Ehsan ESTAJI, Wilfried IMRICH,
Rafał KALINOWSKI, Monika PILŚNIAK

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Distinguishing Cartesian Products of Countable Graphs

Ehsan Estaji
Hakim Sabzevari University, Sabzevar, Iran
ehsan.estaji@hsu.ac.ir

Wilfried Imrich
Montanuniversität Leoben, A-8700 Leoben, Austria
imrich@unileoben.ac.at

Rafał Kalinowski and Monika Pilśniak*
AGH University of Science and Technology,
al. Mickiewicza 30, 30-059 Krakow, Poland
kalinowski, pilśniak@agh.edu.pl

Abstract

The distinguishing number $D(G)$ of a graph $G$ is the minimum number of colors needed to color the vertices of $G$ such that the coloring is preserved only by the trivial automorphism. In this note we improve results about the distinguishing number of Cartesian products of finite and infinite graphs by removing restrictions to prime or relatively prime factors.

Keywords: vertex coloring; distinguishing number; automorphisms; infinite graphs; Cartesian and weak Cartesian products

Mathematics Subject Classifications: 05C25, 05C15, 03E10

1 Introduction

This paper is concerned with automorphisms breaking of Cartesian products of graphs by vertex colorings. Our main focus is on breaking the automorphisms of a graph $G$ with a minimum number of colors. This number is called the distinguishing number $D(G)$ and is defined as the least natural number $d$ such

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that \( G \) has a vertex coloring with \( d \) colors that is only preserved by the trivial automorphism.

The distinguishing number was introduced by Albertson and Collins in [2] and has spawned a wealth of interesting papers. There also is a sizable literature on this problem for the Cartesian product, but not on other products, except for occasional remarks, such as in [5], that the results for the Cartesian product of complete graphs also hold for the direct product, because the structure of the automorphism group is the same. Here we treat the Cartesian product, whereas the extension of the results to the strong and the direct product will appear in [6].

In [7] it was shown that the distinguishing number of the product \( G \square H \) of two finite connected graphs that are relatively prime with respect to the Cartesian product is 2 if \(|G| \leq |H| \leq 2|G| - |G| + 1\). Here we show that \( G \) and \( H \) need not be relatively prime if \( G \square H \) is different from three exceptional graphs that have at most nine vertices.

Furthermore, for countably infinite graphs it was shown in [13] that \( D(G \square H) = 2 \) if \( G \) and \( H \) are connected graphs of infinite diameter. Here we show that the result still holds even when the diameters are not infinite.

2 Preliminaries

We use standard graph theoretic notation, but will denote the order (the number of vertices) and the size (the number of edges) of a graph \( G \) by \(|G|\) and \(|G|\), respectively. Also, we restrict attention to undirected graphs without multiple edges or loops.

The Cartesian product \( G \square H \) has as its vertex set the Cartesian product \( V(G) \times V(H) \). The edge set \( E(G \square H) \) is the set
\[
\{(x, u)(y, v) \mid (xy \in E(G) \land u = v) \lor (x = y \land uv \in E(H))\}.
\]
The product is associative, commutative and \( K_1 \) is a unit.

The graphs \( G \) and \( H \) are called factors of \( G \square H \). We write by \( G^2 \) for the second power \( G \square G \) of \( G \), and recursively define the \( r \)-th Cartesian power of \( G \) as \( G^r = G \square G^{r-1} \). A non-trivial graph \( G \) is called prime if \( G = G_1 \square G_2 \) implies that either \( G_1 \) or \( G_2 \) is the trivial graph. It was proven independently by Sabidussi [12] and Vizing [14] that every connected graph has a prime factor decomposition with respect to the Cartesian product that is unique up to the order and isomorphisms of the factors. Two graphs \( G \) and \( H \) are then called relatively prime if the only common factor of \( G \) and \( H \) is the trivial graph.

