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On measures of nonplanarity of cubic graphs

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Abstract

We study two measures of nonplanarity of cubic graphs G, the genus $g(G)$ and the edge deletion number $ed(G)$. For cubic graphs of small order these parameters are compared with another measure of nonplanarity, the (rectiliniar) crossing number $\overline{cr}(G)$. We introduce operations of connected sum, specified for cubic graphs G, and show that under certain conditions the parameters $g(G)$ and $ed(G)$ are additive (subadditive) with respect to them.

The minimal genus graphs (i.e. the cubic graphs of minimum order with given value of genus g) and the minimal edge deletion graphs (i.e. cubic graphs of minimum order with given value of edge deletion number ed) are introduced. We also provide upper bounds for the order of minimal genus and minimal edge deletion graphs.

1 Introduction

We consider finite graphs without loops and multiple edges. The Kuratowski theorem states that a graph G is planar if and only if it does not contain subgraphs homeomorphic to K_5 and $K_{3,3}$. For cubic graphs, the only forbidden graphs are those which are not homeomorphic to $K_{3,3}$. There are different measures of nonplanarity of a graph. Let us recall their definitions.

For a given connected graph G denote by $g(G)$ the (orientable) genus of G i.e. the minimal genus of an orientable closed connected surface M such that G has an embedding in M. Note that each such embedding is 2-cell. The problem of deciding whether a cubic graph G has the genus $g(G) \leq m$ is known to be NP-complete [18]. There are some upper and lower bounds of $g(G)$ for different classes of graphs G [16]. For cubic graphs G, the precise values of the parameter $g(G)$ are known only for special classes of them (for example, for some snarks *etc.*, see [13, 16]).

Another well known measure of nonplanarity of a graph G is the crossing number $cr(G)$ (the rectilinear crossing number $\overline{cr}(G)$). This is the minimal number of proper double crossings of edges among all immersions of G in the plane (the minimal number of proper double crossings of edges among all rectilinear immersions of G in the plane, respectively). The crossing number of a graph is also NP-complete [4]. Note that, in general, $cr(G)$ and $\overline{cr}(G)$ are different numbers [2]. There are estimations of the parameters $cr(G)$ and $\overline{cr}(G)$ for complete graphs, complete bipartite graphs, and other special classes of graphs (see, for example [7, 17]). The precise values of $cr(G)$ and $\overline{cr}(G)$ are known only for particular nonplanar graphs (for example, for small complete and complete bipartite graphs [14, 17]).

For a given graph G, denote by $ed(G)$ the minimal number of edges in G such that after their deletion the resulting graph becomes planar. The parameter $ed(G)$ is called *the edge deletion number of* G and the corresponding problem of finding the minimal set of edges to be deleted in a graph G is known as MINED. Even for cubic graphs, the problem MINED is known to be NP-complete [8]. Algorithms of computing $ed(G)$, in particular, for cubic graphs, are described in [3, 8, 9].

Comparing with the parameters $g(G)$ and $cr(G)$, there are much more fewer results concerning evaluation of the number $ed(G)$.

Battle *et al.* [1] have shown that the genus of any connected graph is equal to the sum of blocks with respect to its block decomposition. This is perhaps the first known result on additivity of the (orientable) genus of graph. The operation of the vertex amalgamation applied to 2-connected cubic graphs gives a separable graph which contains a vertex of degree 4.

Another operation is the edge amalgamation of graphs G_1 and G_2 [10]. Miller [10] introduced the generalized genus of a graph and showed that it is additive with respect to the edge amalgamation of two graphs. The operation of edge amalgamation does not preserve the class of cubic graphs. In [5], Gross also studied bar-amalgamation of graphs.

In the present paper, we introduce two operations of connected sum of (cubic)graphs. We study additivity properties of genus and edge deletion number with respect to these operations. The first operation, when applied to two 2-connected cubic graphs, results in a 2-connected cubic graph. Similarly, the second operation preserves, in general, the class of 3-connected (or even cyclically 4-edge connected) cubic graphs. Recall that a cubic graph G is called cyclic k-edge connected if no set consisting of fewer than k edges can separate two circuits of G into distinct components. Note that for cubic graphs which contain two separate cycles the values of vertex connectivity, edge connectivity and cyclic k-edge connectivity coincide for $k \leq 3$ but cyclic edge connectivity may be arbitrarily large (see [11]).

For a given graph G, the order of G will be denoted by $|G|$ and the size of G by $\|G\|$. Pegg jr and Exoo [15] introduced the notion of a minimal crossing graph. Recall that for a given natural number k a cubic graph G is called *minimal k-crossing graph* if $|G| = k$ and k is of the minimum order among all cubic graph H with $\overline{cr}(H) = k$. By this analogy, we introduce minimal k-genus and minimal k-edge deletion graphs. We also provide an upper bound for the order of a 2-connected and 3-connected cubic graphs G, which are minimal with respect to the parameters ed or g.

2 Measures of nonplanarity of cubic graphs: small orders

We start by considering the parameters $cr(G)$ and $\overline{cr}(G)$ for small cubic graphs G and compare these numbers with the parameters $g(G)$ and $ed(G)$.

It is easy to see that we have the following inequalities: $g(G) \leq ed(G) \leq cr(G) \leq \overline{cr}(G)$. It can be shown that for cubic graphs the difference between any two of the parameters $g(G), ed(G), cr(G)$ of G can be arbitrarily large. This can be made, for example, by using results of Sections 2 and 3. Moreover, there exist graphs G for which the number $cr(G)$ is less than $\overline{cr}(G)$ (more precisely, $cr(G) = 4$ and $\overline{cr}(G) = m$ for any $m > 4$[2]).

We shall say that a cubic graph G is *minimal genus* for a given value of genus l (or simply minimal l-genus) if G has (orientable) genus equal to l and is of minimum order among all 2-connected cubic graphs with this property. Similarly, for a given number l, a cubic graph G is minimal edge deletion graph with the parameter l, if $ed(G) = l$ and G is of minimum order among all 2-connected cubic graphs with this property.

