Existence results for arbitrarily vertex decomposable trees

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Abstract. In [2], it was shown that the vertex degrees in an arbitrarily vertex decomposable tree are bounded above by 6. We establish several existence and non-existence results for such trees, and improve the upper bound of the vertex degrees to 5, using purely combinatorial methods.

1. Arbitrarily vertex-decomposable trees

1.1. Statement of the problem

Definition 1.1 *A tree T is* arbitrarily vertex-decomposable *if for every partition $n = a_1 + a_2 + \cdots + a_k$ of n into positive integers, there is a partition of the vertex set V_T of T into subsets of size $a_1, a_2, \ldots, a_k$ such that the induced subgraph on each subset is connected. AVT stands for "arbitrarily vertex-decomposable tree."*

Definition 1.2 *A* comet *is a tree T in which exactly one vertex c has degree greater than 2. We call c the* central vertex *of T. An* arm *of a comet T is a connected component of the induced subgraph on $V_T - \{c\}$.*

We denote a comet with central vertex of degree Δ as $S(a_1, a_2, \ldots, a_\Delta)$, where $a_1, a_2, \ldots, a_\Delta$ are the lengths of the arms; thus $|S(a_1, a_2, \ldots, a_\Delta)| = 1 + a_1 + a_2 + \cdots + a_\Delta$. See Figure 1.1.

In this paper for convenience we assume $a_1 \le a_2 \le \cdots \le a_\Delta$. We reserve n to stand for the number of vertices in a tree.

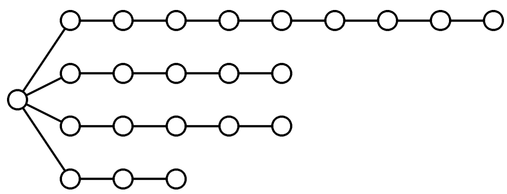

Figure 1.1: *S(3, 5, 5, 9)*

Theorem 1.3 *There is no AVT with $\Delta(T) \geq 7$.*

This is Theorem 13 in [2]. In the same paper the following question appeared.

Conjecture 1.4 *There is no AVT with $\Delta(T) \geq 5$.*

This conjecture remains unproven, yet we show in this paper that there is no AVT with $\Delta(T) \geq 6$.

1.2. Infinite classes of AVTs

Since a path P is obviously an AVT, we have AVTs for every positive integer n.

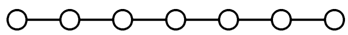

Figure 1.2: *Any path P is an AVT (with $\Delta = 2$)*

Some nontrivial AVTs are shown in the figures 1.3 through 1.5.

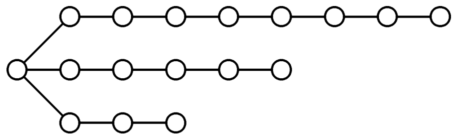

Figure 1.3: *AVT $S(3,5,8)$ with $\Delta = 3$*

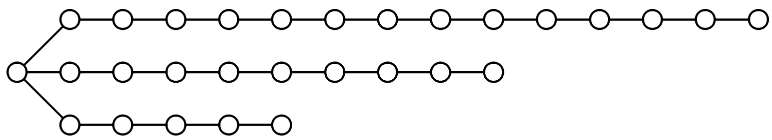

Figure 1.4: *AVT $S(5,9,14)$ with $\Delta = 3$*

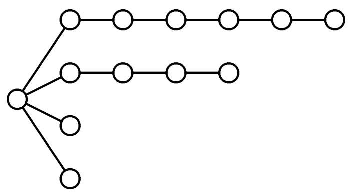

Figure 1.5: *AVT $S(1,1,4,6)$ with $\Delta = 4$*

On the other hand the tree in Figure 1.1 is not AVT; e.g. it can not be decomposed into two components of sizes 11 and 12. Before presenting some infinite classes of AVTs, we introduce the following lemma.

