$\mathbb{Z}_2$-HOMOLOGY OF WEAK 2-PSEUDOMANIFOLDS MAY BE COMPUTED IN $O(n \log^* n)$ TIME

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ABSTRACT. We show that in the class of weak 2-pseudomanifolds with bounded boundaries and coboundaries the Betti numbers with $\mathbb{Z}_2$ coefficients may be computed in time $O(n \log^* n)$ and the $\mathbb{Z}_2$ homology generators in time $O(n(m + \log^* n))$.

1. INTRODUCTION

The task of computing homology may be easily reduced to finding the Smith diagonalization of the matrices of the boundary maps. Unfortunately, the supercubical complexity of the Smith diagonalization results in the failure of this approach in the presence of large input. This causes the demand for specialized, fast homology algorithms. The demand originated about 20 years ago from a few independent sources. Among them are in particular topological methods in: data and image analysis, rigorous numerics of dynamical systems, electromagnetic engineering, material science, robotics. In all these fields the size of input is often measured in millions of elements and more. The problem is additionally complicated by the range of the required output, which varies from Betti numbers, through homology generators, particularly homology generators of minimal size, to matrices of homology maps.

The papers by Donald and Chang [3] and Delfinado and Edelsbrunner [2] indicate that at least in some special situations homology may be computed much quicker than by means of the Smith diagonalization. In particular, Delfinado and Edelsbrunner show that the Betti numbers of simplicial subcomplexes of a triangulation of $\mathbb{S}^3$ may be computed in time $O(n \log^* n)$. This result does not apply to 2-dimensional spaces which cannot be embedded in $\mathbb{S}^3$, for instance the Klein bottle. However, the numerical experiments based on the acyclic subspace

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homology algorithm [7] and the coreduction homology algorithm [8] indicate that at least for low dimensional spaces the homology may be computed fast regardless of the embedding dimension. This, in particular, happens in the case of compact, connected, orientable surfaces. If X is such a surface, then its first Betti number is two minus the Euler characteristic of X amd the other two nonzero Betti numbers are one. Since the Euler characteristic may be computed in time $O(n)$, we conclude that for such surfaces the Betti numbers may be computed in time $O(n)$. Similar analysis applies to nonorientable surfaces. Moreover, G. Vegter and C.-K. Yap [12] proved that the generators of the fundamental group and, via the Hurewicz Theorem, the generators of the first homology group of a surface of genus g may be constructed in time $O(n \log n + ng)$. Recently Erickson and Whittlesey [4] proved that the minimal homology generators of connected, compact, orientable, 2-manifolds with genus g may be computed in time $O(n^2 \log n + n^2 g + ng^3)$.

The aim of this paper is to show that a modified coreduction homology algorithm [8] can find the $\mathbb{Z}_2$-homology of weak 2-pseudomanifolds in $O(n \log^* n)$ time when only Betti numbers are needed and in $O(n(m + \log^* n))$ time when also homology generators are computed with m the number of homology generators. Recall (see [11, Definition 8.1]) that a p-pseudomanifold is a regular CW-complex such that the following two conditions are satisfied

(i) Every cell is a face of some p-cell.
(ii) Every $(p-1)$-cell is a face of exactly two p-cells.
(iii) Any two p-cells may be joined by a sequence of p-cells such that any two consecutive cells in the sequence have a common $(p-1)$-face.

We say that a regular CW-complex is a weak p-pseudomanifold if it satisfies the second property in the definition of a p-pseudomanifold and has no q-cells for $q > p$.

Obviously every triangulated surface is a weak 2-pseudomanifold. By gluing together two vertices of a surface we obtain an example of a 2-pseudomanifold which is not a surface and by gluing two surfaces in a vertex we get an example of a weak 2-pseudomanifold which is not a 2-pseudomanifold.

The ideas of the paper are modelled on the geometry of CW-complexes but are purely combinatorial and applicable to the homology of chain complexes of any origin. For this end we use the language of S-complexes introduced in [8] and in this language we define the counterparts of the concept of p-pseudomanifold and weak p-pseudomanifold.

The organisation of the paper is as follows. In Section 2 we recall the definition and some properties of an S-complex. In the next section we define and study the connected components of an S-complex. The definition of a weak p-pseudomanifold in terms of S-complexes is given in Section 4. In Section 5 we present the gluing algorithm. We then recall the coreduction homology algorithm and study some its properties in Section 6. In the following section we present the concept and properties of a geometric S-complex. The main results of the paper are proved in the last section.

2. S-Complexes.

We begin with recalling from [8] the concept of an S-complex, a reformulation of chain complex suitable for algorithmic setting. Let R be a ring with unity. Given a finite set A let $R(A)$ denote the free module over R generated by A. Let X be a finite set with a gradation X_q such that $X_q = \emptyset$ for all but a finite number of q. Then $R(X_q)$ is a gradation of $R(X)$ in the category of moduli over the ring R. For every element $x \in X$ there exists a unique number q such that $x \in X_q$. This number will be referred to as the *dimension* of x and denoted $\dim x$. For a set subset A of an S-complex X by $(A)_q$ we mean $\{ a \in A \mid \dim(a) = q \}$. We use the notation $\langle \cdot, \cdot \rangle : R(X) \times R(X) \to R$ for the scalar product defined on generators by

$$\langle t, s \rangle = \begin{cases} 1 & t = s, \\ 0 & \text{otherwise} \end{cases}$$

and extend it bilinearly to $R(X) \times R(X)$. Let $\kappa : X \times X \to R$ be a map satisfying

$$\kappa(s,t) \neq 0 \;\Rightarrow\; \dim s = \dim t + 1.$$

We say that (X, κ) is an *S-complex* if $(R(X), \partial^\kappa)$ with *boundary operator*

$$\partial^\kappa : R(X) \to R(X)$$

defined on a generator $s \in X$ by

$$\partial^\kappa(s) := \sum_{t \in X} \kappa(s,t)t$$

is a free chain complex with base X. We also define the dual *coboundary operator*

$$\delta^\kappa : R(X) \to R(X)$$

defined on a generator $t \in X$ by

$$\delta^\kappa(t) := \sum_{s\in X} \kappa(s,t)s.$$

Note that

$$\langle \partial s, t\rangle = \langle s, \delta t\rangle = \kappa(s,t).$$

The map κ will be referred to as the *coincidence index*. If $\kappa(s,t) \neq 0$, then we say that t is a *face* of s and s is a *coface* of t.