In this section, we evaluate or estimate the order of minimal graphs with respect to parameters g and ed for small numbers l. First we count all minimal l-crossing graphs G for small values l. Minimal l-crossing graphs have described up to value $l \leq 8$ in [15]. Note that for $l = 9$ it is unknown any minimal crossing graph G. At present, for $l \geq 10$, there are only some hypothetically minimal l-crossing graphs. Using minimal l-crossing graphs, we find some minimal cubic graphs with respect to parameters ed and g. For cubic graphs of small order we use the notations given in [15].

In the following, we will associate with each 2-cell embedding φ of a graph G in a closed connected oriented 2-manifold M the rotation system Π on G which, in return, determines the embedding φ up to equivalence. We will work in the piece linear category PL. For more detailed information on this subject see the monographs [6] and [12].

1. For $l = 1$ there is a unique minimal crossing graph, the graph $K_{3,3}$. We have obviously $ed(K_{3,3}) = \overline{cr}(K_{3,3}) = cr(K_{3,3}) = g(K_{3,3}) = 1$.

2. For $l = 2$ there are two minimal crossing graphs. These are the Petersen graph P (see Fig.1b) and the graph $CNG2B$ (see Fig.1a). We have obviously $ed(P) = 2$, $g(P) = 1$ and $ed(CNG2B) = g(CNG2B) = 1$;

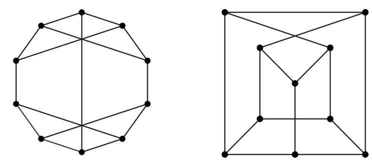

Figure 1: The minimal 2-crossing graphs

Lemma 2.1 *For any cubic graph G of order 12 we have $g(G) \leq 1$.*

Proof. Let G be a connected cubic graph of order 12. We know from [15] that if $|G| \leq 12$, then $\overline{cr}(G) \leq 2$. If the connectivity of G is equal to one, the assertion follows immediately. So we may assume that G is 2-connected. If $\overline{cr}(G) = 1$ we have obviously $g(G) = 1$. If $\overline{cr}(G) = 2$ and the equality reaches via a straight line drawing G in the plane, in which one edge intersects two another edges, the assertion also easily follows.

Assume that we have a drawing of G in the plane with crossings of two pairs of different edges: e_1 and e_2, and f_1 and f_2. Deleting the edges e_1, e_2, f_1 and f_2 from G, we shall obtain a subcubic multigraph H which has a natural embedding φ in the oriented plane. Denote by Π the rotation system on G associated with the embedding φ. Now consider all possible configurations of the induced planar embedding of the (multi)graph H and the positions of the deleted edges with respect to it.

a) There is a face r of the embedding φ which contains two pairs of crossing edges, say e_1 and e_2, and f_1 and f_2. Now there are principally three types of configurations. In the first case (see Fig. 2), we can replace the (oriented) facial circuit dr of Π with three new circuits c_1, c_2, c_3 where $c_1 = (v_5, v_6, v_8, v_3, v_4, v_2, v_3, v_8, v_7), c_2 = (v_1, v_2, v_4, v_5, v_7), c_3 = (v_6, v_1, v_7, v_8)$ which contain four crossing edges e_1, e_2 an f_1, f_2. All other facial circuits of Π remain without changes. As a result, we shall obtain a rotation system Π' on G of genus one.

The second and third types of configurations are shown in Fig. 3 and 4, respectively.

In the first case, we indicate the following circuits: $c_1 = (u_1, u_2, u_3, u_4, u_5, u_6)$,
$c_2 = (u_{12}, u_{11}, u_{10}), c_3 = (u_3, u_2, u_9, u_{10}, u_{11}), c_4 = (u_5, u_4, u_7, u_8)$.

In the second case we choose the following four circuits: $c_1 = (u_3, u_2, u_1), c_2 = (u_6, u_7, u_8, u_9), c_3 = (u_{12}, u_1, u_2, u_{11}, u_{10}, u_9, u_8), c_4 = (u_2, u_3, u_4, u_7, u_6, u_5, u_{11})$.

In both the cases we can complete the family consisting of four circuits to a rotation system R on G which induces the six facial circuits.

An exceptional case of intersection pairs of crossing edges e_1, e_2 and f_1 and f_2 inside the face r

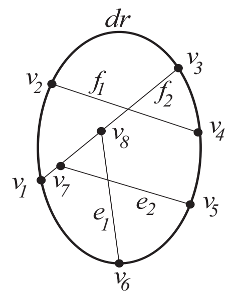

Figure 2:

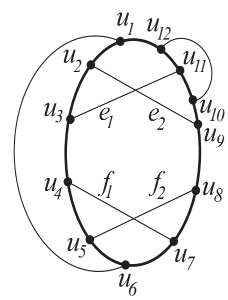

Figure 3:

is shown in Fig. 5. The following family of circuits in G determines a rotation system R of genus one:

$R : \{c_1 = (v_1, v_2, v_3, v_4), c_2 = (v_4, v_3, v_5, v_6, u_6), c_3 = (v_6, u_2, u_3, u_5, u_6), c_4 = (u_2, u_1, u_4, u_3), c_5 = (u_1, u_2, v_6, v_5, v_1, v_4, u_6, u_5), c_6 = (u_1, u_5, u_3, u_4, v_2, v_1, v_5, v_3, v_2, u_4)\}$.

b) There are two faces r_1 and r_2 of the embedding φ such that r_1 contains the crossing of e_1 and e_2, and r_2 contains the crossing of f_1 and f_2.

If r_1 and r_2 are disjoint, the existence of a rotation system Π' on G with 6 circuits is obvious. If r_1 and r_2 have a unique edge in common, we have a configuration shown in Fig. 6. There is a rotation system R on G with 6 facial circuits. We indicate here only four circuits c_1, c_2, c_3 and c_4 which can be completed to a rotation system R of genus one. They are: $c_1 = (u_2, u_3, u_4, u_1), c_2 = (u_1, u_4, u_5, u_{11}, u_{12}), c_3 = (u_{11}, u_5, u_6, u_9, u_{10}), c_4 = (u_9, u_6, u_7, u_8)$.

Assume now that r_1 and r_2 have two edges in common (see Fig. 7).

