

# Period doubling in the Rössler system - a computer assisted proof

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## Abstract

Using rigorous numerical methods we validate a part of the bifurcation diagram for a Poincaré map of the Rössler system [26] - the existence of two period doubling bifurcations and the existence of a branch of period two points connecting them. Our approach is based on the Lyapunov-Schmidt reduction and uses the  $C^r$ -Lohner algorithm [31] to obtain rigorous bounds for the Rössler system.

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## 1 Introduction

The goal of this paper is to show how to produce a piece of the rigorous bifurcation diagram of the periodic orbits for an ODE. We study the Rössler system [26], one of the textbook examples of ODEs generating nontrivial dynamics, for a parameter range containing two period doubling bifurcations.

According to the discussion in the Kuznetsov textbook [16, Section 2.7] there are two extremes in studying bifurcations in dynamical systems. The first one, going back to Poincaré, is to analyze the appearance (branching) of new invariant objects (equilibria or periodic orbits) from the known ones as parameters of the system vary. A good reference for this approach is the textbook by Chow and Hale [7]. On the other extreme is the approach going back to Andronov [2] and Thom [27], which is to study rearrangements (bifurcations) of the whole phase portrait under variations of parameters. It is apparent that the first approach is necessarily one of the initial steps in attempting to describe the bifurcations in the Andronov-Thom sense. In fact in many dimensional systems (even for planar maps like the Hénon map) achieving the complete description

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of the phase space portrait and its changes appears to be hopeless in view of the results on the Henon-like maps [21, 5, 29, 30].

While there exists a vast literature on the bifurcation theory, see for example [4, 7, 8, 16] and the references given there, and also a lot of numerical bifurcation diagrams for various systems can be found in the literature (see for example the references in [16]), there are virtually no rigorous results on bifurcations of periodic orbits for ODEs in dimension three or higher in the situation, when the periodic orbit undergoing the bifurcation is not given to us analytically due to some special symmetries of the system. The basic reason for this is: while numerical experiments and/or normal form computations may clearly show what is happening (in terms of the bifurcations) we usually lack any reasonable rigorous estimates about the observed orbits, which prevents us from turning these observations into rigorous statements. To obtain the necessary estimates, one needs to integrate the variational equations describing the partial derivatives with respect to the initial conditions up to order 3 or higher. This is usually a serious problem for rigorous ODE solvers. It turns out that the naive approach of applying an ODE solver to the system of variational equations does not work, because the methods dealing with the wrapping effect used in the Lohner-type algorithms (the most effective rigorous ODE solvers) [17, 32, 23] break down for such system. As a solution to this problem, the  $C^r$ -Lohner algorithm was proposed in [31] and it is used in the present work.

Concerning the content of the paper regarding the bifurcation theory itself, we were forced to reformulate some well known theorems to make them amenable to computer assisted proofs. It is a common feature of all bifurcation theorems that the bifurcation point (or rather a candidate) and all necessary data like the spectrum and maybe some higher order terms are always given as part of the assumptions. But in a nonlinear system we usually do not have explicitly these data. In fact, the existence of the bifurcation point has to be proved by looking at the behavior of the system in some neighborhood. This forces us to reformulate some bifurcation theorems in a semi-local way and we have to investigate properties of solutions of implicit equations, which are degenerate (due to the presence of bifurcations). This is the reason, why from various approaches to bifurcations we chose the one developed in [7], which is based on the Lyapunov-Schmidt reduction.

In our work we focus on the period-doubling bifurcation of periodic orbits for the Rössler equations. Indeed, we study the Poincaré map for the Rössler system. The paper is organized as follows: in Sections 3, 4 and 5 we discuss the main tools used to produce a validated piece of the bifurcation diagram containing the period doubling bifurcations. In the remaining sections we give some details concerning our results for the Rössler system.

## 2 Basic definitions

By  $\mathbb{N}$ ,  $\mathbb{Z}$ ,  $\mathbb{Q}$ ,  $\mathbb{R}$ ,  $\mathbb{C}$  we denote the set of natural, integer, rational, real and complex numbers, respectively.  $\mathbb{Z}_-$  and  $\mathbb{Z}_+$  are negative and positive integers,

respectively. By  $S^1$  we will denote a unit circle on the complex plane.

For  $\mathbb{R}^n$  we will denote the norm of  $x$  by  $\|x\|$  and if the formula for the norm is not specified in some context, then it means that one can use any norm there. Let  $x_0 \in \mathbb{R}^s$ , then  $B_s(x_0, r) = \{z \in \mathbb{R}^s \mid \|x_0 - z\| < r\}$  and  $B_s = B_s(0, 1)$ .

Let  $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a linear map. By  $\text{Sp}(A)$  we denote the spectrum of  $A$ , which is the set of  $\lambda \in \mathbb{C}$ , such that there exists  $x \neq 0$ , such that  $Ax = \lambda x$ .

For a map  $f : X \rightarrow Y$  by  $\text{dom}(f)$  we will denote the domain of  $f$ . For a map  $F : X \rightarrow X$  we will denote the fixed point set by  $\text{Fix}(F, U) = \{x \in U \mid F(x) = x\}$ .

Let  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ . By  $\pi_i$  we will denote the projection on  $i$ -th coordinate, i.e.  $\pi_i(x) = x_i$ . Analogously for any multiindex  $\alpha = (i_1, i_2, \dots, i_k) \in \mathbb{Z}_+^k$  we define  $\pi_\alpha(x) = (x_{i_1}, x_{i_2}, \dots, x_{i_k})$ . Sometimes the points in the phase space will have coordinates denoted by different letters, for example  $z = (\nu, x, y)$ , then we will index the projection by the names of variables, i.e.  $\pi_{(\nu, x)}(z) = (\nu, x)$  etc.

**Definition 1** Let  $f : \mathbb{R}^n \supset \text{dom}(f) \rightarrow \mathbb{R}^n$  be  $C^1$ . Let  $z_0 \in \text{dom}(f)$ . We say that  $z_0$  is a hyperbolic fixed point for  $f$  iff  $f(z_0) = z_0$  and  $\text{Sp}(Df(z_0)) \cap S^1 = \emptyset$ , where  $Df(z_0)$  is the derivative of  $f$  at  $z_0$ .

**Definition 2** Consider a map  $f : X \supset \text{dom}(f) \rightarrow X$ . Let  $x \in X$ . Any sequence  $\{x_k\}_{k \in I}$ , where  $I \subset \mathbb{Z}$  is a set containing 0 and for any  $l_1 < l_2 < l_3$  in  $\mathbb{Z}$  if  $l_1, l_3 \in I$ , then  $l_2 \in I$ , such that

$$x_0 = x, \quad f(x_i) = x_{i+1}, \quad \text{for } i, i+1 \in I$$

will be called an orbit through  $x$ . If  $I = \mathbb{Z}_- \cup \{0\}$ , then we will say that  $\{x_k\}_{k \in I}$  is a full backward orbit through  $x$ .

**Definition 3** Let  $X$  be a topological space and let the map  $f : X \supset \text{dom}(f) \rightarrow X$  be continuous.

Let  $Z \subset \mathbb{R}^m$ ,  $x_0 \in Z$ ,  $Z \subset \text{dom}(f)$ . We define

$$W_Z^s(z_0, f) = \{z \mid \forall n \geq 0 \ f^n(z) \in Z, \quad \lim_{n \rightarrow \infty} f^n(z) = z_0\},$$

$$W_Z^u(z_0, f) = \{z \mid \exists \{x_n\} \subset Z \text{ a full backward orbit through } z, \text{ such that} \\ \lim_{n \rightarrow -\infty} x_n = z_0\},$$

$$W^s(z_0, f) = \{z \mid \lim_{n \rightarrow \infty} f^n(z) = z_0\},$$

$$W^u(z_0, f) = \{z \mid \exists \{x_n\} \text{ a full backward orbit through } z, \text{ such that} \\ \lim_{n \rightarrow -\infty} x_n = z_0\},$$

$$\text{Inv}^+(Z, f) = \{z \mid \forall n \geq 0 \ f^n(z) \in Z\},$$

$$\text{Inv}^-(Z, f) = \{z \mid \exists \{x_n\} \subset Z \text{ a full backward orbit through } z\},$$

$$\text{Inv}(Z, f) = \text{Inv}^+(Z, f) \cap \text{Inv}^-(Z, f).$$

If  $f$  is known from the context, then we will usually drop it from the notation and use  $W^s(z_0)$ ,  $W_Z^s(z_0)$  etc instead.

**Definition 4** Let  $P : J \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ , where  $J \subset \mathbb{R}$ . For each  $\nu \in J$ , we define  $P_\nu = P(\nu, \cdot)$ . We say that  $P_\nu$  has a period doubling bifurcation at  $(\nu_0, x_0)$  iff there exists  $V = [\nu_1, \nu_2] \times X \subset J \times \mathbb{R}^n$ , such that the following conditions are satisfied

- $(\nu_0, x_0) \in \text{int } V$ ,  $P_{\nu_0}(x_0) = x_0$ .
- there exists a continuous function  $x_{fp} : [\nu_1, \nu_2] \rightarrow \text{int } X$ , such that

$$\text{Fix}(P_\nu, X) = \{x_{fp}(\nu)\}.$$

- there exist two continuous curves  $c_i : [\nu_0, \nu_2] \rightarrow \text{int } X$ ,  $i = 1, 2$ , such that for  $\nu \in [\nu_0, \nu_2]$  there holds

$$\begin{aligned} c_1(\nu_0) &= c_2(\nu_0) = x_{fp}(\nu_0), \\ c_1(\nu) &\neq c_2(\nu), \quad \nu \neq \nu_0, \\ P_\nu(c_1(\nu)) &= c_2(\nu), \quad P_\nu(c_2(\nu)) = c_1(\nu), \\ \text{Fix}(P_\nu^2, X) &= \{c_1(\nu), c_2(\nu), x_{fp}(\nu)\}. \end{aligned}$$

- the dynamics:

for  $\nu \leq \nu_0$

$$\text{Inv}(X, P_\nu) = \{x_{fp}(\nu)\}.$$

For  $\nu > \nu_0$  the maximal invariant set in  $X$   $\text{Inv}(X, P_\nu)$  is equal to

$$\overline{W_X^u(x_{fp}(\nu), P)} \cap \left( \overline{W_X^s(c_1(\nu), P_\nu^2)} \cup \overline{W_X^s(c_2(\nu), P_\nu^2)} \right)$$

and is a one-dimensional connected manifold with boundary points  $c_1(\nu)$ ,  $c_2(\nu)$ .

The objects appearing in the above definition are illustrated on Fig. 1.

### 3 Derivation of the conditions for the occurrence of the period doubling bifurcation

The goal of this section is to present the set of conditions, which guarantee the existence of period doubling bifurcation for a given map, and which can be verified using the rigorous numerics. The main tools used are the Lyapunov-Schmidt reduction [7] and the implicit function theorem.

Assume that we have a parameter dependent map  $z \mapsto P(\nu, z)$ , which apparently undergoes the period doubling bifurcation as the parameter  $\nu$  changes. This means, that a period doubling bifurcation is numerically observed, but we do not assume that it exists. Let  $z_{fp}(\nu)$  be a fixed point curve for  $P$ . We assume that it is of the class  $C^k$ , with  $k \geq 3$  and we can compute it and its all derivatives.

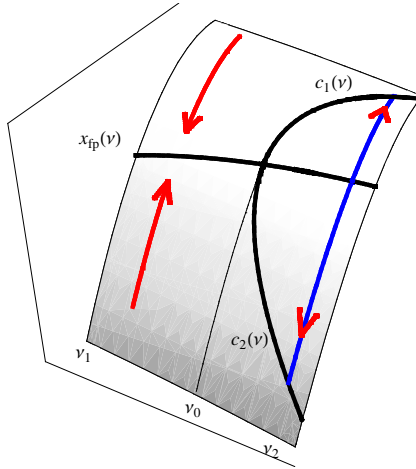


Figure 1: The dynamics in a small neighborhood of the bifurcation point as in the Definition 4. The maximal invariant set for a certain parameter value  $\nu > \nu_0$  is shown in blue (when in color). The arrows show the dynamics in a neighborhood of the bifurcation point.

To prove the existence of the period doubling bifurcation we proceed as in [7]. First we perform the Lyapunov-Schmidt reduction to obtain a function  $G : \mathbb{R} \times \mathbb{R} \supset \text{dom}(G) \rightarrow \mathbb{R}$ , whose zeros correspond to the fixed points and period two points of  $P_\nu$  and then we try to describe the solution set for the equation  $G(\nu, x) = 0$ . Next, through some additional computation of eigenvalues we will be able to decide about the hyperbolicity of bifurcating periodic orbits.

The basic steps of the Lyapunov-Schmidt reduction for  $P^2$  are:

- we choose good coordinates  $(x, y) \in \mathbb{R} \times \mathbb{R}^{n-1}$ . It is desirable to choose  $x$  in the approximate bifurcation direction (in the eigendirection corresponding to the  $-1$  eigenvalue of  $DP_\nu$  at the bifurcation point).
- let  $Z = [\nu_1, \nu_2] \times [x_1, x_2]$  and  $Y \subset \mathbb{R}^{n-1}$  be such that the apparent bifurcation point  $(\nu_0, x_0, y_0)$  belongs to the interior of  $Z \times Y$ .
- we need to show that there exists a function  $y(\nu, x)$ , defined on  $Z$  with the values in  $Y$ , such that

$$y - \pi_y(P_\nu^2(x, y)) = 0 \quad \text{for } (\nu, x, y) \in Z \times Y \text{ iff } y = y(\nu, x). \quad (1)$$

- the bifurcation function  $G : Z \rightarrow \mathbb{R}$  is defined by

$$G(\nu, x) = x - \pi_x(P_\nu^2(x, y(\nu, x))). \quad (2)$$

Now, we have to find the solution set of the following equation

$$G(\nu, x) = 0, \quad (\nu, x) \in Z. \quad (3)$$

Let  $x_{fp}(\nu) = \pi_x(z_{fp}(\nu))$  be the  $x$ -coordinate of the fixed point curve. We assume that  $[\nu_1, \nu_2] \subset \text{dom}(x_{fp})$  and  $x_{fp}([\nu_1, \nu_2]) \subset [x_1, x_2]$ . Therefore we have

$$G(\nu, x_{fp}(\nu)) = 0. \quad (4)$$

The idea of solving (3) goes as follows: we introduce a new bifurcation function

$$g(\nu, x) = \frac{G(\nu, x)}{x - x_{fp}(\nu)} \quad (5)$$

and then we solve equation  $g(\nu, x) = 0$  by the implicit function theorem.

Observe that expression (5) defining  $g(\nu, x)$  contains zero in the denominator. Moreover, usually the exact value of  $x_{fp}(\nu)$  is not known, therefore the formula (5) appears to be useless in rigorous computations. The next lemma will give us an integral representation of  $g$ , which will not contain any singularities and therefore it is well suited for the rigorous numerics.

**Lemma 1** *Assume  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  is  $C^1$ . Let  $x, y \in \mathbb{R}^n$ . Then*

$$F(x) - F(y) = \int_0^1 DF(t(x - y) + y) dt \cdot (x - y)$$

Hence we can define equivalently  $g : [\nu_1, \nu_2] \rightarrow [x_1, x_2]$  by

$$g(\nu, x) = \int_0^1 \frac{\partial G}{\partial x}(\nu, t(x - x_{fp}(\nu)) + x_{fp}(\nu)) dt. \quad (6)$$

We obtain

$$G(\nu, x) = (x - x_{fp}(\nu))g(\nu, x).$$

Therefore, we have to determine the solution set of the following equation

$$g(\nu, x) = 0 \quad (\nu, x) \in Z, \quad (7)$$

where  $g$  is defined in (6).

