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Preprint Nr MD 062
(otrzymany dnia 9 1 2013)

Kraków
2013

On embedding graphs with bounded sum of size and maximum degree

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January 9, 2013

Abstract

We say that a graph is *embeddable* if it is a subgraph of its complement. One of the classic result on graphs embedding says that each graph on n vertices with at most $n-2$ edges is embeddable. The bound on the number of edges cannot be increased because, for example, the star on n vertices is not embeddable. The reason of this fact is the existence of a vertex with very high degree. In this paper we prove that by forbidding such vertices, one can significantly increase the bound on the number of edges. Namely, we prove that if $\Delta(G) + |E(G)| \leq 2n - f(n)$, where $f(n) = o(n)$, then G is embeddable. Our result is asymptotically best possible, since for the star S_n (which is not embeddable) we have $\Delta(S_n) + |E(S_n)| = 2n - 2$. As a corollary we obtain that a digraphs embedding conjecture by Benhocine and Wojda 1985 is true for digraphs with sufficiently many symmetric arcs.

1 Introduction

We deal with finite, simple graphs without loops or multiple edges. The vertex and edge sets of a graph G are denoted by $V(G)$ and $E(G)$. The order of G is the number of vertices of G and is denoted by $|G|$. The size of G is the number of edges of G and is denoted by $||G||$. By $N_G(x)$ we denote the set of vertices adjacent to x in G. The degree (in G) of a vertex x is denoted by $d_G(x)$ and is equal to $|N_G(x)|$. The maximum degree of G is denoted by $\Delta(G)$ and is equal to the maximum among degrees of all vertices of G. For a vertex set X, the set $N_G(X)$ denotes the external neighbourhood of X in G, i.e.

$$N_G(X) = \{y \in V(G) \setminus X : y \text{ is adjacent to some } x \in X\}.$$

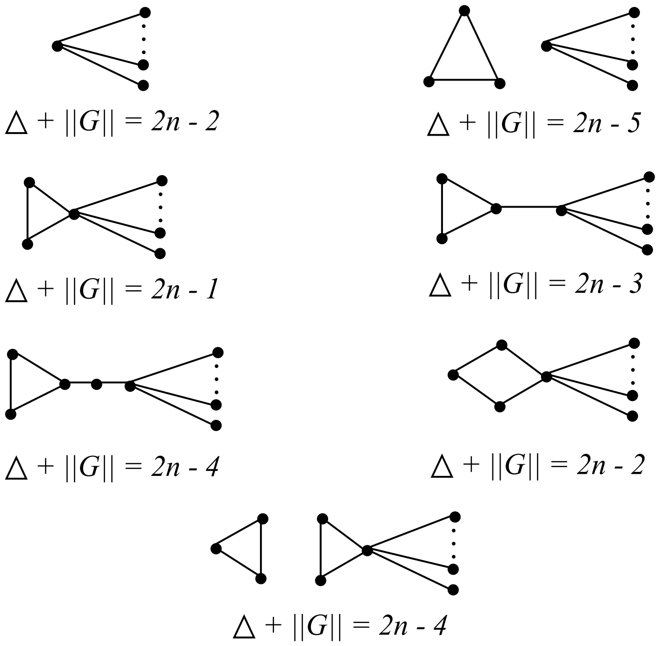
We say that G is *embeddable in its complement* (G is embeddable, in short) if there is a permutation σ on $V(G)$ such that if xy is an edge in G, then $\sigma(x)\sigma(y)$ is not an edge in G. Thus, G is embeddable if and only if G is a subgraph of its complement. If $\sigma(x) \neq x$ for every vertex $x \in V(G)$, then we say that G is *fixed-point-free* embeddable.

One of the classical results in the theory of graph embedding is the following theorem, proved independently in [2, 3, 8].