In this case, we indicate a rotation system R on G with the following six circuits which induces an embedding of G in the torus:

$R : \{c_1 = (u_2, u_1, v_2, v_1), c_2 = (v_5, v_4, u_5, u_4), c_3 = (u_1, u_2, u_3, u_4, u_5, u_6), c_4 = (v_1, v_2, v_3, v_4, v_5, v_6), c_5 =$

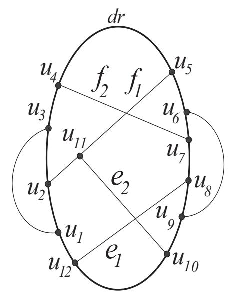

Figure 4:

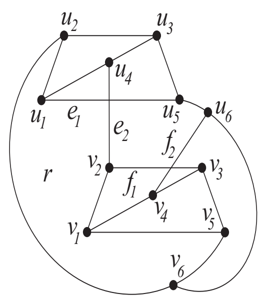

Figure 5:

$(u_2, v_1, v_6, u_6, u_5, v_4, v_3, u_3), c_6 = (u_1, u_6, v_6, v_5, u_4, u_3, v_3, v_2)\}$.

c) It can occur that H is a multigraph with three loops and the pairs of crossing edges of G are situated in the outer face p of the embedding φ. We depict in Fig. 8 such a configuration. In this case, one can easily find rotation systems R of G generating the six circuits. We indicate only four circuits of the rotation R. They are: $c_1 = (u_{10}, u_9, u_{12}, u_{11}), c_2 = (u_{11}, u_{12}, u_3, u_4), c_3 = (u_4, u_3, u_2, u_1), c_4 = (u_5, u_6, u_7)$. The rotation system R induces an embedding of G in the torus.

Note that the case when one pair of crossing edges of G is inside p and the other one is inside a region bounded by a loop of H is not admissible by the assumption that the graph G is 2-connected.$\diamond$

3. For $l = 3$ there are eight crossing minimal graphs. Here we count them according to [15]: $CNG3A$, $CNG3B, CNG3D, CNG3E$, $CNG3F, CNG3H$, the graph $GP(7, 2)$ and the Heawood graph H (see Fig. 9).

Lemma 2.2 *For any* 3*-crossing cubic graph* G *we have* $ed(G) \leq 2$.

Proof. The proof of the assertion uses drawing each such graph in the plane with 3 crossings. We omit here technical details of this checking and left them to the reader as an exercise.$\diamond$

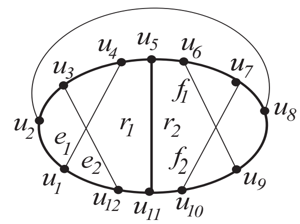

Figure 6:

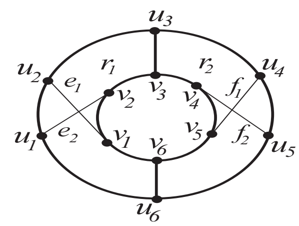

Figure 7:

Note also that $g(CNG3A) = 2$. The proof of this fact will be given in Section 3. It follows that $CNG3A$ is a minimal 2-genus graph. Note that the cyclical connectivity $\zeta(G)$ of the graph $CNG3A$ is equal to 3. By direct computation, all remaining seven 3-crossing graphs have genus equal to one.

4. For $l = 4$ there are two minimal crossing graphs: 8-crossed prism graph Pr_8 (see Fig.10a) and the Möbius-Kantor graph MK (see Fig.10b). By direct computation we have $ed(MK) = 3$ and $ed(Pr_8) = 2$. Moreover it is known that the Möbius-Kantor graph MK is toroidal [15]. It is not difficult to show that the graph Pr_8 is also toroidal.

5. For $l = 5$ there are two minimal crossing graphs: the Pappus graph Pap (see Fig. 11a)) and the graph $CNG5B$ (see Fig. 11b)). By direct computation, we have $ed(Pap) = 3$ and $ed(CNG5B) = 2$. It is known that the Pappus graph is toroidal [15]. It is easy to show that $g(CNG5B) \leq 2$.

3 Additivity and minimal cubic graphs

In this section, we introduce two operations on graphs and establish some additivity properties of parameters ed and g with respect to them, in the case of cubic graphs. The first operation is the connected sum of graphs and the second one is the double (crossed) connected sum of them. We also provide some upper bounds for the order of minimal edge deletion and genus graphs for a given

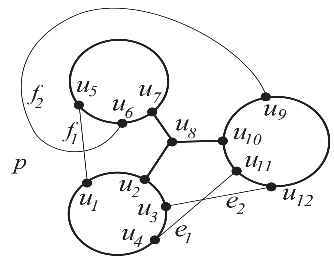

Figure 8:

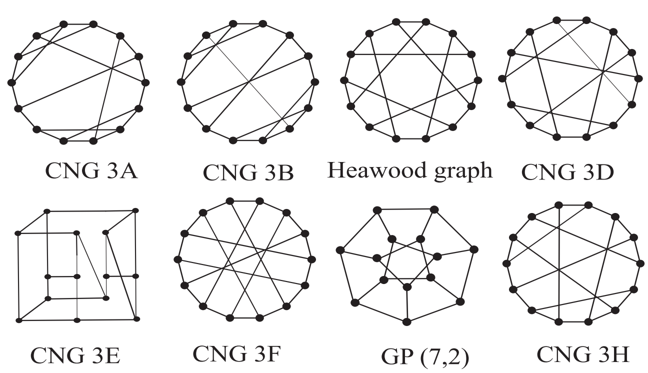

Figure 9: The minimal 3-crossing graphs

value l of the parameters ed and g, respectively.

Let G_1 and G_2 be the 2-connected cubic graphs with distinguished edges e in G_1 and f in G_2. Let u_1, u_2 be the vertices of e and v_1, v_2 the vertices of f, respectively. Remove from G_1 the edge e, and from G_2 the edge f. Take the disjoint sum G of resulting graphs, $G = (G_1 - e) \sqcup (G_2 - f)$, and joint in G the pairs of vertices: u_1 with v_1, and u_2 with v_2, respectively. Denote the resulting graph by $G_1 \star G_2$. We shall say that $G_1 \star G_2$ is the connected sum of the graphs G_1 and G_2 with respect to the pair of edges e and f. Note $G_1 \star G_2$ is also 2-connected cubic graph.