Let \( G = G_1 \square G_2 \square \ldots \square G_r \) and \( v \in V(G) \). Then the subgraph \( G^v_1 \) of \( G \) induced by the vertex set
\[
\{(g_1, g_2, \ldots, g_{i-1}, x, g_{i+1}, \ldots, g_r) \mid x \in G_i\}
\]
is called the \( G_1 \)-layer through \( v \). Clearly, every \( G_1 \)-layer is isomorphic to \( G_1 \).
The automorphism group of the Cartesian product of connected graphs is precisely described by the following theorem proven independently by Imrich and Miller [4, 10]. We use the description from [3, Theorem 6.10].

**Theorem 1** Suppose \( \phi \) is an automorphism of a connected graph \( G \) with prime factor decomposition \( G = G_1 \square G_2 \square \ldots \square G_r \). Then there is a permutation \( \pi \) of \( \{1, 2, \ldots, r\} \) and an isomorphism \( \phi_i : G_{\pi(i)} \rightarrow G_i \) for every \( i \) such that

\[
\varphi(x_1, x_2, \ldots, x_r) = (\varphi_1(x_{\pi(1)}), \varphi_2(x_{\pi(2)}), \ldots, \varphi_r(x_{\pi(r)})).
\]  

(1)

There are two important special cases. In the first \( \pi \) is the identity permutation and only one \( \phi_i \) is nontrivial. Then the mapping \( \varphi^*_i \) defined by

\[
\varphi^*_i(x_1, \ldots, x_r) = (x_1, \ldots, x_{i-1}, \varphi_i(x_i), x_{i+1}, \ldots, x_r)
\]

is an automorphism and we say that \( \varphi^*_i \) is induced by \( \varphi_i \in \text{Aut}(G_i) \). Clearly \( \varphi^*_i \) preserves every \( G_i \)-layer and preserves every set of \( G_j \)-layers for fixed \( j \).

The second case is the transposition \( \varphi_{i,j} \) of isomorphic factors \( G_i \cong G_j \). If we assume that \( G_i = G_j \), where \( i < j \), then \( \varphi_{i,j} \) can be defined by

\[
\varphi_{i,j}(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_k) = (x_1, \ldots, x_j, \ldots, x_i, \ldots, x_k).
\]

It is called a transposition of isomorphic factors and interchanges the set of \( G_i \)-layers with the set of \( G_j \)-layers.

The automorphisms induced by the automorphisms of the factors, together with the transposition of isomorphic factors, generate \( \text{Aut}(G) \). Thus every automorphism \( \varphi \in \text{Aut}(G) \) permutes the sets of \( G_i \)-layers in the sense that \( \varphi \) maps the set of \( G_i \)-layers into the set of \( G_{\pi(i)} \)-layers, where \( \pi \) is the permutation from Equation 1.

We now extend the definition of the Cartesian product to arbitrarily many factors. Given an index set \( I \) and graphs \( G_i, i \in I \), we let the Cartesian product

\[
G = \square_{i \in I} G_i
\]

be defined on the vertex consisting of all functions \( x : i \rightarrow x_i \) with \( x_i \in V(G_i) \), where two vertices \( x \) and \( y \) are adjacent if there exists \( \kappa \in I \) such that \( x_\kappa y_\kappa \in E(G_\kappa) \) and \( x_i = y_i \) for \( i \in I \setminus \{\kappa\} \).

For finite \( I \) we obtain the usual Cartesian product, which is connected if and only if all factors are connected. However, the Cartesian product \( G = \square_{i \in I} G_i \) of infinitely many non-trivial connected graphs is disconnected. The connected components are called weak Cartesian products, and we denote the connected component of a vertex \( a \in V(G) \) by

\[
\sqcap_{i \in I} a G_i.
\]
Clearly, $\square_{i \in I} G_i = \square_{b \in I} G_i$ if and only if $a$ and $b$ differ in only finitely many coordinates.

Note that the distance $d(x, y)$ between two vertices $x$ and $y$ that differ in $k$ coordinates is $k$. Hence, there are vertices of arbitrarily large distance in any weak Cartesian product $G$ of infinitely many non-trivial factors. We say the $G$ has infinite diameter.

