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Preprint Nr MD 091
(otrzymany dnia 11.09.2017)

Kraków 2017

Palette index of regular complete multipartite multigraphs

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September 11, 2017

Abstract

Let G be a multigraph, C a set of colors and $f : E(G) \to C$ a proper edge coloring of G. The palette of a vertex $v \in V(G)$ is the set $S_f(v) = \{f(vw) : vw \in E(G)\}$. The palette index of G is the minimum cardinality of the set $\{S_f(v) : v \in V(G)\}$ taken over all proper edge colorings of G. In the paper there is determined the palette index of $\lambda K_{p\times q}$, the complete multipartite multigraph with p parts of cardinality q and with the constant edge multiplicity equal to λ.

Let G be a multigraph and let $f : E(G) \to C$ be a proper edge-coloring of G, where C is a set of *colors*. The minimum number of colors needed to properly color edges of G is called the *chromatic index* of G and is denoted by $\chi'(G)$. Edges colored with the same color form a *color class*. Due to Vizing [6] we know that $\chi'(G) \le \Delta(G) + \mu(G)$, where $\mu(G)$ is the maximum number of edges in G with the same endvertices. So, if G is a graph, then either $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G) + 1$ and G is called either *class* 1 or *class* 2.

Let $l, m \in \mathbb{Z}$. By $[l, m]$ we denote the *integer interval* of all integers z satisfying $l \le z \le m$. If $m \ge 2$, we use $|l|m|$ to denote the unique

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$k \in [0, m-1]$ satisfying $k \equiv l \pmod{m}$. If G_1, G_2 are vertex-disjoint graphs, the *join* of G_1 and G_2 is the graph $G_1 \oplus G_2$ with $V(G_1 \oplus G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \oplus G_2) = E(G_1) \cup E(G_2) \cup \{x_1x_2 : x_1 \in V(G_1), x_2 \in V(G_2)\}$. We use the standard notation $K_{p\times q}$ for a complete balanced p-partite graph with each part of cardinality q. Let $V(K_{p\times q}) = \bigcup_{i=1}^{p} V_i$, where $V_i \cap V_j = \emptyset$ whenever $i \neq j$. Let $V_i = \{v_{i,j} : \ j \in [1, q]\}$ for each $i \in [1, p]$. Let X be a finite set. By K_X we denote the complete graph with the vertex set X; similarly, D_X is the *discrete* (edgeless) graph with the vertex set X. If $|X| \geq 3$ and C is an arbitrary (but fixed) cycle with $V(C) = X$, then C_X will be the graph with $V(C_X) = X$ and $E(C_X) = E(C)$.

The chromatic index of complete multipartite graphs was determined by Hoffman and Rodger [2] using the notion of an *overfull* graph, *i.e.*, a graph G with $|E(G)| > \Delta(G)\lfloor |V(G)|/2 \rfloor$.

Theorem 1. *The chromatic index of a complete multipartite graph K is $\Delta(K)$ if K is not overfull and $\Delta(K)+1$ if K is overfull.*

The following result concerning the chromatic index of regular graphs was obtained by De Simone and Galluccio [5]:

Theorem 2. *If G is a k-regular graph of even order n with $k \geq \frac{n}{2}$ and there are graphs G_1, G_2 with $G_1 \oplus G_2 \cong G$, then G is a class 1 graph.*

The *palette* of a vertex $v \in V(G)$ with respect to an edge coloring f of G is the set $S_f(v)$ of colors (under f) of edges of G incident to v. For a given multigraph G, the minimum number of palettes taken over all possible proper edge colorings of G is called the *palette index* of G and is denoted by $\check{s}(G)$. A coloring for which this minimum is attained is called *palette-minimum.*

The palette index was introduced in Horňák *et al.* [3] and further investigated in Bonvicini and Mazzuoccolo [1]. The next three basic results for this graph invariant were proven in [3].

Proposition 3. *The palette index of a graph G is 1 if and only if G is regular and class 1.*

Lemma 4. *If a graph G is regular, then $\check{s}(G) \neq 2$.*

Theorem 5. *Let n be a positive integer. Then*

$$\check{s}(K_n) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod 2 \text{ or } n = 1, \\ 3 & \text{if } n \equiv 3 \pmod 4, \\ 4 & \text{if } n \equiv 1 \pmod 4 \text{ and } n \geq 5. \end{cases}$$

Proposition 3 and Lemma 4 can easily be extended to multigraphs. Indeed, in the former case there is (almost) nothing to do and in the latter one the proof works for multigraphs in the same way as for graphs.

