in W \oplus \Pi_{k=m+1}^{\infty}[x_k^-, x_k^+]} |F_k(x)|$, then $\sum_k f_k^2 < \infty$.

C4: For every $u, d \in W \oplus \Pi_{k=m+1}^{\infty}[x_k^-, x_k^+]$, such that for some $k > m$

$$u_k = x_k^+ \quad \text{and} \quad d_k = x_k^- \tag{6}$$

we have

$$F_k(u) \cdot F_k(d) < 0. \tag{7}$$

Definition 2.2. Let $m, M \in \mathbb{N}$ with $m \leq M$. A pair of compact sets $N \subset W \subset X_m$ and a sequence of pairs $\{x_k^\pm \in \mathbb{R} \mid x_k^- < x_k^+\}$ form *the topologically self-consistent a priori bounds* for equation (5) if W and $\{x_k^\pm\}$ are the self-consistent a priori bounds for equation (5) and the following condition holds

C5: For all $x \in \partial N$ and for all $u \in \Pi_{k=m+1}^\infty [x_k^-, x_k^+]$

$$P_m F(x+u) \neq 0$$

Let $y = \sum_{i=m+1}^M \frac{x_i^- + x_i^+}{2} \cdot e_i$. We define a modified projection operator P_k^* for $k \geq m$ by

$$P_k^*(x) = P_k(x) + (I - P_k)y. \tag{8}$$

Observe that for $k > M$ we have $P_k^* = P_k$. The main reason to introduce the projection P_k^* is to have the following property

$$P_k^*(W \oplus \Pi_{i=m+1}^\infty [x_i^-, x_i^+]) \subset W \oplus \Pi_{i=m+1}^\infty [x_i^-, x_i^+]. \tag{9}$$

Theorem 2.3. *Assume that $N \subset W$ and $\{x_k^\pm\}$ are the topologically self-consistent a priori bounds for equation (5). Assume that*

$$\deg(P_m \circ F \circ P_m^* : X_m \to X_m, \operatorname{int} N, 0) \neq 0. \tag{10}$$

Then there exists $x \in N \oplus \Pi_{i=m+1}^\infty [x_i^-, x_i^+]$ such that $F(x) = 0$.

Proof. For $k \geq M$ consider a k-th Galerkin projection of F given by

$$x \in X_k \mapsto P_k F(x). \tag{11}$$

Consider a set

$$N^k = N \oplus \Pi_{i=m+1}^k [x_i^-, x_i^+]. \tag{12}$$

Obviously $N^k \subset W \oplus \Pi_{i=m+1}^\infty [x_k^-, x_k^+]$, hence by **C3** it follows that $N^k \subset \operatorname{dom}(F)$ and $P_k F$ is continuous on N^k.

For $i > m$ we define $\alpha_i \in \{-1, 1\}$ as follows. Let $x \in W \oplus \Pi_{i=m+1}^\infty [x_k^-, x_k^+]$ be any point such that $x_k = x_k^+$

$$\begin{aligned} \alpha_i &= 1, \qquad \text{if } F_i(x) > 0, \\ \alpha_i &= -1, \qquad \text{otherwise.} \end{aligned}$$

It is easy to see from **C4** that this definition does not depend on the choice of x.

We will compute the degree $\deg(P_k F, \operatorname{int} N^k, 0)$ using a homotopy linking $P_k F$ with the map G defined as follows

$$\begin{aligned} P_m G(x) &= P_m(F(P_m^*(x))), \\ G_i(x) &= g_i(x_i) e_i, \qquad \text{for } i > m \end{aligned}$$

where $g_i : [x_k^-, x_k^+] \to \mathbb{R}$ is given by

$$g_i(z) = \alpha_i \left(z - \frac{x_k^+ + x_k^-}{2} \right).$$

The homotopy H is given by the following conditions

$$\begin{aligned} P_m H(t,x) &= P_m F((1-t)x + t P_m^*(x)) \\ H_i(t,x) &= (1-t) F_i(x) + t g_i(x_i), \qquad \text{for } i = m+1, \ldots, k. \end{aligned}$$

Let us observe that $H(0,x) = P_k F(x)$ and $H(1,x) = G(x)$. It remains to show that $\deg(H(t,\cdot), \operatorname{int} N^k, 0)$ is defined and does not depend on t.

For this it is enough to show that

$$H(t,x) \neq 0, \qquad \text{for } x \in \partial N^k \text{ and } t \in [0,1]. \tag{13}$$

Observe that if $x \in \partial N^k$, then one of the following conditions is satisfied

$$P_m x \in \partial N, \tag{14}$$
$$x_i = x_i^{\pm}, \qquad \text{for some } i = m+1, \dots, k. \tag{15}$$

Assume (14) holds. Since $P_m((1-t)x + tP_m^*(x)) = P_m(x) \in \partial N$ the condition **C5** implies $P_m H(t,x) \neq 0$, hence $P_k H(t,x) \neq 0$.

Now assume (15). Without any loss of generality we can assume that $x_i = x_i^+$ and $\alpha_i = 1$. The proof for the other cases is analogous. This means that $F_i(x) > 0$ and $g_i(x_i) > 0$. Hence $H_i(t,x) > 0$ and $P_k F(x) \neq 0$.

From multiplicative property of the local degree and our assumption about the degree of $P_m F$ it follows that

$$\begin{aligned}\deg(G, \operatorname{int} N^k, 0) = \deg(P_m F P_m^*, \operatorname{int} N, 0) \cdot \Pi_{i=m+1}^k \deg(g_i, (x_i^-, x_i^+), 0) = \\ \deg(P_m F P_m^*, \operatorname{int} N, 0) \cdot \Pi_{i=m+1}^k \alpha_i \neq 0.\end{aligned}$$

This means that there exists a point $x^k \in \operatorname{int} N^k$, such that $P_k F(x^k) = 0$. Passing to the limit as in the proof of Theorem 2.16 in [23] we obtain x^* such that $F(x^*) = 0$. $\square$

3. **The issue of local uniqueness.** Assume H is a real Hilbert space, $F : H \supset \operatorname{dom}(F) \to H$ and $V = W \oplus \Pi_{k=m+1}^\infty [a_k^-, a_k^+]$ form the self-consistent a-priori bounds, and we assume that W is convex. (In fact we need only the conditions C1,C2,C3).

We need two additional conditions about the derivatives of F

F1: for every i and j $\frac{\partial F_i}{\partial x_j} : V \to \mathbb{R}$ is continuous

F2: Let $d_{ij} = \max_{x \in V} \left| \frac{\partial F_i}{\partial x_j}(x) \right|$. Then for every i and every $x, y \in V$ the series

$$\sum_{j=1}^\infty d_{ij} \sup_{x,y \in V} |x_j - y_j|$$

converges.

Lemma 3.1. *For every $i = 1, 2, \dots$ and $x, y \in V$ we have*

$$F_i(x) - F_i(y) = \sum_{j=1}^\infty c_{ij}(x_j - y_j), \tag{16}$$

where $c_{ij} \in \left[\frac{\partial F_i}{\partial x_j}(V)\right]_I$, which is defined by

$$\left[\frac{\partial F_i}{\partial x_j}(V)\right]_I = \{y \in \mathbb{R} \mid \exists x \in V \; y = \frac{\partial F_i}{\partial x_j}(x)\}.$$

Moreover, if $x \to y$ then $c_{ij} \to \frac{\partial F_i}{\partial x_j}(y)$.

Proof. Let us fix an i and for any n we consider the map $F_i : X_n \to \mathbb{R}$. We have for any $x, y \in X_n$

$$F_i(x) - F_i(y) = \sum_{j=1}^{n} \left(\int_0^1 \frac{\partial F_i}{\partial x_j}(y + t(x-y))dt \right) \cdot (x_j - y_j) \tag{17}$$

Let us take any $x, y \in V$, then from (17) it follows immediately that

$$F_i(P_n x) - F_i(P_n y) = \sum_{j=1}^{n} \left(\int_0^1 \frac{\partial F_i}{\partial x_j}(P_n y + t(P_n x - P_n y))dt \right) \cdot (x_j - y_j) \tag{18}$$

We want now to pass to the limit $n \to \infty$ in the above equation. From **C3** it follows that $F_i(P_n x) \to F_i(x)$ and $F_i(P_n y) \to F_i(y)$. We will show that

$$\lim_{n\to\infty} \sum_{j=1}^{n} \left(\int_0^1 \frac{\partial F_i}{\partial x_j}(P_n y + t(P_n x - P_n y))dt \right) \cdot (x_j - y_j) = \tag{19}$$
$$\sum_{j=1}^{\infty} \left(\int_0^1 \frac{\partial F_i}{\partial x_j}(y + t(x-y))dt \right) \cdot (x_j - y_j)$$

Let us fix any $\epsilon > 0$. Observe that from **F2** it follows immediately that there exists n_0 such that for $n \geq n_0$ we have

$$\left| \sum_{j=n_0+1}^{\infty} \left(\int_0^1 \frac{\partial F_i}{\partial x_j}(y + t(x-y))dt \right) \cdot (x_j - y_j) \right| < \epsilon$$
$$\left| \sum_{j=n_0+1}^{n} \left(\int_0^1 \frac{\partial F_i}{\partial x_j}(P_n y + t(P_n x - P_n y))dt \right) \cdot (x_j - y_j) \right| < \epsilon.$$

Hence for $n \geq n_0$ we obtain

$$\left| \sum_{j=1}^{n} \left(\int_0^1 \frac{\partial F_i}{\partial x_j}(P_n y + t(P_n x - P_n y))dt \right) \cdot (x_j - y_j) - \right.$$
$$\left. \sum_{j=1}^{\infty} \left(\int_0^1 \frac{\partial F_i}{\partial x_j}(y + t(x-y))dt \right) \cdot (x_j - y_j) \right| \leq 2\epsilon +$$
$$\left| \sum_{j=1}^{n_0} \left(\int_0^1 \frac{\partial F_i}{\partial x_j}(P_n(y + t(x-y)))dt - \int_0^1 \frac{\partial F_i}{\partial x_j}(y + t(x-y))dt \right) \cdot (x_j - y_j) \right|$$

To finish the proof of (19) observe that from **F1** and the compactness of V it follows that for each j the functions $[0,1] \ni t \mapsto \frac{\partial F_i}{\partial x_j}(P_n(y + t(x-y)))$ converge uniformly on $[0,1]$ to the function $[0,1] \ni t \mapsto \frac{\partial F_i}{\partial x_j}(y + t(x-y))$.

Hence we obtain

$$F_i(x) - F_i(y) = \sum_{j=1}^{\infty} \left(\int_0^1 \frac{\partial F_i}{\partial x_j}(y + t(x-y))dt \right) \cdot (x_j - y_j). \tag{20}$$

To finish the proof observe that $\int_0^1 \frac{\partial F_i}{\partial x_j}(y + t(x-y))dt \in \left[\frac{\partial F_i}{\partial x_j}(V)\right]_I$. $\square$

Lemma 3.2. *Assume that for every i*

$$\min_{x\in V}\left|\frac{\partial F_i}{\partial x_i}(x)\right| > \sum_{j\neq i}\max_{x\in V}\left|\frac{\partial F_i}{\partial x_j}(x)\right|, \tag{21}$$

then F is an injection on V. Hence V contains at most one zero of F.

Proof. Let i_0 be such that for all i

$$|x_{i_0} - y_{i_0}| \geq |x_i - y_i|. \tag{22}$$

From Lemma 3.1 it follows that

$$\begin{aligned}|F_{i_0}(x) - F_{i_0}(y)| \geq |c_{i_0i_0}||x_{i_0} - y_{i_0}| - \sum_{j\neq i_0}|c_{i_0j}||x_j - y_j| \geq \\ \geq |x_{i_0} - y_{i_0}|\left(|c_{i_0i_0}| - \sum_{j\neq i_0}|c_{i_0j}|\right) \geq \\ \geq |x_{i_0} - y_{i_0}|\left(\min_{x\in V}\left|\frac{\partial F_i}{\partial x_i}(x)\right| - \sum_{j\neq i}\max_{x\in V}\left|\frac{\partial F_i}{\partial x_j}(x)\right|\right) > 0,\end{aligned}$$

hence $F_{i_0}(x) \neq F_{i_0}(y)$. □

3.1. **Block decomposition.** For Lemma 3.2 to apply, the matrix $\frac{\partial F}{\partial x}$ has to be dominated by the diagonal terms. This can be achieved by an approximate diagonalization in case of real eigenvalues, but to handle complex eigenvalues we need a slight generalization of Lemma 3.2. To this end following [26] we introduce the notion of the block decomposition of H.

Definition 3.3. A decomposition of H, into a sum of subspaces is called *a block decomposition of H* if the following conditions are satisfied

- **1.:** $H = \overline{\bigoplus_i H_i}$,
- **2.:** for every i $h_i = \dim H_i \leq h_{\max} < \infty$,
- **3.:** for every i $H_i = \langle e_{i_1}, e_{i_2}, \dots, e_{i_{h_i}}\rangle$,
- **4.:** If $\dim H = \infty$, then there exists k such that for $i > k$ $h_i = 1$.

For the block decomposition of H we adopt the following notation, which makes a distinction between the blocks and one dimensional subspaces spanned by $\langle e_i\rangle$. For the blocks we use $H_{(i)} = \langle e_{i_1}, \dots e_{i_k}\rangle$, where $(i) = (i_1, \dots i_k)$. The symbol H_i will always mean the subspace generated by e_i. For one-dimensional block (i) we adopt the following convention, the only element of (i) will be denoted by the same letter i.

For a given block decomposition of H and a block (i), we set

$$\dim\,(i) = \dim\,H_{(i)}.$$

For any $x \in H$ by $x_{(i)}$ we will denote the projection of x onto $H_{(i)}$. For any l and $(i) = (i_1, \dots, i_k)$ we will say that $(i) \leq l$ if $i_s \leq l$ for all $s = 1, \dots, k$ and we say that $(i) > l$ if $i_s > l$ for all $s = 1, \dots, k$.

On each component $H_{(i)}$ we will use the norm induced from H. By $P_{(i)}$ we will denote an orthogonal projection onto $H_{(i)}$. By $\mathrm{Lin}(H_{(i)}, H_{(j)})$ we denote the set of all linear maps from $H_{(i)}$ to $H_{(j)}$ equipped with the operator norm $|A| = \max_{|v|=1, v\in H_{(i)}}|Av|$.

Assume that we have a block decomposition of H and conditions **F1, F2** are satisfied. Then it is easy to see that the following two conditions are satisfied

F1': for every (i) and (j) the map

$$\frac{\partial F_{(i)}}{\partial x_{(j)}} : V \to \mathrm{Lin}(H_{(i)}, H_{(j)}) \approx \mathbb{R}^{\dim\,(i) \times \dim\,(j)}$$

is continuous,

F2': Let $n_{(i)(j)} = \max_{x\in V} \left|\frac{\partial F_{(i)}}{\partial x_{(j)}}(x)\right|$. Then for every (i) and every $x, y \in V$ the series

$$\sum_{(j)} n_{(i)(j)} \sup_{x,y\in V} |x_{(j)} - y_{(j)}|$$

converge.

Definition 3.4. Assume that $A(x) \in \mathbb{R}^{n\times m}$ is a matrix depending on x. We define $[A(V)] \subset \mathbb{R}^{n\times m}$ by

$$C \in [A(V)] \quad \text{iff} \quad C_{ij} \in [\inf_{x\in V} A_{ij}(x), \sup_{x\in V} A_{ij}(x)]. \tag{23}$$

Definition 3.5. For any linear map $A \in \mathrm{Lin}(\mathbb{R}^n, \mathbb{R}^m)$ we define

$$\inf(A) = \min_{|v|=1, v\in\mathbb{R}^n} |Av|. \tag{24}$$

For a matrix valued function $A(x)$ we set

$$\inf(A(V)) = \inf_{C\in[A(V)]} \inf(C). \tag{25}$$

We have the following easy

Lemma 3.6. *Let $A \in Lin(\mathbb{R}^n, \mathbb{R}^n)$ be an isomorphism. Then $|A^{-1}| = \frac{1}{\inf A}$.*

The following theorem is a direct generalization of Lemma 3.2 and the proof is essentially the same.

Theorem 3.7. *Assume that we have a block decomposition of H. Assume* **F1'** *and* **F2'** *are satisfied and for every (i)*

$$\inf\left(\frac{\partial F_{(i)}}{\partial x_{(i)}}(V)\right) > \sum_{(j)\neq(i)} \max_{x\in V}\left|\frac{\partial F_{(i)}}{\partial x_{(j)}}(x)\right|, \tag{26}$$

then F is an injection on V. Hence V contains at most one zero of F.

4. **The issues of stability and instability.**

4.1. **A criterion for nonlinear instability.**

Definition 4.1. For any square matrix A by $S(A)$ we denote a symmetric part of A, $S(A) = (A + A^t)/2$.

For any square symmetric matrix A we define

$$\mu_{inf}(A) = \min_{|v|=1}(Av|v),$$
$$\mu_{sup}(A) = \max_{|v|=1}(Av|v).$$

For a matrix valued function $A(x)$ we set

$$\mu_{inf}(A(Z)) = \inf_{C\in[A(Z)]} \mu_{inf}(C),$$
$$\mu_{sup}(A(Z)) = \sup_{C\in[A(Z)]} \mu_{sup}(C).$$

Remark 1. It is easy to see that $\mu_{sup}(A)$ coincides with the logarithmic norm $\mu(A)$ based on Euclidean norm, which was used in [26].

Theorem 4.2. *Let $V = W \oplus \Pi[a_k^-, a_k^+]$ be the self-consistent a priori bounds and $N \subset V$ be an isolating block for a fixed point p. We assume that the logarithmic norms for Galerkin projections are all uniformly bounded on V - hence we have the existence and the uniqueness of classical solutions in V (see [24, 26]).*

*Assume that we have a block decomposition and let the block (i_0) consists of all exit directions. Assume that **F1** and **F2** are satisfied.*

Let

$$d_{(i_0)} = \mu_{\inf}\left(S\left(\frac{\partial F_{(i_0)}}{\partial x_{(i_0)}}(Z)\right)\right) - \sum_{(i)\neq(i_0)}\left|\frac{\partial F_{(i_0)}}{\partial x_{(i)}}(Z)\right|$$

$$a_{(i_0)(i)} = \left(\mu_{\inf}\left(S\left(\frac{\partial F_{(i_0)}}{\partial x_{(i_0)}}(Z)\right)\right) - \sum_{(i)\neq(i_0)}\left|\frac{\partial F_{(i_0)}}{\partial x_{(i)}}(Z)\right| \right.$$
$$\left. -\mu_{sup}\left(S\left(\frac{\partial F_{(i)}}{\partial x_{(i)}}(Z)\right)\right) - \sum_{(i)\neq(j)}\left|\frac{\partial F_{(i)}}{\partial x_{(j)}}(Z)\right|\right),$$

If

$$\begin{aligned} d_{(i_0)} &> 0 \\ a_{(i_0)(i)} &> 0, \qquad \forall (i) \neq (i_0) \end{aligned}$$

then p is unstable, i.e. there exists $\epsilon > 0$ such that for any $\delta > 0$ there exist x_δ, such that $|x_\delta - p| < \delta$, and $t_\delta > 0$ such that $|\varphi(t_\delta, x_\delta) - p| > \epsilon$.

