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Abstract

Let $G = (V, E)$ be a graph of order n. A distance magic labeling of G is a bijection $\ell\colon V \to \{1, 2, \ldots, n\}$ for which there exists a positive integer k such that $\sum_{x\in N(v)} \ell(x) = k$ for all $v \in V$, where $N(v)$ is the open neighborhood of v. In this paper we deal with circulant graphs $C(1, p)$. The circulant graph $C_n(1, p)$ is the graph on the vertex set $V = \{x_0, x_1, \ldots, x_{n-1}\}$ with edges (x_i, x_{i+p}) for $i = 0, \ldots, n-1$ where $i + p$ is taken modulo n. We completely characterize distance magic graphs $C_n(1, p)$ for p odd. We also give some sufficient conditions for p even. Moreover, we also consider a group distance magic labeling of $C_n(1, p)$.

1 Introduction

All graphs considered in this paper are simple finite graphs. Consider a simple graph G whose order we denote by $n = |G|$. Write $V(G)$ for the vertex set and $E(G)$ for the edge set of a graph G. The *open neighborhood* $N(x)$ of a vertex x is the set of vertices adjacent to x, and the degree $d(x)$ of x is $|N(x)|$, the size of the neighborhood of x. By C_n we denote a cycle on n vertices.

In this paper we investigate distance magic labelings, which belong to a large family of magic type labelings. Generally speaking, a magic type labeling of a graph $G(V, E)$ is a mapping from V, E, or $V \cup E$ to a set of

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labels which most often is a set of integers or group elements. Then the weight of a graph element is typically the sum of labels of the neighboring elements of one or both types. If the weight of each element is required to be equal, then we speak about magic-type labeling; when the weights are all different (or even form an arithmetic progression), then we speak about an anti-magic-type labeling. Probably the best known problem in this area is the *anti-magic conjecture* by Hartsfield and Ringel [10], which claims that the edges of every graph except K_2 can be labeled by integers $1, 2, \ldots, |E|$ so that the weight of each vertex is different.

A comprehensive dynamic survey of graph labelings is maintained by Gallian [9]. A more detailed survey related to our topic by Arumugam at al. [1] was published recently.

A *distance magic labeling* (also called *sigma labeling*) of a graph $G = (V, E)$ of order n is a bijection $\ell\colon V \to \{1, 2, \ldots, n\}$ with the property that there is a positive integer k (called the *magic constant*) such that

$$w(x) = \sum_{y \in N_G(x)} l(y) = k \text{ for every } x \in V(G),$$

where $w(x)$ is the *weight* of vertex x. If a graph G admits a distance magic labeling, then we say that G is a *distance magic graph*.

It is worth mentioning that finding an r-regular distance magic labeling turns out equivalent to finding equalized incomplete tournament $\mathrm{EIT}(n, r)$ [8]. In an *equalized incomplete tournament* $\mathrm{EIT}(n, r)$ of n teams with r rounds, every team plays exactly r other teams and the total strength of the opponents that team i plays is k. Thus, it is easy to notice that finding an $\mathrm{EIT}(n, r)$ is the same as finding a distance magic labeling of any r-regular graph on n vertices.

The following observations were independently proved:

Observation 1 ([11], [12], [14], [15]) *Let G be an r-regular distance magic graph on n vertices. Then $k = \frac{r(n+1)}{2}$.*

Observation 2 ([11], [12], [14], [15]) *There is no distance magic r-regular graph with r odd.*

The following cycle-related results were proved by Miller, Rodger, and Simanjuntak, and by Rao, Singh, and Parameswaran, respectively.

Theorem 3 ([12]) *The cycle C_n of length n is distance magic if and only if $n = 4$.*

Theorem 4 ([13]) *The cartesian product $C_n \square C_m$, $n, m \geq 3$ is distance magic if and only if $n = m \equiv 2 \pmod 4$.*

Circulant graphs are another interesting family of vertex-transitive graphs. These graphs arise in various settings; for instance, they are the Cayley graphs over the cyclic group of order n. The circulant graph $C_n(s_1, s_2, \ldots, s_k)$ is the graph on the vertex set $V = \{x_0, x_1, \ldots, x_{n-1}\}$ with edges (x_i, x_{i+s_j}) for $i = 0, \ldots, n-1$, $j = 1, \ldots, k$ where $i + s_j$ is taken modulo n. Moreover, since there is no distance magic r-regular graph for r odd by Observation 2, we can assume that $n > 2s_k + 1$.

It was shown in [2] that a graph $C_n(1, 2)$ is not distance magic unless $n = 6$. For p odd the following theorem was proved:

Theorem 5 ([2]) *If p is odd, then $C_n(1, 2, \ldots, p)$ is a distance magic graph if and only if $2p(p+1) \equiv 0 \pmod n$, $n \geq 2p+2$ and $\frac{n}{\gcd(n,p+1)} \equiv 0 \pmod 2$.*

In this paper we consider the corresponding problem for circulant graphs $C_n(1, p)$. The motivation for considering circulants is a problem stated in [14].

Problem 6 ([14]) *Characterize 4-regular distance magic graphs.*

We will also consider the notion of group distance magic labeling of graphs that was introduced in [7]. A Γ-distance magic labeling of a graph $G(V, E)$ with $|V| = n$ is an injection from V to an Abelian group Γ of order n such that the weight of every vertex $x \in V$ is equal to the same element $\mu \in \Gamma$, called the magic constant. Some families of graphs that are Γ-distance magic were studied in [4, 3, 5, 7]. The following result was proved in [7]:

Theorem 7 ([7]) *The cartesian product $C_n \square C_m$, $n, m \geq 3$ is a $\mathbb{Z}_{nm}$-distance magic graph if and only if nm is even.*

The paper is organized as follows. In the next three sections we consider distance magic circulant graphs $C_n(1, p)$. In the last section we consider a Γ-distance magic labeling.

2 Distance magic graphs $C_n(1, p)$

We start with some observations:

Observation 8 *Let $C_n(1,p)$ be a distance magic graph with magic constant k. Then for any $i \in \{0, 1, \dots, n-1\}$ and any $\gamma \in \mathbb{N}$*

$$\ell(x_i) + \ell(x_{i+p-1}) = \ell(x_{i+\gamma(2p+2)}) + \ell(x_{i+p-1+\gamma(2p+2)}).$$

Proof. Since $C_n(1,p)$ is distance magic we obtain

$$w(x_0) - w(x_1) = w(x_1) - w(x_2) = \dots = w(x_{n-1}) - w(x_0) = 0.$$

Hence

$$w(x_i) - w(x_{i+p+1}) = \ell(x_{i-1}) + \ell(x_{i-p}) - (\ell(x_{i+p+2}) + \ell(x_{i+2p+1})) = 0$$

and

$$w(x_{i+p+1}) - w(x_{i+2p-2}) = \ell(x_{i+p+2}) + \ell(x_{i+2p+1}) - (\ell(x_{i+2p+3}) + \ell(x_{i+3p+2})) = 0$$

for $i \in \{0, 1, \dots, n\}$. So we obtain

$$\ell(x_i) + \ell(x_{i+p-1}) = \ell(x_{i+\gamma(2p+2)}) + \ell(x_{i+p-1+\gamma(2p+2)}).$$

$\blacksquare$

Observation 9 *Let $C_n(1,p)$ be a distance magic graph with magic constant k, then for any $i \in \{0, 1, \dots, n-1\}$ and any $\gamma \in \mathbb{N}$*

$$\ell(x_i) + \ell(x_{i+p+1}) = \ell(x_{i+\gamma(2p-2)}) + \ell(x_{i+p+1+\gamma(2p-2)}).$$

Proof. Since $C_n(1,p)$ is distance magic we obtain

$$w(x_0) - w(x_1) = w(x_1) - w(x_2) = \dots = w(x_{n-1}) - w(x_0) = 0.$$

Hence

$$w(x_i) - w(x_{i+p-1}) = \ell(x_{i+1}) + \ell(x_{i-p}) - (\ell(x_{i+p-2}) + \ell(x_{i+2p-1})) = 0.$$

Also

$$w(x_{i+2p-2})-w(x_{i+3p-3}) = \ell(x_{i+p-2})+\ell(x_{i+2p-1})-(\ell(x_{i+3p-4})+\ell(x_{i+4p-3})) = 0$$

and so on for $i \in \{0, 1, \dots, n\}$. So we obtain

$$\ell(x_i) + \ell(x_{i+p+1}) = \ell(x_{i+\gamma(2p-2)}) + \ell(x_{i+p+1+\gamma(2p-2)}).$$

■

Observations 8 and 9 imply the following corollary.