Proposition 6. *The palette index of a multigraph G is 1 if and only if G is regular and class 1.* ■

Lemma 7. *If a multigraph G is regular, then $\check{s}(G) \neq 2$.* ■

Let λK_n stand for a complete multigraph on n vertices in which the multiplicity of each edge is equal to λ.

Lemma 8. *If λ, n are positive integers, then $\check{s}(\lambda K_n) \leq \check{s}(K_n)$.*

Proof. Let $f : E(K_n) \to C$ be a palette-minimum coloring of a complete graph K_n. To each color $c \in C$ we assign λ "private" colors c_i, $i \in [1, \lambda]$, and we use all of them to color λ parallel edges with endvertices x and y whenever c is the color of the edge xy in K_n. The constructed coloring $f_\lambda : E(\lambda K_n) \to C_\lambda$ with $C_\lambda = \bigcup_{c \in C}\{c_i : i \in [1, \lambda]\}$ then satisfies $S_{f_\lambda}(x) = \bigcup_{c \in S_f(x)}\{c_i : i \in [1, \lambda]\}$ for each $x \in V(\lambda K_n) = V(K_n)$, and hence forms $\check{s}(K_n)$ distinct palettes. ■

In the next theorem we determine the palette index of complete λ-multigraphs.

Theorem 9. *Let λ, n be positive integers. Then*

$$\check{s}(\lambda K_n) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod 2 \text{ or } n = 1, \\ 4 & \text{if } n \equiv 1 \pmod 4 \text{ and either } \lambda \equiv 1 \pmod 2 \text{ or } n = 5, \\ 3 & \text{otherwise.} \end{cases}$$

Proof. By Proposition 6, Lemma 7 and Lemma 8, the assertion immediately holds if $n \not\equiv 1 \pmod 4$. Henceforth we assume that $n \equiv 1 \pmod 4$; in such a case $3 \leq \check{s}(\lambda K_n) \leq 4$.

If either $\lambda \equiv 1 \pmod 2$ or $n = 5$, proceeding by the way of contradiction we suppose that $\check{s}(\lambda K_n) = 3$. Let f be a palette-minimum coloring of λK_n and let P_i, $i = 1, 2, 3$, be three distinct palettes created by f. Let Y_i be the set of all vertices of λK_n with the palette P_i, $i = 1, 2, 3$. Clearly, $|P_i| = \lambda(n-1)$ and $|P_i \setminus P_j| = |P_j \setminus P_i|$ if $i, j \in [1, 3]$, $i \neq j$. Obviously, since n is odd, there is no color belonging to all three palettes. Moreover, $|Y_i| + |Y_j|$ is even whenever $i, j \in [1, 3]$, $i \neq j$. Thus cardinalities of all Y_i's are of the same parity, and then each $|Y_i|$ is odd. Hence each color belongs to exactly two palettes. Therefore, if $\{i,j,k\}=[1,3]$, then $P_i \setminus P_j = P_i \cap P_k$, $P_j \setminus P_i = P_j \cap P_k$, $P_k = (P_i \cap P_k) \cup (P_j \cap P_k)$ and $|P_i \cap P_k| = |P_i \setminus P_j| = |P_j \setminus P_i| = |P_j \cap P_k|$, which (having in mind that $P_i \cap P_j \cap P_k = \emptyset$) leads to $|P_i \cap P_k| = |P_j \cap P_k| = \frac{1}{2}|P_k| = \frac{\lambda(n-1)}{2}$.

If $\lambda \equiv 1 \pmod 2$, let $E_{i,k}$ denote the set of all edges of λK_n joining a vertex of Y_i to a vertex of Y_k. Clearly, $|E_{i,k}|$ is odd. Moreover, for every color c from $P_i \cap P_k$, the number of edges in $E_{i,k}$ colored with c is odd. Since each edge of K_n joining a vertex of Y_i to a vertex of Y_k appears in λK_n with the odd multiplicity λ, the number $\frac{\lambda(n-1)}{2}$ of colors in $P_i \cap P_k$ must be odd, which contradicts the fact that $n-1 \equiv 0 \pmod 4$.