In the case of the period doubling bifurcation we expect solutions of (7) to form a regular curve. The following lemma gives a set of conditions, which implies this fact.

**Lemma 2** *Let  $Z = [\nu_1, \nu_2] \times [x_1, x_2]$ . Assume that  $g : Z \rightarrow \mathbb{R}$  is a  $C^k$ -function,  $k \geq 2$ .*

*Assume that*

$$\frac{\partial^2 g}{\partial x^2}(Z) > 0, \quad (8)$$

$$\frac{\partial g}{\partial \nu}(Z) < 0, \quad (9)$$

$$g(\nu_1, x) > 0, \quad \text{for } x \in [x_1, x_2], \quad (10)$$

$$g(\nu_2, x_1) > 0, \quad (11)$$

$$g(\nu_2, x_2) > 0, \quad (12)$$

$$g(\nu_2, x_0) < 0, \quad \text{for some } x_0 \in (x_1, x_2). \quad (13)$$

Then there exist  $\bar{x}_1, \bar{x}_2$ , such that  $x_1 < \bar{x}_1 < x_0 < \bar{x}_2 < x_2$  and there exists a function  $\nu : [\bar{x}_1, \bar{x}_2] \rightarrow [\nu_1, \nu_2]$  of class  $C^k$ , such that

$$\{(\nu, x) \in Z \mid g(\nu, x) = 0\} = \{(\nu(x), x), x \in [\bar{x}_1, \bar{x}_2]\}.$$

Moreover, there exists  $\bar{x}_0 \in (\bar{x}_1, \bar{x}_2)$  such that

$$\begin{aligned} \nu'(x) &> 0, & x \in (\bar{x}_0, \bar{x}_2), \\ \nu'(x) &< 0, & x \in (\bar{x}_1, \bar{x}_0), \\ \nu(x) &> \nu_1, & x \in [\bar{x}_1, \bar{x}_2], \\ \nu(\bar{x}_1) &= \nu(\bar{x}_2) = \nu_2. \end{aligned}$$

**Proof:** Observe first that from condition (8) it follows that for any given  $\nu \in [\nu_1, \nu_2]$  and any  $c \in \mathbb{R}$  the equation

$$g(\nu, x) = c,$$

has at most two solutions in  $[x_1, x_2]$ .

From this observation and equations (11–13) it follows that there exist  $\bar{x}_1$  and  $\bar{x}_2$ , such that

$$\begin{aligned} x_1 &< \bar{x}_1 < x_0 < \bar{x}_2 < x_2 \\ \{x \in [x_1, x_2] \mid g(\nu_2, x) = 0\} &= \{\bar{x}_1, \bar{x}_2\} \\ g(\nu_2, x) &> 0, & \text{for } x < \bar{x}_1 \text{ or } x > \bar{x}_2 \\ g(\nu_2, x) &< 0, & \text{for } x \in (\bar{x}_1, \bar{x}_2). \end{aligned}$$

From the above conditions and conditions (9) and (10) it follows immediately, that there exists function  $\nu : [\bar{x}_1, \bar{x}_2] \rightarrow [\nu_1, \nu_2]$ , such that

$$\{(\nu, x) \in Z \mid g(\nu, x) = 0\} = \{(\nu(x), x), x \in [\bar{x}_1, \bar{x}_2]\}.$$

By the implicit function theorem, the function  $\nu(x)$  is of class  $C^k$ .

There remains for us to show the existence of a unique minimum of  $\nu(x)$  and its monotonicity properties.

Since  $\nu(\bar{x}_1) = \nu(\bar{x}_2)$ , there exists a critical point  $\bar{x}_0 \in (\bar{x}_1, \bar{x}_2)$ . Let  $y \in (\bar{x}_1, \bar{x}_2)$  be any critical point of  $\nu(x)$ , i.e.  $\dot{\nu}(y) = 0$ . We will show that  $\ddot{\nu}(y) > 0$ .

By differentiating twice equation  $g(\nu(x), x) = 0$  we obtain

$$\begin{aligned} \frac{\partial^2 g}{\partial \nu^2}(\nu(x), x)(\dot{\nu}(x))^2 + 2 \frac{\partial^2 g}{\partial \nu \partial x}(\nu(x), x)\dot{\nu}(x) + \\ \frac{\partial g}{\partial \nu}(\nu(x), x)\ddot{\nu}(x) + \frac{\partial^2 g}{\partial x^2}(\nu(x), x) = 0 \end{aligned}$$

Therefore for  $y$  we have

$$\begin{aligned} 0 &= \frac{\partial g}{\partial \nu}(\nu(y), y)\ddot{\nu}(y) + \frac{\partial^2 g}{\partial x^2}(\nu(y), y), \\ \ddot{\nu}(y) &= - \left( \frac{\partial g}{\partial \nu}(\nu(y), y) \right)^{-1} \frac{\partial^2 g}{\partial x^2}(\nu(y), y) > 0. \end{aligned}$$

We see that all critical points are strong local minima. This implies that the set of critical points consists from just one point.  $\blacksquare$

The model for Lemma 2 is given by the function  $g_1(\nu, x) = x^2 - \nu$  in the neighborhood of the point  $(0, 0)$ . By changing signs of  $\nu$  and  $g$  we obtain the following model functions  $g_2(\nu, x) = \nu + x^2$ ,  $g_3(\nu, x) = \nu - x^2$  and  $g_4(\nu, x) = -\nu - x^2$  for which we can state analogous lemmas.

Now we can formulate a lemma based on the implicit function theorem addressing the assumptions implying intersection of curves solving equation  $G(\nu, x) = 0$ , where  $G$  arises in through the Lyapunov-Schmidt reduction in the context of the period doubling bifurcation.

**Lemma 3** *Let  $Z = [\nu_1, \nu_2] \times [x_1, x_2]$ . Assume that  $G : Z \rightarrow \mathbb{R}$  is a  $C^k$ -function, with  $k \geq 3$ .*

*Assume that there exists a  $C^k$ -function  $x_{fp} : [\nu_1, \nu_2] \rightarrow (x_1, x_2)$ , such that  $G(\nu, x_{fp}(\nu)) = 0$  for  $\nu \in [\nu_1, \nu_2]$ .*

*Assume that*

$$\frac{\partial^3 G}{\partial x^3}(Z) > 0, \quad (14)$$

$$\frac{\partial^2 G}{\partial x \partial \nu}(Z) + \frac{\partial^2 G}{\partial x^2}(Z) x'_{fp}([\nu_1, \nu_2]) \cdot [0, 1] < 0. \quad (15)$$

*We assume that following conditions are satisfied for some  $x_1 \leq \delta_1 < x_{fp}(\nu_1) < \delta_2 \leq x_2$*

$$G(\nu_1, [\delta_2, x_2]) > 0, \quad G(\nu_1, [x_1, \delta_1]) < 0, \quad (16)$$

$$\frac{\partial G}{\partial x}(\nu_1, [\delta_1, \delta_2]) > 0, \quad (17)$$

$$G(\nu_2, x_2) > 0, \quad G(\nu_2, x_1) < 0, \quad (18)$$

$$\frac{\partial G}{\partial x}(\nu_2, x_{fp}(\nu_2)) < 0. \quad (19)$$

*Then there exist  $x_1 < \bar{x}_1 < \bar{x}_2 < x_2$ , such that  $x_{fp}(\nu_2) \in (\bar{x}_1, \bar{x}_2)$  and a function  $\nu : [\bar{x}_1, \bar{x}_2] \rightarrow [\nu_1, \nu_2]$  of class  $C^{k-1}$ , such that*

$$\begin{aligned} \{(\nu, x) \in Z \mid G(\nu, x) = 0\} &= C_{fp} \cup C_{per} = \\ &= \{(\nu, x_{fp}(\nu)), \nu \in [\nu_1, \nu_2]\} \cup \{(\nu(x), x), x \in [\bar{x}_1, \bar{x}_2]\} \end{aligned} \quad (20)$$

*and the intersection of curves  $C_{fp}$  and  $C_{per}$  contains exactly one point.*

*Moreover, there exists  $\bar{x}_0 \in (\bar{x}_1, \bar{x}_2)$  such that*

$$\begin{aligned} \nu'(x) &> 0, & x \in (\bar{x}_0, \bar{x}_2), \\ \nu'(x) &< 0, & x \in (\bar{x}_1, \bar{x}_0), \\ \nu(x) &> \nu_1, & x \in [\bar{x}_1, \bar{x}_2], \\ \nu(\bar{x}_1) &= \nu(\bar{x}_2) = \nu_2. \end{aligned}$$



**Proof:** For the proof we want to apply to Lemma 2. For this end we define  $g$  as in (6).

We start by showing that (14) and (15) imply that  $\frac{\partial^2 g}{\partial x^2}(Z) > 0$  and  $\frac{\partial g}{\partial \nu}(Z) < 0$ , respectively.

We have

$$\frac{\partial^2 g}{\partial x^2}(\nu, x) = \int_0^1 \frac{\partial^3 G}{\partial x^3}(\nu, t(x - x_{fp}(\nu)) + x_{fp}(\nu)) t^2 dt.$$

Hence from (14) we obtain immediately that  $\frac{\partial^2 g}{\partial x^2}(Z) > 0$ .

$$\begin{aligned} \frac{\partial g}{\partial \nu}(\nu, x) &= \int_0^1 \left( \frac{\partial^2 G}{\partial x \partial \nu}(\nu, t(x - x_{fp}(\nu)) + x_{fp}(\nu)) + \right. \\ &\quad \left. \frac{\partial^2 G}{\partial x^2}(\nu, t(x - x_{fp}(\nu)) + x_{fp}(\nu))(1 - t)x'_{fp}(\nu) \right) dt \in \\ &\quad \frac{\partial^2 G}{\partial x \partial \nu}(Z) + \frac{\partial^2 G}{\partial x^2}(Z)x'_{fp}([\nu_1, \nu_2]) \cdot [0, 1]. \end{aligned}$$

This and (15) imply that  $\frac{\partial g}{\partial \nu}(Z) < 0$ .

To obtain condition (10) we need to split the interval  $[x_1, x_2]$  into three parts  $[x_1, \delta_1]$ ,  $[\delta_1, \delta_2]$  and  $[\delta_2, x_2]$ , so that in the middle part we have the zero of  $G(\nu_1, \cdot)$  and we need to use there the integral representation of  $g$ . On the remaining parts it is enough to verify the signs of  $G$ . Hence we see that conditions (16–17) imply (10).

The remaining assumptions in Lemma 2 follow easily from (18–19). Now we use Lemma 2 to obtain the function  $\nu(x)$  and the condition (20).

There remains for us to show that the curves  $C_{fp}$  and  $C_{per}$  defined by (20) intersect in exactly one point. Observe that these curves intersect because the curve  $C_{fp}$  cuts  $Z$  into two pieces and the end points of the second curve belong to different components, which follows directly from the fact that  $x_{fp}(\nu_2) \in (\bar{x}_1, \bar{x}_2)$ .

Now we turn to the question of the uniqueness of the intersection point.

Let  $\alpha, \beta \in [\nu_1, \nu_2]$ ,  $\alpha < \beta$ . For  $t \in [0, 1]$  let  $\nu_t = t\alpha + (1 - t)\beta$  and  $x_t = tx_{fp}(\alpha) + (1 - t)x_{fp}(\beta)$ . Observe that for each  $t \in [0, 1]$  the point  $(\nu_t, x_t)$  belongs to  $Z$ . Let  $\theta \in (\alpha, \beta)$  be such that  $x'_{fp}(\theta) = \frac{x_{fp}(\alpha) - x_{fp}(\beta)}{\alpha - \beta}$ . We have

$$\begin{aligned} \frac{\partial G}{\partial x}(\alpha, x_{fp}(\alpha)) - \frac{\partial G}{\partial x}(\beta, x_{fp}(\beta)) &= \\ \int_0^1 \left( \frac{\partial^2 G}{\partial x \partial \nu}(\nu_t, x_t)(\alpha - \beta) + \frac{\partial^2 G}{\partial x^2}(\nu_t, x_t)(x_{fp}(\alpha) - x_{fp}(\beta)) \right) dt &= \\ \left( \int_0^1 \left( \frac{\partial^2 G}{\partial x \partial \nu}(\nu_t, x_t) + \frac{\partial^2 G}{\partial x^2}(\nu_t, x_t)x'_{fp}(\theta) \right) dt \right) (\alpha - \beta) \in \\ \left( \frac{\partial^2 G}{\partial x \partial \nu}(Z) + \frac{\partial^2 G}{\partial x^2}(Z)x'_{fp}([\nu_1, \nu_2]) \cdot [0, 1] \right) (\alpha - \beta). \end{aligned}$$

Therefore, from the above computations and the assumption (15) it follows that the function  $\nu \mapsto \frac{\partial G}{\partial x}(\nu, x_{fp}(\nu))$  is injective on  $[\nu_1, \nu_2]$ . Observe that from (6) it follows that, if  $(\nu, x_{fp}(\nu)) \in C_{fp} \cap C_{per}$  then  $\frac{\partial G}{\partial x}(\nu, x_{fp}(\nu)) = 0$ , so the intersection of  $C_{fp}$  and  $C_{per}$  contains at most one point. ■

Observe that in the above lemma we cannot make the claim that the intersection point of the curves, which solve equation  $G(\nu, x) = 0$  is exactly in  $(\nu(\bar{x}_0), \bar{x}_0)$ . This can be easily seen in the following example. Let  $G(\nu, x) = (x - 1)(x^2 - \nu)$ ,  $x_1 = -2$ ,  $x_2 = 2$ ,  $\nu_1 = -1$  and  $\nu_2 = 1$ . It is easy to see that all assumptions of Lemma 3 are satisfied, but the intersection of the curves  $(\nu(x) = x^2, x)$  and  $(\nu, x(\nu) = 1)$  is not  $(0, 0)$ . On the other hand in the context of the period doubling bifurcation the intersection point is  $(\nu(\bar{x}_0), \bar{x}_0)$ , but we cannot infer such conclusion from Lemma 3 and we need to use the information about the dynamical origin of the function  $G$ . Now we state the theorem which addresses this issue.