Lemma 1.5 *Let T be the comet $S(a_1, a_2, a_3)$ on n vertices with $\Delta = 3$. Let $b_1, b_2, \ldots, b_k$ be a partitioning of n. Then T can be partitioned into $b_1, b_2, \ldots, b_k$ if either*

1. There exists $i \in \{1,2,3\}$ and $j \in \{1,\ldots,k\}$ s.t. $a_i = b_j$, or
2. There exists $j \in \{1,\ldots,k\}$ such that $b_j > a_1 + a_2$.

Proof.
(1) We will find a partition of T into $b_1, b_2, \ldots, b_k$. Let us take the arm of length a_i to be the partite set of size b_j of V_T. What is left of T is a path of length $n - a_i$ which is an AVT so it can be partitioned into $b_1, \ldots, b_{j-1}, b_{j+1}, \ldots, b_k$. This partitioning together with the set of size b_j forms a partitioning of T into $b_1, b_2, \ldots, b_k$.
(2) We will find a partitioning of T into $b_1, b_2, \ldots, b_k$. Let us take the partite set of size b_j to contain the central vertex and both arms of lengths a_1 and a_2. What is left of T is a path of length $n-1-a_1-a_2$ which is an AVT so it can be partitioned into $b_1, \ldots, b_{j-1}, b_{j+1}, \ldots, b_k$. This partitioning together with the set of size b_j forms a partitioning of T into $b_1, b_2, \ldots, b_k$. □

Theorem 1.6 *Let T be the comet $S(1,1,a)$ on $n = a+3$ vertices. T is an AVT iff n is odd.*

Proof.
($\Rightarrow$) We can easily see that the contrapositive is true: if n is even, then it is obvious that T can not be partitioned into parts of sizes $2, 2, \ldots, 2$.
($\Leftarrow$) If n is odd then in the partitioning of n there has to be at least one part of odd size. We denote this size b. If $b = 1$ the claim follows from Lemma 1.5 part 1. If $b \geq 3$ the claim follows from Lemma 1.5 part 2. □

More generally, we may identify which comets of the form $S(1,a,b)$ are AVTs.

Theorem 1.7 *Let T be the comet $S(1,a,b)$ on n vertices with $1 \leq a \leq b$. T is an AVT iff* $\gcd(a+1, n) = 1$.

Proof.
We denote the arms of T by I, A, and B, of sizes 1, a, and b, respectively, and we denote by c the central vertex of T.
($\Rightarrow$) Again the contrapostive is easy to prove by contradiction. Suppose that $\gcd(a+1, n) = g > 1$. Then write $n = fg$ and consider the partition of n into f equal pieces of size g. Suppose for a contradiction that there is a corresponding parition of T. Since $g|(a+1)$, it follows that c must belong to a partite set all of whose remaining vertices are in A. On the other hand, since $g > 1$, the first arm of T must be contained in the same partite set as the one containing c, a contradiction. So T is not an AVT.
($\Leftarrow$) Suppose that T is not an AVT. Then there is a partition $n = b_1 + \cdots + b_\ell$ with $1 \leq b_1 \leq \cdots \leq b_\ell$ such that there is no corresponding partition of T. If $b_i = 1$ for some i or if $b_i = a$ for some i, then we could find a corresponding partition of T, by Lemma 1.5, part 1. Also, if $b_i \geq a+2$ for some i, then we could find a corresponding partition of T by Lemma 1.5, part 2. So we have $b_i \in \{2, 3, \ldots, a-1, a+1\}$ for all i. Further, we may assume it is not the case that $b_i = a+1$ for all i, since in that case we would have $(a+1)|n$, so $\gcd(a+1, n) > 1$. So in particular, we can say $b_1 \leq a$. It makes sense now to ask for the maximum integer j such that $b_1 + \cdots + b_j \leq a$. Let $d = a + 1 - (b_1 + \cdots + b_j)$. Then $1 \leq d \leq a$. By our choice of j, we have $b_m \geq d$ for all $m > j$. But if $b_m > d$ for some $m > j$, then by partitioning the $b_1 + \cdots + b_j$ vertices on the tail of A into pieces of size $b_1, \ldots, b_j$ and discarding them, we can apply Lemma 1.5, part 2 to what's left of T and conclude that our partition is realizable, a contradiction. Therefore, $b_m = d$ for all $m > j$. Let $s = \ell - j$,