If $n=5$, suppose without loss of generality that $|Y_1| \le |Y_2| \le |Y_3|$, which yields $|Y_1| = |Y_2| = 1$ and $|Y_3| = 3$. Let $Y_1 = \{y_1\}$, $Y_2 = \{y_2\}$, $Y_3 = \{y_3, y_4, y_5\}$, and for $u, v \in V(\lambda K_5)$ let $S_f(u,v)$ be the set of λ colors (under f) of the edges of λK_5 with endvertices u, v. The coloring f is proper, hence $S_f(y_1,y_3) \cap S_f(y_2,y_3) = \emptyset$. Then, however, $S_f(y_4,y_5) \supseteq S_f(y_1,y_3) \cup S_f(y_2,y_3)$, and so $\lambda = |S_f(y_4,y_5)| \ge |S_f(y_1,y_3)| + |S_f(y_2,y_3)| = 2\lambda$, a contradiction.

It remains to consider the case $\lambda \equiv 0 \pmod 2$ and $n \ge 9$. Suppose first that $\lambda = 2$ and $n = 4k+1$, $k \ge 2$. Let $V(\lambda K_n) = V(K_n) = X_1 \cup X_2 \cup X_3$ with $|X_1| = |X_3| = 2k-1$ (≥ 3), $|X_2| = 3$, and $X_i \cap X_j = \emptyset$ whenever $i \ne j$. We express the multigraph $2K_n$ as an edge-disjoint union of graphs $F_i = K_{X_2 \cup X_i}$, $G_i = D_{X_2} \oplus (K_{X_i} - E(C_{X_i}))$, $i = 1,3$, $H_0 = D_{X_1} \oplus D_{X_3}$ and $H = C_{X_1} \oplus C_{X_3}$. All involved graphs are class 1, since $F_i \cong K_{2k+2}$, $i = 1,3$, $H_0 \cong K_{2k-1,2k-1}$, and for G_1, G_3, H we can use Theorem 2: G_i is a $(2k-1)$-regular graph of order $2k+2$ and $2k-1 \ge \frac{2k+2}{2}$, $i = 1,3$, while H is a $(2k+1)$-regular graph of order $4k-2$ and $2k+1 \ge \frac{4k-2}{2}$. Thus, by Proposition 3, the graphs F_1, F_3, H_0, G_1, G_3 and H have palette-minimum colorings with unique palettes R, S, T, R^+, S^+ and T^+ that are (without loss of generality) pairwise disjoint. Then the superposition of all six used colorings yields a proper edge coloring of $2K_n$ with three palettes, namely $R \cup T \cup R^+ \cup T^+$ (for vertices in X_1), $R \cup S \cup R^+ \cup S^+$ (for vertices in X_2) and $S \cup T \cup S^+ \cup T^+$ (for vertices in X_3). Thus, $\check{s}(2K_n) = 3$.

If $\lambda \ge 4$, we split λK_n into $\frac{\lambda}{2}$ edge-disjoint copies of $2K_n$. To obtain a proper edge coloring of λK_n with three palettes each of mentioned copies is properly edge colored using a palette-minimum coloring with a "private" set of colors and the same partition $\{X_1, X_2, X_3\}$ of its vertex set as above. ■

To prove the main result of our paper we shall need three lemmas.

Lemma 10. *If* p, q *are integers with* $\min(p,q) \geq 2$*, then* $\check{s}(K_{p\times q}) \leq \check{s}(qK_p)$.

Proof. Consider a complete multigraph qK_p with $V(qK_p) = [1,p]$ and a palette-minimum coloring $f : E(qK_p) \to C$. For $i, j \in [1,p]$, $i < j$, let $C_{i,j}$ denote the set of q colors used to color q parallel edges of qK_p with endvertices i, j. Use the colors of $C_{i,j}$ in a proper edge coloring $f_{i,j} : E(D_{V_i} \oplus D_{V_j}) \to C_{i,j}$ using the fact that $D_{V_i} \oplus D_{V_j} \cong K_{q,q}$. The superposition of all $f_{i,j}$'s is a proper edge coloring $\bar{f} : E(K_{p\times q}) \to C$, in which $S_{\bar{f}}(v_{i,k}) = S_f(i)$ for every $i \in [1,p]$ and $k \in [1,q]$. As a consequence, $\check{s}(K_{p\times q}) \leq \check{s}(qK_p)$. ■