Proof. Assume that $p = 0$. Consider $Z = \{x \in N \oplus \Pi[a_k^-, a_k^+] \mid |x_i| \leq |x_{i_0}|, \quad \forall (i)\}$.

From Lemma 3.1 if follows that for $x \in Z$ and any block (j) the following condition is satisfied

$$F_{(j)}(x) = F_{(j)}(x) - F_{(j)}(p) \in \sum_{(i)}\left[\frac{\partial F_{(j)}}{\partial x_{(i)}}(Z)\right] \cdot x_{(i)}. \tag{27}$$

Assume that we have an orbit $x(t) \in Z$ for $t \in [0, t_1]$, where t_1 is a supremum of times with this property (we allow for $t_1 = \infty$), then for $t \in [0, t_1]$ we have

$$\begin{aligned} \frac{1}{2}\frac{d|x_{(i_0)}|^2}{dt} = x_{(i_0)} \cdot \frac{dx_{(i_0)}}{dt} \in \\ \left(\left[\frac{\partial F_{(i_0)}}{\partial x_{(i_0)}}(Z)\right] x_{(i_0)} + \sum_{(i)\neq(i_0)}\left[\frac{\partial F_{(i_0)}}{\partial x_{(i)}}(Z)\right] x_{(i)}\right) x_{(i_0)} \geq \\ \mu_{inf}\left(S\left(\frac{\partial F_{(i_0)}}{\partial x_{(i_0)}}(Z)\right)\right) \cdot |x_{i_0}|^2 - \sum_{(i)\neq(i_0)}\left|\frac{\partial F_{(i_0)}}{\partial x_{(i)}}\right| |x_{(i)}||x_{(i_0)}| \geq \\ |x_{(i_0)}|^2\left(\mu_{inf}\left(S\left(\frac{\partial F_{(i_0)}}{\partial x_{(i_0)}}(Z)\right)\right) - \sum_{(i)\neq(i_0)}\left|\frac{\partial F_{(i_0)}}{\partial x_{(i)}}\right|\right) \end{aligned}$$

For any (i) we have

$$\frac{1}{2}\frac{d|x_{(i)}|^2}{dt} = x_{(i)} \cdot \frac{dx_{(i)}}{dt} \in$$
$$\left(\left[\frac{\partial F_{(i)}}{\partial x_{(i)}}(Z)\right] x_{(i)} + \sum_{(i)\neq(j)} \left[\frac{\partial F_{(i)}}{\partial x_{(j)}}(Z)\right] x_{(j)}\right) x_{(i)} \leq$$
$$|x_{(i_0)}|^2 \left(\mu_{sup}\left(S\left(\frac{\partial F_{(i)}}{\partial x_{(i)}}(Z)\right)\right) + \sum_{(i)\neq(j)} \left|\frac{\partial F_{(i)}}{\partial x_{(j)}}\right|\right)$$

From the above inequalities it follows that for any $(i) \neq (i_0)$

$$\frac{1}{2}\left(\frac{d|x_{(i_0)}|^2}{dt} - \frac{d|x_{(i)}|^2}{dt}\right) \geq$$
$$|x_{(i_0)}|^2 \left(\mu_{inf}\left(S\left(\frac{\partial F_{(i_0)}}{\partial x_{(i_0)}}(Z)\right)\right) - \sum_{(i)\neq(i_0)} \left|\frac{\partial F_{(i_0)}}{\partial x_{(i)}}\right| + \right.$$
$$\left. -\mu_{sup}\left(S\left(\frac{\partial F_{(i)}}{\partial x_{(i)}}(Z)\right)\right) - \sum_{(i)\neq(j)} \left|\frac{\partial F_{(i)}}{\partial x_{(j)}}\right|\right).$$

Hence for $t \in [0, t_1]$ and all $(i) \neq (i_0)$ holds

$$\frac{1}{2}\frac{d|x_{(i_0)}|^2}{dt} \geq d_{(i_0)}|x_{(i_0)}|^2, \tag{28}$$
$$\frac{1}{2}\left(\frac{d|x_{(i_0)}|^2}{dt} - \frac{d|x_{(i)}|^2}{dt}\right) \geq a_{(i_0)(i)}|x_{(i_0)}|^2. \tag{29}$$

From(29) it follows that the function $|x_{(i_0)}(t)|^2 - |x_{(i)}(t)|^2$ is increasing for $t \in [0, t_1]$, hence $x(t) \in Z$ for $t \in [0, t_1]$.

Assume that $x(0) \in Z$ and $x_{i_0}(0) \neq 0$. From (28) it follows that $t_1 < \infty$. Let $\epsilon = \mathrm{dist}(p, N^-)$. Observe that $\epsilon > 0$, because the number of exit directions is finite. To finish the proof observe that $x(t_1) \in N^-$. □

Let us comment why the assumptions of Theorem 4.2 can be quite easily satisfied. The condition $d_{(i_0)} > 0$ holds due to the following factors

- $\mu_{\mathrm{inf}}\left(S\left(\frac{\partial F_{(i_0)}}{\partial x_{(i_0)}}(Z)\right)\right) > 0$, we choose the block (i_0) to represent the unstable direction
- $\sum_{(i)\neq(i_0),(i)\leq m} \left|\frac{\partial F_{(i_0)}}{\partial x_{(i)}}(Z)\right|$ is small due to the diagonalization used to produce new coordinates,
- $\sum_{(i)\neq(i_0),(i)>m} \left|\frac{\partial F_{(i_0)}}{\partial x_{(i)}}(Z)\right|$ is small, because the modes with high-wave number have a relatively
  weak influence on the mode with low frequency number (this requires m to be large enough).

To discuss the condition $a_{(i_0)(i)} > 0$ observe first that

$$a_{(i_0)(i)} = d_{(i_0)} - \mu_{sup}\left(S\left(\frac{\partial F_{(i)}}{\partial x_{(i)}}(Z)\right)\right) - \sum_{(i)\neq(j)} \left|\frac{\partial F_{(i)}}{\partial x_{(j)}}(Z)\right|. \tag{30}$$

In the above formula first term $d_{(i_0)}$ is already positive. Observe that for the KS-equation
$-\mu_{sup}\left(S\left(\frac{\partial F_{(i)}}{\partial x_{(i)}}(Z)\right)\right)$ is positive and goes to ∞ as $(i) \to \infty$. It turns out that for the KS-equation the last term tends to $-\infty$ at a slower rate than the second one. Hence there are good chances to verify this condition (see Section 8 for more details).

4.2. **A criterion for the nonlinear stability.** Basically this is contained in Theorem 3.7 in [26]. It is shown there that it is enough to check that, if there exists $l \in \mathbb{R}$, such that for all blocks (i) (see also Remark 1)

$$\mu_{sup}\left(\frac{\partial F_{(i)}}{\partial x_{(i)}}(V)\right) + \sum_{(k),\,(k)\neq(i)} \left|\frac{\partial F_{(i)}}{\partial x_{(k)}}(V)\right| \leq l < 0,$$

then the steady state in $N \oplus \Pi[a_k^-, a_k^+]$ is stable (and attracting).

5. **The continuity of solutions of implicit equations through the self-consistent bounds.** Our goal in this section is to establish the regularity properties of the solution of the equation

$$F(\nu, x) = 0. \tag{31}$$

as a function of ν in the context of the self-consistent a priori bounds and to provide the mathematical basis for the rigorous numerical procedure for the computation of the derivatives of $x(\nu)$, which is required in the bifurcation analysis.

Let us assume additionally (just for simplicity, because this is not essential for the method) that for $k = 1, 2, \ldots$

$$\begin{aligned} \frac{\partial^2 F_k}{\partial \nu^2} &= 0, \\ d^3 F_k &= 0. \end{aligned}$$

By taking the derivatives of (31) with respect to ν we obtain

$$\frac{\partial F_k}{\partial \nu} + \sum_i \frac{\partial F_k}{\partial x_i} x'_i = 0 \tag{32}$$

$$2\sum_i \frac{\partial^2 F_k}{\partial \nu \partial x_i} x'_i + \sum_{i,j} \frac{\partial^2 F_k}{\partial x_i \partial x_j} x'_i x'_j + \sum_i \frac{\partial F_k}{\partial x_i} x_i^{(2)} = 0 \tag{33}$$

$$3\sum_i \frac{\partial^2 F_k}{\partial \nu \partial x_i} x_i^{(2)} + 3\sum_{i,j} \frac{\partial^2 F_k}{\partial x_i \partial x_j} x_i^{(2)} x'_j + \sum_i \frac{\partial F_k}{\partial x_i} x_i^{(3)} = 0 \tag{34}$$

5.1. **The existence and uniqueness issue.** In the computation of the derivatives of the steady states with respect to parameters one has to solve the following linear equation (compare (32)-(34))

$$z + (D + N) \cdot x = 0, \tag{35}$$

where $z, x \in \mathbb{R}^d$ and $D, N \in \mathbb{R}^{d \times d}$, $D + N = \frac{\partial F}{\partial x}$ at a fixed point for the KS-equation (or its Galerkin projection), D is a diagonal matrix and D dominates N in a suitable sense. The dimension d can be infinite. We assume that we have a block decomposition of H.

We introduce the block-infinity norm given by

$$|x|_{b,\infty} = \sup_{(i)} |x_{(i)}|. \tag{36}$$

Let $H_{b,\infty}$ be a completion of H in the above norm.

Theorem 5.1. *Let $d \in \mathbb{N}_+ \cup \{\infty\}$. Consider equation*

$$z + (D + N) \cdot x = 0, \tag{37}$$

where $z, x \in H_{b,\infty}$ and $D, N \in \mathbb{R}^{d\times d}$ (we do not assume that D, N define maps on $H_{b,\infty}$, they are just collections of coefficients indexed by $\mathbb{N}_+^2$).

Assume that there exists $\alpha < 1$ such that

$$\begin{aligned} D_{(i)(j)} &= 0, \qquad \text{if } (i) \neq (j) && (38)\\ \alpha \inf D_{(i)(i)} &> \sum_{(j)} |N_{(i)(j)}|, \quad \text{for all } (i), && (39)\\ \inf D_{(i)(i)} &> \lambda > 0, \quad \text{for all } (i). && (40) \end{aligned}$$

Then equation (37) has a unique solution, $x^\star \in H_{b,\infty}$ and

$$|x^\star|_{b,\infty} \leq \frac{|D^{-1}z|_{b,\infty}}{1-\alpha} \tag{41}$$

Proof. Observe that from (39) it follows immediately that $D_{(i)(i)}$ is an isomorphism (see Def.3.5). From Lemma 3.6 it follows that

$$|D^{-1}_{(i)(i)}| = \frac{1}{\inf D_{(i)(i)}}. \tag{42}$$

Consider the following map

$$P(x) = -D^{-1}z - D^{-1}Nx. \tag{43}$$

We will prove that $P : H_{b,\infty} \to H_{b,\infty}$ is a contraction with the contraction coefficient α.

First of all observe that $\operatorname{dom}(P) = H_{b,\infty}$. Namely we have

$$|D^{-1}z|_{b,\infty} = \sup_{(i)} |D^{-1}_{(i)(i)} z_{(i)}| \leq \sup_{(i)} \frac{1}{\lambda} |z_{(i)}| = \frac{1}{\lambda}|z|_{b,\infty}. \tag{44}$$

It remains to show that also $D^{-1}Nx \in H_{b,\infty}$ and the linear map $D^{-1}N$ is a contraction

$$\begin{aligned} |D^{-1}Nx|_{b,\infty} = \sup_{(i)} \left| D^{-1}_{(i)(i)} \sum_{(j)} \left(N_{(i)(j)} \cdot x_{(j)}\right)\right| \leq \\ \sup_{(i)} |D^{-1}_{(i)(i)}| \cdot \left(\sum_{(j)} |N_{(i)(j)}|\right) \cdot |x|_{b,\infty} = \\ \sup_{(i)} \frac{1}{\inf D_{(i)(i)}} \cdot \left(\sum_{(j)} |N_{(i)(j)}|\right) \cdot |x|_{b,\infty} \leq \alpha |x|_{b,\infty}. \end{aligned}$$

From the Banach contraction principle it follows that the map P has a unique fixed point, x^*. This point is a unique solution to (37) and it is equal to a limit of the sequence $P^n(x)$ for any $x \in H_{b,\infty}$.

To finish the proof observe the ball $B = B(0, \frac{|D^{-1}z|_{b,\infty}}{1-\alpha})$ is mapped by P into itself. Namely, for any $x \in B$ we have

$$|P(x)|_{b,\infty} \leq |D^{-1}Nx|_{b,\infty} + |D^{-1}z|_{b,\infty} \leq \alpha \frac{|D^{-1}z|_{b,\infty}}{1-\alpha} + |D^{-1}z|_{b,\infty} = \frac{|D^{-1}z|_{b,\infty}}{1-\alpha}.$$

□

5.2. **Self-consistent bounds for the derivatives.** We want to solve equation (35) in H (we have the existence and the uniqueness in the larger space $H_{b,\infty}$) over a range of ν, so that a solution for ν belongs to the set L, which forms the self-consistent a priori bounds for (35). We show now that such a solution exists under some mild assumptions, which are easily satisfied for the KS equations.

Let $[Z] \subset H$, $[D], [N]$ be a sets of linear mappings defined densely on H with values in H. We assume that $X_k \subset \mathrm{dom}\,(D)$ and $X_k \subset \mathrm{dom}\,(N)$ for any k and $D \in [D]$ and $N \in [N]$.

For fixed $z \in [Z]$, $D \in [D]$ and $N \in [N]$ we set $F(x) = z + (D+N)x$.

We assume for any $z \in [Z]$, $D \in [D]$ and $N \in [N]$ that the assumptions of Theorem 5.1 are satisfied and additionally the following inequalities hold

$$\begin{aligned}
\gamma &\leq s, \quad \gamma \leq t, \qquad s,t > 5, && (45)\\
\frac{|z_i|}{\inf D_{(i)(i)}} &\leq \frac{\tilde{C}}{i^t}, && (46)\\
\inf D_{(i)(i)} &> \beta i^4, \qquad i > m && (47)\\
|D_{(i)(i)}| &< B i^4, && (48)\\
|N_{ij}| &\leq \frac{iG}{|i-j|^s}, \qquad i \neq j && (49)\\
N_{ii} &= 0 && (50)
\end{aligned}$$

All these assumptions are satisfied for the KS equations as it will be shown later.

Under the above assumptions we are going to construct the set L

$$\begin{aligned}
W_L &= \Pi_{(i)\leq m}\overline{B}_{(i)}(0, r_{(i)})\\
L &= W_L \oplus \Pi_{i=m+1}^{\infty}[x_i^-, x_i^+] \qquad (51)
\end{aligned}$$

which will form for $M = m$ the topologically self-consistent a priori bounds containing the solution of (35) for all $z \in [Z]$, $D \in [D]$ and $N \in [N]$.

We construct L using an algorithm for producing the self-consistent a priori bounds we used for the proof of the existence of fixed points for the KS equations in [23, 26]. As a result of this algorithm we obtain a sequence $L_0 \supset L_1 \supset L_2 \supset \dots$, such that each set L_i is of the form given by (51) and for $i \geq i_0$ L_i forms the topologically self-consistent a priori bounds containing the solution of (35) for all $z \in [Z]$, $D \in [D]$ and $N \in [N]$.

We will use an additional index in the parameters defining L_i, i.e. L_i will be defined according to (51) by setting the values for $x_{i,k}^{\pm}$. We will not change the set W_L.

Algorithm:
Initialization: We define L_0 as follows. Let us fix $\epsilon > 0$. We set

$$\begin{aligned} R &= \frac{(1+\epsilon)}{1-\alpha} \sup_{z\in[Z],D\in[D]} \sup_k \frac{|z_k|}{\inf D_{(k)(k)}} && (52)\\ r_{(i)} &= R, \qquad \text{for } (i) \leq m && (53)\\ x^{\pm}_{0,i} &= R, \qquad \text{for } i > m && (54)\end{aligned}$$

One step: Assume that L_i is defined. We define L_{i+1} as follows: for $k > m$ let b_k be given by

$$b_k = (1+\epsilon) \sup_{x\in L_i,N\in[N],D\in[D]} \frac{|(Nx)_k|}{\inf D_{kk}} + (1+\epsilon) \sup_{z\in[Z],D\in[D]} \frac{|z_k|}{\inf D_{kk}}. \qquad (55)$$

We define $x^{\pm}_{i+1,k}$ as follows

$$\begin{aligned} x^{\pm}_{i+1,k} &= \pm b_k, \qquad \text{if } b_k < x^{+}_{i,k},\\ x^{\pm}_{i+1,k} &= x^{\pm}_{i,k}, \qquad \text{otherwise.}\end{aligned}$$

We set

$$L_{i+1} = W_L \oplus \Pi_{k=m+1}^{\infty}[x^{-}_{i+1,k}, x^{+}_{i+1,k}]. \qquad (56)$$

Lemma 5.2. *For all $z \in [Z]$, $D \in [D]$ and $N \in [N]$ the set L_0 satisfies **C4**, **C5** for all $k > m$, namely $P_k \circ F(x) \neq 0$ for $x \in X_k \cap L_0$ and*

$$\deg(P_m \circ F : X_m \to X_m, \operatorname{int} W_L, 0) \neq 0. \qquad (57)$$

Proof. Let us fix any $z \in [Z]$ $D \in [D]$, $N \in [N]$. We show first that for any block (i) and $x \in L_0$ if $|x_{(i)}| = R$, then

$$\left|D_{(i)(i)}x_{(i)}\right| > |(Nx)_{(i)}| + |z_{(i)}|. \qquad (58)$$

Indeed

$$\begin{aligned} \left|D_{(i)(i)}x_{(i)}\right| &\geq \inf D_{(i)(i)}R,\\ |(Nx)_{(i)}| &\leq \left(\sum_{(j)} \left|N_{(i)(j)}\right|\right) R < \alpha \inf D_{(i)(i)}R,\\ |z_i| &< R(1-\alpha)\inf D_{(i)(i)}.\end{aligned}$$

Hence

$$\left|D_{(i)(i)}x_{(i)}\right| - |(Nx)_{(i)}| - |z_{(i)}| > 0. \qquad (59)$$

Since for $i > m$ $\dim(i) = 1$, hence $D_{ii} \in \mathbb{R} \setminus \{0\}$. If $D_{ii} > 0$, then $F_i(x) > 0$ if $x_i = R$ and $F_i(x) < 0$ if $x_i = -R$. If $D_{ii} < 0$, then $F_i(x) < 0$ if $x_i = R$ and $F_i(x) > 0$ if $x_i = -R$. This proves condition **C4**. Observe that (59) implies that **C5** is satisfied.