**Theorem 4** *Let  $P_\nu : \mathbb{R}^n \supset \text{dom}(P_\nu) \rightarrow \mathbb{R}^n$ , where  $\nu \in I \subset \mathbb{R}$  be one-parameter family of maps of class  $C^k$  ( $k \geq 3$ ), both with respect to the parameter  $\nu$  and  $x \in \mathbb{R}^n$ .*

*Let  $Z = [\nu_1, \nu_2] \times [x_1, x_2]$  and  $Y \subset \mathbb{R}^{n-1}$  be a closure of an open set, such that  $[x_1, x_2] \times Y \subset \text{dom}(P_\nu^2)$  for  $\nu \in [\nu_1, \nu_2]$ . Assume that*

**A1** *for any  $(\nu, x) \in Z$  there exists a unique  $y = y(\nu, x) \in \text{int} Y$ , such that  $y - \pi_y(P_\nu^2(x, y)) = 0$ . Moreover, we assume that  $y : Z \rightarrow Y$  is  $C^k$ .*

**A2** *there exists a  $C^k$ -function  $x_{fp} : [\nu_1, \nu_2] \rightarrow (x_1, x_2)$ , such that for  $\nu \in [\nu_1, \nu_2]$  holds*

$$\text{Fix}(P_\nu, [x_1, x_2] \times Y) = \{(x_{fp}(\nu), y(\nu, x_{fp}(\nu)))\}. \quad (21)$$

**A3** *Let*

$$G(\nu, x) = x - \pi_x(P_\nu^2(x, y(\nu, x))), \quad \text{for } (\nu, x) \in Z.$$

*Assume that  $G$  and  $x_{fp}$  satisfy assumptions of Lemma 3 and let  $\bar{x}_1, \bar{x}_2, \bar{x}_0$  and  $\nu : [\bar{x}_1, \bar{x}_2] \rightarrow [\nu_1, \nu_2]$  be as in the assertion of Lemma 3.*

*Then the fixed point set of  $P_\nu^2$  for  $\nu \in [\nu_1, \nu_2]$ , i.e.*

$$\{(\nu, x, y) \in Z \times Y \mid P_\nu^2(x, y) = (x, y)\}$$

*is equal to the union of the fixed point set for  $P_\nu$*

$$\text{Per}_1 = \{(\nu, x_{fp}(\nu), y(\nu, x_{fp}(\nu))) \mid \nu \in [\nu_1, \nu_2]\}$$

*and the period-2 points set*

$$\text{Per}_2 = \{(\nu(x), x, y(\nu(x), x)) \mid x \in [\bar{x}_1, \bar{x}_2]\}.$$

*For any point  $(\nu(x), x, y(\nu(x), x)) \in \text{Per}_2$  the point  $(\nu(x), P_{\nu(x)}(x, y(\nu(x), x)))$  also belongs to  $\text{Per}_2$  and for  $\xi \in (\nu(\bar{x}_0), \nu_2]$  the set  $\text{Per}_2 \cap \{\nu = \xi\}$  is a period-2 orbit for  $P_\xi$ .*

Sets  $Per_1$  and  $Per_2$  have exactly one common point  $(\nu_b, z_b)$  given by

$$(\nu_b, z_b) = (\nu(\bar{x}_0), (\bar{x}_0, y(\nu(\bar{x}_0), \bar{x}_0))).$$

Moreover, the projections of  $Per_1$  and  $Per_2$  onto  $(\nu, x)$ -plane have exactly one common point  $(\nu_b, x_b)$  given by

$$(\nu_b, x_b) = (\nu(\bar{x}_0), \bar{x}_0).$$

**Proof:** From the construction of the bifurcation function  $G$  and our assumptions we immediately obtain that

$$\{(\nu, x, y) \in Z \times Y \mid P_\nu^2(x, y) = (x, y)\} = Per_1 \cup Per_2.$$

From Lemma 3 we know that projections onto the  $(\nu, x)$ -plane of sets  $Per_1$  and  $Per_2$  intersect exactly in one point, say  $(\bar{\nu}, \bar{x})$ . Observe that the point  $(\bar{\nu}, \bar{x}, y(\bar{\nu}, \bar{x}))$  belongs to the intersection of  $Per_1$  and  $Per_2$ .

We need to show that  $(\bar{\nu}, \bar{x}) = (\nu(\bar{x}_0), \bar{x}_0)$ . We will show that the function  $x \mapsto \nu(x)$  has a local extremum at  $\bar{x}$ . This will imply that  $\bar{x} = \bar{x}_0$ , because by Lemma 3  $\bar{x}_0$  is the only local extremum of  $\nu(x)$ .

We reason by contradiction. Let us assume that  $\nu'(\bar{x}) \neq 0$ . Let  $U = U_\nu \times U_x \times U_y$ , where  $U_\nu \subset [\nu_1, \nu_2]$ ,  $U_x \subset [x_1, x_2]$  and  $U_y \subset Y$ , be a neighborhood of  $(\bar{\nu}, \bar{x}, y(\bar{\nu}, \bar{x}))$ , such that

$$\begin{aligned} P_\nu(x, y) &\in \text{int}([x_1, x_2] \times Y), \quad \text{for } (\nu, x, y) \in U, \\ \nu(a) &\neq \nu(b), \quad \text{for } a, b \in U_x \text{ and } a \neq b. \end{aligned} \quad (22)$$

Such  $U$  exists because  $(\bar{x}, y(\bar{\nu}, \bar{x}))$  is a fixed point for  $P_{\bar{\nu}}$  and  $(\bar{x}, y(\bar{\nu}, \bar{x})) \in \text{int}([x_1, x_2] \times Y)$ .

Let us take  $v \in U_x$ , such that  $v \neq \bar{x}$ . Then  $(v, y(\nu(v), v))$  is not a fixed point for  $P_{\nu(v)}$ . Points  $(v, y(\nu(v), v))$  and  $P_{\nu(v)}(v, y(\nu(v), v))$  are different, both belong to  $Z$  and are period-2 points for  $P_{\nu(v)}$ . Therefore they both belong to  $Per_2$  and

$$\nu(\pi_x P_{\nu(v)}(v, y(\nu(v), v))) = \nu(v). \quad (23)$$

Observe that from the continuity it follows that

$$\lim_{v \rightarrow \bar{x}} \pi_x P_{\nu(v)}(v, y(\nu(v), v)) = \pi_x P_{\nu(\bar{x})}(\bar{x}, y(\nu(\bar{x}), \bar{x})) = \bar{x}.$$

From the above observation it follows that for  $v$  sufficiently close to  $\bar{x}$  the points  $v$  and  $\pi_x P_{\nu(v)}(v, y(\nu(v), v))$  are in  $U_x$ , but in this situation condition (23) contradicts (22). This proves that  $\bar{x} = \bar{x}_0$ .

It remains to prove that for any point  $(\nu(x), x, y(\nu(x), x)) \in Per_2$  the point  $(\nu(x), P_{\nu(x)}(x, y(\nu(x), x)))$  also belongs to  $Per_2$  and for  $\xi \in (\nu(\bar{x}_0), \nu_2]$  the set  $Per_2 \cap \{\nu = \xi\}$  is a period-2 orbit for  $P_\xi$ .

By Lemma 3  $Per_2$  is the union of two curves  $u, l : [\nu(\bar{x}_0), \nu_2] \rightarrow Z \times Y$ , such that

$$u(\nu(\bar{x}_0)) = l(\nu(\bar{x}_0)) = (\bar{x}_0, y(\nu(\bar{x}_0), \bar{x}_0)), \quad (24)$$

$$\pi_x u(\nu_2) = \bar{x}_2, \quad \pi_x l(\nu_2) = \bar{x}_1, \quad (25)$$

$$\frac{d\pi_x(u(\nu))}{d\nu} > 0, \quad \frac{d\pi_x(l(\nu))}{d\nu} < 0. \quad (26)$$

Let us consider the set

$$K = \{\nu \in [\nu(\bar{x}_0), \nu_2] \mid \text{for } \nu(\bar{x}_0) \leq \xi \leq \nu \text{ holds } P_\xi(u(\xi)) = l(\xi)\}. \quad (27)$$

Observe that  $\nu(\bar{x}_0) \in K$ , because  $l(\nu(\bar{x}_0)) = u(\nu(\bar{x}_0))$  is the fixed point for  $P_{\nu(\bar{x}_0)} \in \text{int}([x_1, x_2] \times Y)$ . Hence for  $\xi > \nu(\bar{x}_0)$ ,  $\xi$  close to  $\nu(\bar{x}_0)$  we have  $l(\xi) \in \text{int}([x_1, x_2] \times Y)$  and  $P_\xi(u(\xi)) \in \text{int}([x_1, x_2] \times Y)$ . Therefore  $P_\xi(u(\xi)) = l(\xi)$  and  $\xi \in K$ , because  $P_\xi(u(\xi))$  is a period-2 point in  $\text{int}([x_1, x_2] \times Y)$  different from  $u(\xi)$ . Observe that the same argument applies to  $\xi = \sup K$ . Therefore  $K = [\nu(\bar{x}_0), \nu_2]$ . ■

### 3.1 Hyperbolicity of bifurcating solutions

The Lyapunov-Schmidt projection does not give any direct information about the dynamical character of the bifurcating objects. The required information concerning the hyperbolicity is of course contained in the spectra of  $DP_\nu$  and  $DP_\nu^2$  and its derivatives. Below we present a lemma addressing this issue.

**Lemma 5** *Assume that  $P_\nu : \mathbb{R}^n \rightarrow \mathbb{R}^n$  for  $\nu \in [\nu_1, \nu_2]$  satisfies all assumptions of Theorem 4 and in the sequel we will use all the notation introduced there.*

*Let  $z_{fp}(\nu) = (x_{fp}(\nu), y(\nu, x_{fp}(\nu)))$ .*

**fixed points:** *Assume that there exists  $\epsilon > 0$ , such that for all  $\nu \in [\nu_1, \nu_2]$  holds*

$$\text{Sp}(DP_\nu(z_{fp}(\nu))) = A_\nu \cup B_\nu \cup \{\lambda(\nu)\},$$

*where  $\lambda(\nu) \in \mathbb{R}$  has multiplicity one,  $A_\nu \subset \{\alpha \in \mathbb{C}, |\alpha| < 1 - \epsilon\}$  and  $B_\nu \subset \{\beta \in \mathbb{C}, |\beta| > 1 + \epsilon\}$ . Moreover, we assume that*

$$\lambda(\nu_1) \in (-1, 1), \quad (28)$$

$$\lambda(\nu_2) < -1, \quad (29)$$

$$\frac{d\lambda}{d\nu}(z_{fp}(\nu)) < 0, \quad \nu \in [\nu_1, \nu_2]. \quad (30)$$

*Then the fixed points for  $P_\nu$  on curve  $z_{fp}(\nu)$  are hyperbolic for  $\nu \in [\nu_1, \nu_2] \setminus \{\nu(\bar{x}_0)\}$  and*

$$\dim W^u(z_{fp}(\nu^-), P_{\nu^-}) + 1 = \dim W^u(z_{fp}(\nu^+), P_{\nu^+})$$

*for any  $\nu_1 \leq \nu^- < \nu(\bar{x}_0) < \nu^+ \leq \nu_2$ .*

**period-2 points:** *Assume that there exists  $\epsilon > 0$ , such that on the  $Per_2$  curve (i.e. for  $x \in [\bar{x}_1, \bar{x}_2]$ ) holds*

$$\text{Sp}\left(DP_{\nu(x)}^2(x, y(\nu(x), x))\right) = A_x \cup B_x \cup \{\gamma(x)\}$$

where  $\gamma(x) \in \mathbb{R}$  has the multiplicity one,  $A_x \subset \{\alpha \in \mathbb{C}, |\alpha| < 1 - \epsilon\}$  and  $B_x \subset \{\beta \in \mathbb{C}, |\beta| > 1 + \epsilon\}$ . Moreover, we assume that

$$\frac{d^2\gamma}{dx^2}(x) < 0, \quad x \in [\bar{x}_1, \bar{x}_2] \quad (31)$$

$$0 < \gamma(\bar{x}_1) < 1. \quad (32)$$

Then for  $x \in [\bar{x}_1, \bar{x}_2] \setminus \{\bar{x}_0\}$  the period two points  $z_d(x) = (x, y(\nu(x), x))$  for  $P_{\nu(x)}$  are hyperbolic and

$$\begin{aligned} \gamma(\bar{x}_0) &= 1 \\ 0 < \gamma(x) &< 1, \\ \dim W^s(z_d(x), P_{\nu(x)}^2) &= \dim W^s(z_{fp}(\nu^-), P_{\nu^-}) \end{aligned}$$

for any  $\nu_1 \leq \nu^- < \nu(\bar{x}_0)$  and  $x \in [\bar{x}_1, \bar{x}_2] \setminus \{\bar{x}_0\}$ .

**Proof:** Let us consider first the fixed points part. From conditions (28–30) it follows that there exists a unique  $\bar{\nu} \in [\nu_1, \nu_2]$ , such that  $\lambda(\bar{\nu}) = -1$ . We will show that  $\bar{\nu} = \nu_b (= \nu(\bar{x}_0))$ .

We reason by the contradiction. Assume that  $\lambda(\nu_b) \neq -1$ , then for some small interval  $I$ , containing  $\nu_b$  in the interior, all the fixed points  $z_{fp}(\nu)$  will be a hyperbolic, therefore there will be no period two points for  $P_\nu$  for  $\nu \in I$  in some small neighborhood of the curve  $(\nu, z_{fp}(\nu))$ . But this contradicts the fact (see Thm. 4) that  $(\nu_b, z_b)$  is the intersection point of the fixed point branch with the period-two branch. Hence  $\lambda(\nu) = -1$  for  $\nu \in [\nu_1, \nu_2]$  iff  $\nu = \nu_b$ . Moreover, by conditions (28,30)  $\lambda(\nu) < -1$  for  $\nu_2 \geq \nu > \nu_b = \nu(\bar{x}_0)$  and  $\lambda(\nu) \in (-1, 1)$  for  $\nu_1 \leq \nu < \nu_b$ . This finishes the proof of the fixed point part of the lemma.

For the proof of the second part observe that at the bifurcation point holds

$$\lambda(\nu(\bar{x}_0)) = -1, \quad \gamma(\bar{x}_0) = \lambda(\nu(\bar{x}_0))^2 = 1. \quad (33)$$

We will argue that

$$\frac{d\gamma}{dx}(\bar{x}_0) = 0. \quad (34)$$

From Theorem 4 it follows that for  $v_1 > \bar{x}_0$  and  $v_1 - \bar{x}_0$  is sufficiently small the point  $z_2 = P_{\nu(v_1)}(v_1, y(\nu(v_1), v_1))$  also belongs to the curve  $Per_2$  and its  $x$  projection is converging to  $\bar{x}_0$  for if  $v_1 \rightarrow \bar{x}_0$ . Let  $v_2 = \pi_x z_2$ .

We have

$$v_2 < \bar{x}_0 < v_1, \quad \gamma(v_2) = \gamma(v_1). \quad (35)$$

This implies (34).

Condition (31) and (34) implies that at  $\bar{x}_0$  the function  $\gamma(x)$  achieves the maximum and since  $\gamma(\bar{x}_1) = \gamma(\bar{x}_2)$  we see that

$$\gamma(x) \in (0, 1) \quad \text{for } x \in [\bar{x}_1, \bar{x}_2] \setminus \{\bar{x}_0\}. \quad (36)$$

From the above observation it follows that the number of eigenvalues inside the unit circle is the same for the fixed points for  $\nu < \nu(\bar{x}_0)$  and the period-2 orbits. ■

## 4 Continuation

To apply the tools described in Section 3 in the part regarding the existence of the Lyapunov-Schmidt reduction we need to prove the existence and uniqueness (locally) of a solution of the equation of the form  $f(a, y) = 0$  for a given  $a$ , where  $y \in \mathbb{R}^n$  and  $a$  is a parameter. Similarly, when continuing the fixed point curve or period-2 point curve we have to solve the existence and the local uniqueness of the solution of  $x - P^i(a, x) = 0$ , where  $a$  is the parameter. It turns out that both of the above mentioned tasks, can be handled by the same tools.

In this section we will discuss such tools, the first one consists of classical interval analysis tools: interval Newton method [1, 20, 22] and Krawczyk method [1, 15, 22], which can be seen as clever interval versions of the standard Newton method. These methods work very efficiently in the situation, where the solution sought is well isolated from other solutions and it requires  $C^1$ -estimates, only. The second approach, which is based on the implicit function theorem deals with situation, when we are close to the bifurcation point and therefore there are several solutions close to one another, as in the case of the period doubling we have the fixed point and period two points in a small neighborhood.

### 4.1 Two methods for proving the existence of zeros for a map.

Let  $A \subset \mathbb{R}^n$ . By  $[A]_I$  we will denote an interval enclosure of the set  $A$ , i.e. the smallest set of the form  $[A]_I = [a_1, b_1] \times \cdots \times [a_n, b_n]$ , such that  $A \subset [A]_I$ , where  $a_i, b_i \in \mathbb{R} \cup \{\pm\infty\}$ . By the interval matrix  $M$  we mean a subset of  $\mathbb{R}^{n \times n}$  such that  $[M]_I = M$ . The inverse of an interval matrix  $M$  is defined by

$$M^{-1} = [C^{-1} : C \in M]_I$$

provided each  $C \in M$  is invertible.