Let G_1 and G_2 be any two 3-connected graphs. Take in G_1 a pair of nonincedent edges (e_1, e_2), and in G_2 a pair of edges (f_1, f_2). Denote the vertices of e_1 by u_1, u_2, and the vertices of e_2 by v_1, v_2, respectively. Similarly, let s_1, s_2 be the vertices of f_1, and t_1, t_2 the vertices of f_2. Delete in G_1 the edges e_1 and e_2, and in G_2 the edges f_1 and f_2. Then take a disjoint sum $G = (G_1 - e_1 - e_2) \sqcup (G_2 - f_1 - f_2)$ of two graphs and joint in G the following pairs of vertices: u_1 and s_1, u_2 and s_2, v_1 and t_1, and v_2 and t_2, respectively. Denote the resulting 2-connected graph $G_1 * G_2$ and call it *a double connected sum of* G_1 *and* G_2. The four edges joining the graphs $G_1 - e_1 - e_2$ and $G_2 - f_1 - f_2$ are called the bridge edges of the graph $G_1 * G_2$ and are denoted

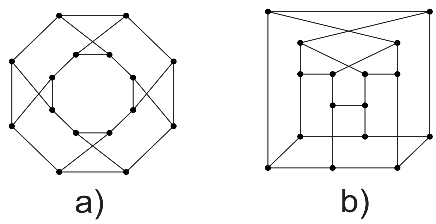

Figure 10: Graphs Pr_8 and MK

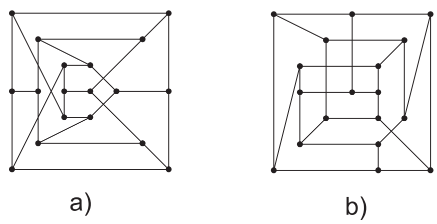

Figure 11: The Pappus graph Pap and the graph $CNG5B$

h_1, h_2, h_3 and h_4 (see Fig. 12).

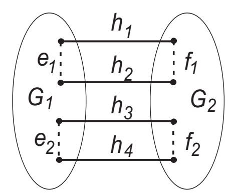

Figure 12: A Double connected sum of graphs G_1 and G_2

If in the above construction, we join the vertices u_1, u_2 with the vertices incident to different edges f_1 and f_2 (then v_1 and v_2 are also joined with the vertices of different edges f_1 and f_2), the resulting cubic graph is called *the crossed connected sum* of G_1 and G_2 and is denoted by $G_1 \sharp G_2$ (see Fig. 13).

It is clear that the operations of double connected sum and crossed connected sums are not determined uniquely and the result $G_1 * G_2$ depends on the distinguished edges of two graphs.

It is naturally to ask whether the (orientable) genus is additive under taking of operations of connected sum and double connected sum of two cubic graphs. In general, the answer is negative. For example, we have $g(K_{3,3}) = 1$ while $g(K_{3,3} \star K_{3,3}) = 1 \neq 2$. Similarly, the genus is not additive subject to the operation of double connected sum of cubic graphs. The following assertions show

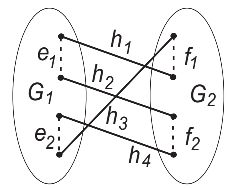

Figure 13: The crossed connected sum of graphs G_1 and G_2

that under certain conditions, the (orientable) genus is subadditive or additive with respect to the operations defined above.

Theorem 3.1 *Let G_1 and G_2 be 2-connected cubic graphs of genus k and l, respectively. Let e and f be distinguished edges of G_1 and G_2, respectively and $G_1 \star G_2$ be the connected sum of G_1 and G_2. Then $g(G_1 \star G_2) \geq g(G_1) + g(G_2) - 1$. Moreover if $g(G_1 - e) = k$ or $g(G_2 - f) = l$, then $g(G_1 \star G_2) = k + l$.*

Proof. We start with proving the second assertion. Denote the two bridge edges of $G_1 \star G_2$ by h_1 and h_2. The inequality $g(G_1 \star G_2) \leq k + l$ is rather obvious and follows from the definition of connected sum of 2-manifolds. Let $\varphi: G_1 \star G_2 \to M$ be a minimal embedding of the graph $G_1 \star G_2$ in a closed orientable surface M. φ is a 2-cell embedding. Denote by Π the rotation system on $G_1 \star G_2$ induced by φ. We have the following possibilities.

1). There are facial circuits c_1 and c_2 of Π such that c_1 contains h_1(twice) and c_2 contains h_2(twice). The corresponding closed faces r_1 and r_2 bounded by c_1 and c_2, respectively, form two handles in M, H_1 and H_2. Then $\chi(M) \leq 0$. Cutting M along the meridians m_1 and m_2 of H_1 and H_2 and pasting the holes by discs, we shall obtain two disjoint closed orientable surfaces, M_1 and M_2. This induces embeddings of the graph $G_1 - e$ in the surface M_1 and the graph $G_2 - f$ in the surface M_2. We have $g(M) = g(M_1) + g(M_2) + 1$. Since $g(G_1 - e) = g(G_1)$ or $g(G_2 - f) = g(G_2)$, the assertion follows.

2). There are two facial circuits c_1 and c_2 of Π each of which contains both the edges h_1 and h_2. Fix an orientation on M. Let r_1 and r_2 be the faces of the embedding φ bounded by c_1 and c_2, respectively. These two faces glued along the edges h_1 and h_2 form a handle. Removing from M the (open) faces r_1, r_2 together with the edges h_1 and h_2, we shall obtain two disjoint 2-manifolds, M_1' and M_2' with boundaries $\partial M_1'$ and $\partial M_2'$, respectively. Elimination of the edges h_1 and h_2 in $G_1 \star G_2$ leads to a surgery of the rotation system Π and induces actually the rotation systems Π_1 and Π_2 on the graphs $G_1 - e$ and $G_2 - f$, respectively. More precisely, instead of the facial circuits c_1 and c_2 in Π we have two cycles, d_1 and d_2, respectively, in Π_1 and Π_2. We thus have

$g(G_1 \star G_2) \geq g(G_1 - e) + g(G_2 - f)$. Let M_1 and M_2 be the surfaces that realize geometrically the rotation systems Π_1 and Π_2, respectively. The subgraphs $G_1 - e$ and $G_2 - f$ are embedded in M_1 and M_2, respectively, in a natural way. By drawing the edge e in the face D_1 bounded by the circuit d_1 and the edge f in the disc D_2 bounded by the circuit d_2, we obtain embeddings of G_1 into M_1 and G_2 into M_2. Therefore we have $g(M) \geq g(G_1) + g(G_2)$.