Lemma 11. *Let* r *be a positive integer. Consider vertex-disjoint cycles* $S = ((s_0)^1, (s_1)^1, \dots, (s_{2r})^1, (s_{2r+1})^1 = (s_0)^1)$ *and* $T = ((t_0)^2, (t_1)^2, \dots, (t_{2r})^2, (t_{2r+1})^2 = (t_0)^2)$ *of length* $2r+1$ *in the complete graph* $K = K_{4r+2}$ *with* $V(K) = \{(i)^j : i \in [0,2r], j \in [1,2]\}$*. If there are* $x, y \in [0, 2r-1]$ *and* $l \in [0,2r]$ *such that either* $t_{y+1} - s_{x+1} = s_y - t_x = l$ *or* $t_{y+1} - s_x = t_y - s_{x+1} = l$*, then the set* $E(S) \cup E(T) \cup \{(i)^1(|i+l|2r+1|)^2 : i \in [0,2r]\}$ *induces a cubic class* 1 *spanning subgraph of the graph* K*.*

Proof. Let G be the graph induced by the mentioned set of edges. Suppose first that $t_{y+1} - s_{x+1} = t_y - s_x = l$. Then $t_y = s_x + l$, $t_{y+1} = s_{x+1} + l$ and $E(G)$ can be partitioned into the following three perfect matchings of G:

$$\{(s_{|x+2i-1|2r+1|})^1(s_{|x+2i|2r+1|})^1 : i \in [1,r]\} \cup \{(s_x)^1(t_y)^2\} \cup$$
$$\{(t_{|y+2i-1|2r+1|})^2(t_{|y+2i|2r+1|})^2 : i \in [1,r]\},$$
$$\{(s_{|x+2i|2r+1|})^1(s_{|x+2i+1|2r+1|})^1 : i \in [1,r]\} \cup \{(s_{x+1})^1(t_{y+1})^2\} \cup$$
$$\{(t_{|y+2i|2r+1|})^2(t_{|y+2i+1|2r+1|})^2 : i \in [1,r]\},$$
$$\{(s_x)^1(s_{x+1})^1\} \cup \{(i)^1(|i+l|2r+1|)^2 : i \in [0,2r] \setminus \{s_x, s_{x+1}\}\} \cup$$
$$\{(t_y)^2(t_{y+1})^2\}.$$

If $t_{y+1} - s_x = t_y - s_{x+1} = l$, we use the fact that $S^{-1} = ((s_{2r+1})^1, (s_{2r})^1, \dots, (s_1)^1, (s_0)^1)$ and T are vertex-disjoint cycles of length $2r+1$ in K, too. ■

Let p be an integer with $p \equiv 1 \pmod 4$ and $p \geq 5$. Further, for $i \in [0, \frac{p-3}{2}]$ and $j \in [1,2]$, let $H^j(i)$ stand for the cycle

$$((i)^j, (|i+1|p-1|)^j, (|i-1|p-1|)^j, (|i+2|p-1|)^j, (|i-2|p-1|)^j, \dots,$$
$$(|i+\tfrac{p-3}{2}|p-1|)^j, (|i-\tfrac{p-3}{2}|p-1|)^j, (|i+\tfrac{p-1}{2}|p-1|)^j, (p-1)^j, (i)^j)$$

on the vertex set $U^j = \{(k)^j : k \in [0, p-1]\}$ of cardinality p. Let G_e^j denote the $\frac{p-1}{2}$-regular graph with $V(G_e^j) = U^j$ and $E(G_e^j) = \bigcup_{k=0}^{(p-5)/4} H^j(2k)$.

Analogously, we define another $\frac{p-1}{2}$-regular graph G_o^j to have $V(G_o^j) = U^j$ and $E(G_o^j) = \bigcup_{k=0}^{(p-5)/4} H^j(2k+1)$. Notice that $E(G_e^j) \cap E(G_o^j) = \emptyset$ and $E(G_e^j) \cup E(G_o^j) = E(K_p^j)$, where the complete graph $K_p^j \cong K_p$ has the vertex set U^j, $j = 1, 2$. Further, we construct two bipartite $\frac{p-1}{2}$-regular graphs B_e, B_o with the bipartition $\{U^1, U^2\}$ and with

$$E(B_e) = \{(i)^1(|i+2k|p|)^2 : i \in [0, p-1], k \in [1, \tfrac{p-1}{2}]\},$$
$$E(B_o) = \{(i)^1(|i+2k-1|p|)^2 : i \in [0, p-1], k \in [1, \tfrac{p-1}{2}]\}.$$

Obviously, $E(B_e) \cap E(B_o) = \emptyset$. Let B be the $(p-1)$-regular bipartite graph with the bipartition $\{U^1, U^2\}$ and with $E(B) = E(B_e) \cup E(B_o)$. Then $B = (D_{U^1} \oplus D_{U^2}) - M$, where $M = \{(i)^1(i)^2 : i \in [0, p-1]\}$ is a perfect matching in the graph $D_{U^1} \oplus D_{U^2} \cong K_{p,p}$. Note that B_e, B_o and B are all class 1 graphs.