**Theorem 6 (Interval Newton Method [1, 20, 22])** *Let  $X \subset \mathbb{R}^n$  be a convex, compact set,  $f: X \rightarrow \mathbb{R}^n$  be smooth and fix a point  $x \in X$ . Let us denote by*

$$N(f, X, x) = \left[ x - [Df(X)]_I^{-1} f(x) \right]_I \quad (37)$$

*the Interval Newton Operator for a map  $f$  on set  $X$  with fixed  $x \in X$ . Then*

- *if  $N(f, X, x) \subset \text{int } X$  then the map  $f$  has unique zero in  $X$ . Moreover, if  $x_*$  is such unique zero of  $f$  in  $X$  then  $x_* \in N(f, X, x)$ .*
- *if  $N(f, X, x) \cap X = \emptyset$  then the map  $f$  has no zeros in  $X$ .*

**Theorem 7 (Interval Krawczyk Method [1, 15, 22])** *Let  $X \subset \mathbb{R}^n$  be a convex, compact set,  $f: X \rightarrow \mathbb{R}^n$  be smooth and fix a point  $x \in X$ . Let  $C \in \mathbb{R}^{n \times n}$  be an isomorphism. Let us denote by*

$$K(f, C, X, x) = [x - Cf(x) + (Id - C \cdot [Df(X)]_I)(X - x)]_I \quad (38)$$

the Interval Krawczyk Operator for a map  $f$  on set  $X$  with fixed  $x \in X$  and matrix  $C$ . Then

- if  $K(f, C, X, x) \subset \text{int } X$  then the map  $f$  has unique zero in  $X$ . Moreover, if  $x_*$  is such unique zero of  $f$  in  $X$  then  $x_* \in K(f, C, X, x)$ .
- if  $K(f, C, X, x) \cap X = \emptyset$  then the map  $f$  has no zeros in  $X$ .

## 4.2 Continuation close to the bifurcation point

The next lemma gives us a tool for extending the curves of period two points further from the bifurcation point. Schematically the situation is shown on Fig. 2.

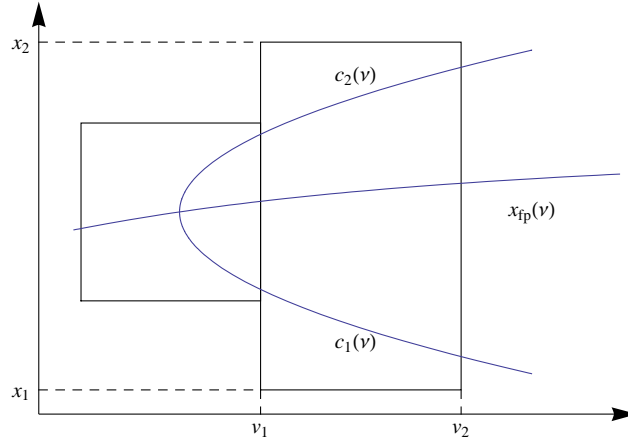


Figure 2: Location of the objects which appear in the Lemma 8.

**Lemma 8** Let  $X = [x_1, x_2]$ . Assume  $f_\nu: \mathbb{R} \times \mathbb{R}^{n-1} \supset X \times Y \rightarrow \mathbb{R}^n$ ,  $\nu \in [\nu_1, \nu_2]$  is  $C^k$  function both with respect to argument and parameter, with  $k \geq 3$ , such that

1. for  $\nu \in [\nu_1, \nu_2]$  there exists unique fixed point  $(x_{fp}(\nu), y_{fp}(\nu))$  for  $f_\nu$  in  $X \times Y$ ,
2. for all  $(\nu, x) \in [\nu_1, \nu_2] \times X$  there exists unique  $y(\nu, x) \in \text{int } Y$  solving equation  $y - \pi_y(f_\nu^2(x, y)) = 0$  and the map  $y: [\nu_1, \nu_2] \times X \rightarrow Y$  is of class  $C^k$ ,

3. the map  $G(\nu, x) = x - \pi_x(f_\nu^2(x, y(\nu, x)))$  satisfies

$$\frac{\partial^3 G}{\partial x^3}(\nu, x) > 0, \quad \text{for } \nu \in [\nu_1, \nu_2], x \in X, \quad (39)$$

$$G(\nu, x_1) < 0, \quad G(\nu, x_2) > 0, \quad \text{for } \nu \in [\nu_1, \nu_2], \quad (40)$$

$$\forall \nu \in [\nu_1, \nu_2] \exists x_- \in (x_{fp}(\nu), x_2) \quad G(\nu, x_-) < 0, \quad (41)$$

$$\forall \nu \in [\nu_1, \nu_2] \exists x_+ \in (x_1, x_{fp}(\nu)) \quad G(\nu, x_+) > 0. \quad (42)$$

Then there exist two  $C^k$  curves  $c_1, c_2: [\nu_1, \nu_2] \rightarrow \mathbb{R}^n$  such that for  $\nu \in [\nu_1, \nu_2]$  holds  $\pi_x(c_1(\nu)) < x_{fp}(\nu) < \pi_x(c_2(\nu))$  and  $c_i(\nu)$  is a period two point for  $f_\nu$ ,  $i = 1, 2$ .

Moreover, if for some  $\nu_0 \in \{\nu_1, \nu_2\}$  holds  $f_{\nu_0}(c_1(\nu_0)) = c_2(\nu_0)$  or  $f_{\nu_0}(c_2(\nu_0)) = c_1(\nu_0)$  then for all  $\nu \in [\nu_1, \nu_2]$

$$f_\nu(c_1(\nu)) = c_2(\nu), \quad f_\nu(c_2(\nu)) = c_1(\nu). \quad (43)$$

**Proof:** The second assumption and (39) imply that for a fixed  $\nu$  the map  $f_\nu^2$  has at most three fixed points in  $X \times Y$ . From the first assumption we know that  $f_\nu$  has unique fixed point  $(x_{fp}(\nu), y_{fp}(\nu))$  in  $X \times Y$ . Therefore any zero of  $G(\nu, \cdot)$ , which is different from  $(\nu, x_{fp}(\nu))$  corresponds to a period two point of  $f_\nu$ . From the continuity of  $G$  and from (40–42) it follows that  $G$  has one zero in each of the intervals  $x_{per}^1(\nu) \in (x_1, x_{fp}(\nu))$  and  $x_{per}^2(\nu) \in (x_{fp}(\nu), x_2)$ . It is easy to see that functions  $x_{per}^i$  are continuous for  $i = 1, 2$ . We set  $c_i(\nu) = (x_{per}^i(\nu), y(\nu, x_{per}^i(\nu)))$ .

We will show the smoothness of  $x_{per}^i$ , which together with assumption that  $y(x, \nu)$  is  $C^k$  implies the smoothness of  $c_i$ . It is enough to show that

$$\frac{\partial G}{\partial x}(\nu, x_{per}^i(\nu)) \neq 0,$$

because then we can apply the implicit function theorem to obtain the required differentiability. Let us fix  $\nu \in [\nu_1, \nu_2]$ . Observe that condition (39) implies that for any fixed  $\nu$  the function  $x \mapsto \frac{\partial G}{\partial x}(\nu, x)$  has at most two zeros in  $[x_1, x_2]$ . From remaining assumptions it is clear that on interval  $[x_1, x_{fp}(\nu)]$  and  $[x_{fp}(\nu), x_2]$  the function  $x \mapsto G(\nu, x)$  has strictly positive maximum and strictly negative minimum, respectively. Therefore these extremal points are zeros of  $\frac{\partial G}{\partial x}(\nu, x)$  and obviously they are different from points  $x_{per}^i(\nu)$ , which are zeros of  $G(\nu, \cdot)$ . Hence we have shown that  $\frac{\partial G}{\partial x}(\nu, x_{per}^i(\nu)) \neq 0$ .

Assume that  $\nu_0 = \nu_1$  (the other case is analogous). From the implicit function theorem it follows that for some  $\nu' > \nu_1$  condition (43) is satisfied for  $\nu_1 \leq \nu < \nu'$ . Let  $\nu_m$  be supremum of such  $\nu' \leq \nu_2$ . It is easy to see that  $\nu_m = \nu_2$ , because at  $\nu_m$  is also satisfied by the continuity and implicit function theorem allows us to extend the range of  $\nu$  satisfying (43) to the right if  $\nu_m < \nu_2$ .  $\blacksquare$



## 5 Extracting the dynamical information from the Lyapunov-Schmidt reduction

As was mentioned already in Section 3.1 the Lyapunov-Schmidt projection does not give us any direct information about the dynamics of bifurcating solutions regarding the invariant manifolds of the bifurcating objects as required by Def. 4. In this section, following the ideas of de Oliveira and Hale [10, 24], we show that the information obtained from the Lyapunov-Schmidt reduction and the spectrum of the bifurcating fixed point curve is enough to say precisely, what is the dynamics in the neighborhood of the bifurcation point.

Our argument follows the ideas from [7, Chapter 9, Thm. 3.1 and 4.2], where an analogous problem was considered for fixed points for ODEs and periodic orbits for periodically forced ODEs. The notion of the central manifold [14] plays a crucial role in this proof.

**Theorem 9** *Let  $P_\nu : \mathbb{R}^n \rightarrow \mathbb{R}^n$  for  $\nu \in [\nu_1, \nu_2]$  be a  $C^k$ -map ( $k \geq 3$ ) both with respect to  $\nu$  and its arguments. Assume that on the set  $K = [\nu_1, \nu_2] \times [x_1, x_2] \times Y$ , where  $Y \subset \mathbb{R}^{n-1}$  is a closure of an open set, we were able to perform the Lyapunov-Schmidt reduction and verify assumptions of Theorem 4. Let  $(\nu_b, z_b)$  be the bifurcation point and  $z_{fp}(\nu) = (x_{fp}(\nu), y(\nu, x_{fp}(\nu)))$  be the fixed point curve for  $P_\nu$  in  $K$ .*

*Let  $v$  be the eigenvector of  $DP_{\nu_b}(z_b)$  corresponding to the eigenvalue  $-1$ . We assume that  $\pi_x v \neq 0$ .*

*Assume that there exists  $\epsilon > 0$ , such that for all  $\nu \in [\nu_1, \nu_2]$  holds*

$$Sp(DP_\nu(z_{fp}(\nu))) = A_\nu \cup B_\nu \cup \{\lambda(\nu)\},$$

*where  $\lambda(\nu) \in \mathbb{R}$  has the multiplicity one,  $A_\nu \subset \{\alpha \in \mathbb{C}, |\alpha| < 1 - \epsilon\}$  and  $B_\nu \subset \{\beta \in \mathbb{C}, |\beta| > 1 + \epsilon\}$ . Moreover, we assume that*

$$\lambda(\nu_1) \in (-1, 1)$$

$$\lambda(\nu_2) < -1.$$

*Then the map  $P$  has a period doubling bifurcation at  $(\nu_b, x_b, y(\nu_b, x_b))$ .*

**Proof:** Let  $\nu : [\bar{x}_1, \bar{x}_2] \rightarrow [\nu_1, \nu_2]$  be the function from assumption A3 of Theorem 4 (in fact of Lemma 3), which is assumed to be satisfied. In the notation used in Theorem 4 we have  $(\nu_b, z_b) = (\nu(\bar{x}_0), (\bar{x}_0, y(\nu(\bar{x}_0), \bar{x}_0)))$ . Let  $c_1(\nu)$  and  $c_2(\nu)$  be, respectively, lower and upper branch of the graph of the function  $x \rightarrow \nu(x)$  giving period-2 points – see Fig. 3.

Let us define a map  $H : K \rightarrow \mathbb{R} \times \mathbb{R}^n$  by

$$H(\nu, z) = (\nu, P(\nu, z)).$$

Consider the spectrum of  $DH(\nu_b, z_b)$ . It is easy to see that  $+1$  is an eigenvalue of  $DH(\nu_b, z_b)$  of multiplicity one,  $\lambda = -1$  has also multiplicity one and all other eigenvalues are off the unit circle.

We apply the center manifold theorem [14, 8, 12] to  $H$  in the neighborhood of  $(\nu_b, z_b)$ . Therefore, there exists a neighborhood  $M$  of  $(\nu_b, z_b)$  and a two-dimensional center manifold  $W^c \subset M$  such that

$$\begin{aligned} \forall (\nu, z) \in W^c \quad \text{if } H^i(\nu, z) \in M \text{ then } H(\nu, z) \in W^c, \text{ for } i = -1, 1, \\ \text{Inv}(M, H) \subset W^c, \end{aligned}$$

$W^c$  is tangent at  $(\nu_b, z_b)$  to the subspace spanned by vectors  $\{(1, 0), (0, v)\} \subset \mathbb{R} \times \mathbb{R}^n$ . Observe that from our assumption about  $v$ , i.e.  $\pi_x(v) \neq 0$ , it follows that we can use on  $W^c$  in the neighborhood of  $(\nu_b, z_b)$  the same coordinates  $(\nu, x)$  as in the Lyapunov-Schmidt reduction. There exists a neighborhood of  $(\nu_b, z_b)$  denoted by  $U = [\tilde{\nu}_1, \tilde{\nu}_2] \times [\tilde{x}_1, \tilde{x}_2] \times \tilde{Y} \subset M \cap K$  and  $C^k$ -functions  $h : [\tilde{\nu}_1, \tilde{\nu}_2] \times [\tilde{x}_1, \tilde{x}_2] \rightarrow \tilde{Y}$  and  $f : [\tilde{\nu}_1, \tilde{\nu}_2] \times [\tilde{x}_1, \tilde{x}_2] \rightarrow \mathbb{R}$  satisfying

$$\begin{aligned} W^c &= \{(\nu, x, h(\nu, x))\}, \\ P(\nu, x, h(\nu, x)) &= (f(\nu, x), h(\nu, f(\nu, x))), \\ \text{Inv}(U, H) &\subset W^c. \end{aligned} \tag{44}$$

Let us stress that the dynamics of  $P_\nu$  in  $W^c$  is one-dimensional, namely that of  $x \mapsto f(\nu, x)$ .

From the Lyapunov-Schmidt reduction, we know that a point  $(\nu, x, y) \in U$  has period one or two with respect to map  $H$  iff  $y = y(\nu, x)$  and  $G(\nu, x) = 0$ . Let  $N = [\tilde{\nu}_1, \tilde{\nu}_2] \times [\tilde{x}_1, \tilde{x}_2]$ . If  $U$  is chosen to be sufficiently close to the bifurcation point, then the set  $N \setminus \{(\nu, x) \mid G(\nu, x) = 0\}$  has four connected components – see Fig 3. Namely,

$$\begin{aligned} A_1 &= \{(\nu, x) \in N \mid ((\nu \leq \nu_b) \text{ and } (x < x_{fp}(\nu))) \text{ or} \\ &\quad ((\nu > \nu_b) \text{ and } (x < c_1(\nu)))\}, \\ A_2 &= \{(\nu, x) \in N \mid ((\nu \leq \nu_b) \text{ and } (x > x_{fp}(\nu))) \text{ or} \\ &\quad ((\nu > \nu_b) \text{ and } (x > c_2(\nu)))\}, \\ B_1 &= \{(\nu, x) \in N \mid (\nu > \nu_b) \text{ and } (x_{fp}(\nu) > x > c_1(\nu))\}, \\ B_2 &= \{(\nu, x) \in N \mid (\nu > \nu_b) \text{ and } (x_{fp}(\nu) < x < c_2(\nu))\}. \end{aligned}$$

We also require that

$$\tilde{x}_1 < c_1(\tilde{\nu}_2) < c_2(\tilde{\nu}_2) < \tilde{x}_2. \tag{45}$$

Such a neighborhood  $N$  exists, since we assumed that Theorem 4 applies. Therefore zeros of the function  $G$  in the neighborhood of the bifurcation point are two intersecting curves, as presented on Fig. 3.