so that $n = b_1 + \cdots + b_j + sd$. We have $sd = n - (b_1 + \cdots + b_j) \geq n - a = 2 + b > a$. So writing $a = qd + r$ for integers q and r with $0 \leq r < d$, we see that $qd \leq a < sd$, so $q < s$; and we can write $a - r = d + d + \cdots + d$, with q pieces of size d. We may partition the qd tailmost vertices of A into q pieces of size d; since we have $(q+1)d > a$, adding another piece of size d to A would swallow all remaining vertices of A together with c; if more, then again our partition would be realizable. Therefore, $d = r + 1$, and $a + 1 = (q+1)d$. Now if $b_1 < d$, then instead of adding another piece of size d to A in the previous argument, we could add a piece of size b_1 followed by a piece of size d, and again realize our partition on T. It follows that $d = b_1 = b_2 = \cdots = b_\ell$, and $d|n$. We have shown that $\gcd(a+1, n) \geq d > 1$. □

We note that in the previous result it is the partitions of n into equal pieces which alone can cause T not to be an AVT. It would be of interest to classify all AVTs of the form $S(a, b, c)$ in a concise way. Compare [1] for related issues, including an investigation of the time-complexity of determining whether $S(a, b, c)$ is an AVT.

2. Known results

One result which is shown in the article [2] is about reduction of the problem from general trees to comets.

Theorem 2.1 *Let T be a tree with a vertex v of degree $\deg(v) \geq 3$ s.t. $T \setminus v$ is a forest in which at least two components are paths. We denote these paths $P_1 = u_1u_2\ldots, u_r$ and $P_2 = v_1v_2\ldots, v_s$. We denote by T'_n the tree which is obtained from T by deleting the edge vv_1 and adding the edge u_rv_1. If T is an AVT then T'_n is also an AVT.*

Proof. It is in [2]. □

Corollary 2.2 *If T is an AVT, then there exists a comet C which is also an AVT, such that $|V_T| = |V_C|$, and such that the maximum vertex degrees of T and C are equal. (That is, any AVT can be "reduced" to an AVT comet.)*

Proof. Is also in [2]. By repeating the construction form Theorem 2.1 we reduce degrees of all nonleaf vertices (except one) to 2. □

Notice the Corollary 2.2 gives us a tool not only for simplifying AVTs from arbitrary AVTs to comets, but its contrapositive is a tool for generating classes of trees which are not AVT.

Corollary 2.3 *Let T be a tree with an edge uv s.t. $T \setminus uv$ is a forest in which at least one component is a path. We denote this path $P = a_1a_2\ldots, a_r$. We denote by T'_n the tree which is obtained from T by deleting the edge uv and adding the edge a_ax where x is any nonleaf vertex in $T \setminus P$. If T is not an AVT then also T'_n is not an AVT.*

Proof. This is the contrapositive of Theorem 2.1. □

3. Classes of graph which are not AVT

To prove the Conjecture 1.4 we can focus on comets as shown in Corollary 2.2. It would be enough to show that all comets with $\Delta = 5$ are not AVTs. The claim would follow immediately from Corollary 2.3.

In the following sections we will construct several classes of comets which are *not* AVT. We can use them to construct other classes with larger Δ.

To show that a tree T is not AVT it is enough to find one partitioning P of n such that T can't be partitioned into P. The most promising candidates for such partitionings are partitionings into *equal* or *almost equal* parts.

