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On vertex stability with regard to complete k-partite graphs

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Abstract

A graph G is called H-stable if $G - u$ contains a subgraph isomorphic to H for arbitrary chosen $u \in V(G)$. The minimum size among the sizes of all H-stable graphs is denoted by $\mathrm{stab}(H)$. It is known that $\|H\| + \Delta_H$ and $\|H\| + |H|$ are, respectively, the lower and upper general bound for $\mathrm{stab}(H)$, satisfied for every graph H. We give the exact values of $\mathrm{stab}(H)$ and characterize all H-stable graphs with minimal size for H being any complete k-partite graph, which is a generalization of the results of Dudek and Żak regarding to complete bipartite graphs. In particular, we show that, dependently on the orders of components of partition of H, $\mathrm{stab}(H)$ is equal to the lower or the upper general bound (no in-between value is possible).

Keywords: vertex stability, minimal stable graphs, complete k-partite graphs.
MSC: 05C35, 05C60

1 Introduction

Consider a network of sensors (processors, transmitters etc.). We require that given configuration of connections between the sensors is assured even in the case of a failure of one of them. Assuming that the connections between sensors are more costly than the sensors we are interested in establishing the structure of a fault-tolerant network of minimal cost with respect to the given configuration.

More formally, we consider only simple graphs without loops, multiple edges and isolated vertices. We are using the standard notation of graph theory [2] and some of the notation introduced in [3]. Let H be any graph with set of vertices $V(H)$ and set of edges $E(H)$. A graph G is said to be (H, k)- vertex stable if G contains a subgraph isomorphic to H after removing any k of its

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vertices. If $k = 1$ we say shortly that G is H-stable. Moreover, $\mathrm{stab}(H)$ denotes the minimum of sizes of all H-stable graphs. The order and the size of H are denoted by n and m respectively.

The exact values of $\mathrm{stab}(H)$ are known some basic classes of graphs, e.g. K_n, $K_{p,q}$ [5]. Moreover, the exact value of $\mathrm{stab}(C_n)$ is known for some infinite classes of n [1]. The (H, k)-stable graphs of minimal size were characterized for H being $C_3, C_4, K_4, K_{1,m}$ [3], K_5 [6] and K_n for k large enough [7]. The general bounds of the value of $\mathrm{stab}(H)$ are following.

Proposition 1 ([5]). *Let H be any graph with n vertices and m edges. Then*

$$m + \Delta_H \leq \mathrm{stab}(H, 1) \leq \min\{m + n, 2m\} \tag{1}$$

□

Remark 1. A star ($K_{m,1}$) is the only graph for which the general lower bound is equal to the upper one (1). Therefore $\mathrm{stab}(K_{m,1}) = 2m$ [3].

The $K_{n,n}$-stable and $K_{n,n+1}$-stable graphs of minimal size were characterized in [4]. That results was generalized [5] to all complete bipartite graphs as follows.

Theorem 2 ([5]). *Let $p \geq q \geq 2$. Then for $H = K_{p,q}$*

$$\mathrm{stab}(H) = \begin{cases} pq + p & \text{for } p - q = 1 \\ pq + p + q & \text{for } p - q \neq 1. \end{cases}$$

Moreover, in [5] all $K_{p,q}$-stable graphs of minimal size was characterized. Namely, if $p = q + 1 > 2$ then $K_{p,p}$ is the only $K_{p,q}$-stable graph of minimal size. Otherwise, if $p \geq 4, q \geq 2$ and $p \geq q$ then the only $K_{p,q}$-stable graph of minimal size are $G_1 = K_{p,q} * K_1$ and $G_2 = K_{p+1,q+1} - e$, where e is any edge of $K_{p+1,q+1}$.

Keeping the assumption that $H = K_{p,q}$ with $p \geq q \geq 2$ we can formulate (2) in the following way,

$$\mathrm{stab}(H) = \begin{cases} m + \Delta_H & \text{for } p - q = 1 \\ m + n & \text{for } p - q \neq 1. \end{cases}$$

Observe that $\mathrm{stab}(K_{p,q})$ achieves exactly the lower or the upper bound of (1) and no in-between value of is possible. Before we show that this property holds in more general case of k-partite complete graphs with $k \geq 2$ we prove two useful lemmas.