3). There is a unique facial circuit c of Π which contains both the edges h_1 and h_2 twice. Now we proceed just as in the case 1). After surgery of the surface M we shall obtain two disjoint surfaces, M_1 and M_2, such that $g(M) = g(M_1) + g(M_2) + 1$. Moreover, G_1 has embedding in M_1 or G_2 has embedding in M_2. Since $g(G_1 - e) = g(G_1)$ or $g(G_2 - f) = g(G_2)$ we have $g(M) \geq g(G_1) + g(G_2)$ completing the proof of the second assertion.

The first assertion follows directly from the above proof through the careful analysis of the cases 1)-3). $\diamond$

Corollary 3.1 *Let G_1 be a 2-connected cubic graphs with the distinguished edges e. Let e' be a distinguished edge of the graph $K_{3,3}$ and $H = G_1 \star K_{3,3}$ be a connected sum of G_1 and $K_{3,3}$ subject to the edges e and e'. If e is inessential in G_1, then $g(H) = g(G_1) + 1$.*

Corollary 3.2 *If H is a minimal l-genus graph in the class of 2-connected graphs, then $|H| \leq 8l-2$.*

Theorem 3.2 *Let G_1 be a 3-connected cubic graph with the pair of distinguished edges e_1 and e_2 and G_2 be a cyclically 4-edge connected cubic graph with the pair of distinguished edges f_1 and f_2. Assume that $g(G_1 - e_1) = g(G_1)$ or $g(G_1 - e_2) = g(G_1)$ and $g(G_2 - f_1 - f_2) \geq g(G_2) - 1$. Then $G_1 * G_2$ is a 3-edge connected graph and $g(G_1 * G_2) \geq g(G_1) + g(G_2) - 1$.*

Proof. Let g be an embedding of the graph $G_1 * G_2$ in a surface M of minimal genus. We can cut the surface M along k disjoint nonparallel cycles $c_1, \ldots, c_k$, $k \leq 4$, which cross the bridge edges h_1, h_2, h_3 and h_4. As a result, we obtain two (connected) submanifolds M_1 and M_2 such that $\partial M_1 = \partial M_2 = \sqcup_{i=1}^{k} c_i$ and $G_2 - f_1 - f_2$ is embedded in M_1 and $G_2 - f_1 - f_2$ is embedded in M_2. Pasting the connected components c_i of ∂M_1 by discs we obtain a closed surface M_1'. In the same way we obtain from M_2 a closed surface M_2'. By assumptions, we have $g(M_1') \geq g(G_1) - 1$ and $g(M_2') \geq g(G_2) - 1$. Suppose that $g(G) = g(G_1) + g(G_2) - 2$. This can occur only if the following equality folds: $g(M) = g(M_1') + g(M_2')$ i.e. M_1' and M_2' are joined with one tube in M and the number k of cycles c_i is equal to one. But in this case we can draw the edge e_1 and the edge e_2 in the disc D pasted to M_1. This would give embeddings of both the graphs $G_1 - e_1$ and $G_1 - e_2$ in the surface M_1' of genus $g(G_1) - 1$ contradicting to the assumption.$\diamond$

Note also that an analogue of Theorem 3.4 holds also for crossed connected sum of cubic graphs.

Let G_1 be a 2-connected cubic graph with distinguished pair of non incident edges (e_1, e_2) where $e_1 = (u_1, v_1), e_2 = (u_2, v_2)$ and let G_2 be a connected cubic graph with distinguished pair of edges (f_1, f_2) where $f_1 = (u'_1, v'_1), e_2 = (u'_2, v'_2)$. Assume that the following conditions are satisfied:

(i) $g(G_1) = k > 0$ and $g(G_1 - e_1) = k$ or $g(G_1 - e_2) = k$;

(ii) $g(G_2) = 1$ and $g(G_2 - f_2 - f_1) = 1$ or $g(G_2) = 0$ and for any plain embedding of $G_2 - f_2 - f_1$ there is no facial circuit c' containing the four vertices u'_1, v'_1, u'_2, v'_2 and the only possibility that the two facial circuits c'_1, c'_2 cover all these vertices is that one of them contains the vertices u'_1, u'_2 and the other one contains the vertices v'_1, v'_2.

For a moment, let $G_1 \sharp G_2$ denote the crossed connected sum of cubic graphs G_1 and G_2 in which the vertices of the pair $\{u_1, v_1\}$ are joined to the vertices of the pair $\{u'_1, u'_2\}$ and the vertices of the pair $\{u_2, v_2\}$ to the vertices of the pair $\{v'_1, v'_2\}$.

Theorem 3.3 *Let G_1 and G_2 be connected cubic graphs that satisfy conditions (i) and (ii). Assume that G_1 is 3-connected and G_2 is cyclically 4-edge connected. Then $G_1 \sharp G_2$ is 3-connected graph and $g(G_1 \sharp G_2) = k + 1$.*

Proof. The first assertion follows from the definition of the crossed connected sum of cubic graphs and its proof uses standard graph-theoretical tools. We omit here technical details.

It remains to prove the second assertion. Suppose contrary, that $g(G_1 \sharp G_2) \leq k$. Let φ be a minimal embedding of $G_1 \sharp G_2$ into an orientable surface M of genus k. Consider the embedding ψ of the subgraph $G_1 - e_1 - e_2$ into M induced by the embedding φ.

Let $N(G_1)$ be an open regular neighborhood of the polyhedron $\psi(G_1 - e_1 - e_2)$ in M. The image $\varphi(G_2 - f_1 - f_2)$ is contained in one connected component of the 2-manifold $M_2 = M \setminus N(G_1)$, say s. The component s cannot be a disc (i.e. a face of the embedding ψ). Indeed otherwise the bridge edges of H would join the four vertices from $G_2 - f_1 - f_2$ to four vertices of $G_1 - e_1 - e_2$ in a disc. But this is impossible by condition (ii). Therefore s contains tubes (i.e. is a submanifold with nontrivial fundamental group). It follows that $g(G_1 \sharp G_2) \geq k$.

