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Preprint Nr MD 094
(otrzymany dnia 30.10.2017)

Kraków
2017

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Keywords: configuration space, PL-manifold, deformation retraction, equivariant map, polyhedron, collar, discretized configuration space

AMS classification: 57Q91

Abstract

In this paper, we prove that the configuration space $F_n(M)$ of n particles in a compact connected PL-manifold M with nonempty boundary ∂M is homotopy equivalent to the configuration space $F_n(\mathrm{Int}M)$ where $\mathrm{Int}M = M \setminus \partial M$. We formulate some generalization of this result for polyhedra. The similar results has been obtained independently for topological manifolds by C.Zapata in [11], by using somewhat different techniques.

We also adress the question of whether a compact PL-manifold M can be approximated up to homotopy type by discrete configuration spaces defined combinatorially via a simplicial subdivision of M.

1 Introduction

Let X be a topological space and X^k its k-fold cartesian product, $k \geq 2$. Define the (complete) diagonal D of X^k as follows: $D = \{(x_1, \dots, x_k) \in X^k | x_i = x_j \text{ for some } i \neq j\}$.

For a given topological space X denote by $F_k(X)$ the space $X^k \setminus D$, the configuration space of k particles in X without collisions. The topology of classical classification spaces $F_k(\mathbf{R}^n)$ has been extensively studied by many authors (see for example, [6, 8] for backgrounds). A fundamental work on this topic is the monograph of Falell and Husseini [7], in which the case of sphere $X = S^m$ is also treated. The homology structure of $F_k(\mathbf{R}^n)$ was described, for example, in [3]. It is also known that configuration spaces are not homotopy invariant even for closed manifolds (see [10]). In this paper, we will show that if the homotopy equivalence of manifolds is given by a deformation retraction of one onto another inside a collar of the boundary of the first manifold, it descends to the deformation of corresponding configuration spaces.

2 Configuration spaces of compact manifolds with boundary

In this section, we compare configuration spaces of a compact connected manifold M with nonempty boundary ∂M and the open manifold $\mathrm{Int}(M)$. In the following, we assume that M is endowed with a smooth or PL-structure.

Let D^n be a closed n-dimensional disc in $\mathbf{R}^n$. The proof of the following lemma uses the techniques developed by S.Crowley and A.Skopenkov in [5].

Lemma 2.1 *For each positive integer k the space $F_k(D^n)$ is a deformation retract of the space $F_k(\mathbf{R}^n)$. Moreover, there is a Σ_k-equivariant deformation retraction of $F_k(\mathbf{R}^n)$ onto $F_k(D^n)$.*

Proof. Let S^n be n-dimensional sphere, $S^n = \mathbf{R}^n \cup \{\infty\}$. Decompose S^n into two half-spheres S_0 and S_∞ where $S_0 = \{w \in \mathbf{R}^n : |w| \leq 1\}$ and $S_\infty = \mathrm{cl}(S^n \setminus S_0)$. Consider the subspace $R = S^n \setminus \{0\}$ of S^n which is homeomorphic obviously to $\mathbf{R}^n$. There is a Σ_k-equivariant deformation retraction g_t of $F_k(R)$ on $F_k(S_\infty)$. To show this, consider in $\mathbf{R}^n \subset S^n$ a closed disc D_2 of radius 2 centered in 0. Then S_0 is identified with a closed unite disc D_1.

Take any $x = (x_1, \ldots, x_k) \in F_k(R) \setminus F_k(S_\infty)$ and any j such that $min_s\{|x_s|\} = |x_j|$. Denote $|x_j|$ by ρ. We have obviously $1 > \rho > 0$. Take a piece linear function $h\colon [0, 2] \to [0, 2]$ such that $h(0) = 0$ $h(2) = 2$ and $h(\rho) = 1$. Define a map $f\colon F_k(R) \to F_k(S_\infty)$ as follows:
1)$f_s(x_1, \ldots, x_k) = x_s$ if no $x_i, i = 1, \ldots, k$, is not in $\mathrm{Int}(D_1), s = 1, \ldots, k$,;
2)$f_s(x_1, \ldots, x_k) = \frac{x_s}{|x_s|}\frac{2-2\rho+|x_s|}{2-\rho}$ if $(x_1, \ldots, x_k) \in F_k(R) \backslash F_k(S_\infty)$ and $x_s \in D_2$; 3) $f_s(x_1, \ldots, x_k) = x_s$ if $(x_1, \ldots, x_k) \in F_k(R) \setminus F_k(S_\infty)$ and x_s is not in $\mathrm{Int}(D_2)$.