Given $A \subset X$ we put

$$\begin{aligned}
\operatorname{bd}_X A &:= \{\, t \in X \mid \kappa(s,t) \neq 0 \text{ for some } s \in A \,\},\\
\operatorname{cbd}_X A &:= \{\, s \in X \mid \kappa(s,t) \neq 0 \text{ for some } t \in A \,\}.
\end{aligned}$$

In the sequel we drop the braces in $\operatorname{bd}_X\{s\}$ and $\operatorname{cbd}_X\{s\}$ in the case of a singleton $\{s\} \subset X$ and write $\operatorname{bd}_X s$ and $\operatorname{cbd}_X s$.

By the homology of an S-complex (X,κ) we mean the homology of the chain complex $(R(X), \partial^\kappa)$ and we denote it by $H(X,\kappa) := H(R(X), \partial^\kappa)$. The kernel and image of ∂^κ, i.e. the module of cycles and boundaries are denoted by $Z(X,\kappa)$ and $B(X,\kappa)$ respectively. We drop κ and write $H(X)$ and ∂ whenever κ is clear from the context. However, to emphasize the ring R used we often write $H(X,R)$ for $H(X,\kappa)$ with $\kappa : X \times X \to R$. The same convention applies to $Z(X,\kappa)$ and $B(X,\kappa)$. By $[z]_X \in H(X,\kappa)$ we mean the homology class of a cycle z and we write $[z]$ when X is clear from the context. For $R = \mathbb{Z}_2$ and a set $A \subset X$ we identify A with the chain $\sum_{a\in A} a$.

Note that when κ is given explicitly, for instance in the form of a matrix, then the S-complex is simply a chain complex with a fixed basis. However, in the context of an S-complex we assume that κ is given implicitly, via some coding of the elements of X. In particular, every simplicial complex and every cubical complex is an example of an S-complex (see [8])

A subset X' of an S-complex X is called *regular* if for all $s, u \in X'$ and $t \in X$

$$t \in \operatorname{bd}_X s \text{ and } u \in \operatorname{bd}_X t \text{ implies } t \in X'. \tag{1}$$

Proposition 2.1. *(see* [8, Theorem 3.1]*) If X' is a regular subset of an S-complex X then (X', κ') where $\kappa' := \kappa|_{X'\times X'}$ is also an S-complex.*

If the map κ is clear from the context then by $\partial_{X'}$ we mean $\partial^{\kappa|_{X'\times X'}}$.

We say that $X' \subset X$ is *closed* in X if $\operatorname{bd}_X X' \subset X'$. We say that $X' \subset X$ is *open* in X if $X \setminus X'$ is closed in X.

Proposition 2.2. *(see* [8, Theorem 3.2]*) If $X' \subset X$ is closed in X, then X' and $X \setminus X'$ are regular.*

Given $A \subset X$ we define the *geometric boundary* of A by

$$\operatorname{gbd} A := \bigcup_{i=1}^{\infty} \operatorname{bd}^n A.$$

It is straightforward to observe that the geometric boundary of any subset of an S-complex is closed.

As in [8] we say that a pair (a, b) of elements of S is an *elementary coreduction pair* or briefly a coreduction pair if $\kappa(b, a)$ is invertible in R and $\operatorname{bd}_S b = \{a\}$. From [8, Theorem 4.1] and [8, Corollary 3.6] we get the following proposition.

Proposition 2.3. *If (a, b) is a coreduction pair in an S-complex X then $X' := X \setminus \{a, b\}$ is a regular subset of X and $H(X)$ is isomorphic to $H(X')$.* □

Let $M > 0$ be a fixed integer. By $\mathcal{S}_M$ we denote the class of S-complexes X such that for each $a \in X$ the cardinalities of $\operatorname{bd} a$ and $\operatorname{cbd} a$ are bounded by M.

3. Connected components of S-complexes

We say that two elements a, b of an S-complex X are *adjacent* if $\kappa(a, b) \neq 0$ or $\kappa(b, a) \neq 0$. This defines a symmetric relation on X.

`<<COMMENT>> Dodana definicja wymiaru sciezki` !!

A *path* P between $a, b \in X$ is a sequence $a = p_1, p_2, \ldots, p_k = b$ of elements in X such that p_i is adjacent to p_{i+1} for $i = 1, \ldots, k-1$. We say that such a path has *length* k. By the *dimension* of a path we mean the maximum dimension of its elements.

The reflexive and transitive closure of the adjacency relation is an equivalence relation. The equivalence classes of this relation will be referred to as the *connected components* of X.

`<<COMMENT>> Drobna modyfikacja definicji.` !!

We say that an S-complex X is *connected* if it is non-empty and has exactly one connected component. We denote by $\mathcal{C}(X)$ the collection of connected components of X and we put $\mathcal{C}_p(X) := \{ A_p \mid A \in \mathcal{C}(X) \}$. Given an element $x \in X$ we denote by $\operatorname{cc}_X(b)$ the connected component of X to which x belongs.

Proposition 3.1. *A connected component of an S-complex X is a closed S-complex in X.*

Proof: It is straightforward to verify that a connected component of an S-complex is closed in X, therefore the conclusion follows immediately from Proposition 2.1 and Proposition 2.2. □

Lemma 3.2. *Assume $Y \subset X$ is a connected component of X. Then the inclusion $\iota : Y \to X$ induces the monomorphism*

$$\iota_* : H(Y) \to H(X).$$

Proof: From Proposition 3.1 we know that Y is closed in X. Therefore, by [8, Theorem 3.4] ι_* is a well defined homomorphism. Let $z \in Z_k(Y)$ and assume $\iota_*[z]_Y = 0$. Then $[z]_X = 0$, so there exists a $c \in R(X_{k+1})$ such that $\partial c = z$. We may write c as $c = c_{X\setminus Y} + c_Y$ where $c_{X\setminus Y} \in R((X\setminus Y)_{k+1})$ and $c_Y \in R(Y_{k+1})$. Hence $\partial c = \partial c_{X\setminus Y} + \partial c_Y$ and consequently $\partial c_{X\setminus Y} = -\partial c + \partial c_Y$. Since Y is a connected component of X, $X \setminus Y$ is a sum of connected components and by Proposition 3.1 the sets Y and $X \setminus Y$ are closed in X. Since $\partial c = z \in R(Y_k)$, we see that $-\partial c + \partial c_Y \in R(Y_k)$. However $\partial c_{X\setminus Y} \in R((X \setminus Y)_k)$ which is possible only when $-\partial c + \partial c_Y = 0$. Therefore $\partial c = \partial c_Y = z$ and $[z]_Y = 0$. Thus ι_* is a monomorphism. □