Corollary 10 *If $C_n(1,p)$ is a distance magic graph, then for any $i \in \{0, 1, \dots, n-1\}$ and for any $\alpha, \gamma \in \mathbb{N}$*

$$\begin{aligned}
\ell(x_i) + \ell(x_{i+(2\alpha+1)(p-1)}) &= \ell(x_{i+\gamma(2p+2)}) + \ell(x_{i+(2\alpha+1)(p-1)+\gamma(2p+2)}),\\
\ell(x_i) + \ell(x_{i+(2\alpha+1)(p+1)}) &= \ell(x_{i+\gamma(2p-2)}) + \ell(x_{i+(2\alpha+1)(p+1)+\gamma(2p-2)}).
\end{aligned}$$

Proof. Since $C_n(1,p)$ is distance magic we obtain by Observation 8

$$\begin{aligned}
&\ell(x_i) + \ell(x_{i+p-1}) = \ell(x_{i+\gamma(2p+2)}) + \ell(x_{i+p-1+\gamma(2p+2)}),\\
&\ell(x_{i+p-1}) + \ell(x_{i+2(p-1)}) = \ell(x_{i+p-1+\gamma(2p+2)}) + \ell(x_{i+2(p-1)+\gamma(2p+2)}),\\
&\ell(x_{i+2(p-1)}) + \ell(x_{i+3(p-1)}) = \ell(x_{i+2(p-1)+\gamma(2p+2)}) + \ell(x_{i+3(p-1)+\gamma(2p+2)}),\\
&\vdots\\
&\ell(x_{i+2\alpha(p-1)}) + \ell(x_{i+(2\alpha+1)(p-1)}) = \ell(x_{i+2\alpha(p-1)+\gamma(2p+2)}) + \ell(x_{i+(2\alpha+1)(p-1)+\gamma(2p+2)}).
\end{aligned}$$

Alternatively subtracting and summarizing the above equations we obtain

$$\ell(x_{i+2(p-1)}) + \ell(x_{i+3(p-1)}) = \ell(x_{i+2(p-1)+\gamma(2p+2)}) + \ell(x_{i+3(p-1)+\gamma(2p+2)}).$$

Using Observation 9 by similar arguments we obtain

$$\ell(x_i) + \ell(x_{i+(2\alpha+1)(p+1)}) = \ell(x_{i+\gamma(2p-2)}) + \ell(x_{i+(2\alpha+1)(p+1)+\gamma(2p-2)}).$$

■

Theorem 11 *If $C_n(1,p)$ is distance magic then $\frac{n}{\gcd(n,p+1)} \equiv 0 \pmod 2$ and $\frac{n}{\gcd(n,p-1)} \equiv 0 \pmod 2$.*

Proof. Let k be a magic constant for $C_n(1,p)$. It is well known that if $a,b \in \mathbb{Z}_n$ and $\gcd(a,n) = \gcd(b,n)$, then a and b generate the same subgroup of $\mathbb{Z}_n$, that is, $\langle a \rangle = \langle b \rangle$. Suppose that $\frac{n}{\gcd(n,p+1)} \equiv 1 \pmod 2$, then we have $\gcd(n,p+1) = \gcd(n,2p+2)$ and $\langle 2(p+1) \rangle = \langle p+1 \rangle$. Hence, $p+1 = 2c(p+1)$ for some $c \geq 1$. Then we use Lemma 8, set $\gamma = c$, $i = 0, p-1$ and obtain respectively:

$$\ell(x_0) + \ell(x_{p-1}) = \ell(x_{2c(p+1)}) + \ell(x_{p-1+2c(p+1)}) = \ell(x_{p+1}) + \ell(x_{2p}),$$

$$\ell(x_{p-1}) + \ell(x_{2p-2}) = \ell(x_{p-1+2c(p+1)}) + \ell(x_{2p-2+2c(p+1)}) = \ell(x_{2p}) + \ell(x_{3p-1}).$$

Since $N(x_i) = \{x_{i-p}, x_{i-1}, x_{i+1}, x_{i+p}\}$ and $C_n(1,p)$ is distance magic, we obtain:

$$\ell(x_0) + \ell(x_{p-1}) = \ell(x_{p+1}) + \ell(x_{2p}) = \frac{k}{2},$$

$$\ell(x_{p-1}) + \ell(x_{2p-2}) = \ell(x_{2p}) + \ell(x_{3p-1}) = \frac{k}{2}.$$

Therefore $\ell(x_0) = \ell(x_{2p-2})$ and we have a contradiction, because $n > 2p+1$.

Suppose that $\frac{n}{\gcd(n,p+1)} \equiv 1 \pmod 2$, then we have $\gcd(n,p+1) = \gcd(n,2p+2)$ and $\langle 2(p+1) \rangle = \langle p+1 \rangle$. Hence, $p+1 = 2c(p+1)$ for some $c \geq 1$. Then we use Lemma 8, set $\gamma = c$, $i = 0, p-1$ and obtain respectively:

$$\ell(x_0) + \ell(x_{p-1}) = \ell(x_{2c(p+1)}) + \ell(x_{p-1+2c(p+1)}) = \ell(x_{p+1}) + \ell(x_{2p}),$$

$$\ell(x_{p-1}) + \ell(x_{2p-2}) = \ell(x_{p-1+2c(p+1)}) + \ell(x_{2p-2+2c(p+1)}) = \ell(x_{2p}) + \ell(x_{3p-1}).$$

Since $N(x_i) = \{x_{i-p}, x_{i-1}, x_{i+1}, x_{i+p}\}$ and $C_n(1,p)$ is distance magic, we obtain:

$$\ell(x_0) + \ell(x_{p-1}) = \ell(x_{p+1}) + \ell(x_{2p}) = \frac{k}{2},$$

$$\ell(x_{p-1}) + \ell(x_{2p-2}) = \ell(x_{2p}) + \ell(x_{3p-1}) = \frac{k}{2}.$$

Therefore $\ell(x_0) = \ell(x_{2p-2})$ and we have a contradiction, because $n > 2p+1$ and then the labeling ℓ is not a bijection.