For automorphisms we have the following theorem of Imrich and Miller [4, 10].

**Theorem 2** Every connected graph is uniquely representable as a weak Cartesian product of connected prime graphs.

Again every $\varphi \in \text{Aut}(G)$ can be represented in the form

$$\varphi(x)_i = \varphi_i(x_{\pi(i)}),$$

where $i \in I$, $\varphi_i \in \text{Aut}(G_i)$, and $\pi$ is a permutation of $I$. As in the finite case all automorphisms of a weak Cartesian product are generated by automorphisms induced by the factors and transpositions of isomorphic factors.

## 3 Finite Cartesian products

The distinguishing number of the Cartesian powers of finite graphs has been thoroughly investigated. It was first proved by Albertson [1] that if $G$ is a connected prime graph, then $D(G^k) = 2$ whenever $k \geq 4$, and, if $|V(G)| \geq 5$, then $D(G^k) = 2$ for $k \geq 3$. Next, Klavžar and Zhu showed in [9] that for any connected graph $G$ with a prime factor of order at least 3 the distinguishing number $D(G^k) = 2$ for $k \geq 3$. Both results were obtained using the Motion Lemma [11]. Finally, Imrich and Klavžar [7] provided the complete solution for the problem of the distinguishing number of the Cartesian powers of connected graphs.

**Theorem 3** [7] Let $G$ be a connected graph and $k \geq 2$. Then $D(G^k) = 2$ except for the graphs $K_2^2, K_3^2, K_3^2$ whose distinguishing number is three.

Their proof is based on the algebraic properties of the automorphism group of the Cartesian product of graphs.

In the same paper Imrich and Klavžar considered the Cartesian product of distinct factors and obtained a sufficient condition when the distinguishing number of the Cartesian product of two relatively prime graphs equals 2.

**Theorem 4** [7] Let $G$ and $H$ be connected, relatively prime graphs such that

$$|G| \leq |H| \leq 2^{|G|} - |G| + 1.$$

Then $D(G \square H) \leq 2$. 

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They also proved several lemmas that will be useful in this paper.

**Lemma 5** [7] Let $G$ and $H$ be two connected, relatively prime graphs such that $2 \leq D(G) \leq 3$ and $D(H) = 2$. Then $D(G \square H) = 2$.

**Lemma 6** [7] Let $G$ and $H$ be two connected graphs such that $G$ is prime, $2 \leq |G| \leq |H| + 1$ and $D(H) = 2$. Then $D(G \square H) = 2$.

We now prove a generalization of Theorem 4 for not necessarily relatively prime graphs.

**Theorem 7** Let $G$ and $H$ be connected graphs such that

$$|G| \leq |H| \leq 2^{|G|} - |G| + 1.$$ 

Then $D(G \square H) \leq 2$ unless $G \square H \in \{K_2^2, K_2^3, K_3^2\}$.

**Proof.** The case when $G$ and $H$ are relatively prime was settled in Theorem 4. Let then $G$ and $H$ have at least one common factor. Let $G = G_1^{l_1} \square \ldots \square G_r^{l_r}$ and $H = H_1^{k_1} \square \ldots \square H_r^{k_r}$ be the prime factor decompositions of $G$ and $H$. Assume that the first $r$ prime factors are common, i.e., $G_i = H_i, i = 1, \ldots, r$. Define

$$G_c = G_1^{l_1} \square \ldots \square G_r^{l_r}, \quad H_c = H_1^{k_1} \square \ldots \square H_r^{k_r}.$$ 