Lemma 3.1 *Let T be the comet $S(a_1, a_2, \ldots, a_\Delta)$ with $\Delta \geq 3$. If $n = k(a_i + 1)$ for some i with $1 < i \leq \Delta$, then T is not arbitrarily vertex-decomposable.*

Proof. Consider the partitioning of $n = (a_i + 1) + (a_i + 1) + \cdots + (a_i + 1)$ (k parts). Suppose for a contradiction that T can be partitioned into k components of size $(a_i + 1)$. The component containing the leaf vertex of the arm of length a_i has to contain the central vertex. The component containing the leaf vertex of the arm of length a_1 has also to contain the central vertex. Thus they are in the same component of size at least $a_1 + a_i + 1 > a_i + 1$. But this is a contradiction since all components must have size $a_i + 1$. □

Whenever we can partition n into equal size parts, we expect restrictions on how to partition the corresponding tree. The following lemma is based on this idea.

Lemma 3.2 *Let T be comet $S(a_1, a_2, \ldots, a_\Delta)$ with $\Delta \geq 3$. Suppose $n = k \cdot l$ where $1 < l \leq k$. If $a_i < l$ and $\sum_{j=1}^{i} a_j \geq l$ for some i, $1 < i \leq \Delta$, then T is not arbitrarily vertex-decomposable.*

Proof. Consider the partition $n = l + l + \cdots + l$ (k parts). We claim that T can not be partitioned into k components of size l. We can prove it by contradiction using the same argument as in the proof of Theorem 3.1. □

Theorem 3.3 *Let T be the comet $S(a_1, a_2, \ldots, a_\Delta)$ with $\Delta \geq 3$. If $a_2 \leq k(a_1 - 1)$ and $n \geq k(a_2 + 1)$ for some positive integer k then T is not arbitrarily vertex-decomposable.*

Proof. We can write $n = q \cdot (a_2 + 1) + r$ with integers q and r such that $r \leq a_2$. We know that $q \geq k$ since $n \geq k(a_2 + 1)$. We divide r into q parts differing by at most one and we denote them b and $b + 1$. (We can always do this by solving $r = qb + s$ in integers b and s with $0 \leq s < q$; then s is the number of pieces of size $b + 1$.)

Now consider the partition $n = (a_2 + 1 + b + 1) + \cdots + (a_2 + 1 + b + 1) + (a_2 + 1 + b) + \cdots + (a_2 + 1 + b)$. The corresponding partitioning of T cannot be realized. We show this by contradiction. Suppose such a partitioning exists.

The part containing the leaf vertex in the first arm (of length a_1) has to contain also the central vertex. Similarly the part containing the leaf vertex in the second arm (of length a_2) has to contain also the central vertex. So they are in the same part, but this is a contradiction since no part is big enough to contain both arms a_1 and a_2, as we will now show.

There are two possibilities:

1. If $r = qb$ then the biggest part is of size $a_1 + b + 1$. We have

$$\begin{aligned} k(a_1 - 1) \geq a_2 \geq r = qb \geq kb \\ k(a_1 - 1) \geq kb \\ a_1 \geq b + 1. \end{aligned}$$

And both arms together with the central vertex are bigger than the biggest part:

$$a_1 + a_2 + 1 \geq a_2 + b + 1 + 1.$$
$$a_1 + a_2 + 1 > a_2 + b + 1.$$

2. If $r > bk$ then the biggest part is of size $a_1 + b + 2$. Now we have

$$k(a_1 - 1) \geq a_2 \geq r > qb \geq kb$$
$$k(a_1 - 1) > kb$$
$$a_1 > b + 1.$$

And again both arms together with the central vertex are bigger than the biggest part:
$$a_1 + a_2 + 1 > a_2 + b + 2.$$

□

The previous result can be generalized using Corollary 2.3.