It remains to prove (57). Observe that from (58) it follows that for any $\lambda \in [0,1]$ and $x \in L_0$, such that $|x_{(i)} = R$

$$\left|D_{(i)(i)}x_{(i)}\right| > |(\lambda Nx)_{(i)}| + |\lambda z_{(i)}|. \qquad (60)$$

Let us define a homotopy $H : [0,1] \times X_m \to X_m$

$$H(\lambda, x) = H_\lambda(x) = \lambda z + (D + \lambda N)x. \qquad (61)$$

From (60) it follows that

$$P_m \circ H_\lambda(x) \neq 0, \qquad x \in \partial W_L. \tag{62}$$

From the continuation property of the degree it follows that

$$\deg(P_m \circ F, \operatorname{int} W_L, 0) = \deg(P_m \circ H_1, \operatorname{int} W_L, 0) = \deg(P_m \circ H_0, \operatorname{int} W_L, 0). \tag{63}$$

To finish the proof observe $P_m \circ H_0$ is an linear isomorphism. Hence $\deg(P_m \circ H_0, \operatorname{int} W_L, 0) = \pm 1$. □

The following Lemma follows directly from the rule of construction of L_{i+1} starting from L_i.

Lemma 5.3. *For all $z \in [Z]$, $D \in [D]$, $N \in [N]$ and any $i \geq 1$ $i = 1, 2, \dots$ the set L_i satisfies **C4**, **C5** for all $k > m$, namely $P_k \circ F(x) \neq 0$ for $x \in X_k \cap L_i$ and*

$$\deg(P_m \circ F : X_m \to X_m, \operatorname{int} W_L, 0) \neq 0. \tag{64}$$

In view of the above lemma it is clear that to show that the algorithm produces the topologically self-consistent a priori bounds for (35) it is enough to show that for i large enough conditions **C2** and **C3** are satisfied.

As the first step in this direction we have the following

Lemma 5.4. *There exists a constant E_1, such that for all k holds*

$$|x^{\pm}_{1,k}| \leq \frac{E_1}{k^3}. \tag{65}$$

Proof. We have for $k > m$

$$|(Nx)_k| \leq \left(\sum_{j \neq k} |N_{kj}| \right) R \leq kR \sum_{j \neq k} \frac{G}{|k-j|^s} < kRGS(s),$$

where $S(s) = \sum_{j \in \mathbb{Z} \setminus \{0\}} \frac{1}{|j|^s}$. Hence

$$b_k \leq (1+\epsilon) \left(\frac{kRGS(s)}{\beta k^4} + \frac{\tilde{C}}{i^t} \right) \leq \frac{E_1}{k^3}. \tag{66}$$

□

In the sequel we will use several times the following

Lemma 5.5. *Assume $s > 1$, $\gamma > 1$. There exists a constant $S(s, \gamma)$ such that for all $k \in \mathbb{N}_+$ holds*

$$\sum_{j \in \mathbb{N}_+, j \neq k} \frac{1}{|k-j|^s j^\gamma} \leq \frac{C_Q(s,\gamma)}{|k|^{\min(s,\gamma)}}.$$

Lemma 5.6. *There exist i and a constant E_i, such that*

$$|x^{\pm}_{i,k}| \leq \frac{E_i}{k^{\min(s,t)}}. \tag{67}$$

Proof. By induction. As an induction assumption we assume that for $i \geq 1$ holds ($i = 1$ is treated in Lemma 5.4)

$$|x_{i,k}| \leq \frac{E_i}{k^{\gamma_i}}, \qquad \gamma_i \geq 3, \qquad \gamma \leq s. \tag{68}$$

Using Lemma 5.5 we obtain

$$\frac{1}{|D_{kk}|}\sum_j |N_{kj}x_j| + \frac{|z_k|}{|D_{kk}|} \leq \frac{kGE_i}{\beta k^4}\sum_j \frac{1}{|k-j|^s|j|^{\gamma_i}} + \frac{\tilde{C}}{k^t} \leq \frac{GE_iC_Q(s,\gamma_i)}{k^{\gamma_i+3}} + \frac{\tilde{C}}{|k|^t},$$

Since for a suitable constant E_{i+1} we have

$$b_k \leq \frac{GE_iC_Q(s,\gamma_i)}{k^{\gamma_i+3}} + \frac{\tilde{C}}{k^t} \leq \frac{E_{i+1}}{k^{\min(\gamma_i+3,t)}}. \tag{69}$$

Hence we had proven that if $\gamma_i \leq s$, then $\gamma_{i+1} = \min(\gamma_i+3,t)$. □

Lemma 5.7. *If $|x^{\pm}_{i,k}| \leq \frac{E_i}{k^{\gamma_i}}$ and $\gamma_i > 5$, then conditions **C2** and **C3** are satisfied on L_i for $F(x) = z + (D+N)x$.*

Proof. Condition **C2** is manifestly satisfied. For condition **C3** it is enough to prove the convergence statement for f_k.

We have

$$f_k \leq |D_{kk}x_k| + \sum_j |N_{kj}x_j| + |z_k| \leq \frac{E_iBk^4}{k^{\gamma_i}} + GE_ik\sum_j \frac{1}{|k-j|^s|j|^{\gamma_i}} + \frac{B}{\beta}\frac{C}{k^t}.$$

Using Lemma 5.5 we obtain

$$f_k \leq \frac{C_1}{k^{\gamma_i-3}} + \frac{C_2}{k^{\min(s,\gamma_i)-1}} + \frac{}{C_3}k^t \tag{70}$$

for some constants C_1, C_2, C_3. Now if $\gamma_i - 3 > 1/2$, $s-1 > 1/2$ and $t > 1/2$, then $\sum_k f_k^2 < \infty$. □

The following theorem summarizes the above developments

Theorem 5.8. *There exists L of the form given by (51), which for all $z \in [Z]$, $D \in [D]$ and $N \in [N]$ forms the topologically self-consistent a priori bounds for (35). Moreover, the solution (35) belongs to L.*

5.3. **Regularity of solutions of implicit equations.** In this section we state and prove theorems about the regularity of solutions of the equation

$$F(\nu, x) = 0 \tag{71}$$

and (what will be important near the bifurcation points)

$$F_i(\nu, x_1, x) = 0, \qquad i \geq 2. \tag{72}$$

We want to find $x(\nu)$ for (71) and $x(\nu, x_1)$ for (72).

In both cases we assume that the domain of interest is a compact and convex set, Λ, $\Lambda \subset \mathbb{R}$ for (71) and $\Lambda \subset \mathbb{R}^2$ for (72). We assume in both cases that we have constructed $V = W \oplus \Pi_{k>m}[x^-_k, x^+_k]$ the topologically self-consistent a priori bounds, where we were able to verify the uniqueness. Hence we have a function $x(\nu)$ $(x(\nu, x_1))$ for $\nu \in \Lambda$ $((\nu, x_1) \in \Lambda)$.

Through this section we assume:

- bounds on V: $|x_i| \leq \frac{C}{i^s}$, for $x \in V$
- we have a block decomposition

- **F1** and **F2** for F and all its partial derivatives are satisfied on $\Lambda \times V$
- we define the set of matrices $[D]$, $[N]$

$$
\begin{aligned}
[D]_{(i)(i)} &= \left[\frac{\partial F_{(i)}}{\partial x_{(i)}}(\Lambda, V)\right]_I \\
[D]_{(i)(j)]} &= 0, \qquad (i) \neq (j) \\
[N]_{(i)(j)} &= \left[\frac{\partial F_{(i)}}{\partial x_{(j)}}(\Lambda, V)\right]_I, \qquad (i) \neq (j) \\
[N]_{(i)(i)} &= \{0\}
\end{aligned}
$$

- Theorem 5.1 type assumptions:
  there exists $\alpha < 1$ such that

$$\alpha \inf_{D \in [D]} \inf D_{(i)(i)} > \sum_{(j)} \sup_{N \in [N]} |N_{(i)(j)}|, \qquad \text{for all } (i), \tag{73}$$

$$\inf_{D \in [D]} \inf D_{(i)(i)} > \lambda > 0, \qquad \text{for all } (i). \tag{74}$$

- Section 5.2 type assumptions

$$\gamma \leq s, \quad \gamma \leq t, \qquad s, t > 5, \tag{75}$$

$$\inf_{D \in [D]} \inf D_{(i)(i)} > \beta i^4, \qquad i > m \tag{76}$$

$$\sup_{D \in [D]} |D_{(i)(i)}| < B i^4, \tag{77}$$

$$\sup_{N \in [N]} |N_{ij}| \leq \frac{iG}{|i-j|^s}, \qquad i \neq j \tag{78}$$

-

$$\left|\frac{\partial F_i}{\partial \nu}(\Lambda, V)\right| \leq \Theta i^4 \sup_{x \in V} |x_i| \tag{79}$$

$$\left|\frac{\partial F_i}{\partial \nu \partial x_j}(\Lambda, V)\right| = \delta_{i,j} i^4, \qquad \text{for } i > i_0 \tag{80}$$

$$\sum_j \frac{\left|\frac{\partial F_i}{\partial \nu \partial x_j}(\Lambda, V)\right|}{j^s} < \infty, \qquad \text{for all } i \tag{81}$$

Theorem 5.9. *Under the above assumptions the function $x(\nu)$ ($x(\nu, x_1)$) is C^1 on Λ and there exists a constant C_1 such that $|x_i'| \leq \frac{C_1}{i^s}$ ($\left|\frac{\partial x_i}{\partial \nu}\right| \leq \frac{C_1}{i^s}$, $\left|\frac{\partial x_i}{\partial x_1}\right| \leq \frac{C_1}{i^s}$).*

Moreover, for equation (71) for any $\nu_1, \nu_2 \in \Lambda$ and any i

$$\left|\frac{x_i(\nu_1) - x_i(\nu_2)}{\nu_1 - \nu_2}\right| \leq \frac{C_1}{i^s}.$$

In the case of equation (72) for $(\nu_1, y), (\nu_2, y) \in \Lambda$ and $(\nu, y_1), (\nu, y_2) \in \Lambda$ and for any i the following holds

$$\left|\frac{x_i(\nu_1, y) - x_i(\nu_2, y)}{\nu_1 - \nu_2}\right| \leq \frac{C_1}{i^s},$$

$$\left|\frac{x_i(\nu, y_1) - x_i(\nu, y_2)}{y_1 - y_2}\right| \leq \frac{C_1}{i^s}.$$

Proof. We provide the proof essentially for the derivative with respect to ν, only. The proof for (72) is the same hence will be omitted; only the place where there is some difference in estimates will be discussed.

Let us fix $\nu \in \Lambda$ and let $h \neq 0$ be such that $\nu + h \in \Lambda$, then from Lemma 3.1 we obtain for all i

$$0 = F_i(\nu + h, x(\nu + h)) - F_i(\nu, x(\nu)) = c_{i\nu} h + \sum_j c_{ij}(x_j(\nu + h) - x_j(\nu)),$$

where

$$c_{i\nu} \in \left[\frac{\partial F_i}{\partial \nu}(\Lambda, V)\right]_I, \qquad c_{i\nu} \to \frac{\partial F_i}{\partial \nu}(\nu, x(\nu)) \quad \text{for } h \to 0$$

$$c_{ij} \in \left[\frac{\partial F_i}{\partial x_j}(\Lambda, V)\right]_I = [D]_{ij} + [N]_{ij}, \qquad c_{ij} \to \frac{\partial F_i}{\partial x_j}(\nu, x(\nu)) \quad \text{for } h \to 0.$$

Hence the difference ratio $r(h) = \frac{x(\nu+h)-x(\nu)}{h}$ satisfies the following system of equations

$$c_{(i)\nu} + \sum_{(j),(j)\neq(i)} c_{(i)(j)} r_j(h) + c_{(i)(i)} r_{(i)}(h) = 0 \tag{82}$$

The above equation was considered in Section 5.2. To apply Theorem 5.8 to obtain bounds for $r(h)$ we need to provide bounds for $c_{i\nu}$ for i large enough.

Since $c_{i\nu} \in \left[\frac{\partial F_i}{\partial \nu}(\Lambda, V)\right]_I$, then from (79) we have

$$|c_{i\nu}| \leq \Theta i^4 |x_i|, \qquad x \in V. \tag{83}$$

Since $|x_i| \leq \frac{C}{i^s}$ on V, then we obtain

$$\frac{|c_{i\nu}|}{\inf D_{ii}} \leq \frac{i^4 \Theta C}{i^s \beta i^4} = \frac{C\Theta/\beta}{i^s}. \tag{84}$$

Hence (46) is satisfied with $t = s$ and $\tilde{C} = C\Theta/\beta$.

When considering (72) and the partial derivative with respect of x_1, instead of $c_{i\nu}$ we will have the term $c_{i1} \in \left[\frac{\partial F_i}{\partial x_j}(\Lambda, V)\right]_I = [N]_{i1}$.

From (78) it follows that for some $\tilde{G}$

$$|c_{i1}| \leq \frac{\tilde{G}}{i^{s-1}}, \tag{85}$$

hence

$$\frac{|c_{i1}|}{\inf D_{ii}} \leq \frac{\tilde{G}}{i^{s-1} \beta i^4} = \frac{\tilde{G}/\beta}{i^{s+3}}. \tag{86}$$

Hence (46) is satisfied with $t = s + 3$ and $\tilde{C} = \tilde{G}/\beta$.

From Theorem 5.8 we obtain a topologically self-consistent a priori bounds, L (independent of h), $r(h) \in L$ and $|r_i(h)| \leq \frac{C_1}{i^s}$.

We will prove now that $r(h) \to r^*$, where r^* is a solution of the following equation

$$\frac{\partial F}{\partial \nu}(\nu, x(\nu)) + \frac{\partial F}{\partial x}(\nu, x(\nu)) \cdot r = 0. \tag{87}$$

First of all, this is an equation studied in the previous section, L are also the topologically self-consistent a priori bounds for it and $r^* \in L$.

The set L is compact, it is enough to prove that for any sequence $h_n \to 0$ for $n \to \infty$, such that ratios $r(h_n) \to \tilde{r}$, we have that $\tilde{r}$ satisfies (87).

To prove this observe that from Lemma 3.1 it follows that $c_{i\nu} \to \frac{\partial F_i}{\partial \nu}(\nu, x(\nu))$ and $c_{ij} \to \frac{\partial F_i}{\partial x_j}(\nu, x(\nu))$, hence by passing to the limit we see that indeed $\tilde{r}$ satisfies (87), hence $\tilde{r} = r^*$. Uniqueness of solutions of (87) and continuity of coefficients in this equation with respect to ν, x and r imply continuity of $x'(\nu)$. $\square$

Proceeding inductively it is now easy to prove the following

Theorem 5.10. *Same assumptions as in Theorem 5.9 and additionally* $d^3F = 0$. *Then function* $x(\nu)$ $(x(\nu, x_1))$ *is* C^∞ *on* Λ *and for any* $l \geq 1$ *there exists a constant* C_1 *such that* $|x_i^{(l)}| \leq \frac{C_l}{i^s}$ $\left(\left|\frac{\partial x_i}{\partial \nu}\right| \leq \frac{C_l}{i^s}, \left|\frac{\partial x_i}{\partial x_1}\right| \leq \frac{C_l}{i^s}\right)$.

Moreover, for equation (71) for any $\nu_1, \nu_2 \in \Lambda$ *and any* i

$$\left| \frac{x_i^{(l-1)}(\nu_1) - x_i^{(l-1)}(\nu_2)}{\nu_1 - \nu_2} \right| \leq \frac{C_l}{i^s}.$$

In the case of equation (72) for any $(\nu_1, y), (\nu_2, y) \in \Lambda$ *and* $(\nu, y_1), (\nu, y_2) \in \Lambda$, *any operator,* $D^{(l-1)}$, *of the partial derivatives of order* $l-1$ *and any* i *the following inequalities are satisfied*

$$\left| \frac{D^{(l-1)} x_i(\nu_1, y) - D^{(l-1)} x_i(\nu_2, y)}{\nu_1 - \nu_2} \right| \leq \frac{C_l}{i^s},$$
$$\left| \frac{D^{(l-1)} x_i(\nu, y_1) - D^{(l-1)} x_i(\nu, y_2)}{y_1 - y_2} \right| \leq \frac{C_l}{i^s}.$$

Proof. The proof is by induction.

We will illustrate the induction step for $l = 2$ for equation (72) for the computation of $\frac{\partial x}{\partial \nu \partial x_1}$. This example contains all essential ingredients for the whole proof, (due to the fact that $d^3 F = 0$).

Let us fix $(\nu, y) \in \Lambda$ and let $h \neq 0$ be such that $(\nu, y + h) \in \Lambda$.

For any i we have

$$\begin{aligned} 0 = \frac{\partial F_i}{\partial \nu}(\nu, y+h, x(\nu, y+h)) + \\ \sum_j \frac{\partial F_i}{\partial x_j}(\nu, y+h, x(\nu, y+h)) \cdot \frac{\partial x_j}{\partial \nu}(\nu, y+h) - \\ \frac{\partial F_i}{\partial \nu}(\nu, y, x(\nu, y)) - \sum_j \frac{\partial F_i}{\partial x_j}(\nu, y, x(\nu, y)) \cdot \frac{\partial x_j}{\partial \nu}(\nu, y) = \\ \left(\frac{\partial F_i}{\partial \nu}(\nu, y+h, x(\nu, y+h)) - \frac{\partial F_i}{\partial \nu}(\nu, y, x(\nu, y)) \right) + \\ \left(\sum_j \frac{\partial F_i}{\partial x_j}(\nu, y+h, x(\nu, y+h)) \frac{\partial x_j}{\partial \nu}(\nu, y+h) - \right. \\ \left. \sum_j \frac{\partial F_i}{\partial x_j}(\nu, y, x(\nu, y)) \frac{\partial x_j}{\partial \nu}(\nu, y) \right). \end{aligned}$$

We transform each term in parentheses separately. For the first term we have

$$
\begin{aligned}
&\frac{\partial F_i}{\partial \nu}(\nu, y+h, x(\nu, y+h)) - \frac{\partial F_i}{\partial \nu}(\nu, y, x(\nu, y)) = \\
&\left(\frac{\partial F_i}{\partial \nu}(\nu, y+h, x(\nu, y+h)) - \frac{\partial F_i}{\partial \nu}(\nu, y+h, x(\nu, y))\right) + \\
&\left(\frac{\partial F_i}{\partial \nu}(\nu, y+h, x(\nu, y)) - \frac{\partial F_i}{\partial \nu}(\nu, y, x(\nu, y)\right) = \\
&\sum_j c_{i\nu j} \cdot (x_j(\nu, y+h) - x_j(\nu, y)) + c_{i\nu 1} \cdot h
\end{aligned}
$$

where $c_{i\nu j} \to \frac{\partial F_i}{\partial \nu \partial x_j}(\nu, y, x(\nu, y))$ and $c_{i\nu 1} \to \frac{\partial F_i}{\partial \nu \partial x_1}(\nu, y, x(\nu, y))$ for $h \to 0$.