Theorem 1 ([2, 3, 8]) *Every n-vertex graph having at most $n-2$ edges is embeddable.*

This theorem cannot be improved by raising the size of G since for example a star on n vertices is not embeddable. In [4] and [5] all non-embeddable graphs with order n and size $n-1$ and n are presented, see also [10]. Among the non-embeddable $(n, n-1)$ and (n, n) graphs there are 7 infinite

Figure 1: Infinite families of non-embeddable $(n, n-1)$-graphs and (n, n)-graphs



families, see Figure 1. It is clear from the examples that the strong restriction on the number of edges in Theorem 1 is a result of the existence of a vertex with very high degree. It seems to be very likely that by forbidding such vertices one can significantly improve the bound on the size of a graph in the statement of Theorem 1. We confirm this feeling by proving the following theorem.

Theorem 2 *Let G be an n-vertex graph. If $||G|| + \Delta(G) \leq 2n - 14n^{2/3} - 20$ then G is embeddable.*

Note that the bound in Theorem 2 is nearly best possible. Indeed, it cannot be larger than $2n-6$ which follows from Figure 1, see the second example. In fact, this example can be generalized in the following way.

Example Let $V_1, ..., V_{t-1}$ be pairwise disjoint subsets with $|V_i| = t$ for $i = 1, ..., t-2$ and $|V_{t-1}| = n - t(t-2)$. Furthermore, let $x \in V_{t-1}$. Let G be a graph with $V(G) = V_1 \cup ... \cup V_{t-1}$ such that each V_i, $i = 1, ..., t-2$, induce a clique, V_{t-1} induce a star with center x and there are no other edges in G. Observe that G is not embeddable if n is sufficiently large. Indeed, suppose that σ is an embedding of G. If $\sigma(x) \in V_i$ for some $i \in \{1, ..., t-2\}$, then the remaining vertices of V_i must be images of vertices from different sets V_j, $j \neq t-1$. However, there are not enough sets V_j. Suppose that $\sigma(x) \in V_{t-1}$. If $\sigma(x) \neq x$, then x must be an image of a vertex from some set V_i with $i \in \{1, ..., t-2\}$. Thus, the remaining vertices of V_i have to be mapped on vertices from different sets V_j with $j \neq t-1$. However, there are not enough such sets V_j. Finally, if $\sigma(x) = x$, then the neighbors of x have to be mapped on the vertices from $V_1 \cup ... \cup V_{t-2}$, which is impossible if n is sufficiently large.

Furthermore, $\Delta(G) = n - t(t-2) - 1$, $||G|| = \frac{t(t-1)}{2}(t-2) + n - t(t-2) - 1$. Hence, $\Delta(G) + ||G|| = 2n - 2 + t(t-2)\frac{t-5}{2}$. Therefore, the coefficient 2 in Theorem 2 cannot be increased.

*The author was partially supported by the Polish Ministry of Science and Higher Education.

2 Lemmas

We use the following result from [6].

Lemma 3 ([6]) *Let G be a graph and k, l non-negative integers. If G has an independent set U of cardinality $k + l$ such that*

1. *U has k vertices with degree at most l, and its other vertices have degree at most k,*
2. *the neighborhoods of the vertices of U are pairwise disjoint,*
3. *there is an embedding σ' of $G - U$,*

then there exists an embedding σ of G.

We will need also the following known results.

Theorem 4 ([8]) *Let G_1 and G_2 be graphs of order n with maximum degrees $\Delta(G_1)$ and $\Delta(G_2)$, respectively. If $2\Delta(G_1)\Delta(G_2) < n$, then the complete graph K_n contains edge-disjoint copies of G_1 and G_2.*

Theorem 5 ([9]) *Every graph of order n and size at most $n - 2$ is fixed-point-free embeddable.*

For convenience, let $\alpha(n) = 14n^{2/3}+20$. In many places in the proofs we will use the following observation.