Theorem 3.4 *Let T be the comet $S(a_1, a_2, \ldots, a_\Delta)$ with $\Delta \geq 3$. If there is a positive integer k and an integer i with $1 < i < \Delta$ such that $a_i \leq k(\sum_{j=1}^{i-1} a_j - 1)$ and $n \geq k(a_i + 1)$, then T is not arbitrarily vertex-decomposable.*

Proof. By contradiction. Suppose T satisfying the given conditions is AVT. Then applying Theorem 2.1 we can reattach the shortest arms $a_1, \ldots, a_{i-2}$ to the arm a_{i-1} obtaining an AVT T'_n in which the two shortest arms are of lengths $a'_1 = \sum_{j=1}^{i-1} a_j$ and $a'_2 = a_i$. Since $a_i \leq k(\sum_{j=1}^{i-1} a_j - 1)$, we have $a'_2 \leq k(a'_1 - 1)$. But this is a contradiction with Theorem 3.3 which says that such a graph is not AVT. Thus the assumption was wrong and T is not AVT. □

Corollary 3.5 *If T is an AVT, then we have $a_j > \lfloor \frac{n}{a_j+1} \rfloor (a_1 + a_2 + \cdots + a_{j-1} - 1)$ for all j with $2 \leq j \leq \Delta$.*

Proof. Pick $k = \lfloor \frac{n}{a_j+1} \rfloor$. Then $n \geq k(a_j + 1)$, and since T is an AVT, we must have $a_j > k(\sum_{j=1}^{i-1} a_j - 1)$. □

Corollary 3.6 $a_j > \sqrt{n+1} - 1$ *if* $3 \leq j \leq \Delta$.

Proof. Suppose $3 \leq j \leq \Delta$. By Corollary 3.5, we have $a_j > \lfloor \frac{n}{a_j+1} \rfloor (a_1 + a_2 + \cdots + a_{j-1} - 1) \geq \lfloor \frac{n}{a_j+1} \rfloor \geq \frac{n-a_j}{a_j+1}$, and the result follows. □

Corollary 3.7 $a_\Delta > n/2 - 1$.

Proof. We have $a_\Delta > \lfloor \frac{n}{a_\Delta+1} \rfloor (a_1 + a_2 + \cdots + a_{\Delta-1} - 1) \geq a_1 + a_2 + \cdots + a_{\Delta-1} - 1$. So $2a_\Delta > a_1 + a_2 + \cdots + a_{\Delta-1} + a_\Delta - 1 = n - 2$, and the result follows. □

So far, we have only shown that the arm lengths are large and grow quickly (after the first two, anyway). Next, we will try to show that for some number α, the decomposition $n = \alpha + \alpha + \cdots + \alpha + r$, where $0 \leq r < \alpha$, does not correspond to any possible decomposition of T if T has too many arms.

4. Main result

Theorem 4.1 *If $T = S(a_1, a_2, \dots, a_\Delta)$ is an AVT, then $\Delta \le 5$.*

Proof. Suppose for a contradiction that T is an AVT and $\Delta \ge 6$. Set $a = a_3$ and $b = a_4$. If $a_5 < 3a_4$ then set $c = a_6$; otherwise, set $c = a_5$. We claim that $c \ge 3b$ and $b \ge 3a$. To see this, first note that $a_6 > a_5 > a_4$ by Corollary 3.5. So if $a_4 > n/3 - 1$, then we would have $n = 1 + a_1 + a_2 + \cdots + a_\Delta \ge 1 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 > 3 + 3(n/3 - 1) = n$, a contradiction. Thus $a_4 \le n/3 - 1$, so by Corollary 3.5, we have $a_4 > \lfloor \frac{n}{a_4+1} \rfloor (a_1 + a_2 + a_3 - 1) \ge 3(a_3 + 1)$. So $b \ge 3a$. If $a_5 \ge 3a_4$, then $c = a_5 \ge 3a_4 = 3b$. Otherwise, we have $a_5 < n - 1 - a_1 - a_\Delta < n - 2 - (n/2 - 1)$ by Corollary 3.7, so $a_5 < n/2 - 1$, so $a_5 > 2a_4$ by Corollary 3.5; by Corollary 3.5 again, we have $a_6 > a_1 + a_2 + a_3 + a_4 + a_5 - 1 > 3a_4$, so in this case also, $c \ge 3b$.