On each of these components the function  $d(\nu, x) = x - f(\nu, f(\nu, x))$  must have a constant sign. Observe that on  $A_2$  we have

$$x - f(\nu, f(\nu, x)) > 0, \quad \text{for } (\nu, x) \in A_2 \tag{46}$$

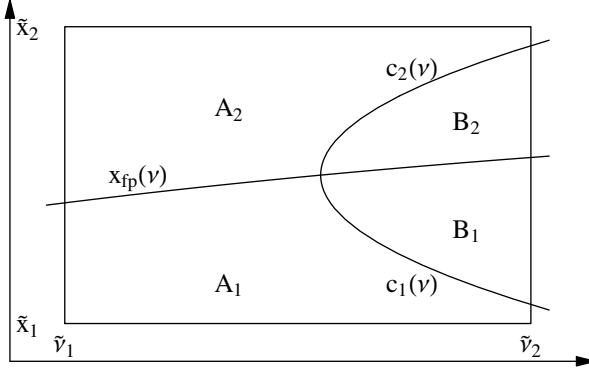


Figure 3: Location of sets  $A_i, B_i$  with respect to zeros of  $G$ .

because  $z_{fp}(\tilde{\nu}_1)$  is attracting on  $W^c$  and we consider the second iterate. Analogously we obtain

$$x - f(\nu, f(\nu, x)) < 0, \quad \text{for } (\nu, x) \in A_1. \quad (47)$$

For the component  $B_2$  we have

$$x - f(\nu, f(\nu, x)) < 0, \quad \text{for } (\nu, x) \in B_2,$$

because  $x_{fp}(\tilde{\nu}_2)$  is repelling on  $W^c$  and we consider the second iterate. Analogously

$$x - f(\nu, f(\nu, x)) > 0, \quad \text{for } (\nu, x) \in B_1.$$

For a subset  $Z \subset N$  by  $Z_\nu$  we will denote  $Z_\nu = \{x : (\nu, x) \in Z\}$ . Observe that for each  $\nu \in [\tilde{\nu}_1, \tilde{\nu}_2]$  holds

$$f_\nu^2((A_i)_\nu) \cap [\tilde{x}_1, \tilde{x}_2] \subset (A_i)_\nu, \quad f_\nu^2((B_i)_\nu) \cap [\tilde{x}_1, \tilde{x}_2] \subset (B_i)_\nu \quad i = 1, 2. \quad (48)$$

For the proof of (48) observe that map  $l(\nu, x) = (\nu, f(\nu, x))$  maps connected components of  $N \setminus \{G(\nu, x) = 0\}$  into connected components, i.e. for any  $S \in \{A_1, A_2, B_1, B_2\}$  there exists  $T = T(S) \in \{A_1, A_2, B_1, B_2\}$ , such that

$$(l(S) \cap N) \subset T(S), \quad (49)$$

because

$$l(G^{-1}(0) \cap N) \cap N = G^{-1}(0) \cap N = l^{-1}(G^{-1}(0) \cap N) \cap N.$$

Observe that the relevant eigenvalue of  $DP_\nu$  at  $z_{fp}(\nu)$  describing the dynamics on  $W^c$  is  $\lambda(\nu)$ , which is real and since we consider the second iterate we see that in the neighborhood of the fixed point curve we have points mapped into the same component. This together with (49) proves (48).

From the above considerations we obtain for  $\nu \leq \nu_b$

$$\begin{aligned} x_{fp}(\nu) < f(\nu, f(\nu, x)) < x, & \quad \text{for } x_{fp}(\nu) < x \leq \tilde{x}_2, \\ x_{fp}(\nu) > f(\nu, f(\nu, x)) > x, & \quad \text{for } x_{fp}(\nu) > x \geq \tilde{x}_1. \end{aligned}$$

The above conditions, (44) and nonexistence of other period two points in  $U$  imply that

$$\text{Inv}(\pi_{x,y}U, P_\nu) = \{z_{fp}(\nu)\}, \quad \text{for } \nu \in [\tilde{\nu}, \nu_b].$$

For  $\nu \in (\nu_b, \tilde{\nu}_2]$  we have

$$\begin{aligned} x_{fp}(\nu) < x < f(\nu, f(\nu, x)) < c_2(\nu), & \quad \text{for } x_{fp}(\nu) < x < c_2(\nu), \\ x_{fp}(\nu) > x > f(\nu, f(\nu, x)) > c_1(\nu), & \quad \text{for } x_{fp}(\nu) > x > c_1(\nu), \\ c_2(\nu) < f(\nu, f(\nu, x)) < x, & \quad \text{for } x > c_2(\nu), \\ c_1(\nu) > f(\nu, f(\nu, x)) > x, & \quad \text{for } x < c_1(\nu). \end{aligned}$$

The above conditions, conditions (44–45) and nonexistence of other period two points in  $U$  imply that for  $\nu \in (\nu_b, \tilde{\nu}_2]$

$$\text{Inv}(\pi_{x,y}U, P_\nu) = \{(x, h(\nu, x)) \mid x \in [c_1(\nu), c_2(\nu)]\}.$$

■

We would like to stress here that, contrary to all previous theorems and lemmas, in the proof of the above theorem we prove the statements about the invariant manifold of bifurcating orbits from Definition 4 on some set  $U$ , whose size we do not control, whereas it is given by the range of the existence of the central manifold. In principle, this range can be inferred from the proof of the center manifold theorem, but it will be an interesting task to develop a computable approach, which will allow to rigorously prove these facts on the whole set  $V$ . Such task will require explicit estimates about the central manifold in the region very close to the bifurcation and some other tools, may be of Conley index type [18], further away from the bifurcation.

## 6 Application to the Rössler system.

Consider an autonomous ODE in  $\mathbb{R}^3$  called the Rössler system [26]

$$\begin{cases} \dot{x} &= -y - z \\ \dot{y} &= x + by \\ \dot{z} &= b + z(x - a) \end{cases} \quad (50)$$

The classical parameter values (considered by Rössler) are  $a = 5.7$  and  $b = 0.2$ . For the remainder of this paper we fix  $b = 0.2$ .

The system (50) has been extensively studied in the literature and is treated as one of the classical examples of systems generating chaotic attractors. Yet, the number of rigorous results concerning it is very small. In Fig. 4 we show a

numerically obtained bifurcation diagram for periodic orbits on section  $x = 0$  with  $b = 0.2$  and  $a$  as parameter. We see that, when the parameter  $a$  increases from 2 to 5.7, one observes a cascade of period doubling bifurcations. In Fig. 5 we show some periodic orbits for different values of  $a$ . Our goal in this section is to validate the part of the bifurcation diagram in Fig.4 containing two first period doublings using the approach introduced in the previous sections.

Let us list the few known rigorous results about (50). Pilarczyk (see [25] and references given there) gave a computer assisted proof of the following facts: for  $a = 2.2$  there exists a periodic orbit, for  $a = 3.1$  there exists two periodic orbits. However from his proof one cannot infer any information about the dynamical character of these orbits. He constructs suitable isolating neighborhoods, which have an index of an attracting or a hyperbolic orbit with one unstable direction, but no such claim can be made about the periodic orbit proved to exist. In fact we do not even know, whether this orbit is unique.

Finally, for the classical parameter values  $b = 0.2$  and  $a = 5.7$  the system is chaotic [34] in the following sense: a suitable Poincaré map has an invariant set  $S$  and the dynamics on  $S$  for the second iterate of the Poincaré map is semiconjugated to the full shift map on two symbols.

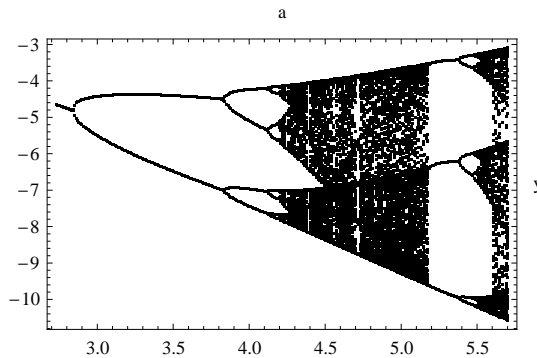


Figure 4: Bifurcation diagram for the Rössler system.

Before proceeding any further we need to introduce some notation. Let  $\Pi = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0, x' > 0\}$  be a Poincaré section. Since for  $u \in \Pi$  the first coordinate is equal to zero we will use the remaining two coordinates  $u = (y, z)$  to represent a point on  $\Pi$ . For a fixed parameter value  $a > 0$  by  $P_a = (P_{a,y}, P_{a,z}): \Pi \rightarrow \Pi$  we will denote the corresponding Poincaré return map. By  $P$  we will denote the map defined by  $P(a, y, z) = (a, P_{a,y}(y, z), P_{a,z}(y, z))$ .

Apparently the first period doubling bifurcation is observed for  $a \approx 2.832445$  and the second one for  $a \approx 3.837358$ . In the remainder of this section we discuss the computer assisted proof of the existence of both these bifurcations. In our presentation we will discuss the first one more in details, while for the second one we will just state relevant lemmas and estimates.

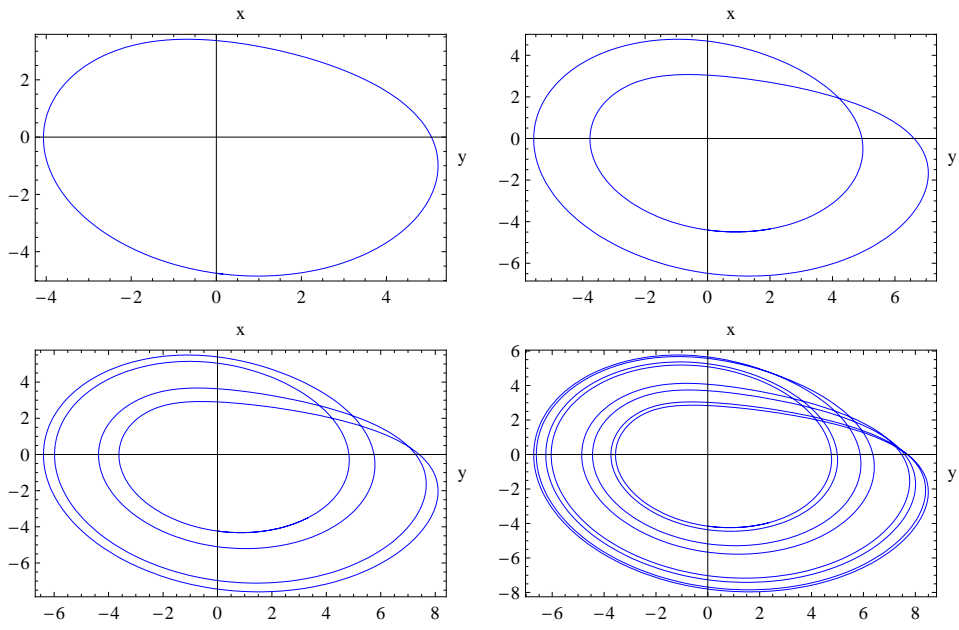


Figure 5: Periodic orbits corresponding to fixed point, period two point, period four point and period eight point for the Poincaré map of the Rössler system (50). Parameter values are  $a = 2.8$ ,  $a = 3.5$ ,  $a = 4$  and  $a = 4.2$ .

Let  $u_0 = (y_0, z_0)$  be an approximate fixed point for  $P_{a_0}$ , i.e. we set

$$u_0 = (y_0, z_0) = (-4.7946653021070986256, 0.052488098609082899093) \quad (51)$$

and put

$$M = \begin{bmatrix} 0.99999765967819775891 & -0.9582095926217468751 \\ 0.0021634782474835700244 & -0.28606708410382636343 \end{bmatrix} \quad (52)$$

The columns of  $M$  are normalized approximate eigenvectors of  $DP_{a_0}(u_0)$ , where first column corresponds to the eigenvalue close to  $-1$  and the second one to the eigenvalue close to zero. On section II we choose new coordinates  $(\tilde{y}, \tilde{z}) = M^{-1}((y, z) - u_0)$  and since, in the sequel, we will use only the new coordinates we will drop the tilde.

Define

$$\begin{aligned} A &= [a_1, a_2] = [2.83244, 2.832446] \\ Y &= [y_1, y_2] = 1.3107 \cdot [-1, 1] \cdot 10^{-3} \\ Z &= [z_1, z_2] = 1.3107 \cdot [-1, 1] \cdot 10^{-4} \end{aligned} \quad (53)$$

Now our goal is present the proof of the following theorem.

**Theorem 10** *The map  $P_a$  has a period doubling bifurcation at some point  $(a, y, z) \in \text{int}(A \times Y \times Z)$ .*

**Remark 11** *The existence of period doubling bifurcation is a local phenomenon. In fact the sets  $A, Y, Z$  can be chosen to be smaller which speed up the proof (13 minutes versus 87 minutes), namely we were able to prove the existence of period doubling bifurcation in the set*

$$\begin{aligned} A &= [a_1, a_2] = [2.83244, 2.832445028] \\ Y &= [y_1, y_2] = [-1, 1] \cdot 10^{-4} \\ Z &= [z_1, z_2] = [-1, 1] \cdot 10^{-5} \end{aligned}$$

*However, the choice of a larger set facilitates the proof of the existence of the connecting branch of period two points between first and second period doubling bifurcation, because decreasing  $a_2$  results in the eigenvalue of period-two points to be very close to 1, which makes it very difficult to rigorously continue it.*

## 6.1 The existence of fixed point curve.

**Lemma 12** *There exists a function  $(y_{fp}, z_{fp}): A \rightarrow Y \times Z$  of class  $C^\infty$  such that for  $(a, y, z) \in A \times Y \times Z$  there holds*

$$P_a(y, z) = (y, z) \quad \text{iff} \quad (y, z) = (y_{fp}(a), z_{fp}(a))$$

and

$$y'_{fp}(A) \subset [-1.3336825610133946629, -1.3275439332565022177] \quad (54)$$

**Proof:** The proof, which is computer assisted, consists from two parts, in the first one we prove the existence of the fixed point curve and in the second part we establish the estimate (54).

For the first part, we use the Interval Newton Method (Theorem 6) and the  $C^1$ -Lohner algorithm [32] to prove that for  $a \in A$  there exists a unique fixed point  $(y_{fp}(a), z_{fp}(a))$  for  $P_a$  in  $Y \times Z$ . In computations we insert the whole set  $A \times Y \times Z$  as an initial condition in our routine, which computes the Interval Newton Operator and obtain that the for all  $a \in A$  the fixed point  $(y_{fp}(a), z_{fp}(a))$  belongs to the set

$$N := N(\text{Id} - P_a, Y \times Z) = \left[ \begin{array}{l} [-2.838378938597049559, 3.2727784971172813446] \cdot 10^{-5} \\ [-4.8121450471307824034, 4.2979575521536656939] \cdot 10^{-6} \end{array} \right]^T \quad (55)$$

To obtain (54) we apply  $C^1$ -Lohner algorithm [32] to the system

$$\begin{cases} \dot{x} &= -y - z \\ \dot{y} &= x + by \\ \dot{z} &= b + z(x - a) \\ \dot{a} &= 0 \end{cases} \quad (56)$$

with  $b = 0.2$  in order to compute a bound for  $y'_{fp}(A)$ . Differentiating

$$P(a, y_{fp}(a), z_{fp}(a)) = (a, y_{fp}(a), z_{fp}(a))$$

with respect to  $a$  we obtain

$$y'_{fp} = \frac{\frac{\partial P_{a,y}}{\partial a} \left(1 - \frac{\partial P_{a,z}}{\partial z}\right) + \frac{\partial P_{a,z}}{\partial a} \cdot \frac{\partial P_{a,y}}{\partial z}}{\left(\frac{\partial P_{a,y}}{\partial y} - 1\right) \left(\frac{\partial P_{a,z}}{\partial z} - 1\right) - \frac{\partial P_{a,y}}{\partial z} \frac{\partial P_{a,z}}{\partial y}} \quad (57)$$

where the partial derivatives of  $P$  are evaluated at  $(y_{fp}(a), z_{fp}(a))$ .