Suppose now $\frac{n}{\gcd(n,p-1)} \equiv 1 \pmod 2$, then $\gcd(n, p-1) = \gcd(n, 2p-2)$. Thus, $p-1 = c2(p-1)$ for some $c \geq 1$. Then we use Lemma 9, set $\gamma = c$, $i = 0, p+1$ and obtain respectively:

$$\ell(x_0) + \ell(x_{p+1}) = \ell(x_{2c(p-1)}) + \ell(x_{p+1+2c(p-1)}) = \ell(x_{p-1}) + \ell(x_{2p}),$$

$$\ell(x_{p+1}) + \ell(x_{2p+2}) = \ell(x_{p+1+2c(p-1)}) + \ell(x_{2p+2+2c(p-1)}) = \ell(x_{2p}) + \ell(x_{3p+1}).$$

Since $N(x_i) = \{x_{i-p}, x_{i-1}, x_{i+1}, x_{i+p}\}$ and $C_n(1,p)$ is distance magic, we obtain:

$$\ell(x_0) + \ell(x_{p+1}) = \ell(x_{p-1}) + \ell(x_{2p}) = \frac{k}{2},$$

$$\ell(x_{p-1}) + \ell(x_{2p+2}) = \ell(x_{2p}) + \ell(x_{3p+1}) = \frac{k}{2}.$$

Therefore $\ell(x_0) = \ell(x_{2p+2})$. For $n \neq 2p+2$ we obtain a contradiction, since the labeling ℓ. For $n = 2p+2$ we have $\frac{n}{\gcd(n,p+1)} \equiv 0 \pmod 2$, a contradiction. ■

From the above theorem the below observation easily follows:

Observation 12 *If $C_n(1,p)$ is a distance magic circulant graph, then $n \equiv 0 \pmod 2$. Moreover, when p is odd, then $n \equiv 0 \pmod 8$.*

Proof. If n is odd, then $\frac{n}{\gcd(n,p+1)}$ is odd and $C_n(1,p)$ cannot be distance magic by Theorem 11. When p is odd, one of $p-1$, $p+1$ is congruent to 2 modulo 4 and we can write $\{p-1, p+1\} = \{2q_1, 2^t q_2\}$, where $t \geq 2$ and q_1, q_2 are both odd. Let $n = 2^s q_3$, where q_3 is odd. Because $\frac{n}{\gcd(n,p+1)} \equiv 0 \pmod 2$ and $\frac{n}{\gcd(n,p-11)} \equiv 0 \pmod 2$, we must have $s > t \geq 2$. Hence $2^s \geq 8$ and $n \equiv 0 \pmod 8$. ■

Observation 13 *A graph $C_{2p+2}(1,p)$ is distance magic.*

Proof. Let $\ell(x_i) = i+1, \ell(x_{i+p+1}) = 2p+2-i$ for $i = 0, 1, \ldots, p$. Notice that $w(x_i) = \ell(x_{i-p}) + \ell(x_{i+1}) + \ell(x_{i-1}) + \ell(x_{i+p}) = 4p+6$ for every $x_i \in V(C_{2p+2}(1,p))$. ■

2.1 $C_n(1, 2p'+1)$ distance magic graphs

Theorem 14 *If p is odd and $C_n(1,p)$ is distance magic, then $p^2-1 \equiv (0 \bmod n)$.*

Proof. Let k be a magic constant for $C_n(1,p)$.

Assume first that $p \equiv 3 \pmod 4$. By Observation 9 we obtain

$$\begin{aligned}
&\ell(x_0)+\ell(x_{p+1}) = \ell(x_{\gamma(2p-2)})+\ell(x_{p+1+\gamma(2p-2)}) = k_0,\\
&\ell(x_1)+\ell(x_{p+2}) = \ell(x_{1+\gamma(2p-2)})+\ell(x_{p+2+\gamma(2p-2)}) = k_1,\\
&\vdots\\
&\ell(x_{2p-3})+\ell(x_{3p-2}) = \ell(x_{2p-3+\gamma(2p-2)})+\ell(x_{3p-2+\gamma(2p-2)}) = k_{2p-3},
\end{aligned}$$

for any γ. It implies that

$$\begin{aligned}
&\ell(x_{(p+1)}) = k_0-\ell(x_0),\\
&\ell(x_{2(p+1)}) = k_{p+1}-k_0+\ell(x_0),\\
&\ell(x_{3(p+1)}) = k_{2(p+1) \bmod (2p-2)}+k_0-k_{p+1}-\ell(x_0) = k_4+k_0-k_{p+1}-\ell(x_0).
\end{aligned}$$

Repeating the argument, we get

$$\ell(x_{j(p+1)}) = \sum_{i=0}^{j-1}(-1)^{j-1-i}k_{i(p+1) \bmod (2p-2)}+(-1)^j\ell(x_0).$$

So in particular for $j=p-1$,

$$\ell(x_{p^2-1}) = \sum_{i=0}^{p-2}(-1)^{p-2-i}k_{i(p+1) \bmod (2p-2)}+\ell(x_0) =$$

$$\sum_{i=0}^{(p-1)/2-1}(-1)^{p-2-i}k_{i(p+1) \bmod (2p-2)}+\sum_{i=(p-1)/2}^{p}(-1)^{p-2-i}k_{i(p+1) \bmod (2p-2)}+\ell(x_0)$$

Hence if $p \equiv 3 \pmod 4$, then $\gcd(p+1, 2(p-1)) = 4$ and $\frac{p-1}{2}$ is odd. It is known fact that the order of a subgroup generated by $p+1$ in $\mathbb{Z}_{2p-2}$ is $\frac{p-1}{2}$, namely $|\langle p+1\rangle| = \frac{p-1}{2}$. Moreover, notice that $\frac{p-1}{2}(p+1) \equiv 0 \pmod{(2p-2)}$. It implies that

$$\sum_{i=0}^{(p-1)/2-1}(-1)^{(p-1)/2-1-i}k_{i(p+1) \bmod (2p-2)} = -k_0+k_{p+1}-k_{2(p+1)}+k_{3(p+1)}-\ldots-k_{((p-1)/2-1)(p+1)},$$

$$\sum_{i=(p-1)/2}^{p}(-1)^{p-2-i}k_{i(p+1)\bmod(2p-2)}=k_0-k_{p+1}+k_{2(p+1)}-k_{3(p+1)}+\ldots+k_{((p-1)/2-1)(p+1)}.$$

Hence $\ell(x_{p^2-1})=\ell(x_0)$.

Assume now that $p\equiv 1 \pmod 4$. By Observation 8 we obtain

$$\begin{array}{ll}\ell(x_0)+\ell(x_{p-1}) & =\ell(x_{\gamma(2p+2)})+\ell(x_{p+1+\gamma(2p+2)})=k_0,\\ \ell(x_1)+\ell(x_p) & =\ell(x_{1+\gamma(2p+2)})+\ell(x_{p+2+\gamma(2p+2)})=k_1,\\ \vdots & \\ \ell(x_{2p+1})+\ell(x_{3p}) & =\ell(x_{2p+1+\gamma(2p+2)})+\ell(x_{3p+\gamma(2p+2)})=k_{2p+1},\end{array}$$

for any γ. It implies that

$$\begin{array}{l}\ell(x_{(p-1)})=k_0-\ell(x_0),\\ \ell(x_{2(p-1)})=k_{p-1}-k_0+\ell(x_0),\\ \ell(x_{3(p-1)})=k_{2(p-1)\bmod(2p+2)}+k_0-k_{p+1}-\ell(x_0).\end{array}$$

Repeating the argument, we get

$$\ell(x_{j(p-1)})=\sum_{i=0}^{j-1}(-1)^{j-1-i}k_{i(p-1)\bmod(2p+2)}+(-1)^j\ell(x_0).$$

So in particular for $j=p+1$:

$$\ell(x_{p^2-1})=\sum_{i=0}^{p}(-1)^{p-i}k_{i(p-1)\bmod(2p+2)}+\ell(x_0)=$$

$$\sum_{i=0}^{(p+1)/2-1}(-1)^{p-i}k_{i(p-1)\bmod(2p+2)}+\sum_{i=(p+1)/2}^{p}(-1)^{p-i}k_{i(p-1)\bmod(2p+2)}+\ell(x_0)$$

Hence if $p\equiv 1 \pmod 4$ then $\gcd(p-1,2(p+1))=4$ and $\frac{p+1}{2}$ is odd. As above, we obtain

$$\sum_{i=0}^{(p+1)/2-1}(-1)^{(p+1)/2-1-i}k_{i(p-1)\bmod(2p+2)}=-k_0+k_{p-1}-k_{2(p-1)}+k_{3(p-1)}-\ldots-k_{((p+1)/2-1)(p-1)},$$

$$\sum_{i=(p+1)/2}^{p}(-1)^{p-i}k_{i(p-1)\bmod(2p+2)}=k_0-k_{p-1}+k_{2(p-1)}-k_{3(p-1)}+\ldots+k_{((p+1)/2-1)(p-1)}.$$

Hence $\ell(x_{p^2-1})=\ell(x_0)$. ■

Observation 15 *If p is odd, $p^2 - 1 \equiv 0 \pmod{n}$, $\frac{n}{\gcd(n,p+1)} \equiv 0 \pmod{2}$ and $\frac{n}{\gcd(n,p-1)} \equiv 0 \pmod{2}$, then $C_n(1,p)$ is a distance magic graph.*

Proof. Because we always suppose that $n > 2p$ and the case $n = 2p+2$ was treated in Observation 13, we will assume that $n > 2p+2$.