Next, set $\ell = 1$, $j = \lceil \frac{b\ell}{a} - \frac{b-a}{2a} \rceil$, and $k = \lceil \frac{cj}{b} - \frac{c-b}{2b} \rceil$. We claim that we have an inclusion of closed intervals $[\frac{b}{j}, \frac{b}{j-1/2}] \subseteq [\frac{a}{\ell}, \frac{a}{\ell-1/2}]$. Indeed, this inclusion merely requires that $\frac{a}{\ell} \le \frac{b}{j}$ and $\frac{b}{j-1/2} \le \frac{a}{\ell-1/2}$, which is equivalent to $j \in [\frac{b\ell}{a} - \frac{b-a}{2a}, \frac{b\ell}{a}]$. The length of this last interval is $\frac{b-a}{2a} \ge \frac{3a-a}{2a} = 1$, so there must be an integer inside of it; so our choice of j must in fact be inside of it. Similarly, we find that $[\frac{c}{k}, \frac{c}{k-1/2}] \subseteq [\frac{b}{j}, \frac{b}{j-1/2}]$.

Our next task is to show that there is an integer α in the half-open interval $(\frac{c}{k}, \frac{c}{k-1/2}]$. For this it is sufficient that $\frac{c}{k-1/2} - \frac{c}{k} \ge 1$, which is equivalent to $c \ge k(2k-1)$. So it will be sufficient for our purpose if $k \le \sqrt{\frac{c}{2}}$. We establish this by dividing the claim into two cases.

Case 1. $c = a_5$.

In this case, we have $c = a_5 < n - a_1 - a_2 - a_3 - a_\Delta \le n - 2 - a_3 - a_\Delta < n - 2 - (\sqrt{n} - 1) - (n/2 - 1)$ (using Corollaries 3.6 and 3.7). So $c < n/2 - \sqrt{n}$, and $2c < n - 2\sqrt{n} < (\sqrt{n} - 1)^2 < a^2$ (using Corollary 3.6). But since $\frac{c}{k} \in [\frac{c}{k}, \frac{c}{k-1/2}] \subseteq [\frac{b}{j}, \frac{b}{j-1/2}] \subseteq [\frac{a}{\ell}, \frac{a}{\ell-1/2}] = [a, 2a]$, we have $\frac{c}{k} \ge a$, so $k \le \frac{c}{a} < \frac{c}{\sqrt{2c}} = \sqrt{\frac{c}{2}}$.

Case 2. $c = a_6$.

To arrive in this case, we must have had $a_5 < 3a_4$. If we had $a_5 \le n/3 - 1$, then Corollary 3.5 would imply $a_5 > 3(a_1 + a_2 + a_3 + a_4 - 1) > 3a_4$, a contradiction. Therefore, $a_5 > n/3 - 1$; if $\Delta \ge 7$, then this is impossible, since it would imply $a_5 + a_6 + a_7 > n$. So we can conclude that $\Delta = 6$. Next we need to establish that $a_4 \le n/4 - 1$. Indeed, if $a_4 > n/4 - 1$, then we would have $a_5 < n - 1 - a_1 - a_\Delta \le n - 2 - a_\Delta < n - 2 - (n/2 - 1) = n/2 - 1$, so by Corollary 3.5, we would get $a_5 > 2(a_1 + a_2 + a_3 + a_4 - 1) \ge 2(a_3 + a_4 + 1) > 2(\sqrt{n} + a_4) > 2(\sqrt{n} + n/4 - 1) = n/2 + 2\sqrt{n} - 1$, so $1 + a_5 + a_\Delta > n$, a contradiction. Now by Corollary 3.5, we find $a_4 > 4(a_3 + 1)$, $a_5 > 2(a_1 + a_2 + a_3 + a_4 - 1) > 2(a_3 + a_4 + 1) > 2(a_3 + 4(a_3 + 1) + 1) = 10(a_3 + 1)$, and $a_6 > a_1 + a_2 + a_3 + a_4 + a_5 - 1 > 15(a_3 + 1)$. Therefore, $n \ge 1 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 > 30(a_3 + 1) + a_1 + a_2 \ge 30a_3 + 32$. Since $a_3 > \sqrt{n} - 1$ by Corollary 3.6, we have $n > 30(\sqrt{n} - 1) + 32 = 30\sqrt{n} + 2$. So $n - 30\sqrt{n} > 2$, $(\sqrt{n} - 15)^2 > 227$, $\sqrt{n} > 15 + \sqrt{227}$, and $n > 900$. Now we have $c = a_6 = a_\Delta > n/2 - 1 > 449$, so $c \ge 450$.