It can occur that ∂s consists of one connected component, a circle c. Then $M_1 = \mathrm{cl}(M \setminus s)$ is a 2-manifold with the boundary $\partial M_1 = c$. After gluing a disc D to M_1 along the circle c we shall obtain a surface T of genus $k - 1$. In this case we can draw the edge e_1 (or the edge e_2) in the disc D and obtain an embedding of the graph $G_1 - e_1$ into the surface M_1 contradicting with the equality $g(G_1 - e_1) = k$. We thus exclude such a possibility.

Suppose now that s is glued to the rest of the surface M along two or more circles c_i. The number of circles cannot be bigger than two, otherwise the genus of M would be bigger than k, contradicting to our assumption.

Assume that s has two boundary components, c_1 and c_2. Then s is a cylinder and $g(\mathrm{cl}(M \setminus s)) = k-1$. There are two tubes t_1 and t_2 inside s which contain four bridge edges of the graph $G_1 \sharp G_2$. A tube $t_i, i = 1, 2$, cannot contain three bridge edges h_l, otherwise one circle c'_i would contain three vertices from the set $L = \{u'_1, v'_1, u'_2, v'_2\}$ and the other circle c'_{3-i} contains the remaining vertex, which is impossible by condition (ii).

Therefore the first tube t_1, bounded by c_1 on one side, contains two bridge edges h_1 and h_2 joining the ends of the edge e_1 to the vertices, say u'_1 and u'_2, positioned on the facial circuit c'_1 of $G_2 - f_1 - f_2$. Similarly, the second tube t_2, bounded by c_2 on one side, contains the remaining bridge edges h_3 and h_4 which join the ends of the edge e_2 to the vertices v'_1 and v'_2, positioned on the second facial circuit c'_2 of $G_2 - f_1 - f_2$ (see Fig. 14). In this case we can add the edges $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ to the subgraph $G_1 - e_1 - e_2$ and draw them in the 2-manifold $N(G_1)$. It follows that the graph G_1 admits embedding in a surface of genus $k-1$ contradicting to the condition (i). This completes the proof of the second assertion. ◇

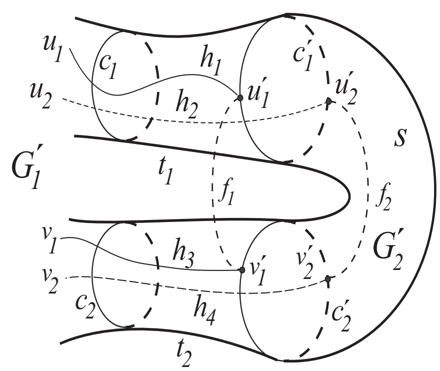

Figure 14:

Note that there are in $G_1 \sharp G_2$ at least two non incident edges which are inessential with respect to genus.

Example 2. Consider the cubic graph H obtained from $K_{3,3}$ by doubling an edge e. Instead of e, we have two edges e_1 and e_2 (see Fig. 15). Take the edges e_1 and e_2 to be distinguished in H. Removing from H the edges e_1 and e_2, we shall obtain a subcubic graph H'.

It is clear that H' admits a unique planar embedding ρ and the graph H satisfies the condition (i) (subject to the pair of edges e_1 and e_2). Moreover the graph H also satisfies the condition (ii) (subject to the pair of edges e_1 and e_2). It follows that $g(H \sharp H) = 2$. Note also that $H \sharp H$ is cyclically 4-edge connected cubic graph.

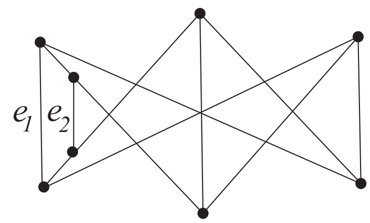

Figure 15: The cubic graph H

Lemma 3.1 *The genus of cubic graph $CNG3A$ is equal to* 2.

Proof. Cut the graph $CNG3A$ across four edges as shown in Fig. 16. We have a decomposition of $CNG3A$ into two planar graphs G_1 and G_2 such that G_1 contains four semiedges e_1, e_2, e_3 and e_4 and G_2 contains four semiedges f_1, f_2, f_3 and f_4.

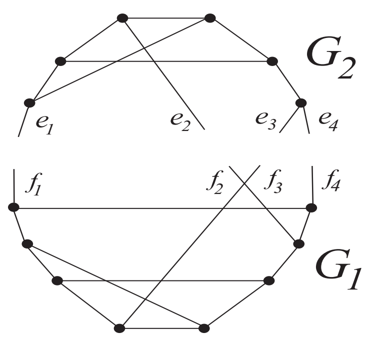

Figure 16:

Suppose that the graph $CNG3A$ is toroidal. Let φ denote embedding of this graph in the torus T. Then φ induces embeddings φ_1 and φ_2 of the subgraphs G_1 and G_2, respectively, in the torus. Let N_1 be an open regular neighborhood of the graph $\varphi_1(G_1)$ and N_2 be an open regular neighborhood of the graph $\varphi_1(G_2)$ in the torus. Then G_1 is contained in one connected component t of the 2-manifold $cl(T \setminus N_2)$ and G_2 is contained in one connected component s of the 2-manifold $cl(T \setminus N_1)$. The component t cannot be a disc since there is no planar embedding of G_1 which contains all semiedges inside the same region r. Similarly the component s is not a disc. Therefore the only possibility to obtain embedding of the graph $CNG3A$ in the torus is as follows. The subgraph G_1 is embedding into a sphere S_1 with two holes, the subgraph G_1 is embedding into a sphere S_2 with two holes and the spheres S_1 and S_2 are joining by two tubes τ_1 and τ_2 which contain four pairs of glued semiedges: $(e_1, f_1), (e_2, f_2), (e_3, f_3)$ and (e_4, f_4). By careful inspection all possibilities we can easily check that this is impossible.◇

Now starting from the graph $CNG3A$ and the graph H in Example 2 and using Theorem 3.5 and Lemma 3.6, we can inductively construct a sequence of 3-connected cubic graphs H_l of order $8l$ with $g(H_l) = l$.