For the second term we obtain

$$
\begin{aligned}
&\sum_j \frac{\partial F_i}{\partial x_j}(\nu, y+h, x(\nu, y+h)) \frac{\partial x_j}{\partial \nu}(\nu, y+h) - \\
&\sum_j \frac{\partial F_i}{\partial x_j}(\nu, y, x(\nu, y)) \frac{\partial x_j}{\partial \nu}(\nu, y) = \\
&\sum_j \left(\frac{\partial F_i}{\partial x_j}(\nu, y+h, x(\nu, y+h)) - \frac{\partial F_i}{\partial x_j}(\nu, y, x(\nu, y+h))\right) \frac{\partial x_j}{\partial \nu}(\nu, y+h) + \\
&\sum_j \left(\frac{\partial F_i}{\partial x_j}(\nu, y, x(\nu, y+h)) - \frac{\partial F_i}{\partial x_j}(\nu, y, x(\nu, y))\right) \frac{\partial x_j}{\partial \nu}(\nu, y+h) + \\
&\sum_i \frac{\partial F_i}{\partial x_j}(\nu, y, x(\nu, y)) \left(\frac{\partial x_j}{\partial \nu}(\nu, y+h) - \frac{\partial x_j}{\partial \nu}(\nu, y)\right) = \\
&h \sum_j c_{ij1} \frac{\partial x_j}{\partial \nu}(\nu, y+h) + \sum_{jk} c_{ijk} \frac{\partial x_j}{\partial \nu}(\nu, y+h)(x_k(\nu, y+h) - x_k(\nu, y)) + \\
&\sum_i \frac{\partial F_i}{\partial x_j}(\nu, y, x(\nu, y)) \left(\frac{\partial x_j}{\partial \nu}(\nu, y+h) - \frac{\partial x_j}{\partial \nu}(\nu, y)\right)
\end{aligned}
$$

where $c_{ij1} \to \frac{\partial^2 F_i}{\partial x_j \partial x_1}(\nu, y, x(\nu, y))$ and $c_{ijk} \to \frac{\partial^2 F_i}{\partial x_j \partial x_k}(\nu, y, x(\nu, y))$ for $h \to 0$.

Hence after multiplication by $1/h$ we obtain the following equation

$$
\begin{aligned}
&\sum_j c_{i\nu j} \cdot \frac{x_j(\nu, y+h) - x_j(\nu, y)}{h} + c_{i\nu 1} + \sum_j c_{ij1} \frac{\partial x_j}{\partial \nu}(\nu, y+h) + \\
&\sum_{jk} c_{ijk} \frac{\partial x_j}{\partial \nu}(\nu, y+h) \frac{x_k(\nu, y+h) - x_k(\nu, y)}{h} + \\
&\sum_i \frac{\partial F_i}{\partial x_j}(\nu, y, x(\nu, y)) \frac{\frac{\partial x_j}{\partial \nu}(\nu, y+h) - \frac{\partial x_j}{\partial \nu}(\nu, y)}{h} = 0
\end{aligned}
$$

Hence similarly as in the proof of Theorem 5.9 we see that
$r_i(h) = \frac{\frac{\partial x_j}{\partial \nu}(\nu, y+h) - \frac{\partial x_j}{\partial \nu}(\nu, y)}{h}$ satisfies an equation of the form

$$
z_i + \sum_i \frac{\partial F_i}{\partial x_j}(\nu, y, x(\nu, y)) r_j(h) = 0. \tag{88}
$$

Hence to conclude the proof it is enough to show that z_i satisfies (46) with $t \geq s$.

We will do it term by term. For i large enough we have

$$\frac{1}{\inf D_{ii}}\left|\sum_j c_{i\nu j}\cdot\frac{x_j(\nu,y+h)-x_j(\nu,y)}{h}\right| \leq$$

$$\frac{C_1}{\beta i^4}\sum_j\frac{|c_{i\nu j}|}{j^s}=\frac{C_1}{\beta i^4}\cdot\frac{i^4}{i^s}=\frac{C_1}{\beta i^s}$$

$$\frac{1}{\inf D_{ii}}|c_{i\nu 1}|=0,$$

$$\frac{1}{\inf D_{ii}}\left|\sum_j c_{ij1}\frac{\partial x_j}{\partial\nu}(\nu,y+h)\right|\leq\frac{\beta CC_1}{i^4}\left(\frac{2i}{(i-1)^s}+\frac{2i}{(i+1))^s}\right)\leq\frac{\overline{C}}{i^{s+3}},$$

$$\frac{1}{\inf D_{ii}}\left|\sum_{jk} c_{ijk}\frac{\partial x_j}{\partial\nu}(\nu,y+h)\frac{x_k(\nu,y+h)-x_k(\nu,y)}{h}\right|\leq$$

$$\frac{\beta}{i^4}\sum_{jk}|c_{ijk}|\frac{C_1^2}{j^sk^s}=\frac{\beta C_1^2 2i}{i^4}\left(\sum_{j=1}^{i-1}\frac{1}{j^s(i-j)^s}+2\sum_{j=1}^{\infty}\frac{1}{j^s(i+j)^s}\right)\leq\frac{\overline{C}_1}{i^{s+3}}$$

□

6. **Bifurcations.**

6.1. **Symmetries and invariant subspaces of the KS-equation.**

We have the following easy

Lemma 6.1. *If $u(t,x)$ is a solution of the KS equations (we ignore the boundary conditions) for some ν, then $\widetilde{u}(t,x)=ku(k^2t,kx)$ is the solution of the KS equations for $\widetilde{\nu}=\frac{\nu}{k^2}$.*

From the above lemma it follows that for odd and periodic boundary conditions in terms Fourier coefficients we obtain the following fact: if $a_k(t)$ is a solution of (3) for some ν then for any $k\in\mathbb{N}$ we obtain a solution of (3) for $\tilde{\nu}=\frac{\nu}{k}$ given by

$$\begin{aligned}\tilde{a}_{nk}(t) &= ka_n(k^2t), \\ \tilde{a}_l(t) &= 0, \quad \text{if } l\notin k\mathbb{N}\ .\end{aligned} \tag{89}$$

Observe that the shift by π

$$\tilde{u}(t,x)=u(t,x+\pi)$$

maps solutions of (1–2) into solutions of the same problem. In terms Fourier coefficients this symmetry, denoted by R, is given by

$$\begin{aligned}a_{2k} &\to a_{2k}\\ a_{2k+1} &\to -a_{2k+1}.\end{aligned}$$

Here are a few simple consequences of the above symmetries

- if $(a_1,a_2,a_3,\dots)$ is a fixed point, then $(-a_1,a_2,-a_3,\dots)$ is a fixed point, too. Moreover the stability of both these points is the same.
- Fixed points $p=(a_1,a_2,a_3,\dots)$, such $a_1\neq 0$ always appear in pairs (p,Rp). They are called unimodal. If $a_1>0$ the it is called a positive unimodal point, otherwise it is called negative unimodal. The same terminology applies to the branches of the unimodal fixed points.

- **even-modal points:** Let $k = 2n$. A symmetric pair of unimodal fixed points

$$(a_1, a_2, a_3, \dots), \quad (-a_1, a_2, -a_3, \dots),$$

gives rise to two k-modal fixed points (see (89)), different by the sign of a_k and accordingly called positive and negative, just as in the case of the unimodal fixed points. Both these points are fixed points of symmetry R, hence their stability may differ.
- **odd-modal points** Let $k > 1$ be odd. A symmetric pair of unimodal fixed points

$$(a_1, a_2, a_3, \dots), \quad (-a_1, a_2, -a_3, \dots),$$

gives rise to two k-modal fixed points These points are mapped one onto another under the symmetry R, hence their stability is the same.

Observe that for each $k \in \mathbb{N}_+$ the space of k-modal functions given by the condition $a_s = 0$ if $s \notin k\mathbb{N}_+$ is invariant.

6.2. **Bifurcations in the KS equation.** Most of the steady state bifurcations in the KS-equations are the bifurcations off the invariant subspace, the k-modal subspace (see [15]). Some of them are intersections of two regular branches (for example an intersection of the trimodal branch with the bi-tri branch for $\nu \approx 0.11039383$), but most of them are the symmetry breaking pitchfork bifurcations. Namely, we have a branch of the steady states laying in a lower dimensional subspace, which bifurcates into three steady states, one being the continuation of the branch in the lower dimensional subspace and two new ones related to one another by the symmetry R. This is the case for the bifurcation which happens when two unimodal solutions collide with the negative bimodal branch for $\nu \approx 0.247833$. The negative bimodal branch belongs to the fixed point set for R and the unimodal solutions are mapped one into another by R.

In an attempt to establish rigorously the existence of the pitchfork bifurcation we use the approach presented in [4]. This means that we perform the Liapunov-Schmidt reduction and then the bifurcation problem is reduced to solving $G(\nu, x) = 0$, where $G : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function.

To be more specific: let $A \in \mathbb{R}^{m \times m}$ be a nonsingular matrix commuting with R defining new coordinate system, such that in the new coordinates the first direction is close to the apparent 'bifurcation direction' and is spanned by odd modes only. To perform the Liapunov-Schmidt reduction we solve the equation

$$\tilde{F}_i(\nu, x_1, x_2, \dots) = 0, \qquad i \geq 2 \tag{90}$$

as a function $y(\nu, x_1) = (x_2, x_3, \dots)(\nu, x_1) = 0$ for $(\nu, x) \in \Lambda \times X_1$. y and its derivatives are computed rigorously using the self-consistent a priori bounds method outlined in Section 5. We define the bifurcation function, G, by
$G(\nu, x) = \tilde{F}_1(\nu, x, y(\nu, x_1))$. Hence to solve equation $\tilde{F}(\nu, x_1, \dots) = 0$ in the set $\Lambda \times X_1 \times V$ (V are the self-consistent a priori bounds obtained in the construction of $y(\nu, x_1)$) it is enough to solve the equation

$$G(\nu, x) = 0. \tag{91}$$

6.3. **The saddle-node and pitchfork bifurcation.** Our goal is to establish an implicit function type theorem for a problem (91) with the explicit bounds for the domain of the existence of the solution. We begin first with a theorem describing the saddle-node bifurcation, which will be later applied in the analysis of the pitchfork bifurcation.

Theorem 6.2. *Let $Z = [\nu_1, \nu_2] \times [-x_1, x_1]$. Assume that $g : Z \to \mathbb{R}$ is a C^2-function and $g(\nu, -x) = g(\nu, x)$.*

Assume that

$$\begin{aligned}
\frac{\partial^2 g}{\partial x^2}(Z) &> 0, && (92)\\
\frac{\partial g}{\partial \nu}(Z) &< 0, && (93)\\
g(\nu_1, 0) &> 0, && (94)\\
g(\nu_2, x_1) &> 0, && (95)\\
g(\nu_2, 0) &< 0. && (96)
\end{aligned}$$

Then there exist $0 < x_0 \leq x_1$ and a function $\nu : [-x_0, x_0] \to [\nu_1, \nu_2]$ of class C^2, such that all solutions of the equation $g(\nu, x) = 0$ in Z belong to the graph of the function $\nu(x)$. Moreover, the following is true

$$\begin{aligned}
\nu(x) &= \nu(-x), \qquad x \in [-x_0, x_0]\\
\nu'(x) &> 0, \qquad x > 0\\
\nu(x_0) &= \nu_2.
\end{aligned}$$

The very easy proof is left to the reader.

The model for Theorem 6.2 is given by the map $g_1(\nu, x) = x^2 - \nu$ in the neighborhood of point $(0, 0)$. By changing signs of ν and g we obtain the model maps $g_2(\nu, x) = \nu + x^2$, $g_3(\nu, x) = \nu - x^2$ and $g_4(\nu, x) = -\nu - x^2$ for which we can state analogous theorems.

Now we turn to an application of the above theorem to the pitchfork bifurcation.

Let $G : \mathbb{R}^2 \to \mathbb{R}$ be C^3 function such that $G(\nu, x) = -G(\nu, -x)$. Consider the problem

$$G(\nu, x) = 0. \tag{97}$$

Since $G(\nu, 0) = 0$, it is natural to define a function

$$g(\nu, x) = \frac{G(\nu, x)}{x} \tag{98}$$

and then solve $g(\nu, x) = 0$ using Theorem 6.2. To do this we need a representation for g other than formula (98), because we cannot handle the singularity in the denominator in computer computations.

We have the following

Lemma 6.3.

$$g(\nu, x) = \int_0^1 \frac{\partial G}{\partial x}(\nu, sx) ds \tag{99}$$

Proof.

$$G(\nu, x) = G(\nu, x) - G(\nu, 0) = \int_0^1 \frac{d}{ds} G(\nu, sx) ds = \int_0^1 \frac{\partial G}{\partial x}(\nu, sx) ds \cdot x$$

□

Now we can formulate a theorem about solutions of (97), which will be used in our bifurcation computations

Theorem 6.4. *Let $Z = [\nu_1, \nu_2] \times [-x_1, x_1]$. Assume that $G : Z \to \mathbb{R}$ is a C^3-function and $G(\nu, -x) = -G(\nu, x)$.*

Assume that

$$\begin{aligned}
\frac{\partial^3 G}{\partial x^3}(Z) &> 0, && (100)\\
\frac{\partial^2 G}{\partial \nu \partial x}(Z) &< 0, && (101)\\
\frac{\partial G}{\partial x}(\nu_1, 0) &> 0, && (102)\\
G(\nu_2, x_1) &> 0, && (103)\\
\frac{\partial G}{\partial x}(\nu_2, 0) &< 0. && (104)
\end{aligned}$$

Then there exist $0 < x_0 \le x_1$ and a function $\nu : [-x_0, x_0] \to [\nu_1, \nu_2]$ of class C^2, such the set of all solutions of the equation $G(\nu, x) = 0$ in Z is the union of the graph of the function $\nu(x)$ and the line $x = 0$. Moreover, the following is true

$$\begin{aligned}
\nu(x) &= \nu(-x), \qquad x \in [-x_0, x_0]\\
\nu'(x) &> 0, \qquad x > 0\\
\nu(x_0) &= \nu_2.
\end{aligned}$$

Proof. We introduce a function g as in Lemma 6.3 and then we rewrite the assumptions of Theorem 6.2 in terms of G. For example to establish (92) and (93) we observe that

$$\begin{aligned}
\frac{\partial^2 g}{\partial x^2} &= \int_0^1 s^2 \frac{\partial^3 G}{\partial x^3}(\nu, sx) ds\\
\frac{\partial g}{\partial \nu} &= \int_0^1 \frac{\partial^2 G}{\partial x \partial \nu}(\nu, sx) ds.
\end{aligned}$$

□

The assumptions of Theorem 6.4 are well suited for the functions, which behave like $G(\nu, x) = ax(x^2 - b\nu)$, where a, b are positive numbers. To handle the case of other sign combinations we proceed as follows.

Let $\epsilon_1 = \pm 1$, $\epsilon_\nu = \pm 1$. Let $Z = [\nu_1, \nu_2] \times [-x_1, x_1]$ and $G : Z \to \mathbb{R}$. We set

$$\widetilde{Z} = \begin{cases} Z & \text{for } \epsilon_\nu = 1,\\ [-\nu_2, -\nu_1] \times [-x_1, x_1] & \text{for } \epsilon_\nu = -1. \end{cases} \tag{105}$$

We define a function, $\widetilde{G}$

$$\widetilde{G} : \widetilde{Z} \to \mathbb{R}, \qquad \widetilde{G}(\nu, x) = \epsilon_1 G(\epsilon_\nu \nu, x). \tag{106}$$

We will describe now how to define ϵ_1 and ϵ_ν, so that the function $\widetilde{G}$ satisfies the assumptions of Theorem 6.4.

Lemma 6.5. *Assume that $0 \notin \frac{\partial^3 G}{\partial x^3}(Z)$ and $0 \notin \frac{\partial^2 G}{\partial \nu \partial x}(Z)$.*

Let ϵ_1 and ϵ_2 be defined as follows

- *if $\frac{\partial^3 G}{\partial x^3}(Z) > 0$, then $\epsilon_1 = +1$*
- *if $\frac{\partial^3 G}{\partial x^3}(Z) < 0$, then $\epsilon_1 = -1$*
- *if $\epsilon_1 \frac{\partial^2 G}{\partial \nu \partial x}(Z) < 0$, then $\epsilon_\nu = +1$*
- *if $\epsilon_1 \frac{\partial^2 G}{\partial \nu \partial x}(Z) > 0$, then $\epsilon_\nu = -1$.*

If $\epsilon_\nu = +1$, then assume that the following conditions are satisfied

$$\begin{aligned} \epsilon_1 \frac{\partial G}{\partial x}(\nu_1, 0) &> 0, \\ \epsilon_1 G(\nu_2, x_1) &> 0, \\ \epsilon_1 \frac{\partial G}{\partial x}(\nu_2, 0) &< 0. \end{aligned}$$

If $\epsilon_\nu = -1$, then assume that the following conditions are satisfied

$$\begin{aligned} \epsilon_1 \frac{\partial G}{\partial x}(\nu_2, 0) &> 0, \\ \epsilon_1 G(\nu_1, x_1) &> 0, \\ \epsilon_1 \frac{\partial G}{\partial x}(\nu_1, 0) &< 0. \end{aligned}$$

Then there exist $0 < x_0 \leq x_1$ and a function $\nu : [-x_0, x_0] \to [\nu_1, \nu_2]$ of class C^2, such the set of all solutions of the equation $G(\nu, x) = 0$ in Z is a union of the graph of the function $\nu(x)$ and the line $x = 0$. Moreover, the following is true

$$\begin{aligned} \nu(x) &= \nu(-x), \qquad x \in [-x_0, x_0] \\ \epsilon_\nu \nu'(x) &> 0, \qquad x > 0 \\ \nu(x_0) &= \nu_2, \qquad \text{if } \epsilon_\nu = +1 \\ \nu(x_0) &= \nu_1, \qquad \text{if } \epsilon_\nu = -1 \end{aligned}$$

Proof. For the proof observe that the function $\widetilde{G}$ satisfies the assumptions of Theorem 6.4. □

6.4. **Intersection of regular branches.** In this section we state the bifurcation theorem, which handles the case of the intersection of two regular branches, one of which is contained in a lower dimensional invariant subspace (for example an intersection of the trimodal branch with the bi-tri branch for $\nu \approx 0.11039383$).

Just as in case of the pitchfork bifurcation we analyze the equation $G(\nu, x) = 0$, where $G(\nu, 0) = 0$ (which corresponds to a solution branch contained in the lower dimensional subspace).