By the assumption $p-1$ and $p+1$ are even and hence one of them is congruent to 0 modulo 4. Moreover, by Observation 12 we know that $n \equiv 0 \pmod{8}$. We will further assume that $p+1 \equiv 0 \pmod{4}$ and leave the other case to the reader, since it is essentially similar.

It is well known that when $a, b \in \mathbb{Z}_n$ and $\gcd(n,a) = \gcd(n,b)$, then $\langle a \rangle = \langle b \rangle$. Obviously, $\gcd(n, \gcd(n,p+1)) = \gcd(n,p+1)$. Hence, $\langle \gcd(n,p+1) \rangle = \langle p+1 \rangle$ and $|\langle \gcd(n,p+1) \rangle| = |\langle p+1 \rangle|$. Because $\frac{n}{\gcd(n,p+1)} \equiv 0 \pmod{2}$ and we assumed that $p+1 \equiv 0 \pmod{4}$, we observe that $\gcd(n,p+1) = 4s$ for some s and that the subgroup $H = \langle p+1 \rangle = \langle 4s \rangle$ of $\mathbb{Z}_n$ is of order $2k$ for some k.

Let us denote by X_j for $j = 0,1,2,\ldots,4s-1$ the set of all vertices whose subscripts belong to the coset $H + j$. First we label vertices of $X_0, X_2, \ldots, X_{4s-2}$. Notice that there are $2s$ of them.

Case 1: $k \equiv 0 \pmod{2}$.

Label the vertices of X_0 as follows:

If $k = 2$, then $\ell(x_0) = 1$, $\ell(x_{p+1}) = n-1$, $\ell(x_{2(p+1)}) = 2$, $\ell(x_{3(p+1)}) = n$.

If $k = 4$, then $\ell(x_0) = 1$, $\ell(x_{p+1}) = n-1$, $\ell(x_{2(p+1)}) = 3$, $\ell(x_{3(p+1)}) = n-3$, $\ell(x_{4(p+1)}) = 4$, $\ell(x_{5(p+1)}) = n-2$, $\ell(x_{6(p+1)}) = 2$, $\ell(x_{7(p+1)}) = n$.

For $k \geq 6$ let:

$\ell(x_0) = 1, \ell(x_{2(p+1)}) = 3, \ell(x_{4(p+1)}) = 5, \ldots, \ell(x_{2i(p+1)}) = 2i+1, \ldots,$
$\ell(x_{(k-4)(p+1)}) = k-3, \ell(x_{(k-2)(p+1)}) = k-1,$
$\ell(x_{k(p+1)}) = k, \ell(x_{(k+2)(p+1)}) = k-2, \ell(x_{(k+4)(p+1)}) = k-4, \ldots,$
$\ell(x_{(2k-4)(p+1)}) = 4, \ell(x_{(2k-2)(p+1)}) = 2,$
and
$\ell(x_{p+1}) = n-1, \ell(x_{3(p+1)}) = n-3, \ell(x_{5(p+1)}) = n-5, \ldots,$
$\ell(x_{(2i+1)(p+1)}) = n-2i-1, \ldots,$
$\ell(x_{(k-3)(p+1)}) = n-k+3, \ell(x_{(k-1)(p+1)}) = n-k+1,$
$\ell(x_{(k+1)(p+1)}) = n-k+2, \ell(x_{(k+3)(p+1)}) = n-k+4 \ldots,$
$\ell(x_{(2k-3)(p+1)}) = n-2, \ell(x_{(2k-1)(p+1)}) = n.$

Notice that a vertex x_m belongs to X_j if the vertex $x_{m-2+i(p+1)}$ for any i belongs to X_{j-2}. The vertices in $X_2, X_4 \ldots, X_{4s-2}$ will be labeled recursively as follows:

$\ell(x_m) = \ell(x_{m-2+(k+1)(p+1)}) + k$ when $\ell(x_{m-2+(k+1)(p+1)}) < \frac{n}{2}$ and
$\ell(x_m) = \ell(x_{m-2+(k+1)(p+1)}) - k$ when $\ell(x_{m-2+(k+1)(p+1)}) > \frac{n}{2}$.

In particular, we have
$\ell(x_{v(p+1)+2z}) = \ell(x_{v(p+1)+z(k+1)(p+1)}) + zk$ when $\ell(x_{v(p+1)+z(k+1)(p+1)}) < \frac{n}{2}$ and
$\ell(x_{v(p+1)+2z}) = \ell(x_{v(p+1)+z(k+1)(p+1)}) - zk$ when $\ell(x_{v(p+1)+z(k+1)(p+1)}) > \frac{n}{2}$.

We notice that the sum of two consecutive labels in each X_j falls into one of three cases. For instance, in X_0 we have

$$\begin{aligned}\ell(x_0) + \ell(x_{(p+1)}) = \ell(x_{2(p+1)}) + \ell(x_{3(p+1)}) = \dots \\ = \ell(x_{(k-2)(p+1)}) + \ell(x_{(k-1)(p+1)}) = n\end{aligned}$$

and also

$$\ell(x_{(k+1)(p+1)}) + \ell(x_{(k+2)(p+1)}) = \dots = \ell(x_{(2k-3)(p+1)}) + \ell(x_{(2k-2)(p+1)}) = n.$$

Then we have

$$\begin{aligned}\ell(x_{(p+1)}) + \ell(x_{2(p+1)}) = \ell(x_{3(p+1)}) + \ell(x_{4(p+1)}) = \dots \\ = \ell(x_{(k-3)(p+1)}) + \ell(x_{(k-2)(p+1)}) = n+2\end{aligned}$$

and also

$$\ell(x_{k(p+1)}) + \ell(x_{(k+1)(p+1)}) = \dots = \ell(x_{(2k-2)(p+1)}) + \ell(x_{(2k-1)(p+1)}) = n+2.$$

Finally, we have

$$\ell(x_{(2k-1)(p+1)}) + \ell(x_0) = \ell(x_{(k-1)(p+1)}) + \ell(x_{k(p+1)}) = n+1.$$

In X_2 we have

$$\begin{aligned}\ell(x_2) + \ell(x_{2+(p+1)}) = \ell(x_{2+2(p+1)}) + \ell(x_{2+3(p+1)}) = \dots \\ = \ell(x_{2+(k-4)(p+1)}) + \ell(x_{2+(k-3)(p+1)}) = n,\end{aligned}$$

$$\begin{aligned}\ell(x_{2+(k+1)(p+1)}) + \ell(x_{2+(k+2)(p+1)}) = \dots \\ = \ell(x_{2+(2k-3)(p+1)}) + \ell(x_{2+(2k-2)(p+1)}) = n,\end{aligned}$$

$$\begin{aligned}\ell(x_{2+(2k-1)(p+1)}) + \ell(x_2) = \ell(x_{2+(p+1)}) + \ell(x_{2+2(p+1)}) = \dots \\ = \ell(x_{2+(k-1)(p+1)}) + \ell(x_{2+k(p+1)}) = n+2,\end{aligned}$$

$$\ell(x_{2+k(p+1)}) + \ell(x_{2+(k+1)(p+1)}) = \dots$$
$$= \ell(x_{2+(2k-2)(p+1)}) + \ell(x_{2+(2k-1)(p+1)}) = n+2.$$