Hence, $G = G_d \square G_c$ and $H = H_d \square H_c$. Now denote

$$n_1 = |G_d|, \quad n_2 = |G_c|, \quad m_1 = |H_d|, \quad m_2 = |H_c|.$$ 

We begin with finding the distinguishing number of the Cartesian product

$$G_c \square H_c = G_1^{l_1+k_1} \square \ldots \square G_r^{l_r+k_r}.$$ 

Due to Theorem 3, for each $i = 1, \ldots, r$, either $D(G_i^{l_i+k_i}) = 2$ or $D(G_i^{l_i+k_i}) = 3$ if $G_i^{l_i+k_i} \in \{K_2^2, K_2^3, K_3^2\}$. The distinguishing number of the Cartesian product of two graphs from the set $\{K_2^2, K_2^3, K_3^2\}$ equals 2 by Theorem 4. It follows from Lemma 5 that $D(G_c \square H_c) = 2$ unless $G_c \square H_c \in \{K_2^2, K_2^3, K_3^2\}$. In this case $D(G_c \square H_c) = 3$.

Now assume that $G \square H \notin \{K_2^2, K_2^3, K_3^2\}$ and consider the graphs $G' = G_d \square G_c \square H_c$ and $H' = H_d$. Clearly, they are relatively prime and $m_1 = |H'| \leq 2^{|G|} - |G'| + 1$. In the remaining part of the proof we consider several cases.

1. If $n_1 n_2 m_2 \leq m_1$, then graphs $G'$ and $H'$ satisfy the conditions of Theorem 4, hence $D(G' \square H') = D(G \square H) \leq 2$.

2. If $n_1 n_2 m_2 > m_1$, then we consider two subcases.

2.1 Let $n_2 m_2 \geq n_1$. 

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2.1.1 If $D(G_c \square H_c) = 2$, then $D(G') = 2$ by repeated application of Lemma 6 since $|G_d| \leq |G_c \square H_c|$.

2.1.2 Let $D(G_c \square H_c) = 3$, i.e., $G_c \square H_c \in \{K_2^2, K_3^2, K_2^3\}$. Recall that $G_d$ and $G_c \square H_c$ are relatively prime, and $|G_d| \leq |G_c \square H_c|$. It is easy to verify that the graphs $G_d$ and $G_c \square H_c$ do not fulfill the assumptions of Theorem 4 only if either $G_d = K_3$ and $G_c \square H_c = K_2^3$, or $G_d = K_2$ and $G_c \square H_c = K_2^4$, or else $n_1 = 1$, and these cases will be considered separately. Otherwise, Theorem 4 implies $D(G') = 2$. Moreover, $|H'| = m_1 < n_1n_2m_2 = |G'|$, so by applying Lemma 6 we arrive at $D(G' \square H') = D(G \square H) = 2$.

2.1.2.1 If $G' = K_3 \square K_3^2$, then the graph $G$ is isomorphic to $K_2 \square K_3$ or to $K_2^2 \square K_3$. Hence, $D(G) = 2$ by Theorem 4, and $D(G \square H_d) = 2$ once more by Theorem 4. Finally, $D(G \square H) = 2$ by Lemma 6.

2.1.2.2 If $G' = K_3 \square K_2^3$, then the graph $G$ is isomorphic to $K_2 \square K_3$. Similarly as in the previous subcase, $D(G) = 2$ and $D(G \square H_d) = 2$ by Theorem 4, and finally we obtain $D(G \square H) = 2$ by Lemma 6.

2.1.2.3 Let $n_1 = 1$. As $|H_c| < |G_c \square H_c|$, we can apply Theorem 4 unless $G_c \square H_c \in \{K_2^2, K_3^2\}$ and $|H_c| \leq 3$. Similarly as in the previous two subcases, one can easily verify that $D(G_c \square H_c \square H_d) = D(G \square H) = 2$.

2.2 Let $n_2m_2 < n_1$. Notice that $4 < n_1$.

2.2.1 Let $|G_d| = n_1 \leq m_1 \leq 2^{\lceil G_d \rceil} - |G_d| + 1$. Here we have $D(G_d \square H_d) \leq 2$ since $G_d$ and $H_d$ are relatively prime and the assumption of Theorem 4 is satisfied. Recall also that $2 \leq D(G_c \square H_c) \leq 3$ and graphs $G_d \square H_d$ and $G_c \square H_c$ are relatively prime. The Cartesian product $G_d \square G_c \square H_d \square H_c = G \square H$ has distinguishing number $2$ by Lemma 5.