We introduce the function g by (98) and we compute it using Lemma 6.3. The task is now reduced to checking if the solution curve for $g(\nu, 0) = 0$ intersects the line $x = 0$ at a nonzero angle and does not make any fold.

Theorem 6.6. *Let $Z = [\nu_1, \nu_2] \times [-x_1, x_1]$. Assume that $G : Z \to \mathbb{R}$ is a C^3-function and $G(\nu, 0) = 0$.*

Assume that the following conditions are satisfied

$$0 \notin \frac{\partial G^2}{\partial x \partial \nu}(Z) \tag{107}$$

$$0 \notin \frac{\partial^2 G}{\partial x^2}(Z), \tag{108}$$

$$\frac{\partial G}{\partial x}(\nu_1, 0) \cdot \frac{\partial G}{\partial x}(\nu_2, 0) < 0 \tag{109}$$

Then the solution of equation $G(\nu, x) = 0$ is a sum of line $x = 0$ and a curve $(\nu(x), x)$ for $x \in [a, b]$, $a < 0 < b$. Moreover $\nu'(x) \neq 0$ for $x \in [a, b]$.

Proof. Let $g(\nu, x) = \frac{G(\nu, x)}{x}$. For the proof it is enough to show that the solution of equation $g(\nu, x) = 0$ can be parameterized by x, as a curve $(\nu(x), x)$ and $\nu'(x) \neq 0$.

Since by Lemma 6.3

$$\frac{\partial g}{\partial \nu}(\nu,x) = \int_0^1 \frac{\partial^2 G}{\partial x \partial \nu}(\nu, sx)ds, \tag{110}$$

hence from (107) it follows that

$$\frac{\partial g}{\partial \nu}(\nu,x) \neq 0, \qquad \text{for } (\nu,x) \in Z. \tag{111}$$

Since Z is connected we see that $\frac{\partial g}{\partial \nu}(\nu,x)$ has constant sign on Z, hence for each $x \in [-x_1, x_1]$ the equation $g(\nu,x)=0$ has at most one solution.

Since by Lemma 6.3

$$g(\nu,0) = \frac{\partial G}{\partial x}(\nu,x), \tag{112}$$

hence it follows from (109) that there exists $\nu_0 \in (\nu_1,\nu_2)$ such that $g(\nu_0,0) = 0$. From the implicit function theorem follows the existence of the function $\nu(x)$ satisfying the assertion of the theorem. It remains to show that $\nu'(x) \neq 0$.

We have $\nu'(x) = -\frac{\partial g}{\partial x}/\frac{\partial g}{\partial \nu}$. From Lemma 6.3 we easily obtain

$$\frac{\partial g}{\partial x} = \int_0^1 s \frac{\partial^2 G}{\partial^2 x}(\nu, sx)ds. \tag{113}$$

From the above equation and (108) we obtain that $\nu'(x) \neq 0$ for $x \in (a,b)$. $\square$

7. **Details for the KS equation - the local uniqueness issue.**

The goal of this section is to derive the formulas necessary to verify the assumptions of Theorem 3.7 for the KS equation.

We can write the KS-equation (see Section 1.1) in the Fourier domain as follows

$$\dot{a}_k = F_k(a) = \lambda_k a_k + N_k(a) \tag{114}$$

$$\lambda_k = k^2(1-\nu k^2) \tag{115}$$

$$N_k(a) = -k\sum_{n=1}^{k-1} a_n a_{k-n} + 2k\sum_{n=1}^{\infty} a_n a_{n+k}. \tag{116}$$

The formal first derivatives of F are given by

$$\frac{\partial N_i}{\partial a_j} = 2ia_{i+j}, \quad \text{for } i=j \tag{117}$$

$$\frac{\partial N_i}{\partial a_j} = -2ia_{i-j} + 2ia_{i+j}, \quad \text{for } j<i \tag{118}$$

$$\frac{\partial N_i}{\partial a_j} = 2ia_{j-i} + 2ia_{i+j}, \quad \text{for } j>i \tag{119}$$

$$\frac{\partial F_i}{\partial a_j} = i^2(1-\nu i^2)\delta_{ij} + 2i\sum_{k\geq 1}(-\delta_{k,i-j} + \delta_{k,i+j} + \delta_{k,j-i})a_k, \tag{120}$$

$$\frac{\partial F_i}{\partial \nu} = -i^4 a_i. \tag{121}$$

For the second derivatives we have the following formulas

$$\frac{\partial^2 F_i}{\partial^2 \nu} = 0, \tag{122}$$

$$\frac{\partial^2 F_i}{\partial x_k \partial \nu} = -i^4 \delta_{i,k}, \tag{123}$$

$$\frac{\partial^2 F_i}{\partial a_j \partial a_k} = \frac{\partial^2 N_i}{\partial a_j \partial a_k} = 2i(-\delta_{k,i-j} + \delta_{k,i+j} + \delta_{k,j-i}). \tag{124}$$

All higher order derivatives vanish.

We start with the verification of **F1** and **F2**.

Theorem 7.1. *Let V be the self-consistent a priori bounds for the KS-equation, such that $|x_k| \leq \frac{C}{k^s}$ for $s > 5$.*

*Then conditions **F1** and **F2** are satisfied on V.*

Proof. Condition **F1** is manifestly satisfied.

From the formulas for $\frac{\partial F_i}{\partial x_j}$ given above and the assumed polynomial decay rate of x_i on V we have

$$d_{ij} \leq 2iC \left(\frac{1}{(j-i)^s} + \frac{1}{(j+i)^s} \right), \quad \text{for } j > i \tag{125}$$

Hence to prove **F2** it is enough to show that the series $\sum_{j=i+1}^{\infty} d_{ij} \frac{C}{j^s}$ converges for $s \geq 4$. Observe that

$$d_{ij} \frac{C}{j^s} = 2iC^2 \left(\frac{1}{(j-i)^s j^s} + \frac{1}{(j+i)^s j^s} \right) \sim 2iC^2 \frac{2}{j^{2s}}. \tag{126}$$

Hence the series in **F2** converges for $s > \frac{1}{2}$. $\square$

The following lemma does not require any proof.

Lemma 7.2. *Let $A : H \to H$ be a linear coordinate change of the form*

$$A : X_m \oplus Y_m \to X_m \oplus Y_m$$
$$A(x \oplus y) = Ax \oplus y.$$

Let $\tilde{F} = A \circ F \circ A^{-1}$ ($\tilde{F}$ is F expressed in new coordinates).

$$\begin{aligned}
\frac{\partial \tilde{F}_i}{\partial x_j} &= \textstyle\sum_{k,l=1}^{m} A_{ik} \frac{\partial F_k}{\partial x_l} A^{-1}_{lj} && \text{for } i \leq m \text{ and } j \leq m \\
\frac{\partial \tilde{F}_i}{\partial x_j} &= \textstyle\sum_{k \leq m} A_{ik} \frac{\partial F_k}{\partial x_j} && \text{for } i \leq m \text{ and } j > m \\
\frac{\partial \tilde{F}_i}{\partial x_j} &= \textstyle\sum_{l \leq m} \frac{\partial F_i}{\partial x_l} A^{-1}_{lj} && \text{for } i > m \text{ and } j \leq m \\
\frac{\partial \tilde{F}_i}{\partial x_j} &= \frac{\partial F_i}{\partial x_j} && \text{for } i > m \text{ and } j > m \\
\frac{\partial \tilde{F}_k}{\partial \nu} &= \textstyle\sum_{i=1}^{m} A_{ki} \frac{\partial F_i}{\partial \nu}, && \text{if } k \leq m \\
\frac{\partial \tilde{F}_k}{\partial \nu} &= \frac{\partial F_k}{\partial \nu}, && \text{if } k > m.
\end{aligned}$$

Consider now the KS equation. We assume that V is the self-consistent a priori bounds for a fixed point. Let the numbers $m < M$ be as in conditions **C1,C2,C3**. We assume that $a_k^{\pm} = \pm \frac{C}{k^s}$ for $k > M$ (as in [23]).

Let $A \in \mathbb{R}^{m \times m}$ be a coordinate change around an approximate fixed point in X_m for m-dimensional Galerkin projection of (114). This matrix induces a coordinate change in H. It is optimal to choose A so that the m-dimensional Galerkin projection of F is very close to the diagonal matrix (or to the block diagonal one when the complex eigenvalues are present).

We will use the new coordinates in H. We also change the norm so that the new coordinates are orthogonal. We define the splitting of $P_m H$ into blocks which are either 2-dimensional (the case of the complex eigenvalue) or one-dimensional (the real eigenvalue). For the instability consideration we may glue several 'unstable blocks' into one - see Section 8. For the KS equation there was no need to consider more complicated situations such as the nontrivial Jordan cells. For $(i) > m$ all blocks are 1-dimensional (these coordinates are not affected by our coordinate change).

We would like to derive the formula for

$$\Delta_{(i)} := \inf\left(\frac{\partial \tilde{F}_{(i)}}{\partial x_{(i)}}(V)\right) - \sum_{(j)\neq(i)} \max_{x\in V}\left|\frac{\partial \tilde{F}_{(i)}}{\partial x_{(j)}}(x)\right|$$

We introduce $S(l)$ and $S_{ND}((i))$ by

$$S(l) = \sum_{k\geq l} \max_{a\in V} |a_k|, \tag{127}$$

$$S_{ND}((i)) = \sum_{(j)\neq(i)} \max_{x\in V}\left|\frac{\partial \tilde{F}_{(i)}}{\partial x_{(j)}}(x)\right|. \tag{128}$$

Since many times in the estimates we need to estimate $\sum_{k=l}^{\infty}\frac{1}{k^s}$ we elevate one such estimate to the lemma status.

Lemma 7.3.

$$\sum_{k=l}^{\infty}\frac{1}{k^s} < \int_{l-1}^{\infty}\frac{dx}{x^s} = \frac{1}{(s-1)(l-1)^{s-1}} \tag{129}$$

We will estimate $S(l)$ from the above using the following

Lemma 7.4. *Assume that $|a_k(V)| \leq \frac{C}{k^s}$ for $k > M$, $s > 1$, then*

$$S(l) < \sum_{k=l}^{M} |a_k(V)| + \frac{C}{(s-1)M^{s-1}}, \quad \text{for } l \leq M$$

$$S(l) < \frac{C}{(s-1)(l-1)^{s-1}}, \quad \text{for } l > M.$$

Proof. It follows immediately from Lemma 7.3. □

We set

$$\overline{S}(l) = \sum_{k=l}^{M} |a_k(V)| + \frac{C}{(s-1)M^{s-1}}, \quad \text{for } l \leq M \tag{130}$$

$$\overline{S}(l) = \frac{C}{(s-1)(l-1)^{s-1}}, \quad \text{for } l > M. \tag{131}$$

Lemma 7.5. *Let $Q \in Lin(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})$, then*

$$|Q| \leq \sum_{i=1}^{n_1}\sqrt{\sum_{k=1}^{n_2} Q_{ik}^2}$$

Lemma 7.6. *If* $(i) = (i_1, i_2, \ldots, i_r)$, $(i) \leq m$. *Then*

$$\begin{aligned} S_{ND}((i)) \quad &\leq \quad \overline{S_{ND}}((i)) := \sum_{j \leq M, j \notin (i)} \sqrt{\sum_{i_l \in (i)} \left(\frac{\partial \tilde{F}_{i_l}}{\partial x_j} \right)^2} \\ &+ \sum_{i_l \in (i)} \sum_{k \leq m} 2|A_{i_l,k}| k \left(\overline{S}(M+1-k) + \overline{S}(M+1+k) \right) \end{aligned}$$

Proof. From Lemma 7.5 it follows that we can ignore the structure for all blocks different from (i) and use the following estimate

$$\left| \frac{\partial \tilde{F}_{(i)}}{\partial x_{(j)}}(x) \right| \leq \sum_{j_s \in (j)} \sqrt{\sum_{i_l \in (i)} \left(\frac{\partial \tilde{F}_{i_l}}{\partial x_{j_s}}(x) \right)^2}. \tag{132}$$

Therefore we have

$$\sum_{(j) \neq (i)} \left| \frac{\partial \tilde{F}_{(i)}}{\partial x_{(j)}}(x) \right| \leq \sum_{j \notin (i)} \sqrt{\sum_{i_l \in (i)} \left(\frac{\partial \tilde{F}_{i_l}}{\partial x_j} \right)^2}. \tag{133}$$

To finish the proof it is enough to show that

$$\sum_{j > M} \sqrt{\sum_{i_l \in (i)} \left(\frac{\partial \tilde{F}_{i_l}}{\partial x_j} \right)^2} \leq \sum_{i_l \in (i)} \sum_{k \leq m} 2|A_{i_l,k}| k (S(M+1-k) + S(M+1+k)).$$

We have

$$\sum_{j > M} \sqrt{\sum_{i_l \in (i)} \left(\frac{\partial \tilde{F}_{i_l}}{\partial x_j} \right)^2} \leq \sum_{i_l \in (i)} \sum_{j > M} \left| \frac{\partial \tilde{F}_{i_l}}{\partial x_j} \right| \tag{134}$$

We have for $i_l \in (i)$ (observe that $i_l \leq m$)

$$\begin{gathered} \sum_{j > M} \left| \frac{\partial \tilde{F}_{i_l}}{\partial x_j} \right| \leq \sum_{j > M} \sum_{k \leq m} |A_{i_l,k}| \left| \frac{\partial F_k}{\partial x_j} \right| = \\ \sum_{k \leq m} |A_{i_l,k}| \sum_{j > M} \left| \frac{\partial F_k}{\partial x_j} \right| \leq \sum_{k \leq m} 2k |A_{i_l,k}| \sum_{j > M} (|a_{j-k}| + |a_{j+k}|) \leq \\ \sum_{k \leq m} 2k |A_{i_l,k}| \left(S(M+1-k) + S(M+1+k) \right). \end{gathered}$$

This finishes the proof. ∎

7.1. **Formulas for 1-dimensional blocks.** Observe that if $\dim(i) = 1$, then $\frac{\partial \tilde{F}_{(i)}}{\partial x_{(i)}} \in \mathbb{R}$, hence

$$\inf \left(\frac{\partial \tilde{F}_{(i)}}{\partial x_{(i)}} \right) = \left| \frac{\partial \tilde{F}_{(i)}}{\partial x_{(i)}} \right| \tag{135}$$

From the above observation and Lemma 7.6 we obtain the following

Lemma 7.7. *Assume* $(i) \leq m$ *and dim* $(i) = 1$.

$$\Delta_i \quad \geq \quad \overline{\Delta}_i := \inf_{x \in V} \left| \frac{\partial \tilde{F}_i}{\partial x_i}(x) \right| - \overline{S_{ND}}(i),$$

where $\overline{S_{ND}}(i)$ *is defined in Lemma 7.6.*

Observe that from our assumptions about the decomposition of H it follows that all blocks (i) such that $(i) > m$ are one-dimensional.

Lemma 7.8. *For* $m < (i) \leq M$ *we have*

$$\begin{aligned} S_{ND}(i) \quad &\leq \quad \overline{S_{ND}}(i) := \sum_{j \leq M} \max_{x \in V} \left| \frac{\partial \tilde{N}_i}{\partial x_j}(x) \right| \\ &\qquad +2i \left(\overline{S}(M+1-i) + \overline{S}(i+M+1) \right) \\ \Delta_i \quad &\geq \quad \overline{\Delta}_i := |i^2(1-\nu i^2)| - \overline{S_{ND}}(i) \end{aligned}$$

Proof. Just as in the proof of Lemma 7.6 we can ignore the block structure here. It is easy to see that

$$\min_{x \in V} \left| \frac{\partial \tilde{F}_i}{\partial x_i}(x) \right| - \sum_{j \neq i} \max_{x \in V} \left| \frac{\partial \tilde{F}_i}{\partial x_j}(x) \right| \geq |i^2(1-\nu i^2)| - \sum_{j=1}^{\infty} \max_{x \in V} \left| \frac{\partial \tilde{N}_i}{\partial x_j}(x) \right| \tag{136}$$

Therefore to finish the proof it is enough to show that

$$\sum_{j=M+1}^{\infty} \max_{x \in V} \left| \frac{\partial \tilde{N}_i}{\partial x_j}(x) \right| < 2i(S(M+1-i) + S(i+M+1)) \tag{137}$$

We proceed as follows

$$\sum_{j=M+1}^{\infty} \max_{x \in V} \left| \frac{\partial \tilde{N}_i}{\partial x_j}(x) \right| = \sum_{j=M+1}^{\infty} \max_{x \in V} \left| \frac{\partial N_i}{\partial x_j}(x) \right| \leq$$

$$\sum_{j=M+1}^{\infty} 2i(|a_{j-i}(V)| + |a_{i+j}(V)|) \leq 2i(S(M+1-i) + S(M+1+i))$$

∎

Lemma 7.9. *For* $(i) > m$ *we have*

$$\begin{aligned} S_{ND}(i) \quad &\leq \quad \overline{S_{ND}}(i) := 2im \left(\overline{S}(i-m) + \overline{S}(i+1) \right) \max_{k,l=1,\dots,m} |A^{-1}_{kl}| \\ &\qquad +2i \left(\overline{S}(i+m+1) + 2\overline{S}(1) \right) \\ \Delta_i \quad &\geq \quad \overline{\Delta}_i := |i^2(1-\nu i^2)| - \overline{S_{ND}}(i) \end{aligned}$$

Proof. Just as in the proof of Lemma 7.6 we can ignore the block structure here. It is easy to see that

$$\min_{x \in V} \left| \frac{\partial \tilde{F}_i}{\partial x_i}(x) \right| - \sum_{j \neq i} \max_{x \in V} \left| \frac{\partial \tilde{F}_i}{\partial x_j}(x) \right| \geq |i^2(1-\nu i^2)| - \sum_{j=1}^{\infty} \max_{x \in V} \left| \frac{\partial \tilde{N}_i}{\partial x_j}(x) \right| \tag{138}$$

Therefore to finish the proof it is enough to show that

$$\sum_{j=1}^{m} \max_{x\in V} \left| \frac{\partial \tilde{N}_i}{\partial x_j}(x) \right| \leq 2im(S(i-m)+S(i+1)) \max_{k,l=1,\dots,m} |A^{-1}_{kl}|, \tag{139}$$

$$\sum_{j=m+1}^{\infty} \max_{x\in V} \left| \frac{\partial \tilde{N}_i}{\partial x_j}(x) \right| \leq 2i(S(i+m+1)+2S(1)). \tag{140}$$

To prove (139) observe that

$$\begin{aligned}
\sum_{j=1}^{m} \max_{x\in V} \left| \frac{\partial \tilde{N}_i}{\partial x_j}(x) \right| = \sum_{j=1}^{m} \max_{x\in V} \left| \sum_{l=1}^{m} \frac{\partial N_i}{\partial x_l}(x) A^{-1}_{lj} \right| \leq \\
\sum_{j=1}^{m}\sum_{l=1}^{m} 2i(|a_{i-l}(V)|+|a_{i+l}(V)|)|A^{-1}_{lj}| \leq \\
2i\sum_{j=1}^{m}(S(i-m)+S(i+1)) \max_{k,l=1,\dots,m} |A^{-1}_{kl}| = \\
2im(S(i-m)+S(i+1)) \max_{k,l=1,\dots,m} |A^{-1}_{kl}|.
\end{aligned}$$

To prove (140) we proceed as follows

$$\begin{aligned}
\sum_{j=m+1}^{\infty} \max_{x\in V} \left| \frac{\partial \tilde{N}_i}{\partial x_j}(x) \right| = \sum_{j=m+1}^{\infty} \max_{x\in V} \left| \frac{\partial N_i}{\partial x_j}(x) \right| \leq \\
\sum_{m<j<i} (2i(|a_{i-j}(V)|+|a_{i+j}(V)|)) \\
+2i|a_{2i}(V)| + \sum_{j>i} 2i(|a_{j-i}(V)|+|a_{i+j}(V)|) \leq \\
2i\left(\sum_{j>m} |a_{i+j}(V)| + \sum_{m<j<i} |a_{i-j}(V)| + \sum_{j>i} |a_{j-i}(V)| \right) < \\
2i\left(S(i+m+1)+2S(1)\right)
\end{aligned}$$

□

The following lemma shows that how to handle the case of large i.