Lemma 3.3. *If X^1, X^2 are two different connected components of an S-complex X, $c \in R(X^1)$ and $x \in X^2$, then $\langle \partial c, x\rangle = 0$.*

Proof: Assume by contrary that $\langle \partial c, x\rangle \neq 0$. Since $c = \sum_{y\in X^1} \langle c, y\rangle y$, we see that

$$0 \neq \langle \partial c, x\rangle = \sum_{y\in X^1} \langle c, y\rangle \langle \partial y, x\rangle.$$

Therefore $\langle \partial y, x\rangle = \kappa(y, x) \neq 0$ for some $y \in X^1$. We get from Proposition 3.1 that $x \in X^1$, a contradiction. □

The homology of an S-complex splits as the direct sum of the homologies of its connected components. More precisely, we have the following theorem.

Theorem 3.4. *Let X be an S-complex with connected components*

$$X^1, X^2, \dots, X^n.$$

Then

$$H(X) \cong \bigoplus_{i=1}^{n} H(X^i). \tag{2}$$

Proof: From Lemma 3.2 we get monomorphisms

$$\iota^i_* : H(X^i) \to H(X)$$

for $i = 1, \dots, n$. We will show that the required isomorphism is

$$\iota_* : \bigoplus_{i=1}^{n} H(X^i) \ni (\xi_i)_{i=1}^n \to \sum_{i=1}^{n} \iota^i_*(\xi_i) \in H(X).$$

It is straightforward to observe that ι_* is a monomorphism. To see that it is an epimorphism take $[z] \in H(X)$. Then $z = \sum_{i=1}^n z_i$ where z_i is a chain in X^i. We will show that z_i is a cycle in X^i. Indeed, if $\partial_{X^{i_0}} z_{i_0} \neq 0$ for some i_0, then $\langle \partial_{X^{i_0}} z_{i_0}, y\rangle \neq 0$ for some $y \in X^{i_0}$. However, by Lemma 3.3 $\langle \partial_{X^{i_k}} z_{i_k}, y\rangle = 0$ for $k \neq 0$, therefore

$$\langle \partial_X z, y\rangle = \langle \partial_X z_{i_0}, y\rangle = \langle \partial_{X^{i_0}} z_{i_0}, y\rangle \neq 0,$$

a contradiction. It follows that

$$\iota_*([z_i]_{X^i})_{i=1}^n = [z]_X.$$

□

We refer to an S-complex X as *p-faceless* if for all $q \leq p$ we have $X_q = \emptyset$. A 0-faceless S-complex X will be also referred to as *vertexless.*

Lemma 3.5. *Let X be a $(p-2)$-faceless S-complex and let $a \in X_{p-1}$ for some $p > 0$. Then a and $X \setminus a$ are S-complexes, the map*

$$\bar\partial_p : H_p(X \setminus a, \mathbb{Z}_2) \ni [z] \to [\partial_X z] \in H_{p-1}(a, \mathbb{Z}_2).$$

is well defined, $H_k(X, \mathbb{Z}_2) = 0$ for $k \notin \{\, p-1, p\,\}$ and

$$H_p(X, \mathbb{Z}_2) \cong \begin{cases} H_p(X \setminus a, \mathbb{Z}_2) & \text{if } \bar\partial_p = 0, \\ \ker \bar\partial_p & \text{otherwise,} \end{cases} \tag{3}$$

$$H_{p-1}(X, \mathbb{Z}_2) \cong \begin{cases} H_{p-1}(a, \mathbb{Z}_2) \oplus H_{p-1}(X \setminus a, \mathbb{Z}_2) & \text{if } \bar\partial_p = 0, \\ H_{p-1}(X \setminus a, \mathbb{Z}_2) & \text{otherwise.} \end{cases} \tag{4}$$

Proof: First observe that a is closed in X, because X is $(p-2)$-faceless. Therefore a and $X \setminus a$ are S-complexes by Proposition 2.1 and Proposition 2.2. For $z, z' \in Z_p(X \setminus a, \mathbb{Z}_2)$ such that $[z]_{X\setminus a} = [z']_{X\setminus a}$ we will show that $\bar\partial_p[z]_{X\setminus a} = \bar\partial_p[z']_{X\setminus a}$. Since the homology classes of z and z' in $X \setminus a$ coincide, there exists a $\mathbb{Z}_2$-chain c in $X \setminus a$ such that $\partial_{X\setminus a} c = z - z'$. Therefore we get

$$\partial_X z - \partial_X z' = \partial_X \partial_{X\setminus a} c = \partial_{X\setminus a}\partial_{X\setminus a} c + \langle \partial_X \partial_{X\setminus a} c, a\rangle a = \langle \partial_X \partial_{X\setminus a} c, a\rangle a.$$

Hence $[\partial_X z]_a = [\partial_X z']_a$ and consequently $\bar\partial_p$ is well defined.

Now consider the long exact sequence (see [8, Theorem 3.4])

$$0 \xrightarrow{\iota_p} H_p(X, \mathbb{Z}_2) \xrightarrow{\pi_p} H_p(X \setminus a, \mathbb{Z}_2) \xrightarrow{\bar\partial_p} \tag{5}$$

$$H_{p-1}(a, \mathbb{Z}_2) \xrightarrow{\iota_{p-1}} H_{p-1}(X, \mathbb{Z}_2) \xrightarrow{\pi_{p-1}} H_{p-1}(X \setminus a, \mathbb{Z}_2) \xrightarrow{\bar\partial_{p-1}} 0.$$