Recalling from above that $a_5 > n/3 - 1$ and $a_4 > a_5/3 > n/9 - 1/3$, we find $c = a_6 = n - 1 - a_1 - a_2 - a_3 - a_4 - a_5 < n - 3 - (\sqrt{n} - 1) - (\frac{n}{3} - 1) - (\frac{n}{9} - \frac{1}{3}) = n - \frac{n}{3} - \frac{n}{9} - \sqrt{n} - \frac{2}{3} < \frac{5n}{9} - \sqrt{n} = \frac{5}{9}(n - \frac{9}{5}\sqrt{n}) = \frac{5}{9}((\sqrt{n} - \frac{9}{10})^2 - \frac{81}{100}) < \frac{5}{9}(\sqrt{n} - \frac{9}{10})^2$. So $\frac{9}{5}c < (\sqrt{n} - \frac{9}{10})^2$, $\sqrt{n} - \frac{9}{10} > \sqrt{\frac{9c}{5}}$, and $a > \sqrt{n} - 1 > \sqrt{\frac{9c}{5}} - \frac{1}{10}$.

Next, note that $j = \lceil \frac{b+a}{2a} \rceil < \frac{b+a}{2a} + 1 = \frac{b}{2a} + \frac{3}{2}$ and $k = \lceil \frac{cj}{b} - \frac{c-b}{2b} \rceil < \frac{cj}{b} - \frac{c-b}{2b} + 1 < \frac{c}{b}(\frac{b}{2a} + \frac{3}{2}) - \frac{c-b}{2b} + 1 = \frac{c}{2a} + \frac{3c}{2b} - \frac{c}{2b} + \frac{3}{2} = \frac{c}{2a} + \frac{c}{b} + \frac{3}{2}$.

Since $b \geq 3a$, we have

$$\begin{aligned}
k &< \frac{c}{2a} + \frac{c}{3a} + \frac{3}{2} \\
&= \frac{5c}{6a} + \frac{3}{2} \\
&< \frac{5c}{6\left(\sqrt{\frac{9c}{5}} - \frac{1}{10}\right)} + \frac{3}{2} \\
&= \frac{\sqrt{c}}{1.2\left(\sqrt{\frac{9}{5}} - \frac{1}{10\sqrt{c}}\right)} + \frac{3}{2} \\
&< \frac{\sqrt{c}}{1.2\left(\sqrt{\frac{9}{5}} - \frac{1}{200}\right)} + \frac{3}{2},
\end{aligned}$$

where the last inequality holds since we know $c \geq 450$. Set $\omega = 1.2\left(\sqrt{\frac{9}{5}} - \frac{1}{200}\right)$. Now $\frac{\sqrt{c}}{\omega} + \frac{3}{2} \leq \sqrt{\frac{c}{2}}$ if and only if $\sqrt{c}(\frac{1}{\sqrt{2}} - \frac{1}{\omega}) \geq \frac{3}{2}$, which is true for $c \geq 450$. This finishes Case 2.