Lemma 7.10. *If for some $n>m$, $\overline{\Delta}_n>0$ and $1-\nu n^2<0$, then*

$$\overline{\Delta}_i > \overline{\Delta}_j > 0 \qquad \text{for} \qquad i>j, \quad j\geq n \tag{141}$$

Proof. From Lemma 7.9 it follows that

$$\overline{\Delta}_i = i\left((\nu i^3 - i) - 2m(\overline{S}(i-m)+\overline{S}(i+1))a - 2(\overline{S}(i+m+1)+2\overline{S}(1))\right)$$

where $a=\max_{k,l=1,\dots,m}|A^{-1}_{kl}|$.

Hence

$$\overline{\Delta}_i = i\left((\nu i^3 - i) - f(i)\right), \tag{142}$$

where $f(i)$ is a positive decreasing function of i. It is easy to see that the function $i\mapsto(\nu i^3-i)$ is increasing and positive for $i\geq n$. □

In the computation of the derivatives of the steady states with respect to the parameters (see Theorem 5.1) we will be interested in the ratio

$$r((i)) = \frac{\overline{S}_{ND}((i))}{\inf \frac{\partial \tilde{F}_{(i)}}{\partial x_{(i)}}}. \tag{143}$$

We have the following

Lemma 7.11. *For $i > m$ the function $r(i)$ is decreasing.*

Proof. From Lemma 7.9 it follows that for

$$\begin{aligned} f(i) &= 2m(\overline{S}(i-m) + \overline{S}(i+1))a + 2(\overline{S}(i+m+1) + 2\overline{S}(1) \\ S_{ND}(i) &= if(i) \end{aligned}$$

where $a = \max_{k,l=1,\dots,m} |A^{-1}_{kl}|$ and f is a decreasing function of i.

Since $r(t) = \frac{f(i)}{i^3(\nu - \frac{1}{i^2})}$, the assertion follows immediately. □

7.2. **Complex block.** We need to introduce a new notation. Let Q be a 2×2-matrix. We define the a new matrix $K(Q)$

$$K(Q) = \left[\begin{array}{cc} (Q_{11}+Q_{22})/2, & (Q_{12}-Q_{21})/2 \\ (Q_{21}-Q_{12})/2, & (Q_{11}+Q_{22})/2 \end{array} \right] \tag{144}$$

Observe that $K(Q)$ has the following form

$$K(Q) = \left[\begin{array}{cc} \alpha, & \beta \\ -\beta, & \alpha \end{array} \right]. \tag{145}$$

Obviously $K(K(Q)) = K(Q)$, the eigenvalues of $K(Q)$ are $\lambda_{1,2} = \alpha \pm \mathrm{i}\beta$ and

$$|K(Q)| = \inf(K(Q))) = \sqrt{\alpha^2 + \beta^2}. \tag{146}$$

The difference $Q - K(Q)$, where Q is 2-dimensional block obtained from linearization, measures how good this linearization really is. In our applications to fixed points for the KS equations it is usually very small, which is achieved by taking a small set V and m large enough. To measure this difference we will use the function $\delta(Q)$ given by

$$\delta(Q) = \max_{i,j=1,2} |K(Q)_{ij} - Q_{ij}| \tag{147}$$

Lemma 7.12. *Let $Q \in \mathbb{R}^{n \times n}$ and $|Q_{ij}| \leq \epsilon$, then $|Q| \leq n\epsilon$.*

Proof. Let $x = (x_1, \dots, x_n)$, $|x| = 1$. We have

$$\begin{aligned} |Qx|^2 = (Q_{11}x_1 + \dots + Q_{1n}x_n)^2 + (Q_{21}x_1 + \dots + Q_{2n}x_n)^2 + \dots \leq \\ n\epsilon^2(|x_1| + \dots + |x_n|)^2 \leq n^2\epsilon^2|x|^2. \end{aligned}$$

□

Lemma 7.13. *Let $Q \in \mathbb{R}^{2 \times 2}$, then*

$$\inf(Q) \geq \inf(K(Q)) - 2\delta(Q)$$

Proof. Since

$$\inf(Q) = \inf(K(Q) + (Q - K(Q))) \geq \inf(K(Q)) - |Q - K(Q)|, \tag{148}$$

then the assertion follows from Lemma 7.12. □

Lemma 7.14. *If* $(i) = (i_1, i_2)$, $(i) \leq m$. *Then*

$$\Delta_{(i)} \quad \geq \quad \overline{\Delta}(i) := \inf K \left[\frac{\partial \tilde{F}_{(i)}}{\partial x_{(i)}}(V) \right] - 2 \max_{x \in V} \delta \left(\frac{\partial \tilde{F}_{(i)}}{\partial x_{(i)}}(x) \right) - \overline{S_{ND}}((i))$$

Proof. From Lemma 7.13 it follows immediately that

$$\inf \left(\frac{\partial \tilde{F}_{(i)}}{\partial x_{(i)}}(V) \right) \geq \inf \left(K \left(\frac{\partial \tilde{F}_{(i)}}{\partial x_{(i)}}(V) \right) \right) - 2 \max_{x \in V} \delta \left(\frac{\partial \tilde{F}_{(i)}}{\partial x_{(i)}}(x) \right). \tag{149}$$

□

8. **Details for the KS-equation - the instability.** In this section we provide the formulas for the verification of the assumptions of Theorem 4.2.

We assume that we have the coordinates introduced in Section 7 and the same block decomposition. For the purpose of the proof of Theorem 4.2 we modify slightly this block decomposition as follows,

- let $(i_1), \ldots, (i_s)$ be all the blocks such that all diagonal elements of $\frac{\partial \tilde{F}_{(i)}}{\partial x_{(i)}}$ are positive. Observe there are only finitely many such blocks (at most m).
- we create a new block $(i_0) = (i_1) \cup \cdots \cup (i_s)$. This means that in the block decomposition of H we have $H_{(i_0)} = H_{(i_1)} \oplus \cdots \oplus H_{(i_s)}$.

Observe that the sum of non-diagonal elements appearing in Theorem 4.2 was already computed in Section 7, but the block decomposition is slightly different now. It reduces to the fact that there on $H_{(i_0)}$ the sum-norm was used and now we have to use the Euclidean norm.

The next lemma addresses the computation of μ_{inf} and μ_{sup}

Lemma 8.1. *Let* $A = S\left(\left[\frac{\partial \tilde{F}_{(i)}}{\partial x_{(i)}})(V) \right] \right)$. *Then*

$$\mu_{sup}(A) = \max_{j=1,\ldots,\dim(i)} \max A_{jj} + \sum_{k \neq j} \max |A_{jk}|,$$

$$\mu_{inf}(A) = \min_{j=1,\ldots,\dim(i)} \min A_{jj} - \sum_{k \neq j} \max |A_{jk}|$$

Proof. We provide the proof for μ_{sup} only. The other case is analogous.

First observe that for any symmetric matrix M

$$\mu_{sup}(M) = \text{largest eigenvalue of } M. \tag{150}$$

Hence

$$\mu_{sup}(A) = \text{largest eigenvalue of } M,\ M \in A,\ M\text{-symmetric}. \tag{151}$$

The assertion now follows from the Gershgorin Theorem (see [20, Property 5.2]). □

The formulas for the computation of $\sum_{(i) \neq (j)} \left| \frac{\partial F_{(j)}}{\partial x_{(i)}}(Z) \right|$ are given in Lemmas 7.7, 7.8, 7.9 and 7.6. Hence it remains to discuss how to verify in the finite computation that $a_{(i_0)(i)} > 0$ holds for all (i).

It turns out that the same analysis as in Lemma 7.10 gives rise to the following

Lemma 8.2. *If for some* $n > m$, $a_{(n)(i_0)} > 0$ *and* $1 - \nu n^2 < 0$, *then*

$$a_{(i)(i_0)} > a_{(j)(i_0)} > 0 \qquad \text{for} \qquad i > j, \quad j \geq n. \tag{152}$$

The file *bifdata.txt* [29] contains data from the proof of the uniqueness, the instability (plus the regularity computation) for the positive bimodal fixed point for $\nu = 0.127 + 10^{-5} \cdot [-1, 1]$. For this steady state the unstable direction is two dimensional and corresponds to a pair of complex eigenvalues.

9. **Algebra of polynomial bounds.**

In the computation various derivatives of the implicit function defined as the z-terms in equation (35) we have the following expressions or the sums thereof (and due to the fact, that F is a second degree polynomial we have only the terms of this type)

$$\sum_j \frac{\partial \tilde{F}_i}{\partial x_j} y_j, \qquad \sum_j \frac{\partial^2 \tilde{F}_i}{\partial \nu \partial x_j} y_j, \qquad \sum_j \frac{\partial^2 \tilde{F}_i}{\partial x_1 \partial x_j} y_j, \qquad \sum_{jk} \frac{\partial^2 \tilde{F}_i}{\partial x_j \partial x_k} y_j w_k, \tag{153}$$

where $|y_i|$ and $|w_i|$ satisfy some decay condition (define the self-consistent a priori bounds) and $\tilde{F}$ and its derivatives are also evaluated on the self-consistent a priori bounds. It turns out that the computation of these terms might be realized by an algebra, which we are going to develop in this section.

Throughout this section we fix a positive integer M.

Definition 9.1. A compact set $Y \subset H$ is called *the polynomial bounds*, if there exist $E > 0$ and $\beta \geq 0$, such that for all $y \in Y$ holds

$$|y_i| \leq \frac{E}{i^\beta}, \qquad \text{for } i > M \ .$$

We will often use the triple (Y, E, β) to denote the polynomial bounds.

We introduce some arithmetic operations on bounds.

Definition 9.2. Assume that (X, E, β) and (Y, G, γ) are polynomial bounds. We define

- $Z = X + Y$ by $Z_i = X_i + Y_i$ for $i \leq M$ and $|z_i| \leq \frac{C_z}{i^t}$ for $i > M$, where $t = \min(\beta, \gamma)$ and $C_Z = \frac{E}{(M+1)^{\beta - t}} + \frac{G}{(M+1)^{\gamma - t}}$
- for $c \in \mathbb{R}$ we define $Z = cY$ by $Z_i = cY_i$.
- $Z = (i)Y$ by $Z_i = iY_i$ for $i \in \mathbb{N}$.

With the above definitions we can define the product of $A \cdot (Y, E, \beta)$, where $A \in \mathbb{R}^{m \times m}$, $m \leq M$. It also makes sense to apply projections P_n and $Q_n = I - P_n$, where $n \leq M$, to the polynomial bounds.

It turns out that using the following functions we can compute all terms in (153)

$$QF_i(y, w) = \sum_{k=1}^{i-1} y_k w_{i-k}, \qquad QI_i(y, w) = \sum_{k=1}^{\infty} y_k w_{k+i}.$$

The goal of the next few lemmas is to define the operations QF and QI on the polynomial bounds.

9.1. **Some estimates.**

Lemma 9.3.

$$\sum_{j > M+k} \frac{1}{(j-k)^s j^\gamma} < \frac{M^{-(s-1/2)}}{\sqrt{2s-1}} \cdot \frac{(M+k)^{-(\gamma - 1/2)}}{\sqrt{2\gamma - 1}},$$

$$\sum_{j > M} \frac{1}{(j+k)^s j^\gamma} < \frac{(M+k)^{-(s-1/2)}}{\sqrt{2s-1}} \frac{M^{-(\gamma - 1/2))}}{\sqrt{2\gamma - 1}}$$

Proof.

$$\sum_{j>M+k} \frac{1}{(j-k)^s j^\gamma} < \int_{M+k}^{\infty} \frac{dx}{(x-k)^s x^\gamma} \leq$$
$$\sqrt{\int_{M+k}^{\infty} \frac{dx}{(x-k)^{2s}}} \cdot \sqrt{\int_{M+k}^{\infty} \frac{dx}{x^{2\gamma}}} = \frac{M^{-(s-1/2)}}{\sqrt{2s-1}} \cdot \frac{(M+k)^{-(\gamma-1/2)}}{\sqrt{2\gamma-1}}.$$

Similarly

$$\sum_{j>M} \frac{1}{(j+k)^s j^\gamma} \leq \int_{M}^{\infty} \frac{dx}{(x+k)^s j^\gamma} \leq$$
$$\left(\int_{M}^{\infty} \frac{dx}{(x+k)^{2s}}\right)^{1/2} \cdot \left(\int_{M}^{\infty} \frac{dx}{x^{2\gamma}}\right)^{1/2} = \frac{(M+k)^{-(s-1/2)}}{\sqrt{2s-1}} \frac{M^{-(\gamma-1/2))}}{\sqrt{2\gamma-1}}$$

□

Lemma 9.4. *Assume $s \geq \gamma$ and $i > 2M+1$. Then*

$$\sum_{k=M+1}^{i-M-1} \frac{1}{(i-k)^\gamma k^s} < \frac{(i-2M-1)2^\gamma}{i^\gamma M^s}$$

Proof. Observe that

$$\sum_{k=M+1}^{i-M-1} \frac{1}{(i-k)^\gamma k^s} < (i-2M-1) \max_{k=M+1,\ldots,i-M-1} \frac{1}{(i-k)^\gamma k^s}.$$

Hence it is enough to estimate from below $\min\{(i-k)^\gamma k^s \mid k = M+1,\ldots,i-M-1\}$. Since the function $x \mapsto (i-x)^\gamma x^s$ on the interval $[0,i]$ increases from 0 at $x=0$ to some maximum value and then decreases to 0 at $x=i$, hence the minimum we look for is achieved for $k = M+1$ or $k = i-M-1$.

For $k = M+1$ we have

$$(i-(M+1))^\gamma (M+1)^s = i^\gamma \left(1 - \frac{M+1}{i}\right)^\gamma (M+1)^s >$$
$$i^\gamma \left(1 - \frac{M+1}{2M+2}\right)^\gamma M^s = i^\gamma \left(\frac{1}{2}\right)^\gamma M^s.$$

For $k = i-M-1$ we obtain

$$(i-(i-M-1))^\gamma (i-M-1)^s >$$
$$(M+1)^\gamma (i-M-1)^\gamma (i-M-1)^{s-\gamma} >$$
$$i^\gamma \left(1 - \frac{M+1}{i}\right)^\gamma M^\gamma M^{s-\gamma} \geq i^\gamma \left(1 - \frac{M+1}{2M+2}\right)^\gamma M^s = i^\gamma \left(\frac{1}{2}\right)^\gamma M^s.$$

Hence

$$\min\{(i-k)^\gamma k^s \mid k = M+1,\ldots,i-M\} > i^\gamma 2^{-\gamma} M^s. \tag{154}$$

□

9.2. **Operation QF.** First observe that the computation of QF_i for $i \leq 2M$ is finite and can be done directly using interval arithmetic.

Lemma 9.5. *Let (Y, E, β) and (W, G, γ) be polynomial bounds. If $y \in Y$ and $w \in W$, then for $i > M$*

$$|QF_i(y, w)| \leq \frac{C_{QF}(Y, W)}{i^{t-1}}, \tag{155}$$

where $t = \min(\beta, \gamma)$ and $C_{QF}(Y, W) = \max(C_1, C_2)$ where

$$\begin{aligned} C_1 &= \sup\{i^{t-1}|QF_i(y, w)| \mid i = M+1, \ldots, 2M,\ y \in Y,\ w \in W\} \\ C_2 &= \frac{G}{(2M+1)^{\gamma-t+1}} \sum_{k=1}^{M} \frac{\sup_{y \in Y} |y_k|}{\left(1 - \frac{k}{2M+1}\right)^{\gamma}} + \\ &\quad \frac{E}{(2M+1)^{\beta-t+1}} \sum_{j=1}^{M} \frac{\sup_{w \in W} |w_j|}{\left(1 - \frac{j}{2M+1}\right)^{\beta}} + \frac{EG2^t}{M^{\max(\beta,\gamma)}} \end{aligned}$$

Proof. For the proof it is enough to estimate QF_i for $i > 2M$.