And we again have

$$\ell(x_{2+(2k-2)(p+1)})+\ell(x_{2+(2k-1)(p+1)}) = \ell(x_{2+(k-2)(p+1)})+\ell(x_{2+(k-1)(p+1)}) = n+1.$$

Now we look at the weights of vertices in X_1. We have

$$w(x_1) = \ell(x_0)+\ell(x_{(p+1)})+\ell(x_{2+(2k-2)(p+1)})+\ell(x_2) = n+(n+2) = 2n+2.$$

Similarly, we have

$$w(x_{1+2(p+1)}) = \cdots = w(x_{1+(k-2)(p+1)}) = n + (n+2) = 2n+2.$$

and also

$$w(x_{1+(k+1)(p+1)}) = \cdots = w(x_{1+(2k-3)(p+1)}) = n + (n+2) = 2n+2.$$

Then we have

$$w(x_{1+(p+1)}) = \ell(x_{(p+1)}) + \ell(x_{2(p+1)}) + \ell(x_2) + \ell(x_{2+(p+1)})$$
$$= (n+2) + n = 2n+2.$$

Similarly, we have

$$w(x_{1+3(p+1)}) = \cdots = w(x_{1+(k-3)(p+1)}) = (n+2) + n = 2n+2.$$

and also

$$w(x_{1+k(p+1)}) = \cdots = w(x_{1+(2k-2)(p+1)}) = (n+2) + n = 2n+2.$$

Finally, the two special vertices are $x_{1+(2k-1)(p+1)}$ and $x_{1+(k-1)(p+1)}$ with

$$w(x_{1+(2k-1)(p+1)})$$
$$= \ell(x_{(2k-1)(p+1)}) + \ell(x_{(p+1)}) + \ell(x_{2+(2k-2)(p+1)}) + \ell(x_{2+(2k-1)(p+1)})$$
$$= (n+1) + (n+1) = 2n+2.$$

and

$$w(x_{1+(k-1)(p+1)}) = \ell(x_{(k-1)(p+1)}) + \ell(x_{k(p+1)}) + \ell(x_{2+(k-2)(p+1)}) + \ell(x_{2+(k-1)(p+1)})$$
$$= (n+1) + (n+1) = 2n+2.$$

Using similar reasoning, one can verify that the weights of all vertices with odd subscripts will be equal to $2n+2$. In particular, using the recursive nature of the labeling, we have

$$\begin{aligned} w(x_{(2i+1)+r(p+1)}) & \\ &= \ell(x_{2i+r(p+1)}) + \ell(x_{2i+(r+1)(p+1)}) \\ &\qquad + \ell(x_{(2i+2)+(r-1)(p+1)}) + \ell(x_{(2i+2)+r(p+1)}) \\ &= \ell(x_{2i-2+(r+k+1)(p+1)}) + k + \ell(x_{2i-2+(r+k+2)(p+1)}) - k \\ &\qquad + \ell(x_{2i+(r+k)(p+1)}) + k + \ell(x_{2i+(r+k+1)(p+1)}) - k \\ &= \ell(x_{2i-2+(r+k+1)(p+1)}) + \ell(x_{2i-2+(r+k+2)(p+1)}) \\ &\qquad + \ell(x_{2i+(r+k)(p+1)}) + \ell(x_{2i+(r+k+1)(p+1)}) \\ &= w(x_{2i-1+(r+k+1)(p+1)}) = 2n+2. \end{aligned}$$

The roles of k and $-k$ may be interchanged, depending on the value of r.

Finally, the vertices in $X_1, X_3, \ldots, X_{4s-1}$ will be labeled as
$\ell(x_m) = \ell(x_{m-1}) + 2sk$ when $\ell(x_{m-1}) < \frac{n}{2}$ and
$\ell(x_m) = \ell(x_{m-1}) - 2sk$ when $\ell(x_{m-1}) > \frac{n}{2}$.

Obviously, ℓ is a bijection. Using similar arguments as above one can check that $w(x_i) = 2(n+1)$ for $x_i \in X_0, X_2, \ldots, X_{4s-2}$.

The case of $p-1 \equiv 0 \pmod 4$ is essentially the same and is left to the reader.

Case 2: $k \equiv 1 \pmod 2$.

Label the vertices of X_0 as follows:
If $k=1$, then $\ell(x_0) = 1$, $\ell(x_{p+1}) = n$.
If $k=3$, then $\ell(x_0) = 1$, $\ell(x_{p+1}) = n-1$, $\ell(x_{2(p+1)}) = 3$, $\ell(x_{3(p+1)}) = n-2$, $\ell(x_{4(p+1)}) = 2$, $\ell(x_{20}) = n$.
For $k \geq 5$ let:
$\ell(x_0) = 1, \ell(x_{2(p+1)}) = 3, \ell(x_{4(p+1)}) = 5, \ldots, \ell(x_{2i(p+1)}) = 2i+1, \ldots,$
$\ell(x_{(k-3)(p+1)}) = k-2, \ell(x_{(k-1)(p+1)}) = k,$
$\ell(x_{(k+1)(p+1)}) = k-1, \ell(x_{(k+3)(p+1)}) = k-3, \ell(x_{(k+5)(p+1)}) = k-5, \ldots,$
$\ell(x_{(2k-4)(p+1)}) = 4, \ell(x_{(2k-2)(p+1)}) = 2,$

and
$\ell(x_{p+1}) = n-1, \ell(x_{3(p+1)}) = n-3, \ell(x_{5(p+1)}) = n-5, \ldots,$
$\ell(x_{(2i+1)(p+1)}) = n-2i-1, \ldots,$
$\ell(x_{(k-4)(p+1)}) = n-k+4, \ell(x_{(k-2)(p+1)}) = n-k+2,$
$\ell(x_{k(p+1)}) = n-k+1, \ell(x_{(k+2)(p+1)}) = n-k+3 \ldots,$
$\ell(x_{(2k-3)(p+1)}) = n-2, \ell(x_{(2k-1)(p+1)}) = n.$

The vertices in X_2 will be labeled as
$\ell(x_m) = \ell(x_{-m+2-2(p+1)}) + k$ when $\ell(x_{-m+2-2(p+1)}) < \frac{n}{2}$ and
$\ell(x_m) = \ell(x_{-m+2-2(p+1)}) - k$ when $\ell(x_{-m+2-2(p+1)}) > \frac{n}{2}$.

Notice that a vertex x_m belongs to X_k if the vertex x_{m+2p-2} belongs to X_{k-4}. The vertices in $X_4, X_6, \ldots, X_{p-3}$ will be labeled recursively as follows.
$\ell(x_m) = \ell(x_{m+2p-2}) + k$ when $\ell(x_{m+2p-2}) < \frac{n}{2}$ and
$\ell(x_m) = \ell(x_{m+2p-2}) - k$ when $\ell(x_{m+2p-2}) > \frac{n}{2}$.