We use the set  $A \times N$ , where  $N$  is defined in (55) as initial condition in our routine which computes partial derivatives of  $P$  and after substituting them to (57) we obtain a bound for  $y'_{fp}$  as in (54).

We used the Taylor method of order 14 and the time step equal to 0.02 to integrate the system (50) in  $\mathbb{R}^3$  for the first part of the proof and the order 10 and the time step 0.01 when we integrate the extended system (56) in the second part.  $\blacksquare$

**Lemma 13** *The eigenvalues  $\lambda_1, \lambda_2 : A \rightarrow \mathbb{R}$  of  $DP_a(y_{fp}(a), z_{fp}(a))$  are given by*

$$\begin{aligned} \lambda_1(a) &= \frac{1}{2} \left( \frac{\partial P_{a,y}}{\partial y} + \frac{\partial P_{a,z}}{\partial z} - s(a) \right), \\ \lambda_2(a) &= \frac{1}{2} \left( \frac{\partial P_{a,y}}{\partial y} + \frac{\partial P_{a,z}}{\partial z} + s(a) \right), \\ s(a) &= \sqrt{\left( \frac{\partial P_{a,y}}{\partial y} - \frac{\partial P_{a,z}}{\partial z} \right)^2 + 4 \frac{\partial P_{a,y}}{\partial z} \frac{\partial P_{a,z}}{\partial y}} \end{aligned}$$



where the partial derivatives of  $P$  are evaluated at  $(y_{fp}(a), z_{fp}(a))$ . Let  $v(a)$  be the normalized eigenvector corresponding to eigenvalue  $\lambda_1(a)$ . Then

$$\begin{aligned}\lambda_1(a_1) &\in [-0.99999781944914578613, -0.99999548919217751131] \\ \lambda_1(a_2) &\in [-1.00000064581599335217, -1.00000064581598072628] \\ \lambda_2(A) &\subset [-0.0013533261367103342071, 0.0013530378340487671934] \\ \lambda'_1(A) &\subset [-0.70107900728585614836, -0.62770519734197127715] \\ v_y(A) &\subset \pm[0.99728887963031764841, 1.0027184248801992439]\end{aligned}$$

where  $v_y$  denotes the  $y$  coordinate of  $v$ .

**Proof:** We leave the derivation of formulas for  $\lambda_1, \lambda_2$  to the reader. We used the  $C^1$ -Lohner algorithm applied to the system (50) in order to compute bounds for  $\lambda_1(a_1)$  and  $\lambda_1(a_2)$ . Since the parameter  $a_2$  has been chosen to be very close to the bifurcation parameter we find difficulties with the verification of condition  $\lambda_1(a_2) < -1$  in computations performed in interval arithmetics based on double precision (52-bit mantissa) boundary value type. In our computations we used interval arithmetics based on float numbers with 150-bit mantissa (MPFR [19] and GMP [9] packages).

Since the eigenvalue  $\lambda_1(a)$  of  $DP_a(y_{fp}(a), z_{fp}(a))$  is given by an explicit formula one can express  $\lambda'_1(a)$  in terms of first and second order partial derivatives of  $P$ . We obtain

$$\begin{aligned}\lambda'_1(a) &= \frac{1}{2} \left( \frac{d}{da} \frac{\partial P_{a,y}}{\partial y} + \frac{d}{da} \frac{\partial P_{a,z}}{\partial z} - s'(a) \right), \\ s'(a) &= \frac{1}{s(a)} \left( \frac{\partial P_{a,y}}{\partial y} - \frac{\partial P_{a,z}}{\partial z} \right) \left( \frac{d}{da} \frac{\partial P_{a,y}}{\partial y} - \frac{d}{da} \frac{\partial P_{a,z}}{\partial z} \right) \\ &\quad + \frac{2}{s(a)} \left( \left( \frac{d}{da} \frac{\partial P_{a,y}}{\partial z} \right) \frac{\partial P_{a,z}}{\partial y} + \left( \frac{d}{da} \frac{\partial P_{a,z}}{\partial y} \right) \frac{\partial P_{a,y}}{\partial z} \right),\end{aligned}$$

where the symbols  $\frac{d}{da} \frac{\partial P_{a,y}}{\partial z}$  and  $\frac{d}{da} \frac{\partial P_{a,z}}{\partial y}$  should be understood as

$$\begin{aligned}\frac{d}{da} \frac{\partial P_{a,z}}{\partial y}(y_{fp}(a), z_{fp}(a)) &= \frac{\partial^2 P_{a,z}}{\partial a \partial y}(y_{fp}(a), z_{fp}(a)) \\ &\quad + \frac{\partial^2 P_{a,z}}{\partial y^2}(y_{fp}(a), z_{fp}(a)) y'_{fp}(a) \\ &\quad + \frac{\partial^2 P_{a,z}}{\partial y \partial z}(y_{fp}(a), z_{fp}(a)) z'_{fp}(a)\end{aligned}$$

and  $y'_{fp}(a)$  and  $z'_{fp}(a)$  can be computed as in (57). Next, we applied the  $C^2$ -Lohner algorithm [31] to the extended system (56) in order to compute a bound for the first and the second order partial derivatives of  $P$  and in consequence a bound for  $\lambda'_1(A)$ .

We inserted  $A \times N$ , where  $N$  is defined in (55), as the initial condition in our routine, which computes the partial derivatives of the Poincaré map up to

	order	step
$\lambda_2(A), \lambda'_1(A)$	10	0.03
$\lambda_1(a_1)$	10	0.1
$\lambda_1(a_2)$ - 150-bit precision	14	0.05

Table 1: Parameters of the  $C^1 - C^2$ -Lohner algorithms.

second order. In these computations we simultaneously computed bounds for  $\lambda_2(A)$  and  $\lambda'_1(A)$ . The parameter settings of the Taylor method used in the computations are listed in Table 1. ■

## 6.2 The existence of Lyapunov-Schmidt reduction.

**Lemma 14** *For all  $(a, y) \in A \times Y$  there exists a unique  $z = z(a, y) \in Z$  such that*

$$P_{a,z}^2(y, z) = z \quad \text{iff} \quad z = z(a, y) \quad (58)$$

and the map  $z: A \times Y \rightarrow Z$  is smooth of class  $C^\infty$ . Moreover, the map  $G: A \times Y \rightarrow \mathbb{R}$  defined by

$$G(a, y) = y - P_{a,y}^2(y, z(a, y))$$

satisfies

$$\frac{\partial^3 G}{\partial y^3}(A \times Y) \subset [1.8296823158090675943, 7.2204769494502958338] \quad (59)$$

$$\frac{\partial^2 G}{\partial y^2}(A \times Y) \subset [-0.2084557586786322414, 0.2080871792788867581] \quad (60)$$

**Proof:** Let us fix  $(a, y) \in A \times Y$  and define a function  $V_{a,y}: Z \rightarrow \mathbb{R}$  by  $V_{a,y}(z) = z - P_{a,z}^2(y, z)$ . The computer assisted proof of this Lemma consists of the following steps

- We divide uniformly the interval  $A$  onto 30 parts. For each subinterval  $\bar{A}$  in this covering we proceed as follows.
- We verified that for all  $(a, y) \in \bar{A} \times Y$  the function  $V_{a,y}$  has exactly one zero in  $Z$ . Notice that  $Z$  is centered at zero – see (53). Put

$$\begin{aligned} \bar{Z} = N(V_{\bar{A},Y}, Z, 0) &= 0 - \left(1 - \frac{\partial}{\partial z} P_{\bar{A},z}^2(Y \times Z)\right)^{-1} \left(0 - P_{\bar{A},z}^2(Y, 0)\right) \\ &= P_{\bar{A},z}^2(Y, 0) \left(1 - \frac{\partial}{\partial z} P_{\bar{A},z}^2(Y \times Z)\right)^{-1}. \end{aligned}$$

Using the Interval Newton Method (Theorem 6) we obtain that if  $\bar{Z} \subset Z$  then for each  $(a, y) \in \bar{A} \times Y$  there exists unique  $z(a, y) \in \bar{Z}$  such that

$V_{a,y}(z(a,y)) = 0$ . This defines the unique map  $z: \bar{A} \times Y \rightarrow Z$  which is smooth by implicit function theorem and which satisfies (58). The subdivision of the set  $A$  onto 30 equal parts is the minimal one for which we were able to obtain inclusion  $\bar{Z} \subset Z$  for each  $\bar{A}$  in the subdivision and such that  $\frac{\partial^3 G}{\partial y^3}(A \times Y) > 0$  – see below.

- Let  $\bar{Z}$  denote a bound for  $z(\bar{A}, Y)$  resulting from the previous step. Differentiating  $z(a,y) - P_{a,z}^2(y, z(a,y)) = 0$  with respect to  $y$  we obtain

$$\begin{aligned} \left(1 - \frac{\partial P_{a,z}}{\partial z}\right) \frac{\partial z}{\partial y} &= \frac{\partial P_{a,z}}{\partial y}, \\ \left(1 - \frac{\partial P_{a,z}}{\partial z}\right) \frac{\partial^2 z}{\partial y^2} &= \frac{\partial^2 P_{a,z}}{\partial z^2} \left(\frac{\partial z}{\partial y}\right)^2 + 2 \frac{\partial^2 P_{a,z}}{\partial y \partial z} \frac{\partial z}{\partial y} + \frac{\partial^2 P_{a,z}}{\partial y^2}, \\ \left(1 - \frac{\partial P_{a,z}}{\partial z}\right) \frac{\partial^3 z}{\partial y^3} &= \frac{\partial^3 P_{a,z}}{\partial z^3} \left(\frac{\partial z}{\partial y}\right)^3 + 3 \frac{\partial^3 P_{a,z}}{\partial y \partial z^2} \left(\frac{\partial z}{\partial y}\right)^2 \\ &\quad + 3 \left(\frac{\partial^2 z}{\partial y^2} \frac{\partial^2 P_{a,z}}{\partial z^2} + \frac{\partial^3 P_{a,z}}{\partial y^2 \partial z}\right) \frac{\partial z}{\partial y} \\ &\quad + 3 \frac{\partial^2 z}{\partial y^2} \frac{\partial^2 P_{a,z}}{\partial y \partial z} + \frac{\partial^3 P_{a,z}}{\partial y^3}. \end{aligned}$$

We see that we can compute all the partial derivatives of  $z(a,y)$  as a functions of the partial derivatives of  $P$ . Hence, the partial derivatives of  $G(a,y) = y - P_{a,y}^2(y, z(a,y))$  can be expressed in terms of the partial derivatives of  $P$ .

Using the  $C^3$ -Lohner algorithm [31] applied to the system (50) with a range of parameter values  $\bar{A}$  and an initial condition  $Y \times \bar{Z}$  we computed bounds of partial derivatives of the Poincaré map  $P$  up to third order and an estimation for  $\frac{\partial^3 G}{\partial y^3}(\bar{A} \times Y)$  and  $\frac{\partial^2 G}{\partial y^2}(\bar{A} \times Y)$ . The estimates (59) and (60) are an interval enclosures of the estimates obtained in each of 30 steps.

We used 6-th order Taylor method with the time step 0.04, both, to verify the existence of  $z(a,y)$  and to compute higher order partial derivatives of  $P$ . ■

	order	step	grid	remarks
$\frac{\partial G}{\partial y}(a_2, y_{fp}(a_2))$	14	0.05	–	150-bit mantissa
$G(a_2, y_1)$	14	0.05	–	150-bit mantissa
$G(a_2, y_2)$	14	0.05	–	150-bit mantissa
$\frac{\partial^2 G}{\partial y \partial a}(A \times Y)$	6	0.05	$5 \times 30$	integration of (56)
$\frac{\partial G}{\partial y}(\{a_1\} \times Y)$	10	0.05	$1 \times 16000$	nonequal parts

Table 2: Parameters of the  $C^0 - C^2$ -Lohner algorithms.

### 6.3 The existence of period doubling bifurcation for $P$ .

**Lemma 15** For  $A = [a_1, a_2]$ ,  $Y = [y_1, y_2]$  the following estimations hold true

$$\frac{\partial G}{\partial y}(a_2, y_{fp}(a_2)) \in [-1.2916325, -1.2916323] \cdot 10^{-6} \quad (61)$$

$$G(a_2, y_1) \in [-1.15, -1.07] \cdot 10^{-13} \quad (62)$$

$$G(a_2, y_2) \in [5.2, 5.21] \cdot 10^{-12} \quad (63)$$

$$\frac{\partial^2 G}{\partial y \partial a}(A \times Y) \subset [-2.421398492231531, -0.278863623843693] \quad (64)$$

$$\frac{\partial G}{\partial y}(\{a_1\} \times Y) \subset [0.83, 16.87] \cdot 10^{-6} \quad (65)$$

**Proof:** The estimations have been obtained using  $C^0 - C^1 - C^2$ -Lohner algorithms [31, 32] applied to the systems (50) and (56). The verification of conditions (61–63) required computations in interval arithmetics based on 150-bit mantissa floating points.

The settings of  $C^0$ - $C^2$ -Lohner methods for the above computations are listed in Table 2. ■

**Proof of Theorem 10:** The assertion follows from Theorems 4, 9 and numerical Lemmas 12, 13, 14, 15.

Indeed, assumptions of Theorem 4 has been verified in

- **A1** – Lemma 14,
- **A2** – from Lemma 12 there exists a fixed point curve  $(y_{fp}, z_{fp}): A \rightarrow (y_1, y_2)$  and from Lemma 14 it has form as desired in **A2**,
- **A3** –  $0 \notin \frac{\partial^3 G}{\partial y^3}(A \times Y)$  because of (59).

From (60), (64) and (54) it follows that  $0 \notin \frac{\partial^2 G}{\partial a \partial y}(A \times Y) + \frac{\partial^2 G}{\partial y^2} y'_{fp}(A) \cdot [0, 1]$ .

Finally, Lemma 15 guarantees that the remaining assumptions of Lemma 3 with  $\varepsilon_1 = \varepsilon_\nu = +1$  and  $[\delta_1, \delta_2] = [y_1, y_2]$ .

Finally, from Lemma 13 we see that the assumptions about the spectrum of  $DP_a(A)$  and an eigenvector  $v(a)$  as desired in Theorem 9 are satisfied. ■

#### 6.4 The existence of second period doubling bifurcation.

In Section 6.3 we gave a computer assisted proof that for some parameter value  $\bar{a}_1 \in [2.83244, 2.832446]$  period doubling bifurcation occurs for  $P_{\bar{a}_1}$ . In this section we use similar arguments in order to prove that  $P_{\bar{a}_2}^2$  has period doubling bifurcation for some  $\bar{a}_2 \in [3.83735812, 3.837358168411]$ .

Since the arguments used to prove the existence of second period doubling bifurcation are the same as in the first period doubling bifurcation we omit the details and we present only the sets and the necessary estimates.

Define

$$\begin{aligned} A_2 &= [a_3, a_4] = [3.83735812, 3.837358168411], \\ Y_2 &= [y_3, y_4] = [-1.1, 1.1] \cdot 10^{-6}, \\ Z_2 &= \frac{1}{3}Y_2, \\ u_2 &= (-4.5003284169596655673, 0.043136987520848421584), \\ M_2 &= \begin{bmatrix} 0.99999908059259889903 & 0.82277742767392003653 \\ 0.0013560287448822982113 & -0.56836370794614177182 \end{bmatrix}. \end{aligned}$$

The point  $u_2$  is an approximate period two point for parameter value  $a_4$ , and the columns of matrix  $M_2$  are normalized eigenvectors of  $DP_{a_4}^2$ , where the first column corresponds to eigenvalue close to  $-1$ .