Obviously either $\operatorname{im} \bar\partial_p \cong 0$ or $\operatorname{im} \bar\partial_p \cong H_{p-1}(a, \mathbb{Z}_2)$. In both cases the exact sequence (5) splits into two short exact sequences. In the first case the sequences are

$$0 \xrightarrow{\iota_p} H_p(X,\mathbb{Z}_2) \xrightarrow{\pi_p} H_p(X\setminus a,\mathbb{Z}_2) \xrightarrow{\bar\partial_p} 0, \tag{6}$$

$$0 \to H_{p-1}(a,\mathbb{Z}_2) \xrightarrow{\iota_{p-1}} H_{p-1}(X,\mathbb{Z}_2) \xrightarrow{\pi_{p-1}} H_{p-1}(X\setminus a,\mathbb{Z}_2) \xrightarrow{\bar\partial_{p-1}} 0 \tag{7}$$

and in the other case the sequences are

$$0 \xrightarrow{\iota_p} H_p(X,\mathbb{Z}_2) \xrightarrow{\pi_p} H_p(X\setminus a,\mathbb{Z}_2) \xrightarrow{\bar\partial_p} H_{p-1}(a,\mathbb{Z}_2) \xrightarrow{\iota_{p-1}} 0, \tag{8}$$

$$0 \to H_{p-1}(X,\mathbb{Z}_2) \xrightarrow{\pi_{p-1}} H_{p-1}(X\setminus a,\mathbb{Z}_2) \xrightarrow{\bar\partial_{p-1}} 0. \tag{9}$$

Now, we obtain (3) from (6) and (8) and (4) from (7) and (9). □

4. Weak p-pseudomanifolds.

Now we extend the concept of p-pseudomanifolds to S-complexes.

We say that an S-complex X is a *weak p-pseudomanifold* if $X_q = \emptyset$ for $q > p$ and for each $s \in X_{p-1}$ the cardinality of $\operatorname{cbd}_X s$ is exactly two.

Lemma 4.1. *Let X be a $(p-2)$-faceless weak p-pseudomanifold. If $a \in X_{p-1}$ is such that $\operatorname{cbd} a = \{ b_1, b_2 \}$ for some $b_1 \neq b_2$, then for the map $\bar\partial_p$ defined in Lemma 3.5 we have*

$$\bar\partial_p \neq 0 \textit{ if and only if } cc_{X\setminus a}(b_1) \neq cc_{X\setminus a}(b_2).$$

Proof: Assume $\bar\partial_p \neq 0$. There exists a $\mathbb{Z}_2$-chain A in $X \setminus a$ such that $\bar\partial_p[A] \neq 0$. Therefore

$$0 \neq \langle \partial_X A, a\rangle = \langle A, \delta_X a\rangle = \langle A, b_1\rangle + \langle A, b_2\rangle.$$

It follows that exactly one of the two elements b_1, b_2 belongs to A, i.e.

$$cc_{X\setminus a}(b_1) \neq cc_{X\setminus a}(b_2).$$

The proof of the reverse implication is analogous. □

Theorem 4.2. *If X is a connected $(p-2)$-faceless weak p-pseudomanifold then*

$$H_p(X,\mathbb{Z}_2) = [X_p].$$

Proof: Let $X_p = \{ x_1, \dots, x_n \}$, $X_{p-1} = \{ y_1, \dots, y_m \}$ and $c = \sum_{i=1}^n \epsilon_i x_i$ for some $\epsilon_i \in \mathbb{Z}_2$. We will show that c is a nonzero cycle if and only if

$\epsilon_i = 1$ for every $i \in \{1, 2, \dots, n\}$. For this end observe that

$$\begin{aligned} \partial c &= \sum_i \epsilon_i \partial x_i \\ &= \sum_i \epsilon_i \sum_j \kappa(x_i, y_j) y_j \\ &= \sum_j \Big(\sum_i \epsilon_i \kappa(x_i, y_j) \Big) y_j \end{aligned}$$

and the latter is zero if and only if

$$\sum_i \epsilon_i \kappa(x_i, y_j) = 0 \tag{10}$$

for every $j \in \{1, 2, \dots, m\}$.

Since X is a weak p-pseudomanifold, for every $j \in \{1, 2, \dots, m\}$ there exist exactly two indices $i_0(j), i_1(j)$, such that

$$\kappa(x_{i_0(j)}, y_j) \neq 0 \text{ and } \kappa(x_{i_1(j)}, y_j) \neq 0$$

and consequently the equation (10) becomes

$$\epsilon_{i_0(j)} \kappa(x_{i_0(j)}, y_j) + \epsilon_{i_1(j)} \kappa(x_{i_1(j)}, y_j) = 0$$

or

$$\epsilon_{i_0(j)} + \epsilon_{i_1(j)} = 0. \tag{11}$$

Therefore, if $\epsilon_i = 1$ for all i, then equation (10) is obviously satisfied, because of the $\mathbb{Z}_2$ coefficients we use. To prove the opposite implication, assume by contrary that there exist two nonempty subsets I_0, I_1 of $I := \{1, \dots, n\}$ such that $I_0 \cup I_1 = I$ and $\epsilon_i = q$ for $i \in I_q$, $q \in \{0, 1\}$. Since X is connected, for some $i_0 \in I_0$ and $i_1 \in I_1$ there exists a path $P = \{p_i\}_{i=1}^k \subset X_p \cup X_{p-1}$ between x_{i_0} and x_{i_1}. Without loss of generality we may assume that P has length 3. Then $p_2 \in X_{p-1}$, in particular $p_2 = y_j$ for some $j \in \{1, 2, \dots m\}$. Since X is a weak p-pseudomanifold, we get $i_0 = i_0(j)$ and $i_1 = i_1(j)$. It follows from (11) that $\epsilon_{i_0} + \epsilon_{i_1} = 0$, However, by the choice of I_0 and I_1, we have

$$\epsilon_{i_0} + \epsilon_{i_1} = 0 + 1 = 1,$$

and we get a contradiction. Therefore X_p is the only nontrivial p-cycle in X and since there are no q-chains in X for $q > p$, the conclusion follows. □

Corollary 4.3. *If X is a $(p-2)$-faceless weak p-pseudomanifold then*

$$H_p(X, \mathbb{Z}_2) \cong \bigoplus_{A \in \mathcal{C}_p(X)} [A].$$

Algorithm 5.1. GetZ2Generators

```
function GetZ2Generators(S-complex X, integer p)
begin
   S := empty structure for disjoint sets;
   1:  foreach b in X_p do S.makeSet(b);
   2:  foreach a in X_{p-1} do begin
      (b_1, b_2) := cbd(a);
      if S.find(b_1)=S.find(b_2) then
         S.makeSet(a);
      else
         S.union(b_1, b_2);
   end;
   return sets from S;
end;
```