Proposition 6 *Let G be a graph of order n such that $||G|| + \Delta(G) \leq 2n - \alpha(n)$. If G' is a graph that arises from G by deleting m vertices and at least $2m$ edges, then $||G'|| + \Delta(G') \leq 2n' - \alpha(n')$, where n' is the order of G'.*

Proof. Note that $\alpha(n)$ is increasing with respect to n. Thus,

$$||G'|| + \Delta(G') \leq 2n - \alpha(n) - 2m = 2(n - m) - \alpha(n) \leq 2n' - \alpha(n').$$

□

Lemma 7 *Let G be a graph of order n such that $||G|| + \Delta(G) \leq 2n - \alpha(n)$. If $n \leq 2744$, then G is embeddable.*

Proof. Note that if $n \leq 2744$ then $2n - 14n^{2/3} - 20 \leq n - 20$. Hence G is embeddable by Theorem 1. □

Lemma 8 *Let G be a graph of order n such that $||G|| + \Delta(G) \leq 2n - \alpha(n)$. If $\Delta(G) \leq 37$, then G is embeddable.*

Proof. If $n \leq 2744$, then G is embeddable by Lemma 7. So we may assume that $n \geq 2745$. Note that if $\Delta(G) \leq 37$, then $2\Delta^2(G) < 2745 \leq n$. Hence G is embeddable by Theorem 5. □

A *starry tree* is a graph H such that (1) $V(H)$ can be partitioned into four sets V_1, V_2, V_3 and $\{x\}$ that each induce a tree, (2) there is at least one edge incident to x, (3) all edges not belonging to the trees induced by V_1, V_2 and V_3 are incident to x and (4) there are not edges between x and $V_2 \cup V_3$. A vertex x we call *a middle vertex* of H. Note that a starry tree is not connected.

Lemma 9 *Every starry tree admits an embedding such that its middle vertex is the image of one of its neighbors.*

Proof. Let H be a starry tree. The proof is by induction on $|T_1|+|T_2|+|T_3|$. If $|T_1|+|T_2|+|T_3| = 3$, then the existence of an embedding as required is obvious. Assume that $|T_1| + |T_2| + |T_3| \geq 4$. We distinguish two cases:
Case 1. There exists a leaf l in T_1 such that the middle vertex x is adjacent to l.
Case 2. All the leaves of T_1 are not adjacent to x.

Consider Case 1. Let $u \in V(T_2)$, $v \in V(T_3)$ be vertices such that $T_2 - u$ as well as $T_3 - v$ either is disconnected or has at most one vertex. Thus, by Theorem 1 (or trivially in the latter situation), there is an embedding σ_2 of $T_2 - \{u\}$ and there is an embedding σ_3 of $T_3 - \{v\}$.

Suppose first that $|T_1| = 1$ with $V(T_1) = \{l\}$. Then, the product $(l, x, v, u)\sigma_2\sigma_3$ is an embedding as required of H. Suppose next that $|T_1| = 2$ with $V(T_1) = \{l, l'\}$. Then, $(l, x, u)(l', v)\sigma_2\sigma_3$ is an embedding as required of H.

So we may assume that $|T_1| \geq 3$. Let l' be the neighbor of l in T_1. Let $y \in V(T_1)$ such that $T_1 - \{l, l', y\}$ either is disconnected or has at most one vertex. Thus, by Theorem 1 (or trivially in the latter situation), there is an embedding σ_1 of $T_1 - \{l, l', y\}$. Then the product $(l, x, u, y)(l', v)\sigma_1\sigma_2\sigma_3$ is an embedding as required of H.

Consider Case 2. Let L be the set of the leaves of T_1, $L = \{l_1, ..., l_s\}$. Note, that $|T_1| \geq 3$. Suppose first that all the leaves of T_1 have a common neighbor y. Since H is a starry tree (so there is at least one edge incident to x) and the leaves of T_1 are not joined with x, xy is an edge of H. Let $H' = H - L$. Clearly, H' is a starry tree. Thus, by the induction hypothesis there is an embedding as required σ' of H'. Furthermore, since y is the only neighbor of x in H', we have $\sigma'(y) = x$. In particular y is not a fixed point of σ'. Thus the product $(l_1)...(l_s)\sigma'$ (i.e. $l_1, ..., l_s$ are fixed points) is an embedding as required of H.