Note that if $2 \geq |x_s| > \rho$ where x_s is the s coordinate of $(x_1, \ldots, x_k)$, then $2 \geq |f_s(x_1, \ldots, x_k)| > 1$. The map f_s simply constrict the s-th coordinate vector x_s of $(x_1, \ldots, x_k) \in F_k(R) \setminus F_k(S_\infty)$ in accordance to the piece linear function h.

The map f_s obviously extends to a homotopy g_t^s via the following formula:

$g_t^s(x_1, \ldots, x_s, \ldots, x_k) = (1-t)(x_1, \ldots, x_s, \ldots, x_k) + tF_s(x_1, \ldots, x_s, \ldots, x_k)$. This gives the desired Σ_k-equivariant deformation retraction $g_t\colon F_k(R) \times I \to F_k(S_\infty)$.$\diamond$

Let M be a connected, compact and smooth or PL manifold with the nonempty boundary ∂M.

Theorem 2.1 *For each $k \geq 1$ the configuration space $F_k(M)$ is Σ_k-equivariantly homotopy equivalent to the configuration space $F_k(\mathrm{Int} M)$.*

Proof. Let $R_1, \ldots, R_m$ be the connected components of ∂M. Moreover let $C_i, i = 1, \ldots, m$ be the closed collars of $R_1, \ldots, R_m$, respectively, in M where $C_i \cong R_i \times [0, 2], i = 1, \ldots, m$, and $C_i \cap C_j = \emptyset$ if $i \neq j$ and R_i is identified with $R_i \times \{0\}$. We can also identify $U = \cup_i^m (R_i \times [0, 1))$ with a collar of ∂M in M and call it a small (open) collar of ∂M in M. Therefore each $y_i \in C_i$ can be uniquely represented as $y_i = (x_i, \tau_i)$ where $x_i \in R_i$ and $0 \le \tau_i \le 2$. Now we define a deformation retraction of the manifold $F_k(\mathrm{Int}M)$ onto the manifold $F_k(M \setminus U) \cong F_k(\mathrm{cl}(M \setminus (\cup_{i=1}^m R_i \times [0,1])))$. Note that the manifold $F_k(M \setminus U)$ is homeomorphic to $F_k(M)$.

Take any $y = (y_1, \dots, y_k) \in F_k(\mathrm{Int}M)$. Let $y_{i_1}, \dots y_{i_r}$ be the coordinates of y that belong to the small collar U of ∂M. Assume that the set of such the coordinates is nonempty. Let $y_{i_s} = (x_{i_s}, \tau_s)$ for each $1 \le s \le r$ where $x_{i_s} \in R_{j_s}$ for some $j_s, 1 \le j_s \le m$. Take any l such that $min_s\{\tau_s\} = \tau_l$ where s runs from 1 to r. Denote τ_l by ρ. We have obviously $1 > \rho > 0$. To continue, take a piece linear function $h\colon [0,2] \to [0,2]$ as in the proof of Lemma 2.1, so that $h(0) = 0$ $h(2) = 2$ and $h(\rho) = 1$.

Define a map f $F_k(\mathrm{Int}M)$ onto $F_k((M \setminus U)$ as follows:

1)$f_s(y_1, \dots, y_k) = y_s$ if no $y_i, i = 1, \dots, k$, belongs to U;

2)$f_s(y_1, \dots, y_k) = y_s$ if some $y_i, i = 1, \dots, k$, belongs to U and y_s does not belong to $\cup_i^m (C_i \setminus \partial C_i)$;

3)$f_s(y_1, \dots y_k) = (x_s, \frac{2-2\rho+\tau_s}{2-\rho})$, if $(y_1, \dots y_k) \in F_k(\mathrm{Int}M) \setminus F_k(M \setminus U)$ and $y_s \in C_i \setminus \partial C_i$ for some i, where $y_i = (x_i, \tau_i), x_i \in R_i, 0 < \tau_i < 2, s = 1, \dots, k$.