Consider $(p-1)$-regular graphs L_e, L_o and $\frac{3p-3}{2}$-regular graphs $Ł_e$, $Ł_o$ on the vertex set $U^1 \cup U^2$ with $E(L_e) = E(G_e^1) \cup E(G_o^2) \cup E(B_e)$, $E(L_o) = E(G_o^1) \cup E(G_e^2) \cup E(B_o)$, $E(Ł_e) = E(L_e) \cup E(B_o)$ and $E(Ł_o) = E(L_o) \cup E(B_e)$.

Lemma 12. *If p is an integer with $p \equiv 1 \pmod 4$ and $p \geq 5$, then $\check{s}(L_e) = \check{s}(L_o) = \check{s}(Ł_e) = \check{s}(Ł_o) = 1$.*

Proof. Let $J_e(i)$ denote the cubic spanning subgraph of L^e with $E(J_e(i)) = E(H^1(2i)) \cup E(H^2(\frac{p-3}{2} - 2i)) \cup \{(k)^1(|k-1-4i|p)^2 : k \in [0, p-1]\}$ and $i \in [0, \frac{p-5}{4}]$. Let $x = p-2$ and $y = 0$. If $H^1(2i) = ((s_0)^1, (s_1)^1, \dots, (s_{p-1})^1, (s_p)^1 = (s_0)^1)$ and $H^2(\frac{p-3}{2} - 2i) = ((t_0)^2, (t_1)^2, \dots, (t_{p-1})^2, (t_p)^2 = (t_0)^2)$, then $s_x = |2i - \frac{p-3}{2}|p-1| = 2i + \frac{p+1}{2}$, $s_{x+1} = 2i + \frac{p-1}{2}$, $t_y = \frac{p-3}{2} - 2i$ and $t_{y+1} = \frac{p-1}{2} - 2i$. Since $t_{y+1} - s_x = -1 - 4i = t_y - s_{x+1}$, by Lemma 11 the graph $J_e(i)$ is class 1. Moreover, for any $l \in \mathbb{Z}$ the set $P(l) = \{(k)^1(|k-l|p|)^2 : k \in [0, p-1]\}$ is a perfect matching of the graph L_e. Therefore, $\{J_e(i) : i \in [0, \frac{p-5}{4}]\} \cup \{P(3 + 4i) : i \in [0, \frac{p-5}{4}]\}$ is a partition of $E(L_e)$ into subsets inducing regular class 1 spanning subgraphs of L_e. Thus L_e itself is a regular class 1 graph, and, by Proposition 3, $\check{s}(L_e) = 1$. Any superposition of palette-minimum colorings of L_e and B_o using disjoint sets of colors then shows that $\check{s}(Ł_e) = 1$.

It is easy to see that the mapping $(i)^j \mapsto (|i+1|p|)^{3-j}$, $i \in [0, p-1]$, $j \in [1, 2]$, defines an isomorphism from L_e onto L_o and from $Ł_e$ onto $Ł_o$ as well; so, $\check{s}(L_o) = \check{s}(Ł_o) = 1$. ■

Theorem 13. *Let p, q be integers with $\min(p, q) \geq 2$. Then*

$$\check{s}(K_{p \times q}) = \begin{cases} 1 & \text{if } pq \equiv 0 \pmod 2, \\ 3 & \text{otherwise.} \end{cases}$$

Proof. We consider several cases according to the properties of the pair (p,q). We have $|E(K_{p\times q})| = q^2\binom{p}{2}$, $\Delta(K_{p\times q})| = (p-1)q$ and $|V(K_{p\times q})| = pq$, hence the graph $K_{p\times q}$ is overfull if and only if $\frac{pq}{2} > \lfloor\frac{pq}{2}\rfloor$.