So we may assume that there are $l_1, l_2 \in L$ with neighbors (in T_1) y_1, y_2, respectively, such that $y_1 \neq y_2$, $y_1 \neq l_2$ and $y_2 \neq l_1$ (recall that $|T_1| \geq 3$ in this case). Let $H'' = H - \{l_1, l_2\}$. Clearly, H'' is a starry tree. Hence, by the induction hypothesis, there is an embedding as required σ'' of H''. Then $(l_1, l_2)\sigma''$ is an embedding as required of H if $\sigma''(y_1) = y_1$ or $\sigma''(y_2) = y_2$. Otherwise, $(l_1)(l_2)\sigma''$ is an embedding as required of H. □

Lemma 10 *Let G be a graph with minimum order n such that G is a non-embeddable graph with $||G|| + \Delta(G) \leq 2n - \alpha(n)$. Then G has no isolated vertices.*

Proof. Suppose for a contradiction, that y is an isolated vertex of G. By Lemma 8, there is $x \in G$ with $\deg x \geq 38$. Let $G' = G - \{x, y\}$. By Proposition 6, $||G'|| + \Delta(G') \leq 2|G'| - \alpha(|G'|)$. Thus, by the minimality assumption there is an embedding σ' of G'. Then $(xy)\sigma'$ is an embedding of G, a contradiction. □

Lemma 11 *Let G be a graph with minimum order n such that G is a non-embeddable graph with $||G|| + \Delta(G) \leq 2n - \alpha(n)$. If two vertices of G of degree 1 have different neighbors then G has at most 20 vertices of degree 1.*

Proof. Let V_1 denote the set of all vertices of G with degree 1. Suppose for a contradiction, that $|N(V_1)| \geq 2$ and $|V_1| > 20$. By Lemma 8 we may assume that G contains a vertex x with $\deg x \geq 38$. Let $x_1, x_2 \in V_1$ and y_1, y_2, $y_1 \neq y_2$, be the neighbors of x_1 and x_2 respectively.

Note that y_1 and y_2 cover at most 7 edges. Indeed, otherwise $G' := G - \{x_1, x_2, y_1, y_2\}$ arises from G by deleting 4 vertices and at least 8 edges. Hence, $||G'|| + \Delta(G') \leq 2|G'| - \alpha(|G'|)$, by Proposition 6. Thus, by the minimality assumption there is an embedding σ' of G'. Then, $(x_1, y_1, x_2, y_2)\,\sigma'$ is an embedding of G. On the other hand, if $\deg y_1 = 1$, then $G'' := G - \{x, x_1, y_1\}$ also satisfies $||G''|| + \Delta(G'') \leq 2|G''| - \alpha(|G''|)$ by Proposition 6. Hence, by the minimality assumption there is an embedding σ'' of G''. Then $(x, x_1, y_1)\,\sigma''$ is an embedding of G. The same argument holds if $\deg y_2 = 1$.

Therefore, we may assume that $2 \leq \deg y_1 \leq 6$ and $2 \leq \deg y_2 \leq 6$, and x is not a neighbor of any vertex from V_1. Moreover, $\deg y_1 + \deg y_2 \leq 8$ if y_1y_2 is an edge of G, and $\deg y_1 + \deg y_2 \leq 7$ otherwise. In particular, y_2 has at most $7 - \deg y_1$ neighbors in V_1. Analogously, every vertex other than y_1 of G has at most $7 - \deg y_1$ neighbors in V_1. Let $V_1' \subset V_1$ be the set of all vertices of degree 1 which are at distance equal to 1 or 2 from y_1. Let $V_1'' = V_1 \setminus V_1'$. Thus, $|V_1'| \leq (\deg y_1 - 1)(7 - \deg y_1) + 1$. Hence, $|V_1''| \geq |V_1| - (\deg y_1 - 1)(7 - \deg y_1) - 1$. Since every vertex other than y_1 of G has at most $7 - \deg y_1$ neighbors in V_1, we have

$$|N(V_1'')| \geq \frac{|V_1| - (\deg y_1 - 1)(7 - \deg y_1) - 1}{7 - \deg y_1}.$$