Corollary 3.3 *If H is minimal l-genus graph in the class of 3-connected cubic graphs, then $|H| \leq 8l$.*

Denote by $\chi'(G)$ the chromatic index of the graph G. A cubic graph G is called colorable if $\chi'(G) = 3$, otherwise G is called uncolorable (i.e. $\chi'(G) = 4$) or a weak snark. A weak snark which is cyclically 4-edge connected and whose girth is at least five is called a snark [13].

The Petersen graph is a simplest example of a snark. Using the operation of dot product (see Fig. 17), one obtains from any two snarks of orders k and l, respectively, a a bigger snark of order $k+l-2$. Note that the dot product $G_1 \cdot G_2$ of two cubic graphs G_1 and G_2 is defined non uniquely.

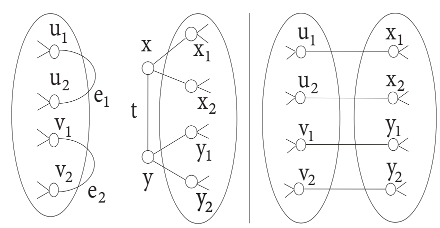

Figure 17: The dot product of two snarks

In [13], the authors consider different powers P^k of the Petersen graph P and study their genus. A kth power P^k of the Petersen graph P is defined inductively: $P^k = P \cdot P^{k-1}$, where $\cdot$ denote a dot product of the cubic graphs. Since the dot product of two cubic graphs is defined non uniquely, there are several powers P^n of the snark P for each natural number $n \geq 2$.

In [13], the authors construct for each pair (k, n) of natural numbers k and n, where $k \leq n$ and $k, n \geq 1$, the powers P^n such that $g(P^n) = k$. Since the order of P^n is equal to $8n + 2$ we have the following upper bound for the order of minimal l-minimal graphs: $g(l) \leq 8l + 2$. This estimation is slightly weaker than the one given above.

This is an open problem to evaluate the number $ed(P^n)$ of the powers P^n of the Petersen graph P such that $g(P^n) = k$.

Now we consider how change the parameter ed of cubic graphs when we apply to them operations of connected and double connected sum of graphs. A simple example shows that this parameter is not additive under the connected sum of cubic graphs. It suffice to consider the graphs $K_{3,3} * K_{3,3}$

and $K_{3,3} \star K_{3,3}$. Indeed we have $ed(K_{3,3}) = 1$ and $ed(K_{3,3} \star K_{3,3}) = 1$. Moreover $ed(K_{3,3} * K_{3,3}) = 1$ for appropriate choice of pairs of the non incident edges in the first and second copies of $K_{3,3}$. However under certain conditions an analogue of additivity property holds also for the parameter ed.

Let G be a cubic graph and e and f are two distinguished edges of G. We shall say that the edge e is inessential (the pair $\{e, f\}$ of edges is inessential) if $ed(G-e) = ed(G)$ ($ed(G-e-f) \geq ed(G)-1$, respectively).

Theorem 3.4 *Let G_1 and G_2 be two 2-connected cubic graphs with the distinguished edges e in G_1 and f in G_2, respectively. Let $ed(G_1) = k > 0$ and $ed(G_2) = l > 0$. Then $ed(G_1 \star G_2) \geq k + l - 1$. Moreover if $ed(G_1 - e) = k$ and $ed(G_2 - f) = l$, then $ed(G_1 \star G_2) = k + l$.*

Proof. Denote the vertices of e in G_1 by v_1 and v_2 and the vertices of f in G_2 by u_1 and u_2. Put $ed(G_1 \star G_2) = m$. Let $E = \{e_1, \ldots, e_m\}$ be the minimal set of edges in $G_1 \star G_2$ such that $G - E$ is planar. Assume that E contains neither the edge $t_1 = (u_1, v_1)$ nor the edge $t_2 = (u_2, v_2)$. There is a path p_1 joining u_1 to u_2 in $G_1 - \{e, e_1, \ldots, e_m\}$ or a path p_2 in $G_2 - \{f, e_1, \ldots, e_m\}$. It follows that $|E(G_1 - e) \cap E| \geq k$ or $|E(G_2 - f) \cap E| \geq l$. Moreover $|E(G_1 - e) \cap E| \geq k - 1$ and $|E(G_2 - f) \cap E| \geq l - 1$, from what the inequality $|E| \geq k + l - 1$ follows.

Assume that E contains one of the edges t_1, t_2. Then E contains at least $k - 1$ edges of $G_1 - e$ and $l - 1$ edges of $G_2 - f$, and the first assertion follows.

The second assertion of the theorem follows directly from the definitions of the connected sum of cubic graphs and the minimal edge deletion set.$\diamond$

Theorem 3.5 *Let G_1 be a 3-connected cubic graph with $ed(G_1) = k > 0$ and G_2 a cyclically 4-edge connected cubic graph with $ed(G_2) = l > 0$. Let $\{e_1, f_1\}$ be a pair of distinguished non incident edges in G_1 and $\{e_2, f_2\}$ a pair of non incident distinguished edges in G_2. Assume that in both the pairs each edge is inessential. Then $G_1 * G_2$ is a 3-connected cubic graph and $ed(G_1 * G_2) \geq k + l$.*

Proof. Let $e_1 = (u_1, u_1'), f_1 = (v_1, v_1'), e_2 = (u_2, u_2')$ and $f_2 = (v_2, v_2')$. The bridge edges in $H = G_1 * G_2$ are the following: $h_1 = (u_1, u_2), h_2 = (u_1', u_2'), h_3 = (v_1, v_2), h_4 = (v_1', v_2')$. The subgraph of the graph H formed by four bridge edges is denoted by B. Put $G_1' = G_1 - e_1 - f_1$ and $G_2' = G_1 - e_2 - f_2$. Let $R = \{r_1, \ldots r_s\}$ be a minimal edge deletion set in H. The planar graph $H - R$ is connected. We have the following three possibilities:

1) R contains two or three bridge edges h_i from B. Since $ed(G_1') \geq k - 1$ and $ed(G_2') \geq l - 1$, it follows that $ed(H) \geq k + l$.