2.2.2 Now, let $m_1 > 2^{\lceil G_d \rceil} - |G_d| + 1$. We get the sequence of inequalities

$$m_1 < n_1n_2m_2 \leq n^*_1 < 2^{\lceil G_d \rceil} - |G_d| + 1 < m_1$$

that gives a contradiction.

2.2.3 Let $n_1 > m_1$. Consider the graphs $G'' = G_d$ and $H'' = G_c \square H_d \square H_c$. Observe that $|G''| = n_1 < n_2m_1m_2 = |H''|$ since $n_1n_2 \leq m_1m_2$. Moreover, we have

$$|H''| = n_2m_1m_2 < n_1m_1 < n^*_2 < 2^{\lceil G'' \rceil} - |G''| + 1.$$  

The graphs $G''$ and $H''$ are relatively prime. Therefore, we can apply Theorem 4 to obtain $D(G'' \square H'') = D(G \square H) \leq 2$.

This completes the proof. □
4 Infinite Cartesian products

It was shown in [8] that the distinguishing number of Cartesian product $G \square H$ of two graphs of the same, but arbitrary, cardinality is 2 if $G$ and $H$ are either relatively prime or prime and isomorphic. In this section we remove the condition that $G$ and $H$ are relatively prime or isomorphic if $G$ and $H$ are both countable. That is, we will prove:

**Theorem 8** Let $G$ and $H$ be countably infinite, connected graphs. Then $D(G \square H) \leq 2$.

It generalizes the result from [8] for countably infinite graphs, which we now state and prove for the sake of completeness.

**Theorem 9** Let $G$ and $H$ be two countably infinite, connected graphs that are relatively prime, or prime and isomorphic. Then $D(G \square H) \leq 2$.

**Proof.** Suppose $G$ and $H$ satisfy the assumptions of the theorem. Let $V(G) = V(H) = \mathbb{N}$. We color the vertices $(i, j) \in V(G \square H)$ black if $1 \leq j \leq i$, and white otherwise. Then all vertices of $G^{(1,1)}$ are black. But, because every $H$-layer $H^{(1,1)}$ has $i$ black vertices, each $H$-layer has only finitely many black vertices. Hence, the set of $G$-layers cannot be interchanged with the set of $H$-layers. Furthermore, notice that every $G$-layer $G^{(1,i)}$ has $i - 1$ white vertices. Thus any two $G$ layers have a different number of white vertices and any two $H$-layers different numbers of black vertices. Thus every color-preserving automorphism must fix all $H$-layers and all $G$-layers. The only automorphism with this property is the identity automorphism. □

If both $G$ and $H$ are complete, then we obtain $D(K_{\omega} \square K_{\omega}) = 2$ as a special case. This was shown in [5] with essentially the same coloring.

We first note that Theorem 8 is true if at least one of the graphs $G$ and $H$ has infinitely many factors because of the following theorem of Smith, Tucker and Watkins.

**Theorem 10** [13] If $G$ and $H$ are countably infinite, connected graphs of infinite diameter, then $D(G \square H) = 2$.

**Corollary 11** Let $G$ and $H$ be connected graphs. If $H$ has infinitely many non-trivial factors, then $D(G \square H) = 2$.

**Proof.** If $H$ has infinitely many non-trivial factors, then this is also true for $G \square H$. Hence, we can represent $G \square H$ as a product $G' \square H'$, where both $G'$ and $H'$ are weak Cartesian products with infinitely many factors. Since both graphs must have infinite diameter we get $D(G \square H) = 2$. □
Proof of Theorem 8. By Corollary 11 we only have to consider the case where the prime factorizations of both $G$ and $H$ consist of only finitely many factors. Thus both $G$ and $H$ contain at least one infinite prime factor. Let $G'$ and $H'$ be infinite prime divisors of $G$ and $H$, respectively. Their product is 2-distinguishable by Theorem 9, and hence $G \Box H$ is the product of the countably infinite, 2-distinguishable graph $G' \Box H'$ with finitely many prime graphs, say $A_1, \ldots, A_k$, which can be finite or infinite.