Therefore, if $|V_1| \geq (\deg y_1 - 1)(7 - \deg y_1) + 1 + (\deg y_1 - 1)(7 - \deg y_1) + 1$ then $|N(V_1'')| \geq \deg y_1$, so we can find an independent set $W \subset V_1$ of $\deg y_1$ vertices of degree 1 that have different neighbors and are at distance at least 3 from y_1. It is easy to check that the above statement is true if $|V_1| \geq 20$ since the largest number of vertices of degree 1 is needed when $\deg y_1 = 4$.

Consider now a graph $G''' := G - (W \cup \{x, x_1, y_1\})$. Note that in order to obtain G''' we remove from G, $\deg y_1 + 3$ vertices and at least $\deg y_1 + (\deg y_1 + \deg x - 1) \geq 2(\deg y_1 + 3)$ edges. Therefore, by Proposition 6, $||G'''|| + \Delta(G''') \leq 2|G'''| - \alpha(|G'''|)$. Hence, by the minimality assumption, there is an embedding σ''' of G'''. Furthermore, $(x, x_1)\sigma'''$ is an embedding of $G - (W \cup \{y_1\})$. Then, by Lemma 3, there is an embedding of G, a contradiction. $\square$

3 Proof of Theorem 2

Proof. Assume that G is a counterexample to Theorem 2 with minimum order n. By Lemma 7, $n \geq 2745$ and, by Lemma 8, $\Delta(G) \geq 38$. Moreover, by Lemma 10, G has no isolated vertices. Let $k = \lfloor n^{1/3} \rfloor$. Let S denote a most numerous independent set consisted of vertices of degrees $2, ..., k$ which have pairwise disjoint sets of neighbors. By Proposition 6, $||G - S|| + \Delta(G - S) \leq 2|G - S| + \alpha(|G - S|)$. Thus, if $S \neq \emptyset$, then, by the minimality assumption, $G - S$ is embeddable. Hence, by Lemma 3 (with $l = k$),

$$|S| < 2k. \tag{1}$$

Clearly, (1) holds also if $S = \emptyset$. Thus,

$$|N(S)| < 2k^2 \leq 2n^{2/3}. \tag{2}$$

Let $V_j := \{v \in V(G) \setminus N(S) : d(v) = j\}$. By the definition of S, every vertex from $V_2 \cup ... \cup V_k$ has a neighbor in $N(S)$. Furthermore, the number n_k of vertices of degree greater than k does not exceed $4n^{2/3}$ because $2||G|| = \sum_{v \in V(G)} d_G(v) < 4n$. Therefore

$$|N(N(S))| \geq |V_2 \cup ... \cup V_k| \geq n - |V_1| - n_k - |N(S)| \geq n - |V_1| - 4n^{2/3} - |N(S)|. \tag{3}$$

Now, let $U = N(S) \cup \{x\}$ if x is a common neighbor of all vertices of degree 1. Otherwise let $U = N(S)$. Hence $U \neq \emptyset$. Indeed, in the former case $x \in U$. In the latter we also have that $U \neq \emptyset$ for otherwise

$$4n - 28n^{2/3} - 40 \geq 2||G|| = \sum_{u \in V(G)} d(u) \geq 20 + (n - 20)n^{1/3} \tag{4}$$

because in this case there are at most 20 vertices of degree 1, see Lemma 11. However, for $n \geq 2745$ inequality (4) is false. Furthermore, by (3), we have