2) R does not contain any bridge edge h_i. First note that if the subgraph $G_1' - R$ is disconnected, then $\|G_1' \cap R\| \geq k$. Indeed, suppose contrary that $\|G_1' \cap R\| \leq k-1$. Then we can add some edge r_i from R to $G_1' - R$ to obtain a planar subgraph U_1 of G_1'. But this contradicts to the assumption that $ed(G_1') \geq k-1$. Similarly, if the subgraph $G_2' - R$ is disconnected, then $\|G_2' \cap R\| \geq l$.

Consider a planar drawing g of the connected graph $H - R$. Let g_1 and g_2 be the planar embeddings of the subgraphs $G_1' - R$ and $G_2' - R$, respectively, induced by g. Now the proof of the assertion reduces to considering the following three subcases.

(i) both the subgraphs $G_1' - R$ and $G_2' - R$ are connected. Since $G_1' - R$ is connected, the plane subgraph $D_2 = (G_2' \cup B) - R$ is contained in a face μ of the 2-cell embedding g_1 of the plane graph $G_1' - R$. This means that the vertices u_1, u_1', v_1, v_1' of $G_1' - R$ are situated on the same facial circuit c, the circuit that bounds the face μ. We can draw the edge e_1 (or the edge f_1) in the face μ and obtain a planar embedding of the subgraph $G_1 - (f_1 \cup R)$ (see Fig. 18). Since the edge f_1 is inessential in G_1 we have $\|G_1' \cap R\| \geq k$. In the same way we can prove that $\|G_2' \cap R\| \geq l$. It follows that $\|R\| \geq k + l$.

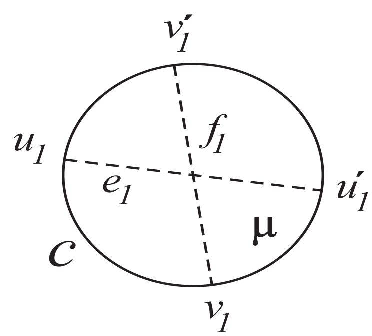

Figure 18:

(ii) both the subgraphs $G_1' - R$ and $G_2' - R$ are disconnected. Then $\|G_2' \cap R\| \geq l$ and $\|G_1' \cap R\| \geq k$, so $\|R\| = k + l$;

(iii) one of the subgraphs $G_1' - R$ and $G_2' - R$ is connected and the other is disconnected. Suppose for instance that $G_1' - R$ is connected and $G_2' - R$ is disconnected. Since $G_2' - R$ is disconnected, we have $\|G_2' \cap R\| = l$. If $\|G_1' \cap R\| = k$ we have $\|R\| = k + l$. Suppose that $\|G_1' \cap R\| = k - 1$. The plane subgraph $D_1 = (G_1' \cup B) - R$ is contained in a connected region s of the planar embedding of the graph $G_2' - R$. Note that s may not be a 2-cell. (see Fig. 19). In any case, we can draw the edge e_1 (or the edge f_1) in the region s and obtain a planar embedding of the subgraph $G_1 - (R \cup f_1)$ (the subgraph $G_1 - (R \cup e_1)$, respectively). But this means that $ed(G_1 - f_1) = k - 1$ contradicting to the assumption. Therefore $\|G_1' \cap R\| = k$ and $\|R\| = k + l$.

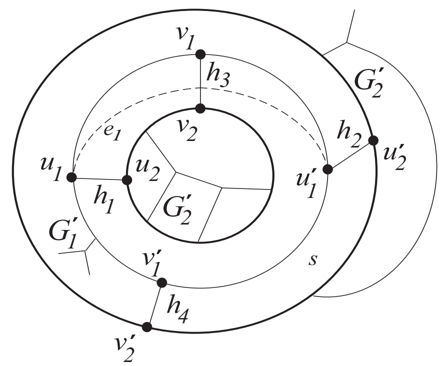

Figure 19:

3) R contains one bridge edge h_i. If $\|G'_2 \cap R\| \geq l$ or $\|G'_1 \cap R\| \geq k$, the assertion follows. Suppose that $\|G'_2 \cap R\| = l - 1$ and $\|G'_1 \cap R\| = k - 1$. By the same arguments as in the case 2), we conclude that both the graphs $G'_1 - R$ and $G'_2 - R$ are connected. Let g be a planar embedding of the connected graph $H - R$. The embedding g induces planar embeddings g_1 and g_2 of the subgraphs $G'_1 - R$ and $G'_1 - R$, respectively. Since $G'_1 - R$ is connected, the plane subgraph $D_2 = (G'_2 \cup B) - R$ is contained in a face γ of the plane graph $G'_2 - R$. This means that three vertices of $G'_1 - R$ from the set $\{u_1, u'_1, v_1, v'_1\}$ are situated on the same facial circuit c, the circuite that bounds the face γ. Therefore we can draw the edge e_1 or the edge f_1 in the face γ which gives in planar embedding of the subgraph $G_1 - (f_1 \cup R)$ ($G_1 - (e_1 \cup R)$, respectively). Since the edges e_1 and f_1 are inessential in G_1 we have $\|G'_1 \cap R\| \geq k$, contradicting to our assumption.

We thus conclude that in any case $\|R\| \geq k + l$. We have proved that $ed(H) \geq k + l$. The first assertion of the theorem is rather obvious. $\diamond$

Note that any bridge edge of the graph H is inessential.

Theorem 3.9 can be used in constructing 3-connected or even cyclically 4-edge connected cubic graphs G of order $6n$ such that $ed(G) \geq n$ for any natural number $n > 0$. We can start from the Möbius-Kantor graph MK and pick one edge e in it. Replacing the edge e with two parallel edges, e' and e'', we obtain a cubic graph G with two distinguish edges, e' and e''. Taking two copies of G, the cubic graphs G' and G_1, and applying to them the operation of double connected sum, we shall obtain a 3-connected cubic graph H_2 of order 36. By Theorem 3.9, we have $ed(H_2) = 6$. Continuing this iteration process, we obtain a sequence of cyclically 4-edge connected cubic graphs H_n of order $18n$ with $ed(H_n) = 3n$.

Corollary 3.4 *If H is an l-edge deletion minimal graph in the class of* 3*-connected and cyclically* 4*-edge connected cubic graphs, then $|H| \leq 6l$.*

Question. We provided above some upper bounds for the order of edge deletion minimal (cubic) graphs and genus minimal (cubic) graphs. What about lower bounds for these graph parameters?

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