We have

$$|QF_i(y, w)| \leq \sum_{k=1}^{M} |y_k| \frac{G}{(i-k)^{\gamma}} + \sum_{k=i-M}^{i-1} \frac{E}{k^{\beta}} |w_{i-k}| + \sum_{k=M+1}^{i-M-1} \frac{EG}{k^{\beta}(i-k)^{\gamma}}.$$

For each term we have following estimates

$$\sum_{k=1}^{M} |y_k| \frac{G}{(i-k)^{\gamma}} = \frac{G}{i^{\gamma}} \sum_{k=1}^{M} \frac{|y_k|}{\left(1 - \frac{k}{i}\right)^{\gamma}} \leq \frac{G}{i^{\gamma}} \sum_{k=1}^{M} \frac{|y_k|}{\left(1 - \frac{k}{2M+1}\right)^{\gamma}}.$$

Analogously for the second term (exchange $y \leftrightarrow w$ and $j = i - k$)

$$\sum_{k=i-M}^{i-1} \frac{E}{k^{\beta}} |w_{i-k}| \leq \frac{E}{i^{\beta}} \sum_{j=1}^{M} \frac{|w_j|}{\left(1 - \frac{j}{2M+1}\right)^{\beta}}.$$

For the third term from Lemma 9.4 we obtain

$$\sum_{k=M+1}^{i-M-1} \frac{EG}{k^{\beta}(i-k)^{\gamma}} < \frac{iEG2^t}{i^t M^{\max(\beta,\gamma)}}.$$

Hence we have shown that for $i > 2M$

$$\begin{aligned} |QF_i(y, w)| \leq i^{-(t-1)} \Bigg(&\frac{G}{(2M+1)^{\gamma-t+1}} \sum_{k=1}^{M} \frac{|y_k|}{\left(1 - \frac{k}{2M+1}\right)^{\gamma}} + \\ &\frac{E}{(2M+1)^{\beta-t+1}} \sum_{j=1}^{M} \frac{|w_j|}{\left(1 - \frac{j}{2M+1}\right)^{\beta}} + \frac{EG2^t}{M^{\max(\beta,\gamma)}} \Bigg). \end{aligned}$$

∎

9.3. **Operation QI.**

Lemma 9.6. *Let (Y,E,β) and (W,G,γ) be polynomial bounds. If $y \in Y$ and $w \in W$, then for $i \leq 2M$*

$$QI_i(y,w) \in \sum_{k=1}^{M} y_k w_{i+k} + [-1,1] r_i,$$

where

$$r_i = \frac{GE}{(M+i)^{\gamma-1/2} M^{\beta-1/2} \sqrt{(2\beta-1)(2\gamma-1)}}$$

Proof. Use Lemma 9.3 to estimate $\sum_{k=M+1}^{\infty} \frac{1}{k^\beta (i+k)^\gamma}$. □

Lemma 9.7. *Let (Y,E,β) and (W,G,γ) be polynomial bounds. If $y \in Y$ and $w \in W$, then for $i > M$*

$$|QI_i(y,w)| \leq \frac{C_{QI}(Y,W)}{i^{\gamma-1}},$$

where $C_{QI} = \max(C_1, C_2)$

$$\begin{aligned} C_1 &= \sup\{ i^{\gamma-1} |QI_i(y,w)| \quad | \quad i = M+1, \dots, 2M,\ y \in Y,\ w \in W \} \\ C_2 &= \frac{G}{2M+1} \sum_{k=1}^{M} \sup_{y \in Y} |y_k| + \frac{GE}{(3M+1)^{1/2} M^{\beta-1/2} \sqrt{(2\beta-1)(2\gamma-1)}} \end{aligned}$$

Proof. For the proof it is enough to compute $QI_i(y,w)$ for $i > 2M$.

We have

$$|QI_i(y,w)| \leq \sum_{k=1}^{M} |y_k| \frac{G}{(i+k)^\gamma} + \sum_{k>M} \frac{EG}{k^\beta (i+k)^\gamma}.$$

For the first term we have the following estimate

$$\sum_{k=1}^{M} |y_k| \frac{G}{(i+k)^\gamma} = \frac{G}{i^\gamma} \sum_{k=1}^{M} \frac{|y_k|}{\left(1+\frac{k}{i}\right)^\gamma} < \frac{G}{i^\gamma} \sum_{k=1}^{M} |y_k|. \tag{156}$$

For the second term using Lemma 9.3 we obtain

$$\begin{aligned} \sum_{k>M} \frac{EG}{k^\beta (i+k)^\gamma} &< \frac{GE}{(M+i)^{\gamma-1/2} M^{\beta-1/2} \sqrt{(2\beta-1)(2\gamma-1)}} \leq \\ & \frac{1}{i^{\gamma-1}} \frac{GE}{(3M+1)^{1/2} M^{\beta-1/2} \sqrt{(2\beta-1)(2\gamma-1)}}. \end{aligned} \tag{157}$$

By combining the above equations together we obtain for $i > 2M$

$$\begin{aligned} |QI_i(y,w)| \leq \frac{1}{i^{\gamma-1}} \Bigg(\frac{G}{2M+1} \sum_{k=1}^{M} |y_k| + \\ \frac{GE}{(3M+1)^{1/2} M^{\beta-1/2} \sqrt{(2\beta-1)(2\gamma-1)}} \Bigg) \end{aligned}$$

□

9.4. **Various sums.** Let A be a coordinate change as in Lemma 7.2. We define functions $\tilde{F}$ and $\tilde{N}$ by

$$\tilde{F} = A \circ F \circ A^{-1}, \qquad \tilde{N} = A \circ N \circ A^{-1}.$$

The following lemma does not require any proof.

Lemma 9.8. *Same assumptions as in Lemma 7.2.*

$$\begin{aligned}
\frac{\partial^2 \tilde{F}_i}{\partial \nu \partial x_j} &= \sum_{k \le m, s \le m} A_{ik} \frac{\partial^2 F_k}{\partial \nu \partial x_s} A^{-1}_{sj}, \qquad i \le m, j \le m \\
\frac{\partial^2 \tilde{F}_i}{\partial \nu \partial x_j} &= \sum_{k \le m} A_{ik} \frac{\partial^2 F_k}{\partial \nu \partial x_j}, \qquad i \le m, j > m \\
\frac{\partial^2 \tilde{F}_i}{\partial \nu \partial x_j} &= \sum_{s \le m} \frac{\partial^2 F_i}{\partial \nu \partial x_s} A^{-1}_{sj}, \qquad i > m, j \le m \\
\frac{\partial^2 \tilde{F}_i}{\partial \nu \partial x_j} &= \frac{\partial^2 F_i}{\partial \nu \partial x_j}, \qquad i > m, j > m \\
\frac{\partial^2 \tilde{F}_i}{\partial x_j \partial x_k} &= \sum_{s \le m, r \le m, l \le m} A_{is} \frac{\partial^2 F_s}{\partial x_r \partial x_l} A^{-1}_{rk} A^{-1}_{lj}, \qquad i \le m, j \le m, k \le m \\
\frac{\partial^2 \tilde{F}_i}{\partial x_j \partial x_k} &= \sum_{s \le m, r \le m} A_{is} \frac{\partial^2 F_s}{\partial x_r \partial x_j} A^{-1}_{rk}, \qquad i \le m, j > m, k \le m, \\
\frac{\partial^2 \tilde{F}_i}{\partial x_j \partial x_k} &= \sum_{s \le m} A_{is} \frac{\partial^2 F_s}{\partial x_k \partial x_j}, \qquad i \le m, j > m, k > m, \\
\frac{\partial^2 \tilde{F}_i}{\partial x_j \partial x_k} &= \sum_{r \le m, l \le m} \frac{\partial^2 F_i}{\partial x_r \partial x_l} A^{-1}_{rk} A^{-1}_{lj}, \qquad i > m, j \le m, k \le m \\
\frac{\partial^2 \tilde{F}_i}{\partial x_j \partial x_k} &= \sum_{r \le m} \frac{\partial^2 F_i}{\partial x_r \partial x_j} A^{-1}_{rk}, \qquad i > m, j > m, k \le m \\
\frac{\partial^2 \tilde{F}_i}{\partial x_j \partial x_k} &= \frac{\partial^2 F_i}{\partial x_j \partial x_k}, \qquad i > m, j > m, k > m
\end{aligned}$$

Let Y and W be polynomial bounds. We now turn to the computation of $\sum_{kj} \frac{\partial^2 \tilde{F}_i}{\partial x_j \partial x_k} y_j w_k$, where $y \in Y$ and $w \in W$. We would like to stress here that $\frac{\partial^2 \tilde{F}_i}{\partial x_j \partial x_k}$ are constants, hence there is no need to specify their arguments. Observe that

$$\begin{aligned}
\sum_{kj} \frac{\partial^2 \tilde{F}_i}{\partial x_j \partial x_k} y_j w_k &= \sum_{s \le m} A_{is} \left(\sum_{kj} \frac{\partial^2 F_s}{\partial x_j \partial x_k} \overline{y}_j \overline{w}_k \right), \qquad \text{for } i \le m \\
\sum_{kj} \frac{\partial^2 \tilde{F}_i}{\partial x_j \partial x_k} y_j w_k &= \sum_{kj} \frac{\partial^2 F_i}{\partial x_j \partial x_k} \overline{y}_j \overline{w}_k, \qquad \text{for } i > m,
\end{aligned}$$

where $\overline{y}_i = \sum_{j=1}^m A^{-1}_{ij} y_j$ for $i \le m$ and $\overline{y}_i = y_i$ otherwise. $\overline{w}$ is defined analogously.

Hence it is enough to derive the formulas for $\sum_{kj} \frac{\partial^2 F_i}{\partial x_j \partial x_k} \overline{y}_j \overline{w}_k$. We have

$$\sum_{j,k} \frac{\partial^2 F_i}{\partial x_j \partial x_k} \overline{y}_j \overline{w}_k = -2i \sum_{j=1}^{i-1} \overline{y}_j \overline{w}_{i-j} + 2i \sum_{j=1}^{\infty} \overline{y}_j \overline{w}_{j+i} + 2i \sum_{k=1}^{\infty} \overline{y}_{k+i} \overline{w}_k = \\ -2iQF_i(\overline{y}, \overline{w}) + 2iQI_i(\overline{y}, \overline{w}) + 2iQI_i(\overline{w}, \overline{y}).$$

Hence

$$\sum_{kj} \frac{\partial^2 \tilde{F}_i}{\partial x_j \partial x_k} y_j w_k \in \big(A \cdot \big(-2(i)QF(A^{-1}Y, A^{-1}W) + \\ 2(i)QI(A^{-1}Y, A^{-1}W) + 2(i)QI(A^{-1}W, A^{-1}Y)\big)\big)_i \tag{158}$$

Now we derive a formula for $\sum_{j=1}^{\infty} \frac{\partial \tilde{N}_i}{\partial x_j}(\nu, y) w_j$.

We have

$$\sum_{j=1}^{\infty} \frac{\partial \tilde{N}_i}{\partial x_j}(\nu, y) w_j = \sum_{k=1}^{m} A_{ik} \left(\sum_{j=1}^{\infty} \frac{\partial N_k}{\partial x_j}(\nu, y) \overline{w}_j \right), \qquad \text{for } i \leq m$$

$$\sum_{j=1}^{\infty} \frac{\partial \tilde{N}_i}{\partial x_j}(\nu, y) w_j = \sum_{j=1}^{\infty} \frac{\partial N_i}{\partial x_j}(\nu, y) \overline{w}_j, \qquad \text{otherwise.}$$

Hence it is enough to have an expression for $\sum_{i=1}^{\infty} \frac{\partial N_k}{\partial x_i} \overline{w}_i$.

Using the formulas for the derivatives for the vector field of the KS equation we obtain

$$\sum_{j=1}^{\infty} \frac{\partial N_i}{\partial x_j}(\nu, y) w_j = -2i \sum_{j=1}^{i-1} y_j w_{i-j} + 2i \sum_{j=1}^{\infty} y_j w_{i+j} + 2i \sum_{j=1}^{\infty} y_{i+j} w_j = \\ -2iQF_i(y, w) + 2iQI_i(y, w) + 2iQI_i(w, y).$$

Summarizing we have shown that

$$\sum_{j=1}^{\infty} \frac{\partial \tilde{N}_i}{\partial x_j}(\nu, y) w_j \in \big[A \cdot \big(-2(i)QF(Y, \overline{W}) + \\ 2(i)QI(Y, \overline{W}) + 2(i)QI(\overline{W}, Y)\big)\big]_i. \tag{159}$$

From the above formula it follows immediately that

$$\sum_{j=1}^{\infty} \frac{\partial \tilde{F}_i}{\partial x_j}(\nu, y) w_j \in \big[A \cdot \big((i^2(1 - \nu i^2))\overline{W} - 2(i)QF(Y, \overline{W}) + \\ 2(i)QI(Y, \overline{W}) + 2(i)QI(\overline{W}, Y)\big)\big]_i. \tag{160}$$

10. **Details for the KS-equation - the regularity issue.** We assume that we have a coordinate change A as Lemma 7.2 and, as in Section 7, we assume that we have a block decomposition of H and V representing the topologically self-consistent bounds for

$$F(\nu, x) = 0 \tag{161}$$

for $\nu \in \Lambda = [\nu_0 - \delta, \nu_0 + \delta]$. We assume that we have the uniqueness property of solutions (161) for $\nu \in \Lambda$. This defines the function $x(\nu)$. In Section 5 it was shown that $x(\nu)$ is C^∞.

As in the proof of the uniqueness in Section 7 we will perform all computations for the function $\tilde{F} = A \circ F \circ A^{-1}$. Similarly we define $\tilde{N}$ as $A \circ N \circ A^{-1}$.

We have to solve

$$z + \frac{\partial \tilde{F}}{\partial x} \cdot y = 0, \tag{162}$$

where z is obtained from the implicit differentiation of $F(\nu, x) = 0$ and depends upon the particular partial derivative we are willing to compute.

The standing assumptions is this section are

- $V = N \oplus \Pi_{k=m+1}^{M}[a_k^-, a_k^+] \oplus \Pi_{k>M}\left[\frac{-C}{k^s}, \frac{C}{k^s}\right]$ are the topologically self-consistent a priori bounds for (161) for $\nu \in \Lambda$
- $Y = \Pi_{(i)\leq m}\overline{B_{(i)}(a_{(i)}, R_{(i)})} \oplus \Pi_{k=m+1}^{M}[y_k^-, y_k^+] \oplus \Pi_{k>M}\left[\frac{-E}{k^\gamma}, \frac{E}{k^\gamma}\right]$ is a candidate for the self-consistent bounds a priori for equation (162). We assume only that γ is large enough, so that conditions **C1,C2,C3** are satisfied.

Hence (V, C, s) and (Y, E, γ) are polynomial bounds.

10.1. **Estimates for the linear part in the equation for the derivatives of an implicitly defined function.** We split the linear term in equation (162) into $D + N$ as follows

$$\begin{aligned} z_{(i)} + \frac{\partial \tilde{F}_{(i)}}{\partial x_{(i)}} \cdot y_{(i)} + \sum_{(j),(j)\neq(i)} \frac{\partial \tilde{F}_{(i)}}{\partial x_{(j)}} \cdot y_{(j)} &= 0, \qquad \text{for } (i) \leq m, \\ z_{(i)} + i^2(1-\nu i^2) \cdot y_i + \sum_j \frac{\partial \tilde{N}_i}{\partial x_j} \cdot y_j &= 0, \qquad \text{for } i > m. \end{aligned} \tag{163}$$

In what follows we will provide the estimates for the infinite sums appearing in the above equation, which can be directly inserted in the computer program.

Basically we have two ranges of coordinates $i \leq M$ and $i > M$. For $i \leq M$ we try to obtain quite tight bounds, as those are coordinates that matter, hence they will be computed for each i separately. Hence in the sum

$$\sum_{(j),(j)\neq(i)} \frac{\partial \tilde{F}_{(i)}}{\partial x_{(j)}} \cdot y_{(j)} = \sum_{(j)\leq M,(j)\neq(i)} \frac{\partial \tilde{F}_{(i)}}{\partial x_{(j)}} \cdot y_{(j)} + \sum_{(j)>M} \frac{\partial \tilde{N}_{(i)}}{\partial x_{(j)}} \cdot y_{(j)}, \tag{164}$$

the finite part will be computed by direct interval evaluation for each (i) and we need to derive the formula for infinite sum (see Lemma 10.1).

For $i > M$ we need a uniform expression valid for all $i > M$ in the form $\frac{G}{i^{\gamma-2}}$. We obtain it from formula (159).

The following Lemma gives an expression for infinite sum part of non-diagonal term in (163) for low wave numbers

Lemma 10.1. *Assume $i \leq M$. Then for any $x \in V$ and $y \in Y$*

$$\begin{aligned} \sum_{j>M} \frac{\partial \tilde{N}_i}{\partial x_j}(x) \cdot y_j \subset [A \cdot (-2(i)QF(V, (I-P_M)Y) + \\ 2(i)QI(V, (I-P_M)Y) + 2(i)QI((I-P_M)Y, V))]_i \end{aligned}$$

Proof. Use formula (159), observe that $A^{-1}(I - P_M)V = (I - P_M)V$. □

In the next lemma we provide a formula for 2-dimensional block.

Lemma 10.2. *Let $(i) = (i_1, i_2) \leq m$. Let for $k = 1, 2$*

$$t_k = \left| \sum_{j=1, j\notin(i_1,i_2)}^{M} \frac{\partial \tilde{F}_{i_k}}{\partial x_j} y_j \right| + r_{i_k},$$

where r_i have been defined in Lemma 10.1.

Then

$$\left| \sum_{(j),(j)\neq(i)} \frac{\partial \tilde{F}_{(i)}}{\partial x_{(j)}} y_{(j)} \right| \leq \sqrt{t_1^2 + t_2^2}.$$

10.2. **Estimates for the constant terms in the equation for the derivatives of the implicit function.** In the computation of various derivatives of the implicit function defined as z in equation (35) we have the following expressions or the sums thereof (and due to the fact, that F is a second degree polynomial we have only the terms of this type)

$$\frac{\partial \tilde{F}_i}{\partial \nu}(\nu,x), \qquad \frac{\partial \tilde{F}_i}{\partial x_1}(\nu,x), \qquad \sum_j \frac{\partial^2 \tilde{F}_i}{\partial \nu \partial x_j}(\nu,x) y_j,$$
$$\sum_j \frac{\partial^2 \tilde{F}_i}{\partial x_1 \partial x_j}(\nu,x) y_j, \qquad \sum_{jk} \frac{\partial^2 \tilde{F}_i}{\partial x_j \partial x_k}(\nu,x) y_j w_k, \tag{165}$$

for $y \in Y$, $w \in W$ and $x \in V$, where Y, W, V are polynomial bounds.

From Lemma 7.2 we have

Lemma 10.3. *For any $\nu \in \Lambda$ and $x \in V$ holds*

$$\frac{\partial \tilde{F}_k}{\partial \nu}(\nu,x) \in -\left(A \cdot (i^4) V\right)_k$$

Lemma 10.4. *Let $(Y, E = 0, \gamma = 0)$ be such that $Y_1 = \{1\}$ and $Y_i = \{0\}$ for $i > 1$. Then for any $\nu \in \Lambda$ and $x \in V$ holds*

$$\frac{\partial \tilde{F}_i}{\partial x_1}(\nu,x) \in \left[\sum_{k=1}^{\infty} \frac{\partial \tilde{F}_i}{\partial x_k}(\Lambda, V) Y \right]_i.$$

The expression on the right hand side is given by formula (160).

From Lemma 9.8 and formulas for $\frac{\partial^2 F_i}{\partial \nu \partial x_j}$ we obtain

Lemma 10.5. *For any $y \in Y$ holds*

$$\sum_j \frac{\partial^2 \tilde{F}_i}{\partial \nu \partial x_j} y_j \in \left[-A \cdot \left((i^4)(A^{-1}Y)\right)\right]_i$$

Lemma 10.6. *Let $(W, G = 0, \gamma = 0)$ be such that $W_1 = \{1\}$ and $W_i = \{0\}$ for $i > 1$. Then for any $\nu \in \Lambda$, $x \in V$ $y \in Y$ holds*

$$\sum_j \frac{\partial \tilde{F}_i}{\partial x_1 \partial x_j}(\nu,x) y_j \in \left[A \cdot \left(\sum_{k,j} \frac{\partial \tilde{F}_i}{\partial x_k \partial x_j} y_j w_k \right) \right]_i,$$

and the sum on the right hand side is given by formula (158).