Proof: The result follows immediately from Theorem 3.4 and Theorem 4.2. □

5. The gluing algorithm

The gluing algorithm which we present in this section is based on the standard disjoint-set data structure which maintains a collection $\mathtt{S} = \{ S_1, \dots, S_k \}$ of disjoint sets. Each set in S is identified by a representative, which is a member of the set (see [1, Chapter 21]). The following operations may be performed on the structure S

- $\mathtt{S}.\operatorname{makeSet}(x)$ - creates a new set whose only member (and thus representative) is x,
- $\mathtt{S}.\operatorname{find}(x)$ - returns a pointer to the representative of the (unique) set containing x,
- $\mathtt{S}.\operatorname{union}(x, y)$ - unites the sets that contain x and y into a new set that is the union of these two sets.

Lemma 3.5 leads to the following iterative algorithm for computing homology groups of $(p-2)$-faceless weak p-pseudomanifolds.

Let $(a_1, a_2, \dots, a_k)$ be the sequence of elements of X_{p-1} in the order in which they appear in the second **foreach** loop in Algorithm 5.1. Let $X^0 = X_p$ and for $i = 1, 2, \dots k$ put

$$X^i := X_p \cup \{ a_1, a_2, \dots, a_i \}.$$

Let $\mathtt{S}^i$ denote the contents of the variable $\mathtt{S}$ at the end of i-th iteration of the loop labeled 2 for $i > 0$ and at the beginning of this loop for $i = 0$.

Lemma 5.2. *For $i = 0, \ldots, k$ the set X^i is an S-complex.*

Proof: We proceed by induction on i. For $i = k$ we have $X = X^k$, so X^k is an S-complex. Assume that the lemma is true for some $i \in \{1, 2, \ldots, k\}$. We will show that it holds also for $i - 1$. By the induction assumption X^i is an S-complex. Moreover, a_i is closed in X^i, hence X^{i-1} is open in X^i. By Proposition 2.1 and Proposition 2.2 the conclusion holds for i. $\square$

Lemma 5.3. *For $i = 0, 1, 2, \ldots, k$ and for all $S \in \mathtt{S}^i$ we have $S \subset X_p^i$ or $S \subset X_{p-1}^i$.*

Proof: We proceed by induction on i. Consider first the case $i = 0$. At the beginning of the second **foreach** loop no $\mathtt{S}.\operatorname{union}(\cdot,\cdot)$ operation is applied yet to the structure $\mathtt{S}$. Therefore $\mathtt{S} = \{\, \{\, a \,\} \mid a \in X_p \,\}$ and the lemma holds true. Assume that it is true for $i-1$. Let $S \in \mathtt{S}^i$. If $S \in \mathtt{S}^{i-1}$ then the conclusion holds by the induction assumption. If $S \notin \mathtt{S}^{i-1}$ then $S = \{\, a_i \,\}$ or $\operatorname{cbd} a_i = \{\, b_1, b_2 \,\} \subset S$. In the first case the lemma is obviously true. In the second case S is the union of $S_1 = \mathtt{S}^{i-1}.\operatorname{find}(b_1)$ and $S_2 = \mathtt{S}^{i-1}.\operatorname{find}(b_2)$, but by the induction assumption $S_1, S_2 \subset X_p^i$ therefore $S = S_1 \cup S_2 \subset X_p^i$. $\square$

Lemma 5.3 allows us to define the dimension $\dim S$ of $S \in \mathtt{S}^i$ as the common dimension of the elements of S. We put

$$\mathtt{S}_q^i := \{\, S \in \mathtt{S}^i \mid \dim S = q \,\}.$$

Lemma 5.4. *For $i = 0, 1, 2, \ldots, k$ and for all $u, v \in X_p$*

$$\operatorname{cc}_{X^i}(u) = \operatorname{cc}_{X^i}(v) \tag{12}$$

if and only if

$$\mathtt{S}^i.\operatorname{find}(u) = \mathtt{S}^i.\operatorname{find}(v). \tag{13}$$

Proof: We proceed by induction on i. At the beginning of the second **foreach** loop no $\mathtt{S}.\operatorname{union}(\cdot,\cdot)$ operation is applied yet to the structure $\mathtt{S}$. Therefore $\mathtt{S} = \{\, \{\, a \,\} \mid a \in X_p \,\}$ and the lemma holds true for $i = 0$. Thus fix $i > 0$ and assume the conclusion holds true for $j < i$.

Let $u, v \in X_p$. First observe that properties (12) and (13) are monotone with respect to i in the sense that if the property hold for some i then it holds for $i+1$, because the algorithm only glues sets.

Assume $\operatorname{cc}_{X^i}(u) = \operatorname{cc}_{X^i}(v)$. If $\operatorname{cc}_{X^{i-1}}(u) = \operatorname{cc}_{X^{i-1}}(v)$, the conclusion follows from the induction assumption and the monotonicity of (13).

Otherwise $\mathtt{S}^{i-1}.\,\mathrm{find}(u) \neq \mathtt{S}^{i-1}.\,\mathrm{find}(v)$. However, then the operation $\mathtt{S}.\,\mathrm{union}(\cdot,\cdot)$ occurs, therefore $\mathtt{S}^{i}.\,\mathrm{find}(u) = \mathtt{S}^{i}.\,\mathrm{find}(v)$. This proves that (12) implies (13).