Lemma 3. *Let H be a graph such that $\delta_H > 1$. If G is H-stable graph of minimal size, and $|G| = n + s$ then*

$$||G|| = \mathrm{stab}(H) \geq \begin{cases} m + \Delta_H & \text{if } s = 1 \\ m + \Delta_H + (s-1)\delta_H - \binom{s}{2} & \text{if } 2 \leq s \leq \delta_H \\ m + \Delta_H + \frac{1}{2}(s-1)(\delta_H - 1) & \text{if } \delta_H + 1 \leq s \leq \Delta_H \\ m + \frac{1}{2}\left(\Delta_H + (s-1)\delta_H\right) & \text{if } \Delta_H + 1 \leq s. \end{cases} \tag{2}$$

Proof. The case $s = 1$ is a straightforward consequence of (1). Then we only have to show the cases with $s > 1$. First, observe that if H contains t vertices of degree Δ_H then G contains at least $t + 1$ vertices of degree at least Δ_H. Consider now the vertices included in $V(G) \setminus V(H')$, where H' is some subgraph of G isomorphic to H. Each of them is of degree at least δ_H [3], and moreover (at least) one of them has degree greater or equal to Δ_H . Therefore

$$\sum_{v \in V(G) \setminus V(H')} \deg_G(v) \geq \Delta_H + (s-1)\delta_H. \tag{3}$$

Now we use (3) to assess the number edges in G. Each edge incident with some vertex of $V(G) \setminus V(H')$ is counted once or twice in (3).

(i) $2 \leq s \leq \delta_H + 1$. At most $\binom{s}{2}$ edges incident to the vertices of $V(G) \setminus V(H')$ are counted twice in (3).

(ii) $\delta_H + 1 \leq s \leq \Delta_H$. There are at least Δ_H edges incident with some vertex $u \in V(G) \setminus V(H')$ and at least $\frac{1}{2}(s-1)(\delta_H - 1)$ edges incident with the vertices of $V(G-u) \setminus V(H')$.

(iii) $\delta_H + 1 \leq s \leq \Delta_H$. All the edges can be counted twice in (3).

□

Lemma 4. *Let H be a graph such that $\delta_H > 1$. If G is H-stable graph of minimal size, and $|G| = n + s$. Then*

$$||G|| = \mathrm{stab}(H) \geq \begin{cases} m + \Delta_H & \text{if } s = 1 \\ m + \Delta_H + \delta_H - 1 & \text{if } 2 \leq s \leq \delta_H \\ m + \Delta_H + \frac{1}{2}\delta_H(\delta_H - 1) & \text{if } \delta_H + 1 \leq s \leq \Delta_H \\ m + \frac{1}{2}\left(\Delta_H + \Delta_H \delta_H\right) & \text{if } \Delta_H + 1 \leq s. \end{cases} \tag{4}$$

where the exact bounds of (4) *can be achieved only for $s = 1, s = 2, s = \delta_H + 1$ and $s = \Delta_H + 1$.*

Proof. It is a simple consequence of Lemma 3 and fact that the expressions of right-side part of (2) are increasing (with respect to the domain) functions of variable s. □

2 Complete k-partite graphs

Theorem 5. *Let H be a complete k-partite graph $H = K_{n_1 n_2 \ldots n_k}$ with $k \geq 2$ and $n_1 \geq n_2 \geq \ldots \geq n_k$ such that $H \neq K_{m,1}$. Then*

$$\mathrm{stab}(H) = \begin{cases} m + \Delta_H & \text{for } n_1 = n_2 = \ldots = n_{k-1} = n_k + 1 \\ m + n & \text{otherwise} \end{cases}$$

Proof. Since the theorem is proved for $k = 2$ [5] we assume that $k \geq 3$.
I. Let $n_1 = n_2 = \ldots = n_{k-1} = n_k + 1$. It can be easily checked that $G = K_{n_1 n_1 \ldots n_1}$ is H-stable and $||G|| = m + \Delta_H$. Due to (1) we know that there is no H-stable graph of smaller size which completes the proof of this case.
II. Now consider any k-partite complete graph H different than defined in I. Let G be an H-stable graph. First, due to Lemma 4 and facts that $\delta_H = n - n_1$ and $\Delta_H = n - n_k$, we can observe that if $|G| \geq n+2$ then $||G|| \geq m + n$. Therefore we assume that $|G| = n + 1$.

Now let us transform our problem to equivalent one. Since we are assuming that $|G - x| = n$ then, in fact, $V(G - x) = V(H')$ for any $x \in V(G)$ where $H' \subset G - x$ is isomorphic to H. Therefore

$$(G - x \supset H') \Leftrightarrow (\overline{H'} \supset \overline{G - x}).$$

Now we are interested in maximizing the size of $\overline{G}$ such that $\overline{G - x}$ is isomorphic with some subgraph of $\overline{H}$ for arbitrary chosen x.