Case 1: If $pq \equiv 0 \pmod 2$, then $K_{p\times q}$ is not overfull, and, by Theorem 1 and Proposition 3, $\check{s}(K_{p\times q}) = 1$.

Case 2: If $pq \equiv 1 \pmod 2$, then the regular graph $K_{p\times q}$ is overfull, so that, by Theorem 1 and Lemma 4, $\check{s}(K_{p\times q}) \geq 3$.

Case 21: If $p \equiv 3 \pmod 4$, then, using Lemma 10 and Theorem 9, $\check{s}(K_{p\times q}) \leq 3$.

Case 22: If $p \equiv 1 \pmod 4$, let $W_j = \{v_{i,j} : i \in [1,p]\}$ for $j \in [1,q]$.

Case 221: If $q \equiv 3 \pmod 4$, we construct a proper edge coloring of $K_{p\times q}$ using a set of colors $\bigcup_{k=1}^{(q-1)/2}(C_k \cup D_k \cup E_k)$, where $\{C_k, D_k, E_k : k \in [1, \frac{q-1}{2}]\}$ is a system of pairwise disjoint sets with $|C_1| = |D_1| = |E_1| = \frac{3p-3}{2}$ and $|C_k| = |D_k| = |E_k| = p-1$ for $k \in [2, \frac{q-1}{2}]$.

First of all, consider a proper edge coloring of a complete graph $K^c_{(q+1)/2} \cong K_{(q+1)/2}$ with $V(K^c_{(q+1)/2}) = [1, \frac{q+1}{2}]$ using colors c_k, $k \in [1, \frac{q-1}{2}]$. Moreover, we require that the edges colored with c_1 are $\{2x-1, 2x\}$, $x \in [1, \frac{q+1}{4}]$. We use the colors of C_1 to color properly (according to Lemma 12) edges of the graph $Ł_e(2x-1, 2x) \cong Ł_e$, in which $v_{i,2x-1}$ and $v_{i,2x}$ play the roles of $(i-1)^1$ and $(i-1)^2$, respectively, $i \in [1,p]$ (for each $x \in [1, \frac{q+1}{4}]$). Further, we use the colors of C_k, $k \in [2, \frac{q-1}{2}]$, to color the edges of the bipartite graph $B(y,z) = (D_{W_y} \oplus D_{W_z}) - \{v_{i,y}, v_{i,z} : i \in [1,p]\} \cong B$ whenever the edge $\{y,z\} \in E(K^c_{(q+1)/2})$ is colored with c_k.

Similarly, we take a proper edge coloring of a complete graph $K^d_{(q+1)/2} \cong K_{(q+1)/2}$ with $V(K^d_{(q+1)/2}) = [\frac{q+1}{2}, q]$ using colors d_k, $k \in [1, \frac{q-1}{2}]$, with the assumption that the color d_1 is used on the edges $\{2x, 2x+1\}$, $x \in [\frac{q+1}{4}, \frac{q-1}{2}]$. The colors of the set D_1 are used to color properly the edges of $Ł_e(2x, 2x+1)$, $x \in [\frac{q+1}{4}, \frac{q-1}{2}]$, and the colors of D_k with $k \in [2, \frac{q-1}{2}]$ are used to color the edges of $B(y,z)$ whenever $\{y,z\} \in E(K^d_{(q+1)/2})$ is colored with d_k.

Finally, consider a proper edge coloring of the graph

$$D_{[1,(q-1)/2]} \oplus D_{[(q+3)/2,q]} \cong K_{(q-1)/2,(q-1)/2},$$

in which the edges $\{x, x+\frac{q+1}{2}\}$, $x \in [1, \frac{q-1}{2}]$, are colored with e_1. The colors of E_1 are used to color the edges of the graph $Ł_o(2x-1, 2x-1+\frac{q+1}{2}) \cong Ł_o$,

$x \in [1, \frac{q+1}{4}]$, as well as the edges of the graph $Ł_e(2x, 2x + \frac{q-1}{2})$, $x \in [1, \frac{q-3}{4}]$. Moreover, the colors of E_k, $k \in [2, \frac{q-1}{2}]$, are used to color the edges of $B(y,z)$ whenever the edge $\{y, z\}$, $y \in [1, \frac{q-1}{2}]$, $z \in [\frac{q+3}{2}, q]$, is colored with e_k.