To prove the opposite implication assume that $\mathtt{S}^{i}.\,\mathrm{find}(u) = \mathtt{S}^{i}.\,\mathrm{find}(v)$. If $\mathtt{S}^{i-1}.\,\mathrm{find}(u) = \mathtt{S}^{i-1}.\,\mathrm{find}(v)$, then the conclusion follows from the induction assumption and the monotonicity of (12). Otherwise $u \in cc_{X^{i-1}}(b_1)$ and $v \in cc_{X^{i-1}}(b_2)$ (or $v \in cc_{X^{i-1}}(b_1)$ and $u \in cc_{X^{i-1}}(b_2)$), and:

$$\mathrm{cc}_{X^i}(u) = \mathrm{cc}_{X^i}(b_1) = \mathrm{cc}_{X^i}(a) = \mathrm{cc}_{X^i}(b_2) = \mathrm{cc}_{X^i}(v)$$

which proves that (13) implies (12). □

Theorem 5.5. *Algorithm 5.1 called with a $(p-2)$-faceless weak p-pseudomanifold X returns a collection of sets $\mathcal{S}$ such that*

$$H(X,\mathbb{Z}_2) \cong \bigoplus_{S\in\mathcal{S}}[S]. \tag{14}$$

Proof: It is sufficient to prove that for $i = 0,1,2,\dots,k$ and for $q \in \{\, p-1, p \,\}$

$$H_q(X^i,\mathbb{Z}_2) \cong \bigoplus_{S\in\mathtt{S}^i_q}[S]. \tag{15}$$

because we get (14) from (15) with $i = k$. Note that by Lemma 5.2 the homology $H_q(X^i,\mathbb{Z}_2)$ is well defined.

We proceed by induction on i. Consider first the case $i = 0$. At the beginning of the second **foreach** loop no $\mathtt{S}.\,\mathrm{union}(\cdot,\cdot)$ operation is applied yet to the structure $\mathtt{S}$. Therefore (15) follows immediately from Corollary 4.3.

Fix $i > 0$ and assume (15) holds for $j < i$. We apply Lemma 3.5 and Lemma 4.1 with $X = X^{i-1}$ and $a = a_i$. Observe that in this case $X \setminus a = X^i \setminus a_i = X^{i-1}$. Let $\mathrm{cbd}\, a_i = \{\, b_1, b_2 \,\}$.

Consider first the case when $\mathtt{S}^{i-1}.\,\mathrm{find}(u) = \mathtt{S}^{i-1}.\,\mathrm{find}(v)$. Then, by Lemma 4.1 and 5.4, $\bar{\partial}_p = 0$. Therefore we get from Lemma 3.5 and the induction assumption

$$\begin{aligned} H_p(X^i,\mathbb{Z}_2) &\cong H_p(X^{i-1},\mathbb{Z}_2) \\ &\cong \bigoplus_{S\in\mathtt{S}^{i-1}_p}[S] \cong \bigoplus_{S\in\mathtt{S}^{i}_p}[S]. \end{aligned} \tag{16}$$

By the same lemma we get

$$\begin{aligned} H_{p-1}(X^i,\mathbb{Z}_2) &\cong H_{p-1}(X^{i-1},\mathbb{Z}_2) \oplus H_{p-1}(a,\mathbb{Z}_2) \\ &\cong \bigoplus_{S\in\mathtt{S}^{i-1}_{p-1}}[S] \oplus H_{p-1}(a,\mathbb{Z}_2). \end{aligned}$$

Since in the considered case the algorithm performs $\mathtt{S}.\operatorname{makeSet}(a)$, we see that

$$H_{p-1}(X^i,\mathbb{Z}_2) \cong \bigoplus_{S\in \mathtt{S}^i_{p-1}} [S]. \tag{17}$$

Consider now the case $\mathtt{S}^{i-1}.\operatorname{find}(u) \neq \mathtt{S}^{i-1}.\operatorname{find}(v)$. Then $\bar{\partial}_p \neq 0$, and consequently

$$\begin{aligned} H_{p-1}(X^i,\mathbb{Z}_2) &\cong H_{p-1}(X^{i-1},\mathbb{Z}_2) \\ &\cong \bigoplus_{S\in \mathtt{S}^{i-1}_{p-1}} [S] \cong \bigoplus_{S\in \mathtt{S}^{i}_{p-1}} [S] \end{aligned} \tag{18}$$

by Lemma 3.5 and the induction assumption. There remains to prove that

$$H_p(X^i,\mathbb{Z}_2) \cong \bigoplus_{S\in \mathtt{S}^i_p} [S].$$

Let $Y_j := cc_{X^{i-1}}(b_j)$ for $j = 1, 2$. Observe that by Lemma 5.4 the sets $(Y_1)_p, (Y_2)_p \in \mathtt{S}^{i-1}_p$. Let $Y := X^{i-1} \setminus (Y_1 \cup Y_2)$. Then

$$H_p(X^{i-1}) = H_p(Y) \oplus H_p(Y_1 \cup Y_2)$$

and by Lemma 3.5

$$\begin{aligned} H_p(X^i,\mathbb{Z}_2) &\cong \ker \bar{\partial}_p \\ &\cong \ker \bar{\partial}_p|_{H_p(Y,\mathbb{Z}_2)} \oplus \ker \bar{\partial}_p|_{H_p(Y_1\cup Y_2,\mathbb{Z}_2)} \\ &\cong H_p(Y,\mathbb{Z}_2) \oplus [(Y_1)_p \cup (Y_2)_p]. \end{aligned}$$

Therefore, by the induction assumption

$$\begin{aligned} H_p(X^i,\mathbb{Z}_2) &\cong \bigoplus_{S\in \mathtt{S}^{i-1}_p \setminus \{\, (Y_1)_p, (Y_2)_p \,\}} [S] \oplus [(Y_1)_p \cup (Y_2)_p] \\ &\cong \bigoplus_{S\in \mathtt{S}^i_p} [S]. \end{aligned}$$

□

Theorem 5.6. *Algorithm 5.1 runs in $O(n \log^* n)$ time, where n denotes the cardinality of the S-complex on input.*

Proof: We call at most $2n$ times operation $\mathtt{S}.\operatorname{makeSet}(\cdot)$ and at most n times operations $\mathtt{S}.\operatorname{union}(\cdot,\cdot)$ and $\mathtt{S}.\operatorname{find}(\cdot)$. Hence, by [1, Theorem 21.13], such a sequence of operations takes $O(n \log^* n)$. □

Algorithm 6.1. Coreduction ([8, Algorithm 6.1])

```
function Coreduction (S-complex S, a generator s)
begin
   Q := empty queue of generators;
   L := empty list of coreduction pairs;
   enqueue(Q,s);
   while Q ≠ ∅ do begin
      s := dequeue(Q);
      if s ∉ S continue;
      if bd_S s contains exactly one element t then begin
         S := S \ {s};
         foreach u ∈ cbd_S t do
            if u ∉ Q then enqueue(Q,u);
         S := S \ {t};
         pushBack(L,(t,s));
      end
      else if bd_S s = ∅ then
         foreach u ∈ cbd_S s do
            if u ∉ Q then enqueue(Q,u);
   end;
   return (S,L);
end;
```

6. Coreduction

Algorithm 6.1 which we use is a simple modification of the coreduction algorithm [8, Algorithm 6.1]. The modification consists in collecting all coreduction pairs in a list. When the algorithm reduces a coreduction pair, then we add the pair to a list L.