$$|N(U)| \geq n - 20 - 4n^{2/3} - |N(S)| \geq n - 6n^{2/3} - 20. \tag{5}$$

Thus, vertices from U cover at least $n - 6n^{2/3} - 20$ edges. Consider now the graph $G - U$. Let T_1,...,T_p denote connected components of $G - U$ which are trees such that each vertex of T_i is adjacent to at most one vertex in $N(S)$. We call these components *minimal components* of $G - U$. Let $R := G - U - V(T_1) - ... - V(T_p)$. Let r denote the sum of the size of R and the number of all vertices in R which are joined (in G) with U by at least two edges. Since R does not contain minimal components, every component of R which is a tree contains a vertex joined with U by at least two edges. On the other hand, every component of R which is not a tree has at least as many edges as vertices. Hence,

$$r \geq |R|. \tag{6}$$

Moreover, r counts all edges in R and some edges between R and $N(S)$ which are not counted in inequality (5), because this inequality counts only the number of vertices in $N(U)$ and ignores the number of connections.

Note that there are exactly $n - |U| - |R| - p$ edges in $\bigcup_{i=1}^{p} T_i$. Below we show that p is greater than or equal to $3|U| + \Delta - |R| + r$. By the assumption and by inequality (5), we have

$$\begin{aligned} 2n - 14n^{2/3} - 20 - \Delta \geq ||G|| &\geq n - 20 - 6n^{2/3} + (n - |U| - p - |R|) + r \\ &\geq 2n - 8n^{2/3} - 20 - p - |R| + r, \end{aligned}$$

because $|U| \leq |N(S)| + 1 \leq 2n^{2/3}$. Thus

$$p \geq 6n^{2/3} - |R| + r + \Delta \geq 3|U| + \Delta - |R| + r. \tag{7}$$

We will now partition $V(G)$ into two sets each of which induce an embeddable subgraph. First we will try to assign to each vertex u of U a minimal component which is connected with u. Let l be the maximum number of minimal components assigned to vertices of U in this way. If $l < |U|$, then we assign an arbitrary minimal component to every remaining vertex of U. Let $\mathcal{M}'$ be the set of minimal components not yet assigned. Now, we assign $2|U|$ different minimal components to vertices from U in such a way that every vertex $u \in U$ has two minimal components in $\mathcal{M}'$ disjoint with u. This is possible because $|\mathcal{M}'| \geq \Delta + 2|U|$. So, we have constructed l starry trees with middle vertices in U. Note, that l is the maximum number of starry trees with middle vertices in U.

Without loss of generality we may assume that we have assigned $T_1, ..., T_{3|U|}$. Let $G' := G[U \cup V(T_1) \cup ... \cup V(T_{3|U|})]$ and $G'' := G - V(G')$. Below we will show that there exists an embedding of G' such that every vertex from U is the image of its neighbor outside of U.

Suppose first that $l = |U|$. Then we pack every starry tree in such a way that the middle vertex is the image of one of its neighbors in the same starry tree (the required embedding exists by Lemma 9). Let σ_i be the required embedding of H_i. We claim that the product $\sigma = \sigma_1 ... . \sigma_{|U|}$ is an embedding of G' as well. Since σ_i is an embedding of H_i, only edges between different starry trees may spoil the embedding of G'. Furthermore, every middle vertex is mapped on a non-middle vertex. Since there are no edges between T_i and T_j for $i \neq j$, the edges between middle vertices do not spoil the embedding. It remains to check the edges of the form xy where x is the middle vertex of some starry tree and y is a non-middle vertex of another starry tree. However, since the middle vertex of each starry tree is the image of one of its neighbors in the same starry tree and this neighbor has no other neighbors outside its minimal component, these edges also do not spoil the embedding.