As above, notice that the sum of two consecutive labels in each X_j falls into one of three cases. For instance, in X_0 we have

$$\begin{aligned}\ell(x_0) + \ell(x_{(p+1)}) = \ell(x_{2(p+1)}) + \ell(x_{3(p+1)}) = \ldots \\ = \ell(x_{(k-3)(p+1)}) + \ell(x_{(k-2)(p+1)}) = n\end{aligned}$$

and also

$$\ell(x_{(k)(p+1)}) + \ell(x_{(k+1)(p+1)}) = \cdots = \ell(x_{(2k-3)(p+1)}) + \ell(x_{(2k-2)(p+1)}) = n.$$

Then we have

$$\begin{aligned}\ell(x_{(p+1)}) + \ell(x_{2(p+1)}) = \ell(x_{3(p+1)}) + \ell(x_{4(p+1)}) = \ldots \\ = \ell(x_{(k-2)(p+1)}) + \ell(x_{(k-1)(p+1)}) = n+2\end{aligned}$$

and also

$$\ell(x_{(k+1)(p+1)}) + \ell(x_{(k+2)(p+1)}) = \cdots = \ell(x_{(2k-2)(p+1)}) + \ell(x_{(2k-1)(p+1)}) = n+2.$$

Finally, we have

$$\ell(x_{(2k-1)(p+1)}) + \ell(x_0) = \ell(x_{(k-1)(p+1)}) + \ell(x_{k(p+1)}) = n+1.$$

In X_2 we have

$$\ell(x_2) + \ell(x_{2+(p+1)}) = \ell(x_{2+2(p+1)}) + \ell(x_{2+3(p+1)}) = \dots \\ \ell(x_{2+(k-3)(p+1)}) + \ell(x_{2+(k-4)(p+1)}) = n,$$

$$\ell(x_{2+k(p+1)}) + \ell(x_{2+(k+1)(p+1)}) = \dots \\ = \ell(x_{2+(2k-3)(p+1)}) + \ell(x_{2+(2k-2)(p+1)}) = n,$$

$$\ell(x_{2+(2k-1)(p+1)}) + \ell(x_2) = \ell(x_{2+(p+1)}) + \ell(x_{2+2(p+1)}) = \dots \\ = \ell(x_{2+(k-4)(p+1)}) + \ell(x_{2+(k-3)(p+1)}) = n+2,$$

$$\ell(x_{2+(k-1)(p+1)}) + \ell(x_{2+k(p+1)}) = \dots \\ = \ell(x_{2+(2k-4)(p+1)}) + \ell(x_{2+(2k-3)(p+1)}) = n+2.$$

And we again have

$$\ell(x_{2+(2k-2)(p+1)})+\ell(x_{2+(2k-1)(p+1)}) = \ell(x_{2+(k-2)(p+1)})+\ell(x_{2+(k-1)(p+1)}) = n+1.$$

Now we look at the weights of vertices in X_1. We have

$$w(x_1) = \ell(x_0)+\ell(x_{(p+1)})+\ell(x_{2+(2k-2)(p+1)})+\ell(x_2) = n+(n+2) = 2n+2.$$

Similarly, we have

$$w(x_{1+2(p+1)}) = \cdots = w(x_{1+(k-2)(p+1)}) = n + (n + 2) = 2n + 2.$$

and also

$$w(x_{1+(k+1)(p+1)}) = \cdots = w(x_{1+(2k-3)(p+1)}) = n + (n + 2) = 2n + 2.$$

Then we have

$$w(x_{1+(p+1)}) = \ell(x_{(p+1)}) + \ell(x_{2(p+1)}) + \ell(x_2) + \ell(x_{2+(p+1)}) \\ = (n+2) + n = 2n + 2.$$

Similarly, we have

$$w(x_{1+3(p+1)}) = \cdots = w(x_{1+(k-3)(p+1)}) = (n+2)+n = 2n+2.$$

and also

$$w(x_{1+k(p+1)}) = \cdots = w(x_{1+(2k-2)(p+1)}) = (n+2)+n = 2n+2.$$

Finally, the two special vertices are $x_{1+(2k-1)(p+1)}$ and $x_{1+(k-1)(p+1)}$ with

$$\begin{aligned} &w(x_{1+(2k-1)(p+1)}) \\ &\quad = \ell(x_{(2k-1)(p+1)}) + \ell(x_{(p+1)}) + \ell(x_{2+(2k-2)(p+1)}) + \ell(x_{2+(2k-1)(p+1)}) \\ &\qquad = (n+1)+(n+1) = 2n+2. \end{aligned}$$

and

$$\begin{aligned} w(x_{1+(k-1)(p+1)}) &= \ell(x_{(k-1)(p+1)}) + \ell(x_{k(p+1)}) + \ell(x_{2+(k-2)(p+1)}) + \ell(x_{2+(k-1)(p+1)}) \\ &= (n+1)+(n+1) = 2n+2. \end{aligned}$$

As in Case 1, using similar reasoning one can verify that the weights of all vertices with odd subscripts will be equal to $2n+2$. ■

By Theorems 11 and Observation 14 and 15 we obtain the following:

Theorem 16 *If p is odd, then $C_n(1,p)$ is distance magic graph if and only if $p^2-1 \equiv 0 \pmod{n}$, $\frac{n}{\gcd(n,p+1)} \equiv 0 \pmod 2$ and $\frac{n}{\gcd(n,p-1)} \equiv 0 \pmod 2$.*

2.2 $C_n(1,2p')$ distance magic graphs

Observation 17 *If p is even, then $C_{2(p^2-1)}(1,p)$ is distance magic.*

Proof. Let $n = 2(p^2-1)$ and $H = \langle p+1 \rangle$ be the subgroup of $\mathbb{Z}_n$ of order $2(p-1)$. Since it was proved that $C_n(1,2)$ is distance magic if and only if $n=6$ (see [2]) we can assume that $p \geq 4$.

As before, denote for $j = 1,2,\ldots,2(p-1)$ by X_j the set of all vertices whose subscripts belong to coset $H+j$ and by ℓ_j the set of all labels of vertices in X_j.

Label the vertices of X_0 as follows:
$\ell(x_0) = 1, \ell(x_{2(p+1)}) = 3, \ell(x_{4(p+1)}) = 5, \ldots, \ell(x_{2i(p+1)}) = 2i+1, \ldots,$

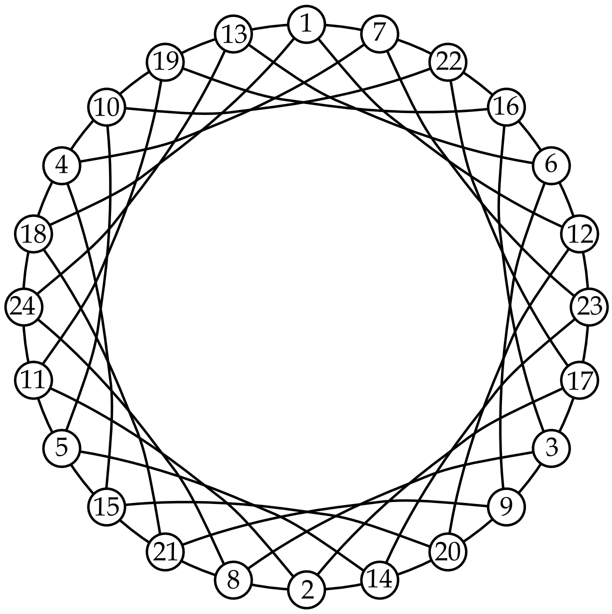

Figure 1: Distance magic labeling for $C_{24}(1,5)$.

$\ell(x_{(p-4)(p+1)}) = p-3, \ell(x_{(p-2)(p+1)}) = p-1,$
$\ell(x_{p(p+1)}) = 2, \ell(x_{(p+2)(p+1)}) = 4,$
$\ell(x_{(p+4)(p+1)}) = 6, \ldots,$
$\ell(x_{(2p-6)(p+1)}) = p-4, \ell(x_{(2p-4)(p+1)}) = p-2,$
and
for $p = 4$ we put $\ell(x_5) = 29$, $\ell(x_{15}) = 30$ and $\ell(x_{25}) = 28$.
For $p \geq 6$
$\ell(x_{p+1}) = 2p^2-3, \ell(x_{3(p+1)}) = 2p^2-5, \ell(x_{5(p+1)}) = 2p^2-7, \ldots,$
$\ell(x_{(2i-1)(p+1)/2}) = 2p^2-2i-3, \ldots,$
$\ell(x_{(p-5)(p+1)}) = 2p^2-p-1, \ell(x_{(p-3)(p+1)}) = 2p^2-p-3,$
$\ell(x_{(p-1)(p+1)}) = 2p^2-2, \ell(x_{(p+1)(p+1)}) = 2p^2-4, \ldots,$
$\ell(x_{(2p-5)(p+1)}) = 2p^2-p+2, \ell(x_{(2p-3)(p+1)}) = 2p^2-p.$

Notice that since p is even then in $\mathbb{Z}_{p+1}$ we have $\langle 2\rangle \cong \mathbb{Z}_{p+1}$ and moreover a vertex x_m belongs to X_k if the vertex $x_{m+(p-1)(p+2)}$ belongs to X_{k+2}. The vertices in $X_1, X_2, \ldots, X_p$ will be labeled recursively as follows:
$\ell(x_m) = \ell(x_{m+(p-1)(p+2)}) + p - 1$ when $\ell(x_{m+(p-1)(p+2)}) < p^2-1$ and
$\ell(x_m) = \ell(x_{m+(p-1)(p+2)}) - p + 1$ when $\ell(x_{m+(p-1)(p+2)}) > p^2-1$.