$\overline{H}$ is a union of k cliques (of orders $n_1, n_2, \ldots, n_k$), hence graph $\overline{G - x}$ has at least k components of connectivity for any $x \in V(G)$. Since each connected graph of order greater than one contains a vertex which can be removed without loosing connectivity, we conclude that graph $\overline{G}$ also consists of at least k components of connectivity (of orders, say, $r_1, \ldots, r_{k+t}$, such that $r_1 \geq \ldots \geq r_{k+t}$ with $t \geq 0$).
Of course

$$n + 1 = n_1 + \ldots + n_k + 1 = r_1 + \ldots r_{k+t}. \tag{5}$$

Consider the multiset $R_j := \{r_1, \ldots, r_{j-1}, r_j - 1, r_{j+1}, \ldots, r_{k+t}\}$. For each $j \in \{1, \ldots k + t\}$ there must exist a partition of R_j into k subsets $R_j^1, \ldots, R_j^k$ such that, due to (5), the sum of elements of R_j^i is equal to n_i.

First assume that $s = 0$ ($\overline{G}$ consists of exactly k components of connectivity). In that case the partition of R_j into k subsets is unique (exact to the labeling of the subsets) - each subset consists just of one element. Then the equality $r_1 = \ldots = r_k$ must be satisfied. Indeed, if $r_j \neq r_l$ then $R_j \neq R_l$ and at least one of R_j, R_l does not correspond to given sequence of clique orders in $\overline{H}$. The equality of all r_i's implies that $n_1 = \ldots = n_{k-1} = n_k + 1$, but this is exactly the case already considered in I, which is exluded in II.

Assume now that $s > 0$. Obviously

$$||\overline{G}|| \leq \binom{r_1}{2} + \ldots + \binom{r_{k+t}}{2}. \tag{6}$$

Consider now the partition of R_{k+t} corresponding to $\overline{G - x}$, where x belongs to the $(k + t)$th component of $\overline{G}$. It is clear that for each $i = 1 \ldots k$ not more than $\binom{n_i}{2}$ edges of $\overline{G - x}$ can be included in a component of $\overline{H}$ of order n_i.

Therefore

$$||\overline{G - x}|| \leq \binom{n_1}{2} + \ldots + \binom{n_k}{2}. \tag{7}$$

(i) If each component of $\overline{G}$ is a clique, $t = 1$, $r_{k+1} = 1$ and $r_i = n_i$ for $i = 1 \dots k$, we obtain

$$||\overline{G}|| = ||\overline{G-x}|| = \binom{n_1}{2} + \dots + \binom{n_k}{2},$$

and consequently

$$||G|| = \binom{n+1}{2} - ||\overline{G}|| = n + \binom{n}{2} - \binom{n_1}{2} - \dots - \binom{n_k}{2} = m + n.$$

The graph G constructed in that way is isomorphic with $K_1 * H$ which is H-stable [5].

(ii) If (i) is not satisfied then some R^l_{k+t} consists of two (or more) elements. Consequently, in lth component of $\overline{H}$ two (or more) disjoint components of $\overline{G-x}$ are included, leaving unused the edges of $\overline{H}$ between them. The number of that unused edges is minimal if there only two disjoint components being cliques. Assuming that the orders of that disjoint cliques are, say a and b, then ab unused edges are in $\overline{H}$. Since the two smallest cliques in $\overline{G-x}$ are of orders not less than 1 and r_{k+t} we obtain that

$$||\overline{G-x}|| \leq \binom{n_1}{2} + \dots + \binom{n_k}{2} - r_{k+t},$$

and, consequently,

$$||\overline{G}|| = ||\overline{G-x}|| + r_{k+t} - 1 \leq \binom{n_1}{2} + \dots + \binom{n_k}{2} - 1$$

which is less then in case (i).

This shows that in case II there is no H-stable graph G containing less than $m + n$ edges which ends the proof. □

Theorem 6. *Let H be a complete k-partite graph $H = K_{n_1 n_2 \dots n_k}$ with $k \geq 3$ and $n_1 \geq n_2 \geq \dots \geq n_k$ such that $H \neq K_{m,1}$ and $H \neq K_3$. Then the only H-stable graph of minimal size is $K_{n_1,\dots,n_1}$ if $n_1 = n_2 = \dots = n_{k-1} = n_k + 1$ and $H * K_1$ otherwise.*

Proof.