Suppose now, that $l < |U|$. Again, we pack every starry tree in such a way that the middle vertex is the image of one of its neighbors. Moreover, since L is maximal, each remaining vertex of U has no neighbors in each of the remaining minimal components (otherwise, we would have an extra starry tree). Hence, by Theorem 5, each of the remaining vertices from U together with three non-trivial minimal components (not involved in any starry tree) can be packed without fixed points. We claim that the product of these embeddings is a proper embedding of G'. Suppose for a contradiction that the image of an edge e in G' coincides with some other edge e' in G'. Using the previous argument, e' must join a vertex $z \in U$ which is not in any starry tree from L with a non-middle vertex of some starry tree H. Moreover, e must join the middle vertex of H with some minimal component which is not in any starry tree from L. However, now we can exchange the two minimal components that contain one of the endvertices of the edges e and e'. This way we obtain more than l starry trees and we get a contradiction. Hence G' is embeddable.

Recall that $r \geq ||R||$. Furthermore, by (5) we have

$$\begin{aligned} ||G''|| = ||R \cup T_{3|U||+1} \cup ... \cup T_p|| &= ||R|| + |T_{3|U|+1}| + ... + |T_p| - (p - 3|U|) \\ &< ||R|| + |T_{3|U|+1}| + ... + |T_p| - (r - |R| + \Delta) - 1 \\ &\leq |R| + |T_{3|U|+1}| + ... + |T_p| - 1 = |R \cup T_{3|U|+1} \cup ... \cup T_p| - 1 = |G''| - 1. \end{aligned}$$

Thus, by Theorem 1, G'' is embeddable.

Let σ', σ'' denote embeddings of G' and G'', respectively. Then $\sigma = \sigma'\sigma''$ is an embedding of G. Suppose for a contradiction that the image of an edge xy in G coincides with some other edge $\sigma(x)\sigma(y)$ in G. Then $x, \sigma(x) \in V(G')$ and $y, \sigma(y) \in V(G'')$. By construction of G' and G'' we have that x and $\sigma(x)$ belong to U. Then we get a contradiction, since the image of every vertex in U is not in U. The embedding σ contradicts the assumption that G was non-embeddable, so we deduce no counterexample to Theorem 2 exists. $\square$

4 Concluding remarks

Let D be a digraph with a vertex set $V(D)$ and an arc set $A(D)$. For a vertex x of $V(D)$ let us denote by $d^+(x)$ the outer degree of x. By $d^-(x)$ we denote the inner degree of x. The degree of a vertex x, denoted by $d(x)$, is defined by $d(x) = d^+(x) + d^-(x)$. If xy and yx belong to $A(D)$, then we say that x and y are joined by a pair of *symmetric arcs*.

Similarly as in case of graphs, we say that D is *embeddable* (in its complement) if there is a permutation σ on $V(D)$ such that if xy is an an arc of D, then $\sigma(x)\sigma(y)$ is not an arc of D.

If a digraph D has only symmetric arcs, then by Theorem 1, D is embeddable if $||D|| \leq 2n-4$. This leads to the following conjecture.

Conjecture 12 ([1]) *Let D be a digraph of order n. If D has at most $2n-4$ arcs, then D is embeddable.*

Conjecture 12 is asymptotically true, see [7]. As a corollary of Theorem 2 we obtain that the conjecture is true for digraphs that have sufficiently many symmetric arcs. Let $d^*(x) = \max\{d^+(x), d^-(x)\}$ and let $\Delta^* = \max\{d^*(x) : x \in V(D)\}$.

Corollary 13 *Let D be a digraph of order n and size m with $m \leq 2n-4$. If the number of pairs of symmetric arcs of D is at least $\Delta^* + 14n^{2/3} + 16$ then D is embeddable.*

Proof. Let s denote the number of pairs of symmetric arcs in D. Construct a graph $G(D)$ by replacing every arc or every pair of symmetric arcs of D by an edge with the same endvertices. Note that $||G(D)|| = m - s$ and $\Delta(G(D)) = \Delta^*$. By the assumption on n and on s we have

$$||G(D)|| + \Delta(G(D)) = m - s + \Delta^* \leq 2n - 4 - (14n^{2/3} + 16 + \Delta^*) + \Delta^* = 2n - 14n^{2/3} - 20$$

Thus, by Theorem 2, G is embeddable. Therefore, D is embeddable, too. $\square$

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