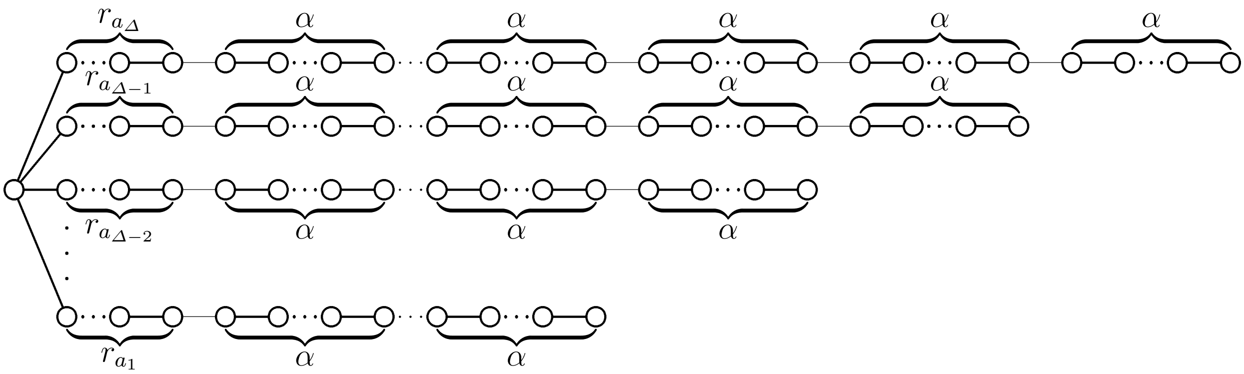

Figure 4.1: *Partitioning of T into parts of size $n = \alpha + \alpha + \cdots + \alpha + r$ ($M = R$ case)*

To summarize, we now have positive integers α, j, k, ℓ such that $\alpha \in (\frac{c}{k}, \frac{c}{k-1/2}] \subseteq [\frac{b}{j}, \frac{b}{j-1/2}] \subseteq [\frac{a}{\ell}, \frac{a}{\ell-1/2}]$. Consider the partition $n = \alpha + \alpha + \cdots + \alpha + r$, where $0 \leq r < \alpha$ (see Figure 4.1). Since T is an AVT, there is a corresponding partition P of V_T. Let A, B, and C denote the vertex sets of the arms of length a, b, and c, respectively. Let M denote the subset of T in our partition P that contains the central vertex of T, and let R denote the unique subset in P of size r. (Note that R may be empty.) First we consider the case that $M = R$. By our choice of partition, the set difference $A - M$ is then partitioned under P into subsets of size α, so α divides $|A - M|$; say $|A - M| = m\alpha$ with $m \in \mathbf{Z}$. Let $r_a = |A \cap M| = a - |A - M|$. Then $0 \leq r_a \leq r < \alpha$, and $a = m\alpha + r_a$, so r_a is the remainder of a modulo α. Since we know that $\alpha \in (\frac{a}{\ell}, \frac{a}{\ell-1/2}]$, where ℓ is an integer, it follows that $r_a \geq \frac{\alpha}{2}$. Since the same conclusion can be drawn about the remainders r_b and r_c, it follows that $r = |R| \geq 1 + r_a + r_b + r_c \geq \frac{3\alpha}{2}$, contradicting that $r < \alpha$.

In case $M \neq R$, at most one arm of T intersects R (exactly one, if R is non-empty). We apply the same argument as above to two of the three arms A, B, C which do not intersect R, and we find $\alpha = |M| \geq 1 + \frac{\alpha}{2} + \frac{\alpha}{2}$, a contradiction. □

References

1. Barth, D., Baudon, O., and Puech, J., *Decomposable trees: a polynomial algorithm for tripodes*, Discrete Applied Mathematics 119 (2002), pp. 205-216.

2. Horňák, Mirko and Woźniak, Mariusz, *Arbitrarily vertex decomposable trees are of maximum degree at most six*, Opuscula Mathematica 23 (2003), pp. 49-62.