As in Observation 15 one can check that the weights of all vertices will be equal to $2n + 2$. ■

3 Group distance magic $C_n(1, p)$

The notion of group distance magic labeling of graphs was introduced in [7]. Let G be a graph with n vertices and Γ an Abelian group with n elements. We call a bijection $g\colon V(G) \to \Gamma$ a Γ-*distance magic labeling* if for all $x \in V(G)$ we have $w(x) = \mu$ for some μ in Γ. Obviously, every graph with n vertices and a distance magic labeling also admits a $\mathbb{Z}_n$-distance magic labeling. The converse is not necessarily true (see, e.g., Theorems 4 and 7).

Recall that any group element $\iota \in \Gamma$ of order 2 (i.e., $\iota \neq 0$ such that $2\iota = 0$) is called an *involution*, and that a non-trivial finite group has elements of order 2 if and only if the order of the group is even. Moreover every cyclic group of even order has exactly one involution. The fundamental theorem of finite Abelian groups states that the finite Abelian group Γ can be expressed as the direct sum of cyclic subgroups of prime-power order. This product is unique up to the order of the direct product. When t is the number of these cyclic components whose order is a power of 2, then Γ has $2t - 1$ involutions. Moreover the sum of all the group elements is equal to the sum of the involutions and the neutral element. Let us denote this sum as $s(\Gamma) = \sum_{g \in \Gamma} g$.

The following lemma was proved in [6] (see [6], Lemma 8).

Lemma 18 ([6]) *Let Γ be an Abelian group.*

(i) If Γ has exactly one involution ι, then $s(\Gamma) = i$.

(ii) If Γ has no involutions, or more than one involution, then $s(\Gamma) = 0$.

We start by proving a general theorem for Γ-distance magic labeling similar to Theorem 2.

Theorem 19 *Let G be an r-regular distance magic graph on n vertices, where r is odd. There does not exists an Abelian group Γ having exactly one involution ι, $|\Gamma| = n$ such that G is Γ-distance magic.*

Proof. Since r is odd, it implies that n is even. Let Γ be an Abelian group of order n having exactly one involution i. Suppose that G is Γ-distance magic. Recall that $ng = 0$ for any $g \in \Gamma$. Let now $w(G) = \sum_{x \in V(G)} w(x) = n \cdot \mu = 0$. On the other hand $w(G) = \sum_{x \in V(G)} \sum_{y \in N(x)} w(y) = rs(\Gamma)$. By Lemma 18 we obtain that $w(G) = r\iota$. Therefore since r is odd, we have $r\iota = \iota$, hence $\iota = 0$, a contradiction. ■

Theorem 19 implies immediately the following observations:

Observation 20 *Let G be an r-regular distance magic graph on $n \equiv 2 \pmod 4$ vertices, where r is odd. There does not exists an Abelian group Γ of order n such that G is Γ-distance magic.*

The condition $n \equiv 2 \pmod 4$ is necessary. For example, a graph $K_{3,3,3,3}$ has a $\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$-distance magic labeling with the magic constant $\mu = (0, 1, 1)$ presented in the table, where columns correspond to the partition sets.

(0,0,0)	(0,1,0)	(0,1,1)	(0,0,1)
(1,1,0)	(1,0,1)	(1,1,1)	(1,0,0)
(2,0,1)	(2,0,0)	(2,1,1)	(2,1,0)

Observation 21 *If G is an r-regular distance magic graph on n vertices, where r is odd, then G is not $\mathbb{Z}_n$-distance magic.*

Another observation can be easily proved.

Observation 22 *If $C_n(1, p)$ is a Γ-distance magic circulant graph for a group Γ, then n is even.*

Proof. Let μ be a magic constant for $C_n(1, p)$. Suppose that n is odd, then $\frac{n}{\gcd(n,p+1)} \equiv 1 \pmod 2$. Thus $\gcd(n, p+1) = \gcd(n, 2p+2)$ and $\langle 2(p+1) \rangle = \langle p+1 \rangle$. Hence, $p+1 = 2c(p+1)$ for some $c \geq 1$. Then we use Lemma 8, set $\gamma = c$, $i = 0, p-1$ and obtain respectively:

$$\ell(x_0) + \ell(x_{p-1}) = \ell(x_{2c(p+1)}) + \ell(x_{p-1+2c(p+1)}) = \ell(x_{p+1}) + \ell(x_{2p}),$$

$$\ell(x_{p-1}) + \ell(x_{2p-2}) = \ell(x_{p-1+2c(p+1)}) + \ell(x_{2p-2+2c(p+1)}) = \ell(x_{2p}) + \ell(x_{3p-1}).$$

Since $N(x_i) = \{x_{i-p}, x_{i-1}, x_{i+1}, x_{i+p}\}$ and $C_n(1,p)$ is Γ-distance magic, we obtain for $i = p$ and $i = 2p-1$:

$$\mu = \ell(x_0) + \ell(x_{p-1}) + \ell(x_{p+1}) + \ell(x_{2p}) = 2(\ell(x_0) + \ell(x_{p-1})),$$

$$\mu = \ell(x_{p-1}) + \ell(x_{2p-2}) + \ell(x_{2p}) + \ell(x_{3p-1}) = 2(\ell(x_{p-1}) + \ell(x_{2p-2})).$$

Therefore $2(\ell(x_0) - \ell(x_{2p-2})) = 0$. Recall that n being odd implies that there does not exists an element $g \neq 0$, $g \in \Gamma$ such that $2g = 0$. Thus $\ell(x_0) = \ell(x_{2p-2})$ and we have a contradiction, because $n > 2p+1$. ■

Theorem 23 *If* $\gcd(n, p+1) = 2k+1$*,* p *and* n *are both even, and* $n = 2r(2k+1)$ *then* $C_n(1,p)$ *has a* $\mathbb{Z}_{2\alpha} \times \mathcal{A}$*-magic labeling for any* $\alpha \equiv 0 \pmod{r}$ *and any Abelian group* $\mathcal{A}$ *of order* $r(2k+1)/\alpha$*.*

Proof. Let $l = r(2k+1)/\alpha$ Since $\Gamma \cong \mathbb{Z}_{2\alpha} \times \mathcal{A}$, thus if $g \in \Gamma$, then we can write that $g = (j, a_i)$ for $j \in \mathbb{Z}_{2\alpha}$ and $a_i \in \mathcal{A}$ for $i = 0, 1, \ldots, l-1$. We can assume that $a_0 = 0 \in \mathcal{A}$. Let $\ell(x) = (\ell_1(x), \ell_2(x))$.

Let $X = \langle p+1 \rangle$ be the subgroup of $\mathbb{Z}_n$ of order $2r$. Let us denote for $j = 1, 2, \ldots, 2k$ by X_j the set of all vertices whose subscripts belong to coset $X + j$.
Notice that $\alpha = rh$ for some h. Let $H = \langle 2h \rangle$ be the subgroup of $\mathbb{Z}_{2\alpha}$ of order r.