Theorem 6.2. *Let $M > 0$ be a fixed integer. Algorithm 6.1 called with an S-complex $X \in \mathcal{S}_M$ and a generator $s \in X_0$ on input returns a pair (Y, L) such that $H(Y)$ is isomorphic to $H(X \setminus v)$ and L is a list of all reduction pairs removed from X by the algorithm. The algorithm runs in time $O(n)$, where n denotes the cardinality of X.*

Proof: The fact that $H(Y)$ is isomorphic to $H(X \setminus v)$ follows immediately from Proposition 2.3. The fact that L is a list of all reduction pairs removed from X is obvious. The complexity analysis of Algorithm 6.1 is the same as of [8, Algorithm 6.1] in [8, Corollary 6.3]. □

Theorem 6.3. *If X on input of Algorithm 6.1 is a weak 2-pseudomanifold, then also Y returned by the algorithm is a weak 2-pseudomanifold.*

Proof: Let us assume that the Algorithm 6.1 reduces a sequence of elementary coreduction pairs $\{ (f_i, c_i) \}_{i=1}^r$ [8, Chapter 4]. We proceed by induction on r to show that $Y = X \setminus \bigcup_{i=0}^r \{ (f_i, c_i) \}$ is a weak 2-pseudomanifold.

For $r = 0$ we have $Y = X$ and the assertion is obvious. Therefore fix an $r > 0$ and assume $Y^j = X \setminus \bigcup_{i=0}^j \{ (f_i, c_i) \}$ is a weak 2-pseudomanifold for $j < r$. There are only three possibilities for $\{ (f_r, c_r) \}$:

(i) $f_r = \emptyset$ and $\dim(c_r) = 0$
(ii) $\dim(f_r) = 0$ and $\dim(c_r) = 1$
(iii) $\dim(f_r) = 1$ and $\dim(c_r) = 2$

We have to show that for all $e \in Y_1^r$ the cardinality of $\mathrm{cbd}_{Y^r} e$ is exactly two. In the cases (i) and (ii) $\mathrm{cbd}_{Y^r} e = \mathrm{cbd}_{Y^{r-1}} e$ for any $e \in Y_1^r$. In the third case $\mathrm{bd}_{Y^{r-1}} c_r = \{ f_r \}$ because it is a coreduction pair. Hence again $\mathrm{cbd}_{Y^r} e = \mathrm{cbd}_{Y^{r-1}} e$ for any $e \in Y_1^r$. It follows by the induction assumption that the cardinality of $\mathrm{cbd}_{Y^r} e$ is two for any $e \in Y_1^r$. Therefore $Y = Y^r$ is a weak 2-pseudomanifold. □

7. Geometric S-complexes.

`<<COMMENT>> Uogolnilem i uproscilem zalozenia. Te wydaja sie bardziej naturalne.` !!

We say that an S-complex X is *geometric* if the following three conditions are satisfied:

(i) $X_q = \emptyset$ for $q < 0$,
(ii) for each $a \in X_1$ the set $\mathrm{bd}\, a$ consists of exactly two elements $a^-, a^+ \in X_0$ such that $\kappa(a, a^-) = -\kappa(a, a^+)$,
(iii) for each $p \geq 2$ and for each $b \in X_p$ the geometric boundary of b is connected.

It is straightforward to observe that an S-complex generated by a regular CW complex is a geometric.

`<<COMMENT>> Oslabilem zalozenie. Wydaje sie, ze tu nie potrzebujemy 2-rozmaitości, tylko wystarczy S-kompleks.` !!

Theorem 7.1. *Algorithm 6.1 called with a geometric, connected S-complex X and a generator $s \in X_0$ on input returns a pair (Y, L) such that Y is a vertexless S-complex.*

Proof: The algorithm deletes the vertex s provided on input. Therefore, it is sufficient to prove that for any $u, v \in X_0$ if the coreduction algorithm deletes u, then it also deletes v. Since X is connected, there exists a path joining u and v. Let $P = \{ p_i \}_{i=1}^k$ be such a path of minimal dimension and let the dimension be q. We claim that q is one. To see this, let $p_j \in X_q$ be a q-dimensional element of P. Then $p_{j-1}, p_{j+1} \in \operatorname{bd} p_j$ and if $q \geq 2$, then by the third property in the definition of a geometric complex there exists a path P' in $\operatorname{bd} p_j$ joining p_{j-1} and p_{j+1}. Therefore, replacing p_j in P by P' we obtain a new path joining u and v of dimension $q-1$, a contradiction.

First assume that $k = 3$. Since $p_1 = u$ is deleted, p_2 is placed in the queue `Q`. Suppose by contrary that $v = p_3$ is not deleted. There are two cases to consider. Either p_2 is deleted by the algorithm or it is not. The other case leads immediately to a contradiction, because then (p_2, p_3) constitutes a coreduction pair, which is removed from X when p_2 is removed from the queue `Q`. Thus assume that p_2 is deleted. Then it is deleted in a coreduction together with its face or its coface. Since $p_2 \in X_1$ and X is geometric, the only face left for a coreduction is p_3, so in this case p_3 is deleted. Thus assume p_2 is deleted together with its coface c. Let T denote the contents of S variable on entering the pass of the while loop on which the pair (p_2, c) is deleted by the algorithm. Since T is an S-complex and $\operatorname{bd}_T c = \{ p_2 \}$,

$$0 = \partial_T \partial_T c = \kappa(c, p_2) \partial_T p_2 = \kappa(c, p_2) \kappa(p_2, p_3) p_3 \neq 0,$$

a contradiction.