On the Poincaré section  $\Pi$  we will use coordinates  $(y, z) = M_2^{-1}(u - u_2)$ , where  $u$  denotes a point in cartesian coordinates. In this subsection we will use only these coordinates.

**Theorem 16** *The Poincaré map  $P_a^2$  has a period doubling bifurcation at some point  $(\bar{a}_2, \bar{y}_2, \bar{z}_2) \in \text{int}(A_2 \times Y_2 \times Z_2)$ .*

The proof is a consequence of the following lemmas (proved with computer assistance)

**Lemma 17** *There exist a function  $(y_{per}, z_{per}): A_2 \rightarrow Y_2 \times Z_2$  smooth of class  $C^\infty$  such that for  $(a, y, z) \in A_2 \times Y_2 \times Z_2$  there hold*

$$P_a^2(y, z) = (y, z) \quad \text{iff} \quad (y, z) = (y_{per}(a), z_{per}(a))$$

and

$$y'_{per}(A_2) \subset [-0.36435039423614490328, -0.36419313389173590956].$$

**Lemma 18** Let  $\lambda_1, \lambda_2 : A \rightarrow \mathbb{R}$  be the eigenvalues of  $DP_a^2(y_{per}(a), z_{per}(a))$  defined by similar formulas as in Lemma 13. Let  $v(a)$  be the normalized eigenvector corresponding to the eigenvalue  $\lambda_1(a)$ . Then

$$\begin{aligned}\lambda_1(a_3) &\in [-0.99999992011934590863, -0.99999992005484927837], \\ \lambda_1(a_4) &\in [-1.00000000000149573159618, -1.00000000000149573159615], \\ \lambda_2(A_2) &\subset [-7.7304566166653588839, 7.7302177026359675856] \cdot 10^{-5}, \\ \lambda_1'(A_2) &\subset [-1.6554066232416912996, -1.6460891324715511974], \\ \pi_y v(A_2) &\subset \pm[0.99984657385734598822, 1.0001534381577519284].\end{aligned}$$

**Lemma 19** For all  $(a, y) \in A_2 \times Y_2$  there exists a unique  $z = z(a, y) \in Z_2$  such that

$$P_{a,z}^4(y, z) = z \quad \text{iff} \quad z = z(a, y)$$

and the map  $z : A_2 \times Y_2 \rightarrow Z_2$  is smooth of class  $C^\infty$ . Moreover, the map  $G : A_2 \times Y_2 \rightarrow \mathbb{R}$  defined by

$$G(a, y) = y - P_{a,y}^4(y, z(a, y))$$

satisfies

$$\begin{aligned}\frac{\partial^3 G}{\partial y^3}(A \times Y) &\subset [11.780861336872181511, 22.544626008881969881], \\ \frac{\partial^2 G}{\partial y^2}(A \times Y) &\subset [-0.12474597648618415136, 0.12474408945310766494].\end{aligned}$$

**Lemma 20** The following estimations hold true

$$\begin{aligned}\frac{\partial G}{\partial y}(a_4, y_{per}(a_4)) &\in [-2.992, -2.991] \cdot 10^{-12}, \\ G(a_4, y_3) &\in [-5.13, -5.11] \cdot 10^{-19}, \\ G(a_4, y_4) &\in [5.21, 5.22] \cdot 10^{-19}, \\ \frac{\partial^2 G}{\partial y \partial a}(A_2 \times Y_2) &\subset [-4.354355790892265432, -2.244876361570084633], \\ \frac{\partial G}{\partial y}(\{a_3\} \times Y_2) &\subset [0.99, 30.98] \cdot 10^{-8}.\end{aligned}$$

Parameter settings of computations involved in proofs of the above lemmas are listed in Table 3.

## 7 Continuation of bifurcation diagram

In the previous sections we proved that the map  $P_a$  has period doubling bifurcations for parameter values  $\bar{a}_1 \in [a_1, a_2] = A$  and  $\bar{a}_2 \in [a_3, a_4] = A_2$  in sets  $Y \times Z$  and  $Y_2 \times Z_2$ , respectively.

Our goal now is to connect these bifurcations with the curve of period two points. More precisely, we prove the following result

	order	step	grid	remarks
$\frac{\partial^3 G}{\partial y^3}(A_2 \times Y_2)$	10	0.04	$150 \times 1$	–
$\frac{\partial^2 G}{\partial y^2}(A_2 \times Y_2)$	10	0.04	$150 \times 1$	–
$\frac{\partial G}{\partial y}(a_4, y_{per}(a_4))$	25	0.05	–	150-bit mantissa
$G(a_4, y_3), G(a_4, y_4)$	25	0.05	–	150-bit mantissa
$\frac{\partial^2 G}{\partial y \partial a}(A_2 \times Y_2)$	10	0.05	$150 \times 1$	integration of (56)
$\frac{\partial G}{\partial y}(\{a_3\} \times Y_2)$	14	0.05	$1 \times 10000$	–

Table 3: Parameters of the  $C^0 - C^3$ -Lohner algorithms in the proof of the existence of second period doubling bifurcation.

**Theorem 21** *There exists a continuous curve*

$$(y_{per}, z_{per}): (\bar{a}_1, \bar{a}_2] \rightarrow \mathbb{R}^2$$

of period two points for  $P_a$ . Moreover,

$$\begin{aligned} (y_{per}(a), z_{per}(a)), P_a(y_{per}(a), z_{per}(a)) &\in Y \times Z \quad \text{for } \bar{a}_1 < a \leq a_2, \\ (y_{per}(a), z_{per}(a)), P_a^2(y_{per}(a), z_{per}(a)) &\in Y_2 \times Z_2 \quad \text{for } a_3 \leq a \leq \bar{a}_2. \end{aligned}$$

Therefore the curve  $(y_{per}, z_{per})$  connects the two bifurcation points for  $a = \bar{a}_1$  and  $a = \bar{a}_2$ .

The proof of the existence of a branch of period two points for  $P$  consists of the following steps.

1. the existence of continuous curve of period two points on intervals  $(\bar{a}_1, a_2]$  and  $[a_3, \bar{a}_2]$  is a consequence of Theorem 10 and Theorem 16, respectively.
2. for parameter values slightly above  $a_2$ ,  $a_2 < a \leq \tilde{a}$ , with  $\tilde{a} - a_2$  small, we extend this curve using Lemma 8, which requires some  $C^3$  estimates (hence it is demanding computationally).
3. for parameters far from  $a_2$  up to  $a_3$ , i.e.  $\tilde{a} < a \leq a_3$ , we verify the existence of period two point curves using the Krawczyk method (Theorem 7), which requires only  $C^1$  estimates.
4. Since we use different methods for proving the existence of segments of period two points curve over some intervals in  $[\bar{a}_1, \bar{a}_2]$  it is necessary to verify that these segments can be glued to produce a continuous curve.

At first, it appears that step 2, requiring costly  $C^3$  computations, is not necessary, because in step 3 we can consider also points close to  $a_2$  using  $C^1$  computations. But it turns out that, while in principle possible, this approach may require very large computation times, because the hyperbolicity is very weak there, due to the fact that one eigenvalue of  $P_{a_2}^2$  is very close to 1.

To deal with this problem we used Lemma 8 to prove that for parameter values slightly above  $a_2$  there exists a continuous branch of period two points. Algorithm 1 is designed to verify assumptions of Lemma 8. In Lemma 22 we prove its correctness.

**Definition 5** Let  $U \subset \mathbb{R}^n$  be a bounded set. We say that  $\mathcal{G} \subset 2^{\mathbb{R}^n}$  is a grid of  $U$  if

1.  $\mathcal{G}$  is a finite set and each  $G \in \mathcal{G}$  is a closed set,
2.  $U \subset \bigcup_{G \in \mathcal{G}} G$ .

In our algorithms, which will be presented below, we always use grids consisting of interval sets, i.e. sets which are cartesian products of intervals, most of the time uniform grids, which are defined as follows.

**Definition 6** Let  $Y = \prod_{i=1}^n Y_i$ , where  $Y_i = [a_i, b_i]$  for  $a_i \leq b_i$  and let  $(g_1, \dots, g_n) \in \mathbb{Z}_+$ . We define a (uniform)  $g_1 \times g_2 \times \dots \times g_n$ -grid for  $Y$  denoted by  $\mathcal{G}(g_1, \dots, g_n, Y)$  as follows.

For any  $(j_1, \dots, j_n) \in \mathbb{Z}_+$ , such that  $j_i \leq g_i$  we set

$$g_{j_1, \dots, j_n} = \prod_{i=1}^n \left[ a_i, a_i + j_i \cdot \frac{b_i - a_i}{g_i} \right]. \quad (66)$$

Then  $\mathcal{G}(g_1, \dots, g_n, Y)$  is a collection of all  $g_{j_1, \dots, j_n}$ .

**Lemma 22** If Algorithm 1 is called with its arguments  $\nu_1, \nu_2, g_1, g_2, g_3, g_x, t, X \times Y$  and  $f_\nu$  and it does not throw an exception then the assumptions of Lemma 8 are satisfied for  $f_\nu, \nu \in [\nu_1, \nu_2]$  on  $X \times Y$ .

**Proof:** The assumption about existence of fixed point curve is verified in lines 15–19 since for all  $\nu \in [\nu_1, \nu_2]$  the Interval Newton Operator satisfies assumptions of Theorem 6.

The existence of Lyapunov-Schmidt reduction together with condition (39) is verified in lines 2–8. In lines 4–6 we see that  $y(\nu, x)$  which solves equation  $y - \pi_y(f_\nu^2(x, y))$  is unique for fixed  $\nu$ , therefore by the implicit function theorem  $y(\nu, x)$  is smooth and we can compute map  $G$  and its partial derivatives.

In lines 9–10 we verify that  $G(\nu_2, \min(X)) < 0$  and  $G(\nu_2, \max(X)) > 0$ . Since  $\frac{\partial G}{\partial \nu}(\nu, \min(X)) > 0$  and  $\frac{\partial G}{\partial \nu}(\nu, \max(X)) < 0$  (lines 11–14) we see that for  $\nu \in [\nu_1, \nu_2]$  holds  $G(\nu, \min(X)) < 0$  and  $G(\nu, \max(X)) > 0$ . Therefore (40) holds true.



---

**Algorithm 1:** verification of assumptions of Lemma 8

---

**Data:**  $[\nu_1, \nu_2]$  - an interval,  $g_1, g_2, g_3, g_x$  - integers,  $t \in (0, 1)$  - float number,  $X \times Y$  - a convex, compact set,  $f_\nu$  - parameterized family of maps

**Result:** If algorithm stops and does not throw an exception then assumptions of Lemma 8 are satisfied

```
1 begin
2    $\mathcal{G}_1 \leftarrow g_1 \times g_x$ -grid for  $[\nu_1, \nu_2] \times X$ ;
3   foreach  $\bar{\nu} \times \bar{X} \in \mathcal{G}_1$  do
4      $y(\bar{\nu}, \bar{X}) \leftarrow$ 
5     IntervalNewtonOperator( $\text{Id}_Y - \pi_Y \circ f_{\bar{\nu}}^2(\bar{X}, \cdot), Y, \text{center}(Y)$ );
6     if not  $y(\bar{\nu}, \bar{X}) \subset \text{int } Y$  then
7        $\lfloor$  throw Lyapunov-Schmidt reduction not verified;
8     if not  $\frac{\partial^3 G}{\partial x^3}(\bar{\nu}, \bar{X}) > 0$  then
9        $\lfloor$  throw condition (39) is not satisfied;
10    if (not  $G(\nu_2, \min(X)) < 0$ ) or (not  $G(\nu_2, \max(X)) > 0$ ) then
11       $\lfloor$  throw condition (40) is not satisfied;
12     $\mathcal{G}_2 \leftarrow g_2$ -grid for  $[\nu_1, \nu_2]$ ;
13    foreach  $\bar{\nu} \in \mathcal{G}_2$  do
14      if (not  $\frac{\partial G}{\partial \nu}(\bar{\nu}, x_1) > 0$ ) or (not  $\frac{\partial G}{\partial \nu}(\bar{\nu}, x_2) < 0$ ) then
15         $\lfloor$  throw condition (40) is not satisfied;
16     $\mathcal{G}_3 \leftarrow g_3$ -grid for  $[\nu_1, \nu_2]$ ;
17    foreach  $\bar{\nu} \in \mathcal{G}_3$  do
18       $(\bar{X}, \bar{Y}) \leftarrow$  IntervalNewtonOperator( $\text{Id} - f_{\bar{\nu}}, X \times Y, \text{center}(X \times Y)$ );
19      if not  $(\bar{X}, \bar{Y}) \subset \text{int}(X \times Y)$  then
20         $\lfloor$  throw fixed points curve not verified;
21       $x_+ \leftarrow (1 - t) \min(X) + t \min(\bar{X})$ ;
22       $x_- \leftarrow (1 - t) \max(X) + t \max(\bar{X})$ ;
23      if (not  $G(\min(\bar{\nu}), x_+) > 0$ ) or (not  $G(\min(\bar{\nu}), x_-) < 0$ ) then
24         $\lfloor$  throw condition (41) or (42) is not satisfied;
25      if (not  $\frac{\partial G}{\partial \nu}(\bar{\nu}, x_+) < 0$ ) or (not  $\frac{\partial G}{\partial \nu}(\bar{\nu}, x_-) > 0$ ) then
26         $\lfloor$  throw condition (41) or (42) is not satisfied;
27 end
```

---

Finally, in lines 15–25 we verify conditions (41–42). Again we verify that for an element of grid  $\bar{\nu}$  it holds  $G(\min(\bar{\nu}), x_+) > 0$  and  $G(\min(\bar{\nu}), x_-) < 0$ . This together with  $\frac{\partial G}{\partial \nu}(\bar{\nu}, x_+) > 0$  and  $\frac{\partial G}{\partial \nu}(\bar{\nu}, x_-) < 0$  proves (41–42). ■

As was mentioned earlier Algorithm 1 is used to prove the existence of period two points curve for parameter values slightly above the first bifurcation - i.e. for parameters close to  $a_2$  where  $G$  has three solutions close to one another. For these parameter values we found difficulties with verifying the existence of period-two points curve using  $C^1$ -computations, only.

For parameter values away from the bifurcation, where all eigenvalues of periodic orbit are well separated from the unit circle, we use the Algorithm 2 based on the Newton interval method and the Krawczyk method, both requiring only  $C^1$ -computations, and which verifies the existence of only one branch of period two points for  $P_a$ .

Before we present this algorithm we need to introduce some notations. Let  $\bar{\Pi} = \{(x, y, z) \in \mathbb{R}^3 : x = 0\}$  be a Poincaré section for (50) and  $\bar{P}_a : \bar{\Pi} \rightarrow \bar{\Pi}$  be corresponding Poincaré map for a system with parameter value  $a$ . Notice that the trajectory can intersect  $\bar{\Pi}$  at a point  $(y, z) \in \bar{\Pi}$  for which  $x' = -y - z$  is positive or negative (if it is equal to zero the Poincaré map is not defined). Hence we have  $\bar{P}_a|_{\Pi} = P_a$ , where  $P_a$  is the Poincaré map for the section  $\Pi = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0, x' > 0\}$  and therefore period two points for  $P_a$  correspond to period four points for  $\bar{P}_a$ . Let us define a map  $F_a : \bar{\Pi}^4 \rightarrow \mathbb{R}^8$  by

$$F_a \begin{bmatrix} (y_1, z_1) \\ (y_2, z_2) \\ (y_3, z_3) \\ (y_4, z_4) \end{bmatrix} = \begin{bmatrix} (y_2, z_2) - \bar{P}_a(y_1, z_1) \\ (y_3, z_3) - \bar{P}_a(y_2, z_2) \\ (y_4, z_4) - \bar{P}_a(y_3, z_3) \\ (y_1, z_1) - \bar{P}_a(y_4, z_4) \end{bmatrix}$$

Algorithm 2 was used to verify the existence of a continuous branch of period two points for  $P_a$  for  $a$  belonging to some interval. Sets  $X_i \times Y_i$  give the size of the neighborhood around a candidate periodic orbit on section  $\bar{\Pi}$ . Lines 4 to 8 constitute a heuristic part and their task is to find a good candidate.