Lemma 12, see below, shows that the product of a connected prime graph with a countably infinite 2-distinguishable graph is also 2-distinguishable. Then the theorem follows by repeated application of Lemma 12.

Lemma 12 Let $G$ and $H$ be connected graphs, where $G$ is finite or infinite, and $H$ countably infinite. If $G$ is prime and $D(H) = 2$, then $D(G \Box H) = 2$.

Proof. We argue similarly as in the proof of Lemma 3.2 in [7]. We color one $H$-layer with a distinguishing 2-coloring $c$. We can assume without loss of generality that $c$ colors infinitely many vertices of $H$ white. Clearly the set of $G$-layers cannot be permuted as all automorphisms of this $H$-layer are broken. If $G$ and $H$ are relatively prime, we color all remaining $H$-layers with distinct 2-colorings. This is possible since $|G| < 2^{|H|}$. Thus all permutations of the $H$-layers are also broken.

If $G$ and $H$ are not relatively prime and $G \neq K_2$, we color all vertices of another $H$-layer black and the remaining $H$-layers such that each layer contains only one black vertex, each of them with a different projection into a white vertex of $H$. Then every $G$-layer is colored with both black and white vertices. If an automorphism maps a $G$-layer into an $H$-layer, then all $G$-layers are mapped into $H$-layers, but one $H$-layer contains only black vertices, hence this is not possible.

Suppose now that $G = K_2$ and that $H$ contains $K_2$ as a factor. Recall that the $k$-th power of $K_2^2$ is the hypercube $Q_k$. Then $G \Box H = K_2 \Box (Q_k \Box H') = Q_{k+1} \Box H'$. Because $K_2$ and $H'$ are relatively prime, the $H'$-layers of $G$ are preserved by every automorphism of $G$.

We now color $G \Box H$. Recall that $G \Box H = K_2 \Box H$ has two $H$-layers, say $H^0$ and $H^1$, both isomorphic to $H$. We color $H^0$ with a distinguishing 2-coloring $c$. This coloring induces 2-colorings of the $H'$-layers that are in $H^0$. These colorings need not be distinguishing colorings of the $H'$-layers, nor need they be different (in the sense that they are equivalent with respect to an automorphism of $H^0$). Both $H^0$ and $H^1$ contain finitely many $H'$-layers, namely $2^k$. Because $H'$ is infinite, it is possible to color the $H'$-layers of $G \Box H$ that are in $H^1$ such that they are

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1The proof of this theorem was roughly outlined in W. Imrich, On the Weak Cartesian Product of Graphs, Topics In Graph Theory. A tribute to A. A. and T. E. Zykov on the occasion of A. A. Zykov’s 90th birthday, University of Illinois at Urbana-Champaign, The personal web page of Professor Alexandr V. Kostochka, http://www.math.uiuc.edu/~kostochk/ (2013), 5663, viewed on August 4, 2015.
pairwise different and different from the 2-colorings of the $H'$-layers in $H^0$ that are induced by $c$.

This means that no automorphism of $G \Box H$ can map an $H'$-layer of $H^1$ into one of $H^0$. Hence, $H^1$, and thus also $H^0$, is preserved. Since $c$ is distinguishing on $H^0$, we infer that we have constructed a distinguishing 2-coloring.

If $k = \aleph_0$, then $G \Box H = K_2 \Box (Q^{\aleph_0} \Box H') \cong Q^{\aleph_0} \Box H'$, and $D(G \Box H) = 2$ by Theorem 9.

**Corollary 13** If $G$ is a countably infinite, connected graph and $2 \leq k \leq \aleph_0$, then $D(G^k) = 2$.

**Proof.** Let $k$ be finite. Then the theorem follows by repeated application of Theorem 8. If $k = \aleph_0$, then the theorem follows from Corollary 11.

**References**