In the resulting proper edge coloring of $K_{p\times q}$ for every $i \in [1, p]$ a vertex $v_{i,j}$ receives the palette $\bigcup_{k=1}^{(q-1)/2}(C_k \cup E_k)$ if $j \in [1, \frac{q-1}{2}]$, the palette $\bigcup_{k=1}^{(q-1)/2}(C_k \cup D_k)$ if $j = \frac{q+1}{2}$ and the palette $\bigcup_{k=1}^{(q-1)/2}(D_k \cup E_k)$ if $j \in [\frac{q+3}{2}, q]$.

Case 222: $q \equiv 1 \pmod 4$

Case 2221: If $q \geq 9$, consider a palette-minimum coloring of a complete multigraph $2K_q$ with colors from the set $R \cup S \cup T \cup R^+ \cup S^+ \cup T^+ = \{a_k : k \in [1, 3q-5]\}$ as constructed in the proof of Theorem 9, where $X_1 = [1, \frac{q-3}{2}]$, $X_2 = [\frac{q-1}{2}, \frac{q+3}{2}]$ and $X_3 = [\frac{q+5}{2}, q]$. We suppose that $a_1 \in R$, $a_2 \in S$, $a_3 \in T$ and edges colored with a_1 have endvertices $2x-1$ and $2x$, $x \in [1, \frac{q-3}{4}]$, those colored with a_2 have endvertices $2x$ and $2x+1$, $x \in [\frac{q-1}{4}, \frac{q-1}{2}]$, and those colored with a_3 have endevertices x and $x + \frac{q+3}{2}$, $x \in [1, \frac{q-3}{2}]$.

We construct a proper edge coloring of $K_{p\times q}$ using a set of colors $\bigcup_{k=1}^{3q-5} A_k$, where $\{A_k : k \in [1, 3q-5]\}$ is a system of pairwise disjoint sets with $|A_k| = p-1$, $k \in [1, 3]$, and $|A_k| = \frac{p-1}{2}$, $k \in [4, 3q-5]$.

First, for $x \in [1, \frac{q-3}{4}]$, we use the colors from A_1 to color properly (according to Lemma 12) the edges of the graph $L_e(2x-1, 2x)$. Similarly, for $x \in [\frac{q-1}{4}, \frac{q-1}{2}]$ we use the colors from A_2 to color properly the edges of $L_e(2x, 2x+1)$. The colors from A_3 are used to color properly the edges of $L_o(2x-1, 2x-1+\frac{q+3}{2})$, $x \in [1, \frac{q-1}{4}]$, as well as those of $L_e(2x, 2x+\frac{q+3}{2})$, $x \in [1, \frac{q-5}{4}]$.

The remaining edges of $K_{p\times q}$ are colored step by step using induced subgraphs $S(y,z) = (D_{W_y} \oplus D_{W_z}) - \{v_{i,y}v_{i,z} : i \in [1,p]\}$ of $K_{p\times q}$ with $y < z$. If no edge of $S(y,z)$ is colored and an edge of $2K_q$ with endvertices y, z is colored with a color a_k, $k \in [4, 3q-5]$, we use the colors of A_k to color properly the edges of the bipartite graph $B_e(y,z) \cong B_e$ with the bipartition $\{W_y, W_z\}$, in which $v_{i,y}$ and $v_{i,z}$ play the roles of $(i-1)^1$ and $(i-1)^2$, respectively, $i \in [1,p]$. If "half" of edges of $S(y,z)$ are already colored using the colors of A_k (which means that there is an edge of $2K_q$ with endvertices y, z colored with a_k), $k \in [1, 3q-5]$, and $a_l \neq a_k$, $l \in [4, 3q-5]$, is the color of the second edge of $2K_q$ with endvertices y, z, we color the edges of $B_o(y,z) \cong B_o$ using the colors of C_l.

In the obtained coloring the vertices of W_j, $j \in [1, \frac{q-3}{2}]$, have the palette consisting of all colors of A_k with $a_k \in R \cup T \cup R^+ \cup T^+$. Analogously, for

W_j with $j \in [\frac{q-1}{2}, \frac{q+3}{2}]$ all k's with $a_k \in R \cup S \cup R^+ \cup S^+$ are involved, while for W_j with $j \in [\frac{q+5}{2}, q]$ all k's with $a_k \in S \cup T \cup S^+ \cup T^+$ appear.