Now fix $k > 3$ and assume that the conclusion holds for all paths of length less than k. Let $P = \{ p_i \}_{i=1}^k$ be a path of length k such that p_1 is deleted. Observe that $p_{k-1} \in X_1$ and $p_{k-2} \in X_0$. Using the induction assumption for path $P_0 = \{ p_i \}_{i=1}^{k-2}$ of length $k-2$ and for path $P_1 = \{ p_i \}_{i=k-2}^{k}$ of length 3 we conclude that p_k is deleted. □

Theorem 7.2. *If X is a geometric, connected S-complex, then $H_0(X)$ is isomorphic to R.*

Proof: First observe that since X is connected, it is nonempty and since it is geometric, $X_0 \neq \emptyset$. Let $v \in X_0$. Then v is closed in X, so we have the following exact sequence

$$0 \to H_1(X) \to H_1(X \setminus v) \xrightarrow{\bar{\partial}_1} H_0(v) \to H_0(X) \to H_0(X \setminus v) \to 0.$$

Let $z \in Z_1(X \setminus v)$. We have

$$\partial_X z = \partial_{X \setminus v} z + \alpha v$$

Algorithm 8.1. WeakPseudomanifoldBettiNumbers

```
function WeakPseudomanifoldBettiNumbers(S-complex X)
begin
   { X^1,...,X^k } := ConnectedComponents(X);
   foreach i ∈ { 1,...,k } do
      a^i := any vertex in X^i_0;
      (Y^i, L^i) := Coreduction(X^i, a);
   S:= GetZ2Generators(⋃_{i=1}^k Y^i, 2);
   β_0 := k;
   β_1 := card S_1;
   β_2 := card S_2;
   return (β_0, β_1, β_2);
end;
```

for some $v \in R$. Since z is a cycle in $X \setminus v$, we get

$$\partial_X z = \alpha v.$$

Consider the augmentation map $\epsilon : R(X_0) \to R$ defined on generator $v \in X_0$ by $\epsilon(v) = 1$.

By assumption (ii) of a geometric S-complex we see that $\epsilon(\partial_X z) = 0$. Therefore

$$\alpha = \epsilon(\alpha v) = \epsilon(\partial_X z) = 0,$$

which means that $\bar{\partial}_1 = 0$. By Theorem 6.2 and Theorem 7.1 the homology of $X \setminus v$ is isomorphic to the homology of a vertexless S-complex, so $H_0(X \setminus v)$ is zero. It follows that $H_0(X)$ is isomorphic to $H_0(v)$ and hence isomorphic to R. □

8. Main results

Now we are ready to present the algorithm for homology groups of a geometric weak 2-pseudomanifold. It is based on Algorithm 5.1 and Algorithm 6.1. We also use `ConnectedComponents` function which computes connected components of an S-complex. Note that the problem of finding the connected components of an S-complex is equivalent to finding the connected components of the graph $G = (X, E)$ where $E = \{\{x, y\} \in X \times X \mid x \text{ is adjacent to } y\}$. For the graph we may use `BFS` or `DFS` approach presented in [1, Chapter 22.3] and in both cases the complexity is linear.

Algorithm 8.3. WeakPseudomanifoldHomology

```
function WeakPseudomanifoldHomology(S-complex X)
begin
   { X^1, ..., X^k } := ConnectedComponents(X);
   L := empty list of coreduction pairs;
   foreach i ∈ { 1, ..., k } do
      a^i := any vertex in X_0^i;
      (Y^i, L^i) := Coreduction(X^i, a);
      L.append(L^i);
   S:= GetZ2Generators(⋃_{i=1}^k Y^i, 2);
   G := ExtractCoreductionGenerators(S, L);
   return G;
end;
```

Theorem 8.2. *Let $M > 0$ be a fixed integer. Algorithm 8.1 called with a geometric weak 2-pseudomanifold $X \in \mathcal{S}_M$ on input returns the Betti numbers of $H(X, \mathbb{Z}_2)$ in time $O(nlog^*n)$, where n denotes the cardinality of X.*

Proof: By Theorem 3.4 and Theorem 7.2 the number β_0 returned by the algorithm is indeed the 0th Betti number of X. Theorems 6.3 and 7.1 imply that the input of algorithm `GetZ2Generators` satisfies the assumptions of Theorem 5.5. Therefore, we get from Theorem 5.5 that β_1 and β_2 are the first and second Betti numbers of X. By [1, Chapter 22.3] the `ConnectedComponents` function may be computed in time $O(n)$. Since Theorem 6.2 implies that the `Coreduction` function calls have complexity $O(n)$, we get from Theorem 5.6 the total complexity of $O(n \log^* n)$. □

We can also get the generators of $H(X, \mathbb{Z}_2)$ via a simple modification of Algorithm 8.1. For this end we need the function `Extract-CoreductionGenerators` which computes $\iota^\alpha(g)$ for all $g \in Y$ where

$$\iota^\alpha = \iota^{(a_1,b_1)} \circ \iota^{(a_2,b_2)} \circ \cdots \circ \iota^{(a_n,b_n)}$$

for $(a_i, b_i) \in L$ and

$$\iota^{(a,b)}(c) := \begin{cases} c - \frac{\langle \partial c, a \rangle}{\langle \partial b, a \rangle} b & \text{if } k = m, \\ c & \text{otherwise.} \end{cases}$$

(see [9]).

Theorem 8.4. *Algorithm 8.3 called with a geometric weak 2-pseudomanifold $X \in \mathcal{S}_M$ on input returns the generators of $H(X, \mathbb{Z}_2)$ in time $O(n(m + \log^* n))$, where n denotes the cardinality of the S-complex on input and m is the number of homology generators.*

Proof: By [1, Chapter 22.3] the `ConnectedComponents` function may be computed in time $O(n)$. By [8, Corollary 6.3] the `Coreduction` function calls have complexity $O(n)$. By Theorem 5.6 `GetZ2Generators` has complexity $O(nlog^*n)$. `ExtractCoreductionGenerators` may be computed in $O(nm)$ (see [9, Theorem 5.1]) which results in the total complexity $O(n(m + \log^* n))$. □

9. Final comments

An implementation by the second author of the coreduction homology algorithm, written in C++, is available from [6] and the web pages of the Computer Assisted Proofs in Dynamics Project [14] and the Computational Homology Project [13]. An implementation by the first author of the adaptation of the coreduction homology algorithm to weak 2-pseudomanifolds presented in this paper is in preparation [5].

By using the elementary coreduction pairs together with elementary reduction pairs it is possible to extend the results of this paper to 2-dimensional S complexes with the property that each edge has at most two elements in its coboundary. The details will be presented in [5].

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