10.3. **Isolation and refinement of bounds for far tail.** Condition **C4** - the isolation condition for $k > M$ is established by the following procedure.

Input data:

- H, k, such that $|z_k| \leq \frac{H}{k^t}$ for $k > M$
- G, w, such that $(Ny)_k < \frac{G}{k^w}$ for $k > M$

- E, γ, such that $|y_k| \leq \frac{E}{k^\gamma}$ for $k > M$

Condition **C4** is satisfied if for all $k > M$, z and y satisfying the above conditions we have

$$k^4(\nu - \frac{1}{k^2})|y_k^\pm| \geq |(Ny)_k + z_k|. \tag{166}$$

The above inequality is implied by the following

$$\left(\nu - \frac{1}{k^2}\right)\frac{E}{k^{\gamma-4}} \geq \frac{G}{k^w} + \frac{H}{k^t}, \qquad \text{for all } k > M, \tag{167}$$

which is equivalent to

$$\left(\nu - \frac{1}{k^2}\right)E \geq Gk^{\gamma-4-w} + Hk^{\gamma-4-t}, \qquad \text{for } k > M\ . \tag{168}$$

Hence **C4** holds if the following conditions are satisfied

$$\gamma - 4 - w \quad \leq \quad 0 \tag{169}$$

$$\gamma - 4 - t \quad \leq \quad 0 \tag{170}$$

$$\left(\nu - \frac{1}{(M+1)^2}\right)E \quad \geq \quad G(M+1)^{\gamma-4-w} + H(M+1)^{\gamma-4-t}. \tag{171}$$

Now assume that conditions (169-171) are satisfied, then we can define new updated bounds $|\tilde{y}_k| \leq \tilde{E}/k^{\tilde{\gamma}}$ by

$$\tilde{\gamma} \quad = \quad \min(w+4, t+4) \tag{172}$$

$$\tilde{E} \quad = \quad \frac{1}{\nu - \frac{1}{(M+1)^2}}\left(\frac{G}{(M+1)^{w+4-\tilde{\gamma}}} + \frac{H}{(M+1)^{t+4-\tilde{\gamma}}}\right) \tag{173}$$

10.4. **Initialization.** Consider equation (163). We want to generate an initial guess for the self-consistent a priori bounds. We have the following input values

- Λ, V, m, M - self consistent a priori bounds for $F(\Lambda, x) = 0$
- a set Z, such that for $z \in Z$ we have $z_{(i)}$ for $(i) \leq M$, $|z_i| \leq \frac{H}{i^t}$ for $i > M$
- interval matrices $[D], [N] \subset \mathbb{R}^{M\times M}$, such that

$$\begin{aligned}
\left[\frac{\partial \tilde{F}_{(i)}}{\partial x_{(i)}}(\Lambda, V)\right]_I &\subset [D]_{(i)(i)}, \qquad \text{for } (i) \leq M \\
[D]_{(i)(j)} &= 0, \qquad \text{if } (i) \neq (j) \\
\left[\frac{\partial \tilde{F}_{(i)}}{\partial x_{(j)}}(\Lambda, V)\right]_I &\subset [N]_{(i)(j)}, \qquad \text{for } (i),(j) \leq M \\
[N]_{(i)(i)} &= 0, \qquad \text{if } (i) \leq N
\end{aligned}$$

We solve the Galerkin projection on X_M of (163) using for example the iterative scheme from the proof of Theorem 5.1.

$$Y^0 = P_M(z), \qquad Y^{n+1} = -[D]^{-1}Z - [D]^{-1}[N]Y^n. \tag{174}$$

Since we already checked the uniqueness of solution $F(\nu, x) = 0$, hence the assumptions of Theorem 5.1 are satisfied, hence the scheme (174) converges quickly to a fixed point (which in our context is a product of M intervals). We stop the iteration when

$$\frac{\rho_H(Y_i^{n+1}, Y_i^n)}{\max(|x|, \quad x \in Y_i^n \cup Y_i^{n+1})} < 0.01, \qquad \text{for } i = 1, \ldots, M, \tag{175}$$

where by $\rho_H(X,Y)$ we denote the Hausdorff distance between the sets X and Y.

Let $\tilde{Y} = \Pi_{i=1}^{M} \left(y_i^c + [-\delta_i, \delta_i]\right)$ be obtained from the above iterative scheme. We define a candidate (an initial guess) for the self-consistent a priori bounds for (163) as follows.

For $1 \leq i \leq j \leq M$ we set

$$\delta_i^j = \max\left(|y_k|, \qquad y \in \tilde{Y},\ i \leq k \leq j\right),$$

Finally the candidate bounds are given by

$$\begin{aligned}
y_i &\in y_i^c + 0.25 \cdot \delta_1^m \cdot [-1,1], && 1 \leq i \leq m \\
y_i &\in y_i^c + 0.25 \cdot \delta_{m+1}^M \cdot [-1,1], && m < i \leq M \\
|y_i| &\leq \frac{E}{i^4}, \quad E = \frac{2H}{(M+1)^{t-4}}, && i > M.
\end{aligned}$$

Let us comment about the bounds for y_i for $i > M$. They are chosen so that we have $\max|z_{M+1}| = \max|y_{M+1}|$ and the decay rate for $|y_i|$ is $\frac{E}{i^4}$.

This procedure worked in most cases - i.e. it produced a candidate, which lead later to the self-consistent a priori bounds. It failed only if $z \approx 0$ (this happens when considering the zero solution) and the coefficient E is very small. In this case when we did not get an isolation starting from the above guess, we produce a new guess by setting $E = 0.01$ and then it always worked.

Observe that the proof of Theorem 5.1 gives a guaranteed good candidate for the self-consistent a priori bounds, but it turns out that the refinement, as described in Section 10.3, of the bounds obtained in this way requires much larger M.

11. **Details of bifurcation computations.** In order to check the assumptions of Theorem 6.4 or Theorem 6.6 we set $G(\nu, x_1) = \tilde{F}_1(\nu, y(\nu, x_1))$, where $y_1 = x_1$ and $y_i(\nu, x_1)$ for $i > 1$ is a solution of system $\tilde{F}_i(\nu, x_1, y) = 0$ for $i > 1$.

We have to compute the following partial derivatives of G for Theorem 6.4

$$\frac{\partial G}{\partial x_1}, \qquad \frac{\partial^2 G}{\partial x_1 \partial \nu}, \qquad \frac{\partial^3 G}{\partial x_1^3}.$$

For Theorem 6.6 we need

$$\frac{\partial G}{\partial x_1}, \qquad \frac{\partial^2 G}{\partial x_1 \partial \nu}, \qquad \frac{\partial^2 G}{\partial x_1^2}.$$

They are given by the following formulas

$$\begin{aligned}
\frac{\partial G}{\partial x_1} &= \sum_{i=1}^{\infty} \frac{\partial \tilde{F}_1}{\partial x_i} \frac{\partial y_i}{\partial x_1} \\
\frac{\partial^2 G}{\partial x_1^2} &= \sum_{i,j=1}^{\infty} \frac{\partial^2 \tilde{F}_1}{\partial x_i \partial x_j} \frac{\partial y_i}{\partial x_1} \frac{\partial y_j}{\partial x_1} + \sum_{i=1}^{\infty} \frac{\partial \tilde{F}_1}{\partial x_i} \frac{\partial^2 y_i}{\partial x_1^2} \\
\frac{\partial^3 G}{\partial x_1^3} &= 3 \sum_{i,j=1}^{\infty} \frac{\partial^2 \tilde{F}_1}{\partial x_i \partial x_j} \frac{\partial^2 y_i}{\partial x_1^2} \frac{\partial y_j}{\partial x_1} + \sum_{i=1}^{\infty} \frac{\partial \tilde{F}_1}{\partial x_i} \frac{\partial^3 y_i}{\partial x_1^3} \\
\frac{\partial^2 G}{\partial x_1 \partial \nu} &= \sum_{i=1}^{\infty} \frac{\partial^2 \tilde{F}_1}{\partial x_i \partial \nu} \frac{\partial y_i}{\partial x_1} + \sum_{i,j=1}^{\infty} \frac{\partial^2 \tilde{F}_1}{\partial x_i \partial x_j} \frac{\partial y_i}{\partial x_1} \frac{\partial y_j}{\partial \nu} + \sum_{i=1}^{\infty} \frac{\partial \tilde{F}_1}{\partial x_i} \frac{\partial^2 y_i}{\partial x_1 \partial \nu}
\end{aligned}$$

All these formulas can be computed using the algebra for polynomial bounds described in Section 9 and then extracting first coordinate.

11.1. **Short description of the procedure for the proof of the existence of bifurcations.** Observe first that both bifurcations theorems (6.4 and 6.6) contain two types of assumptions:

global: a construction of the self-consistent bounds for (90) over $Z = [\nu_1, \nu_2] \times [-a, a]$, on which can evaluate $\frac{\partial^3 G}{\partial x_1^3}(Z)$ and $\frac{\partial^2 G}{\partial \nu \partial x_1}(Z)$ in case of Theorem 6.4, or $\frac{\partial^2 G}{\partial \nu \partial x_1}(Z)$ and $\frac{\partial^2 G}{\partial x_1^2}(Z)$ in case of Theorem 6.6. These partial derivatives of G should not contain zero and this can be achieved by taking the set Z small enough.

Observe that if at this stage we are not able to construct Z on which the global conditions are satisfied, then the proof is inconclusive. We can neither claim or exclude the existence of bifurcation. We can only hope that taking larger m, M and smaller (in diameter) Z will improve the situation.

local: Once we have the set Z over which the global conditions are satisfied we verify the remaining (local) conditions. Observe that each of them involves either some derivatives of G or the value of G at some point (ν_0, x_0). The isolation algorithm presented in [23] applied to (90) and the algorithms for the computation of the partial derivatives $y(\nu, x)$ described in the previous section allow to compute the desired values with an arbitrary accuracy (close to the round-off error) by taking M large enough. Hence we can check whether the local conditions are satisfied or violated (in this case we can rule out the existence of any bifurcation in Z). Hence this part of the proof is conclusive.

The computer procedure performing the proof of the existence of bifurcation works as follows

Input parameters:

- m - the dimension of Galerkin projection, $M = \max(2m, 10)$
- $(\nu_0, x_0) \in \mathbb{R} \times \mathbb{R}^m$ an approximate bifurcation point for m-dimensional Galerkin projection of (3), x_0 is k-modal (it is desirable to take m as a multiple of k)
- (ν_1, ν_2), such that $\nu_0 \in (\nu_1, \nu_2)$. This is our parameter range in the bifurcation theorems

The procedure:

diagonalization: We define the coordinate change A as follows: we approximately diagonalize, $dP_m F(\nu_0, x_0)$, the m-dimensional Galerkin projection of (3) for $\nu = \nu_0$ at $x = x_0$. We choose an eigenvalue, λ_0 , which is the closest to zero (there should be such, otherwise there is no bifurcation nearby). The eigenvector corresponding to λ_0 we choose as our 'bifurcation direction' and it will represent the 1-st coordinate in the new coordinate frame.

We have also to make sure, that the subspace of k-modal functions is contained in the hyperplane $\tilde{x}_1 = 0$. In case of the bimodal branch ($k = 2$) the coordinate change must commute with the symmetry R, from this we obtain the odd-symmetry property of bifurcation function G.

Observe that is easy to satisfy these properties during the diagonalization process, as the subspace of k-modal functions is invariant also for $dP_m F(\nu_0, x_0)$ if x_0 is k-modal itself.

global conditions: Let $X_1 = 10^{-2}$. We set $Z = [\nu_1, \nu_2] \times [-X_1, X_1]$ and we try to verify the global bifurcation conditions on Z. If they are not satisfied, then we set $X_1 = X_1/5$ and try again, till $X_1 < 10^{-5}$, when we decide that we fail.

If we fail then we increase M (hoping for an improvement in the diameters of the relevant partial derivatives of G on Z) and try again.

We do this until we succeed (and we jump to verify the local conditions), or there is no improvement in values of the relevant partial derivatives G on Z (we use some ad-hoc stabilization criterion - for example: the diameter should shrink by a factor at least 0.9).

If we fail then we exit. We can only hope that either the increase of m and the shrinking of the diameter of $[\nu_1, \nu_2]$ will improve things.

local conditions: We evaluate all local conditions (see the discussion at the beginning of this subsection). Consider for example the computation $\frac{\partial G}{\partial x}(\nu_1, 0)$. We keep refining bounds by $y(\nu, x)$ by increasing M till $\frac{\partial G}{\partial x}(\nu_1, 0)$ does not contain 0 or its value stabilize.

The k-modal fixed points were produced from the unimodal fixed point branch using the rescaling described in Lemma 6.1. The solutions on the unimodal branch are all attracting, hence can be easily found either by following the trajectory or by using the Newton method starting at an approximate fixed point for the 2-dimensional Galerkin projection. We picked up the bifurcation values from [12, 15], where the bifurcation parameter was $\alpha = 4/\nu$. It turns out that these values were too crude for our purpose and we refined them by an ad-hoc trial and error approach (this was not automated), until we can finally verify the global bifurcation condition.

The file *bifdata.txt* [29] contains the most relevant numerical data from the proofs of the bifurcations listed in Theorem 1.1.

The program was written in C++ (gnu compiler was used). We used CAPD package[2] to handle the interval arithmetic and graphics. All computations were performed on Windows 98, Pentium III, 450 MHZ computer. We tested the program also under $l$inux.

REFERENCES

[1] G. Arioli and H. Koch, *Computer-assisted methods for the study of stationary solutions in dissipative systems, applied to the Kuramoto-Sivashinski equation*, Arch. Rational Mech. An. **197** (2010), 1033–1051

[2] CAPD - Computer assisted proofs in dynamics, a package for rigorous numerics, Available from: `http://capd.ii.edu.pl/`

[3] L. Cesari, *Functional analysis and Galerkin's method*, Mich. Math. Jour. **11** (1964), 383-414.

[4] S.-N. Chow and J. Hale, "Methods of Bifurcation Theory", Springer–Verlag, New York, 1982

[5] F. Christiansen, P. Cvitanovic and V. Putkaradze, *Spatiotemporal chaos in terms of unstable recurrent patterns*, Nonlinearity **10** (1997), 55–70

[6] P. Collet, J.-P. Eckmann, H. Epstein and J. Stubbe, *Analyticity for the Kuramoto–Sivashinsky equation*, Physica D **67** (1993), 321–326

[7] E.J. Doedel, *AUTO: a programm for the bifurcation analysis of autonomous system*, Congr. Numer. **30** (1981), 265–384

[8] E.J. Doedel and R.C. Paffenroth, *The AUTO2000: command line user interface*, in "Proceedings of the 9-th Python Conference" 2001, 233-241

[9] C. Foias, B. Nicolaenko, G. Sell and R. Temam *Inertial manifolds for the Kuramoto–Sivashinsky equation and an estimate of their lowest dimension*, J. Math. Pures Appl. **67** (1988), 197–226

[10] J. Hyman and B. Nicolaenko, *The Kuramoto–Sivashinsky equation; A bridge between PDEs and dynamical systems*, Physica D **18** (1986), 113–126

[11] Ju. S. Il'yashenko, *Global Analysis of the Phase Portrait for the Kuramoto–Sivashinsky equation.* J. Dyn. Diff. Eq. **4** (1992), 585–615

[12] M. Jolly, I. Kevrekidis and E. Titi, *Approximate inertial manifolds for the Kuramoto–Sivashinsky equation: analysis and computations*, Physica D **44** (1990), 38-60

[13] M. Jolly, R. Rosa and R. Temam, *Evaluating the dimension of an inertial manifold for the Kuramoto–Sivashinsky Equation*, Adv. Differential Equations **5** (2000), 31–66

[14] M. Jolly, R. Rosa and R. Temam, *Acurate computations on inertial manifolds*, SIAM J. Sci. Compt **22** (2001), 2216–2238

[15] I. Kevrekidis, B. Nicolaenko and C. Scovel *Back in saddle again: a computer assisted study of the Kuramoto–Sivashinsky equation,* SIAM J. Appl. Math. **50** (1990), 760–790

[16] Y. Kuramoto and T. Tsuzuki, *Persistent propagation of concentration waves in dissipative media far from thermal equilibrium*, Prog. Theor. Phys., **55** (1976), 365

[17] S. Maier-Paape, U. Miller, K.Mischaikow and T. Wanner, *Rigorous numerics for the Cahn-Hilliard equation on the unit square*, Revista Matematica Complutense **21** (2008), 351-426

[18] R.E. Moore, "Interval Analysis", Prentice Hall, Englewood Cliffs, N.J., 1966

[19] A. Neumeier, "Interval methods for systems of equations", Cambrigde University Press, 1990.

[20] A. Quarteroni, R. Sacco and F. Saleri, "Numerical Mathematics", Texts in Applied Mathematics, 37, Springer

[21] G.I. Sivashinsky, *Nonlinear analysis of hydrodynamical instability in laminar flames – 1. Derivation of basic equations*, Acta Astron. **4** (1977), 1177

[22] C. Scovel, I. Kevrekidis and B. Nicolaenko, *Scaling laws and the prediction of bifurcations in systems modeling pattern formation,* Physics Letters A **130** (1988) ,73–80

[23] P. Zgliczyński and K. Mischaikow, *Rigorous Numerics for Partial Differential Equations: the Kuramoto-Sivashinsky equation*, Foundations of Computational Mathematics **1** (2001), 255-288

[24] P. Zgliczyński, *Trapping regions and an ODE-type proof of existence and uniqueness for Navier-Stokes equations with periodic boundary conditions on the plane*, Univ. Iag. Acta Math., **41** (2003), 89-113

[25] P. Zgliczyński, *On smooth dependence on initial conditions for dissipative PDEs, an ODE-type approach*, J. Diff. Eq. **195** (2003), 271–283

[26] P. Zgliczyński, *Attracting fixed points for the Kuramoto-Sivashinsky equation - a computer assisted proof*, SIAM Journal on Applied Dynamical Systems **1** (2002), 215-235,

[27] P. Zgliczyński, *Rigorous numerics for dissipative Partial Differential Equations II. Periodic orbit for the Kuramoto-Sivashinsky PDE - a computer assisted proof*, Foundations of Computational Mathematics **4** (2004), 157–185

[28] P. Zgliczyński, *Rigorous Numerics for Dissipative PDEs III. An effective algorithm for rigorous integration of dissipative PDEs*, Topological Methods in Nonlinear Analysis **36** (2010), 197-262

[29] the file containing numerical data from the bifurcation proofs, Available from: http://www.ii.uj.edu.pl/~{}zgliczyn/papers/KS/bifdata.txt,

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