Case 2222: If $q = 5$, consider the following subsets of $V(K_{p\times 5})$:

$$S_1 = V_1 \setminus \{v_{1,5}\},\ \ T_1 = \bigcup_{i=2}^{(p+1)/2} \bigcup_{j=1}^{2} \{v_{i,j}\} \cup \bigcup_{i=(p+3)/2}^{p} \bigcup_{j=3}^{5} \{v_{i,j}\},$$

$$S_2 = V_1 \setminus \{v_{1,4}\},\ \ T_2 = \bigcup_{i=2}^{(p+1)/2} \bigcup_{j=3}^{5} \{v_{i,j}\} \cup \bigcup_{i=(p+3)/2}^{p} \bigcup_{j=1}^{2} \{v_{i,j}\}.$$

Let G_i be the subgraph of $K_{p\times 5}$ induced by the set of vertices $S_i \cup T_i$, $i = 1, 2$. Since G_i is a $\frac{5p-5}{2}$-regular graph of the even order $\frac{5p+3}{2}$ with $G_i = D_{S_i} \oplus D_{T_i}$, by Theorem 2 it is a class 1 graph. The set of edges $E(K_{p\times q}) \setminus (E(G_1) \cup E(G_2))$ induces a $\frac{5p-5}{2}$-regular graph G_3 with $V(G_3) = \{v_{1,4}, v_{1,5}\} \cup T_1 \cup T_2$. We shall show that G_3 is a class 1 graph. Then, since the graphs G_1, G_2, G_3 are pairwise edge-disjoint, any superposition of palette-minimum edge colorings of these three graphs using pairwise disjoint sets of colors creates three palettes, namely those for the vertices of $V(G_i) \cap V(G_j)$ with $i \neq j$, *i.e.*, $\{v_{1,1}, v_{1,2}, v_{1,3}\}$, $\{v_{1,4}\} \cup T_1$ and $\{v_{1,5}\} \cup T_2$.

It is easy to see that the following three edge sets M_1, M_2, M_3 are pairwise disjoint perfect matchings of G_3:

$$\bigcup_{i=1}^{(p-1)/2} \bigcup_{j=1}^{5} \{v_{2i,j}v_{2i+1,j}\} \setminus \{v_{2,3}v_{3,3}, v_{p-1,4}v_{p,4}\} \cup \{v_{2,3}v_{p,4}, v_{1,4}v_{3,3}, v_{1,5}v_{p-1,4}\},$$

$$\bigcup_{i=1}^{(p-1)/2} \{v_{2i,1}v_{2i+1,2}, v_{2i,2}v_{2i+1,1}\} \setminus \{v_{2,3}v_{p,3}\} \cup$$

$$\bigcup_{i=0}^{(p-3)/2} \bigcup_{j=3}^{5} \{v_{i+2,j}v_{p-i,j}\} \cup \{v_{1,4}v_{2,3}, v_{1,5}v_{p,3}\},$$

$$\bigcup_{i=0}^{(p-3)/2} \bigcup_{j=1}^{2} \{v_{i+2,j}v_{p-i,j}\} \setminus \{v_{2,3}v_{p,4}, v_{2,4}v_{p,5}, v_{3,3}v_{p-1,4}\} \cup$$

$$\bigcup_{i=0}^{(p-3)/2} \bigcup_{j=3}^{5} \{v_{i+2,j}v_{p-i,3+|j+1|3|}\} \cup \{v_{1,4}v_{2,4}, v_{1,5}v_{p,5}, v_{2,3}v_{3,3}, v_{p-1,4}v_{p,4}\}.$$

Then $G_3 - (M_1 \cup M_2 \cup M_3)$ is a $\frac{5p-11}{2}$-regular bipartite graph with the bipartition $\{\{v_{1,4}\} \cup T_1, \{v_{1,5}\} \cup T_2\}$, which is obviously class 1. ■

By Proposition 6 and Lemma 7, Theorem 13 can easily be extended to the case of (special) regular complete multipartite multigraphs.

Corollary 14. *Let* λ, p, q *be integers with* $\lambda \geq 1$ *and* $\min(p, q) \geq 2$. *Then*

$$\check{s}(\lambda K_{p\times q}) = \begin{cases} 1 & \text{if } pq \equiv 0 \pmod 2, \\ 3 & \text{otherwise.} \end{cases}$$

■

To do: To write a good introduction (not necessarily as a special section). Insert the paper [1] in the References (and mention it in the Introduction).

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