I The case $n_1 = n_2 = \dots = n_{k-1} = n_k + 1$. Let G be a H-stable graph of minimal size, i.e. $||G|| = m + \Delta_H$. If $|G| > n+1$ then, as it was already showed in the proof of Theorem 5, $||G|| \geq m+n > m+\Delta_H$ - a contradiction. Therefore we assume that $|G| = n+1$. It is easy to observe that $\delta_G = \Delta_G = \Delta_H$. Indeed, if $\delta_G \leq \Delta_H - 1 = \delta_H$ then removing some neighbour of a vertex of degree δ_G we obtain a graph of minimal degree less than δ_H, which cannot contain H as a subgraph. On the other hand, if $\Delta_G > \Delta_H$ then $||G|| > \frac{1}{2}(n+1)\Delta_H = m + \Delta_H$ - a contradiction. Therefore $||G - u|| = m$ and, in consequence, $G - u \cong H$ for

arbitrary chosen vertex u. It is clear that the only graph satisfying this property is the complete k-partite graph with all components of partition of order n_1.
II The "otherwise" case. Let G be a H-stable graph of minimal size, i.e. $||G|| = m + n$
1. If $|G| = n + 1$ then, accordingly to the proof of Theorem 5, $G = H * K_1$ is the only H-stable graph of minimal size.
2. Assume, as a contrary, that $|G| = n + s$ with $s > 1$. Due to Lemma 4, after some calculations, we observe that there may exist an H-stable graph of size $m+n$ only if $k = 3$ and $n_2 = n_3 = 1$ and if one of the following cases is satisfied:

a) $s = 2$

b) $s = 3$ with $\Delta_H \geq 3$.

c) $s = 3$ with $\Delta_H = 2$.

We show that, in fact, even in this cases there is no H-stable graph of size $m+n$ and order greater than $n + 1$. First observe that since $k = 3$ and $n_2 = n_3 = 1$ then $\delta_H = 2$ and $\Delta_H = n - 1$. If $\Delta_H = \delta_H = 2$, then, in fact, $H = K_3$ which is excluded in theorem's assumptions (it is easy to observe that the only K_3-stable graph of minimal size are K_4 and $2K_3$). Therefore it is enough to focus only the cases a) and b) assuming that $\Delta_H \geq 3$.

Case a) If $\Delta_G \geq \Delta_H + 1 = n$ then $||G - \bar{u}|| \leq (m + n) - n$, where $\deg_G(\bar{u}) = \Delta_G$. Since $G - \bar{u}$ contains not more than m edges incident with $n+1$ non-isolated vertices it cannot contain H as a subgraph.

Consider now the case $\Delta_G = \Delta_H = n - 1$. Observe that since $n_2 = n_3 = 1$ then there exist vertices u and u' of degree Δ_H in H. If $G - u$ contains H' as a subgraph, where H' is isomorphic to H then there must exist some vertex $v \in V(G - u)$ such that H' is a subgraph of $G - \{u, v\}$. If we show that $\delta_{G-u} = 2$ we obtain a contradiction, because then $||G - \{u, v\}|| \leq ||G - u|| - 2 = m - 1$. Indeed, since H' is a subgraph of G then $V(G) = V(H') \cup \{x, y\}$ and, w.l.o.g. $\deg_G(x) \geq 2$ and $\deg_G(y) \geq \Delta_H = n - 1$. Note that $ux, u'x, uy, u'y \notin E(G)$ (otherwise $\Delta_G > \Delta_H$). Then y is connected with all vertices except u and u', hence each vertex of G except x is of degree at least three. Consequently $\delta_{G-u} \geq 2$.

Case b) Let H' be some copy H being a subgraph of G. Then we may assume that $V(G) = V(H') \cup \{x, y, z\}$ such that $\deg_G(x) \geq 2$, $\deg_G(y) \geq 2$ and $\deg_G(z) \geq \Delta_H = n - 1$. If $xy \notin E(G)$ then $||G|| \geq m + 2 + 2 + n - 1 - 2 > m + n$ - a contradiction.

Therefore assume that $xy \in E(G)$. Since G is a H-stable graph of minimal size then xy is included in some copy of H, say H'', being a subgraph of G. In that case x and y are not in the same component of partition of H'', hence at least one of vertices x, y has degree at least $\Delta_H \geq 3$. Therefore $||G|| \geq m + 2(n - 1) + 2 - \binom{3}{2} > m + n$, a contradiction. $\square$

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