**Lemma 23** *If Algorithm 2 is called with its arguments  $[a_*, a^*]$ ,  $Y_i \times Z_i$ ,  $i = 1, 2, 3, 4$  and  $g$  and does not throw an exception then there exists a continuous curve  $(y_{per}, z_{per}) : [a_*, a^*] \rightarrow \Pi$  such that  $(y_{per}(a), z_{per}(a))$  is period two point for  $P_a$ .*

**Proof:** The existence of fixed point for  $P_a^2$  for all  $a \in [a_*, a^*]$  is verified in lines 12–16. Lines 17–18 guarantee that this is a period two point for  $P_a$ , in fact a unique one in  $U$ .

Uniqueness implies continuity on each  $\bar{a} \in \mathcal{G}$  and due to uniqueness and connectedness of the set  $B$  defined in line 19 we see that they agree on boundaries of  $\bar{a}$ . ■

**Proof of Theorem 21:**

---

**Algorithm 2:** verification the existence of period two points branch.

---

**Data:**  $[a_*, a^*]$  - an interval,  $Y_i \times Z_i$ ,  $i = 1, 2, 3, 4$  - convex, compact sets,  
 $g$  - an integer

**Result:** if algorithm stops and does not throw an exception then there  
exists a continuous branch of period two points for  $P_a$  for  
parameter values  $a \in [a_*, a^*]$

```

1 begin
2    $\mathcal{G} \leftarrow g$ -grid for  $[a_*, a^*]$ ;
3   foreach  $\bar{a} \in \mathcal{G}$  do
4      $a \leftarrow \text{center}(\bar{a})$ ;
5      $u_1 = (y_1, z_1) \leftarrow$  find approximate period two point for  $P_a^2$  using
      standard Newton method;
6      $u_2 = (y_2, z_2) \leftarrow \bar{P}_a(y_1, z_1)$ ;
7      $u_3 = (y_3, z_3) \leftarrow \bar{P}_a(y_2, z_2)$ ;
8      $u_4 = (y_4, z_4) \leftarrow \bar{P}_a(y_3, z_3)$ ;
9      $C \leftarrow$  compute approximate value of  $DF_a(u_1, u_2, u_3, u_4)$ ;
10    if  $C$  is singular then
11       $C \leftarrow \text{Id}$ ;
12     $U \leftarrow (u_1 + Y_1 \times Z_1, u_2 + Y_2 \times Z_2, u_3 + Y_3 \times Z_3, u_4 + Y_4 \times Z_4)$ ;
13     $u \leftarrow (u_1, u_2, u_3, u_4)$ ;
14     $K_{\bar{a}} = (k_{1,\bar{a}}, k_{2,\bar{a}}, k_{3,\bar{a}}, k_{4,\bar{a}}) \leftarrow$ 
      IntervalKrawczykOperator( $F_{\bar{a}}, C^{-1}, U, u$ );
15    if not  $K_{\bar{a}} \subset \text{int } U$  then
16      throw cannot verify the existence of period two point;
17    if  $k_{1,\bar{a}} \cap k_{3,\bar{a}} \neq \emptyset$  then
18      throw the unique fixed point for  $P_{\bar{a}}^2$  in  $k_{1,\bar{a}}$  is not necessary
      period two point for  $P_{\bar{a}}$ ;
19     $B \leftarrow \bigcup_{\bar{a} \in \mathcal{G}} \bar{a} \times k_{1,\bar{a}}$ ;
20    if  $B$  is not connected then
21      throw cannot verify if branch of fixed point curve is continuous
      on interval  $[a_*, a^*]$ ;
22 end

```

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$[\nu_1, \nu_2]$	$g_1$	$g_2$	$g_3$	$g_x$	$t$	$(X_i \times Y_i) \cdot 10^{-4}$
$[a_2, 2.8325]$	8	70	600	150	0.88	$[-99.022, 97.95] \times [-4, 4]$
$[2.8325, 2.8326]$	14	80	300	270	0.7	$[-166.9, 163.9] \times [-4, 4]$
$[2.8326, 2.8327]$	16	60	150	340	0.6	$[-214.59, 209.62] \times [-4, 4]$
$[2.8327, 2.8328]$	17	50	120	340	0.5	$[-253.8, 246.8] \times [-8, 8]$
$[2.8328, 2.8329]$	20	50	100	350	0.5	$[-287.757, 279] \times [-8, 8]$

Table 4: Parameters of the Algorithm 1.

The existence of a continuous curve of period two points on intervals  $(\bar{a}_1, a_2]$  and  $[a_3, \bar{a}_2]$  is a consequence of Theorem 10 and Theorem 16, respectively. Let  $a_s = 2.8329$ . For the parameter values  $[a_2, a_s]$  we verify the existence of period two points branch using Algorithm 1 and for parameter values  $a \in [a_s, a_3]$  we use Algorithm 2.

We have ran Algorithm 1 five times with parameters listed in Table 4 (in each case the map is  $P_a$ ). Since in each case the algorithm had stopped and did not throw an exception we conclude that in each interval of parameters listed in Table 4 there exist two continuous curves  $c_1(a)$ , and  $c_2(a)$  of period two points. Sets  $X_i \times Y_i$  listed in Table 4 are chosen so that

$$Y \times Z \subset X_1 \times Y_1 \subset \dots \subset X_5 \times Y_5, \quad (67)$$

where  $Y \times Z$  is the set used in the proof of the existence of first period doubling bifurcation. Observe also, that since we know that for  $a_2$  holds  $P_{a_2}(c_1(a_2)) = c_2(a_2)$  and  $c_1(a_2), c_2(a_2) \in Y \times Z \subset X_1 \times Y_1$  from Lemma 8 we obtain that  $\{c_1(a), c_2(a)\}$  is period two orbit for  $P_a$ , for  $a \in [a_2, a_s]$ , i.e. the whole interval of parameters covered by intervals listed in the first columns in Table 4. The uniqueness of period two orbit together with condition (67) implies that the curves are continuous on  $(\bar{a}_1, a_s]$ .

One can see that the total number of initial values for which we need compute third order derivatives of  $G$ , which is equal to the sum of  $g_1 g_x$  over all rows in Table 4, is equal to 23200. The total time of computation of this step is ten hours on the Pentium IV, 3GHz processor.

We have run Algorithm 2 with 74 different arguments listed in Table 5. We have chosen the parameters of the Algorithm 2 so that such 74 intervals  $[(a_*)_i, (a^*)_i]$ ,  $i = 1, \dots, 74$  cover the interval  $[a_s, a_3]$ . Notice also, that for parameters  $a$  closer to  $a_s$  we need larger values of  $g$  since the hyperbolicity close to  $a_2$  is very weak. The total number of subintervals used to cover interval  $[a_s, a_3]$  is 614450. In fact this is the longest part of the numerical proof. The total time of computation of this step is 53 hours on the Pentium IV, 3GHz processor. Since in each case Algorithm 2 stops and does not throw an exception we conclude that on each subinterval  $[(a_*)_i, (a^*)_i]$  there exists continuous branch of period two points. We need to show that these curves glue continuously at  $a_i^*$ 's. In fact, this algorithm returns an upper bound for this period two points

branch which is of the form

$$B = \bigcup_{i=1}^{74} B_i \quad (68)$$

where  $B_i$ 's are defined in line 19 of Algorithm 2. We verified that  $B$  is connected - this together with an information that for fixed  $a \in [a_s, a_3]$  there exists a unique period two point  $(y_{per}(a), z_{per}(a))$  such that  $(a, y_{per}(a), z_{per}(a)) \in B$  implies that the curve  $(y_{per}(a), z_{per}(a))$  is continuous on  $[a_s, a_3]$ .

There remains for us to show the continuity of a period two point branch for parameter values  $a = a_s$  and  $a = a_3$ . For  $a = a_s$  we know that there exist period two points  $c_1(a_s), c_2(a_s)$  which belongs to the last set listed in Table 4, i.e.

$$c_1(a_s), c_2(a_s) \in W_1 = u_0 + M \cdot ([-287.757, 279] \times [-8, 8]) \cdot 10^{-4}$$

where  $u_0$  and  $M$  define coordinate system close to the first period doubling bifurcation and are defined in (51–52). On the other hand the estimation for period two point resulting from the Krawczyk method used in the Algorithm 2 is

$$\begin{aligned} W_2 &= (W_2^1, W_2^2), \\ W_2^1 &= [-4.7668051788293892557, -4.7667832743925968586], \\ W_2^2 &= [0.052543190547910088861, 0.052543238016254205369]. \end{aligned}$$

One can verify that  $W_2 \subset W_1$  which obviously means that a period two point  $(y_{per}(a_s), z_{per}(a_s)) \in W_2$  resulting from the Krawczyk method and the Algorithm 2 is one of the points  $c_1(a_s), c_2(a_s)$  resulting from the Algorithm 1. Hence, the curve of period two points is continuous at  $a = a_s$ .

In a similar way we verified continuity at  $a = a_3$ . From the Krawczyk method used in the Algorithm 2 we know that  $(y_{per}(a_3), z_{per}(a_3))$  is a unique period two point in the set

$$\begin{aligned} W_3 &= (W_3^1, W_3^2), \\ W_3^1 &= [-4.5010116820607413146, -4.4996232549240025023], \\ W_3^2 &= [0.043134233933640332703, 0.043140290681655812932]. \end{aligned}$$

On the other hand from Theorem 16 we know that for  $a = a_3$  period two point belongs to the set

$$W_4 = u_2 + M_2 \cdot (Y_2 \times Z_2)$$

where  $M_2, u_2, Y_2, Z_2$  define the set on which we verify the existence of second period doubling bifurcation. One can verify that  $W_4 \subset W_3$  which proves that the branch of period two points is continuous at  $a = a_3$ . ■

$U = [-1, 1] \cdot (1100, 3, 1000, 3000, 1100, 3, 1000, 3000) \cdot 10^{-8}$			
$i$	$[a_*, a^*]$	$g$	$Y_1 \times Z_1 \times Y_2 \times Z_2 \times Y_3 \times Z_3 \times Y_4 \times Z_4$
1	[2.8329, 2.83291]	14000	$U$
2	[2.83291, 2.83292]	13500	$U$
3	[2.83292, 2.83293]	12200	$3U$
4	[2.83293, 2.83294]	11250	$3U$
5	[2.83294, 2.83295]	10400	$3U$
6	[2.83295, 2.83296]	9650	$3U$
7	[2.83296, 2.83297]	9050	$3U$
8	[2.83297, 2.83298]	8500	$3U$
9	[2.83298, 2.83299]	8000	$3U$
10	[2.83299, 2.833]	7550	$3U$
11	[2.833, 2.83301]	7200	$3U$
12	[2.83301, 2.83302]	6800	$3U$
13	[2.83302, 2.83303]	6500	$3U$
14	[2.83303, 2.83304]	6200	$3U$
15	[2.83304, 2.83305]	6000	$3U$
16	[2.83305, 2.83306]	5700	$3U$
17	[2.83306, 2.83307]	5500	$3U$
18	[2.83307, 2.83308]	5300	$3U$
19	[2.83308, 2.83309]	5100	$3U$
20	[2.83309, 2.8331]	4900	$3U$
21	[2.8331, 2.83311]	4800	$3U$
22	[2.83311, 2.83312]	4600	$3U$
23	[2.83312, 2.83313]	4450	$3U$
24	[2.83313, 2.83314]	4300	$3U$
25	[2.83314, 2.83315]	4150	$3U$
26	[2.83315, 2.83316]	4050	$3U$
27	[2.83316, 2.83317]	3950	$3U$
28	[2.83317, 2.83318]	3850	$3U$
29	[2.83318, 2.83319]	3750	$3U$
30	[2.83319, 2.8332]	3650	$3U$
31	[2.8332, 2.8333]	36000	$3U$
32	[2.8333, 2.8334]	29000	$3U$
33	[2.8334, 2.8335]	24000	$3U$
34	[2.8335, 2.8336]	20000	$3U$
35	[2.8336, 2.8337]	18000	$3U$
36	[2.8337, 2.8338]	16000	$3U$
37	[2.8338, 2.8339]	14000	$3U$
38	[2.8339, 2.834]	13000	$3U$

*Table 5 continued on the next page*

<i>Table 5 continued from previous page</i>			
$i$	$[a_*, a^*]$	$g$	$Y_1 \times Z_1 \times Y_2 \times Z_2 \times Y_3 \times Z_3 \times Y_4 \times Z_4$
39	[2.834, 2.8345]	59000	3U
40	[2.8345, 2.835]	42000	3U
41	[2.835, 2.8355]	17500	15U
42	[2.8355, 2.836]	11000	15U
43	[2.836, 2.8365]	8000	15U
44	[2.8365, 2.837]	6200	15U
45	[2.837, 2.8372]	2800	30U
46	[2.8372, 2.8375]	3600	30U
47	[2.8375, 2.838]	5000	30U
48	[2.838, 2.8385]	3600	30U
49	[2.8385, 2.839]	2900	30U
50	[2.839, 2.8395]	2400	30U
51	[2.8395, 2.84]	2100	30U
52	[2.84, 2.841]	3700	30U
53	[2.841, 2.842]	3000	30U
54	[2.842, 2.843]	2500	30U
55	[2.843, 2.844]	2200	30U
56	[2.844, 2.845]	2000	30U
57	[2.845, 2.846]	1700	30U
58	[2.846, 2.848]	3100	30U
59	[2.848, 2.85]	2600	30U
60	[2.85, 2.86]	11000	30U
61	[2.86, 2.87]	6500	30U
62	[2.87, 2.88]	4700	30U
63	[2.88, 2.89]	3700	30U
64	[2.89, 2.9]	3100	30U
65	[2.9, 2.95]	4400	150U
66	[2.95, 3]	2500	150U
67	[3, 3.1]	3500	150U
68	[3.1, 3.2]	2500	150U
69	[3.2, 3.3]	2100	150U
70	[3.3, 3.4]	1800	150U
71	[3.4, 3.5]	1600	150U
72	[3.5, 3.6]	1400	150U
73	[3.6, 3.7]	1500	150U
74	[3.7, $a_3$ ]	2400	150U

Table 5: Parameters of Algorithm 2. The initial set  $U$  is defined in the first line of the table.

## 7.1 Technical data.

In order to compute the Poincaré maps  $P$  and  $P^2$  with their partial derivatives we used the interval arithmetic [13, 20], the set algebra and the  $C^r$ -Lohner algorithm [31] developed at the Jagiellonian University by the CAPD group [6]. The C++ source files of the program with an instruction how it should be compiled and run are available at [28].

All computations were performed with the Pentium IV, 3GHz processor and 2GB RAM under the Ubuntu Intrepid Ibex linux with gcc-4.3.1 and MS Windows XP Professional with gcc-3.4.4. The computations took approximately three days. The main time-consuming part (over 53 hours) is the verification of the existence of connecting branch of period two points between first and second bifurcation.

The program is also available to compile and run in a multithreaded version for multiprocessor computers. In that case computations are as much faster as there are processors free to use for the program. We successfully compiled and ran the program on a computer with 8 processors Quad-Core AMD Opteron(tm) 8354, each 2.2GHz. The program run 2 hours on this computer.

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