Label the vertices of X_0 as follows:

$$\ell(x_{2i(p+1)}) = (2ih, 0),\ \ell(x_{(2i+1)(p+1)}) = (-2ih-1, 0)$$

$i = 0, 1, \ldots, k-1$.
If a subscript m belongs to coset $X + j$, then denote it by m_j. Notice that a vertex x_{m_j} belongs to X_j if the vertex $x_{m_j - p}$ belongs to X_{j-1}. The vertices in $X_1, X_2, X_3 \ldots, X_{2k}$ will be labeled recursively as follows.

$$\ell_1(x_{m_j}) = \begin{cases} \ell_1(x_{m_j-p}) + 1 & \text{if} \quad \ell_1(x_{m_j-p}) \equiv j-1 \pmod{2h} \\ \ell_1(x_{m_j-p}) - 1 & \text{if} \quad \ell_1(x_{m_j-p}) \not\equiv j-1 \pmod{2h} \end{cases}$$

$$\ell_2(x_{m_j}) = \begin{cases} a_{\lfloor j/r \rfloor} & \text{if} \quad \ell_1(x_{m_j}) \equiv 0 \pmod{2} \\ -a_{\lfloor j/r \rfloor} & \text{if} \quad \ell_1(x_{m_j}) \equiv 1 \pmod{2} \end{cases}$$

Obviously ℓ is bijection and $\ell(x_{i-p}) + \ell(x_{i+1}) = (-1, 0)$ or $\ell(x_{i-p}) + \ell(x_{i+1}) = (2h-1, 0)$.

We will consider now two cases:

Case 1. $X = \langle p+1 \rangle = \mathbb{Z}_n$.
Notice that then $l = 1$ and $h = 1$. Suppose that there exists i such that $l(x_{i-p}) + l(x_{i+1}) = (1,0)$ and $l(x_{i-1}) + l(x_{i+p}) = (1,0)$, or similarly $l(x_{i-p}) + l(x_{i+1}) = (-1,0)$ and $l(x_{i-1}) + l(x_{i+p}) = (-1,0)$. Recall that $\ell(x_{2i(p+1)}) + \ell(x_{(2i+1)(p+1)}) = (-1,0)$ and $\ell(x_{(2i+1)(p+1)}) + \ell(x_{(2i+2)(p+1)}) = (1,0)$. It implies that there exists $\beta \in Z$ such that $i-p+2\beta \equiv (i-1) \bmod n$. It means that $2\beta \equiv (p-1) \bmod n$. Since n is even, $p-1$ odd, a contradiction. It follows that $w(x_i) = (0,0)$ for every i and the graph is Γ-distance magic with $\mu = (0,0)$.

Case 2. $X = \langle p+1 \rangle \neq \mathbb{Z}_n$.
Then $\ell(x_{i-p}) + \ell(x_{i+1}) = \ell(x_{i+(j-1)p)}) + \ell(x_{i+jp+1})$ for any $j = 1,2,\dots,2k$. Thus for $j = 2$ we have $\ell(x_{i-p}) + \ell(x_{i+1}) = \ell(x_{i+p}) + \ell(x_{i+2p+1})$. Since $\ell(x_{i-1}) + \ell(x_{i+p}) \neq \ell(x_{i+p}) + \ell(x_{i+2p+1})$ we obtain that $w(x_i) = \ell(x_{i-p}) + \ell(x_{i+1}) + \ell(x_{i-1}) + \ell(x_{i+p}) = (2h-2,0)$.

Thus $\mu = (2h-2,0)$ and the graph is Γ-distance magic. ■

Corollary 24 *If* $\gcd(n,p+1) = 2k+1$*,* p *is even,* $n = 2\alpha(2k+1)$ *and* $\gcd(2k+1,2\alpha) = 1$*, then* $C_n(1,p)$ *has a* $\mathbb{Z}_n$*-magic labeling.*

Proof. By Theorem 23 there exists $\mathbb{Z}_{2\alpha} \times \mathbb{Z}_{2k+1}$-magic labeling. Since $\gcd(2k+1,2\alpha) = 1$, the group $\mathbb{Z}_{2\alpha} \times \mathbb{Z}_{2k+1}$ is isomorphic to the group $\mathbb{Z}_n$. ■

Corollary 25 *If* $n = \alpha(p^2-1)$ *and:*

- $\alpha = 1$ *if* p *is odd*
- $\alpha = 2\beta$ *and* $\gcd(p+1,2\beta) = 1$ *if* p *is even*

then $C_n(1,p)$ *is* $\mathbb{Z}_n$*-distance magic.*

Proof. If $\alpha = 1$ and p is odd, then since $C_n(1,p)$ is distance magic by Theorem 16, then $C_n(1,p)$ is $\mathbb{Z}_n$-distance magic. If $\alpha = 2\beta$, $\gcd(p+1,2\beta) = 1$ and p is even, then $\gcd(n,p+1) = p+1$ and $C_n(1,p)$ has $\mathbb{Z}_n$-magic labeling by Corollary 24. ■

Using the same arguments as in the proof of Theorem 14 we obtain corollary:

Corollary 26 *If p is odd and $2p^2 - 2 \not\equiv (0 \bmod n)$ then $C_n(1,p)$ is not Γ-distance magic for any Abelian group Γ of order n.*

Observation 27 *If $p = 5$, then $C_{p^2-1}(1,p)$ is Γ-distance magic for any Abelian group Γ of order $p^2 - 1$.*

Proof. The fundamental theorem of finite Abelian groups states that the finite Abelian group Γ can be expressed as the direct sum of cyclic subgroups of prime-power order. Since the order of Γ is 24 we have the following possibilities: $\Gamma \cong \mathbb{Z}_8 \times \mathbb{Z}_3 \cong \mathbb{Z}_{24}$, $\Gamma \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4 \cong \mathbb{Z}_6 \times \mathbb{Z}_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_{12}$ and $\Gamma \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \cong \mathbb{Z}_6 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. If $\Gamma \cong \mathbb{Z}_8 \times \mathbb{Z}_3 \cong \mathbb{Z}_{24}$, then since $C_{24}(1,5)$ is distance magic by Theorem 5 it implies that it is $\mathbb{Z}_{24}$-distance magic. We will show now that $C_{24}(1,5)$ is $\mathbb{Z}_6 \times \mathcal{A}$-distance magic for any Abelian group $\mathcal{A}$ of order 4. If $g \in \Gamma$, then we can write that $g = (j, a_i)$ for $j \in \mathbb{Z}_6$ and $a_i \in \mathcal{A}$ for $i = 0, 1, 2, 3$. Define the following labeling.

$$
\begin{array}{lll}
\ell(x_0) = (0, a_0), & \ell(x_8) = (4, a_0), & \ell(x_{16}) = (2, a_0), \\
\ell(x_6) = (5, a_1), & \ell(x_{14}) = (1, a_1), & \ell(x_{22}) = (3, a_1), \\
\ell(x_{12}) = (2, a_2), & \ell(x_{20}) = (0, a_2), & \ell(x_4) = (4, a_2), \\
\ell(x_{18}) = (3, a_3), & \ell(x_2) = (5, a_3), & \ell(x_{10}) = (1, a_3), \\
\ell(x_1) = (0, a_1), & \ell(x_9) = (4, a_1), & \ell(x_{17}) = (2, a_1), \\
\ell(x_7) = (5, a_2), & \ell(x_{15}) = (1, a_2), & \ell(x_{23}) = (3, a_2), \\
\ell(x_{13}) = (2, a_3), & \ell(x_{21}) = (0, a_3), & \ell(x_5) = (4, a_3), \\
\ell(x_{19}) = (3, a_0), & \ell(x_3) = (5, a_0), & \ell(x_{11}) = (1, a_0).
\end{array}
$$

Taking now $a_i = i$ if $\mathcal{A} \cong \mathbb{Z}_4$ or $a_0 = (0,0)$, $a_1 = (0,1)$, $a_2 = (1,1)$, $a_3 = (1,0)$ $\mathcal{A} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ one can check the graph $C_{24}(1,5)$ is $\mathbb{Z}_6 \times \mathcal{A}$-distance magic with the magic constant $(4, 2a_1)$. ■

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