Note that if $\rho < |\tau_s| \le 2$ where $y_s = (x_s, \tau_s)$ is the s-th coordinate of $(y_1, \dots, y_k)$, then $2 \ge |f_s(y_1, \dots, y_s, \dots, y_k)| > 1$. Therefore the map f_s sends the s-th vector $y_s = (x_s, \tau_s)$ of $(y_1, \dots, y_k)$ onto the vector $(x_s, \frac{2-2\rho+\tau_s}{2-\rho})$, in accordance to the piece linear function h. The map $f = (f_1, \dots, f_s, \dots, f_k) : F_k(\mathrm{Int}M) \to F_k(M \setminus U)$ is obviously continuous, so it retracts $F_k(\mathrm{Int}M)$ onto $F_k(M \setminus U)$ where $F_k(M \setminus U)$ is identified with $F_k(M \setminus (\cup_{i=1}^m R_i \times [0,1))$ as it was defined before.

Moreover the map f can be extended to the homotopy g_t of $F_k(\mathrm{Int}M) \to F_k(\mathrm{Int}M)$ with $g_0 = \mathrm{id}_{F_k(\mathrm{Int}M)}$ and $g_1 = f$. The construction of the homotopy g_t runs in the same way as the one in the proof of Lemma 2.1. It follows from definitions and our construction that g_t is Σ_k-equivariant deformatioin retraction of the manifold $F_k(\mathrm{Int}M)$ onto the manifold $F_k(M \setminus U)$. This completes the proof of the theorem.$\diamond$

Remark 2.1 *The assertion of Theorem 2.2 can be strengthened in more general situation. More precisely, let (M, L) be a pair of compact polyhedra where L is a subpolyhedron of M. Assume that L possesses a collar U in M. Then for each k we have: $F_k(M \setminus L)$ deformation retracts onto $F_k(\mathrm{cl}(M \setminus U))$.*

3 Discretized configuration spaces of complexes

Let K be a finite simplicial complex. Denote by $|K|$ the underlying topological space of K which is a polyhedron. For each $k \le n$ the subcomplex $D_n(K)$ of the cell complex K^n is defined in the following way: $D_n(K) = \bigcup \sigma_1 \times \ldots \times \sigma_n$ where the above summation is over all n pairwise disjoint closed cells in K (see [2, 4]). The subcomplex $D_n(K)$ is called the discrete configuration space of the complex K with parameter n. This is the largest cell complex that is contained in the product K^n minus its diagonal $\{(x_1, \ldots, x_n) \in |K|^n : |x_i = x_j \text{ for some } i \neq j\}$.The symmetric group Σ_n acts naturally on $D_n(K)$ by permuting the cells in the product. The polyhedron $|D_n(K)|$ has natural and Σ_n-equivariant embedding in the configuration space $F_n(G)$ for $n \geq 2$.

A.Abrams in [1] studied configuration spaces of graphs. A graph G can considered as 1-complex. A.Abrams proved that for each graph G there is a subdivision G' of G such that the discrete configuration space $D_n(G')$ is homotopy equivalent to the usual configuration space $F_n(G), n \geq 2$.

The problem of cell approximation of the space $F_n(X)$ where X is polyhedron of dimension ≥ 2 has been considered and studied in [4]. For $n = 2$, Hu [9] showed that the configuration spaces $D_2(K)$ and $F_2(K)$ are homotopy equivalent. Moreover he showed that for any finite simplicial complex K there is a Σ_2-equivariant deformation retraction of $F_2(K)$ onto $|D_2(K)|$. In general, the problem can be formulated as follows.

Problem 1. *Let X be a compact connected PL-manifold of dimension $k \geq 2$ and let $n > 2$. Show that there is a subdivision K of X such that the manifold $F_n(X)$ admits a Σ_n-equivariant deformation retraction onto the polyhedron $|D_n(K)|$ or present a counterexample.*

As to our best knowledge, for PL-manifolds of dimension $k \geq 2$, the question of cell approximation of configuration spaces remains open.

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