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A note
on a directed version of the 1-2-3 Conjecture

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Abstract

The least k such that a given digraph $D = (V, A)$ can be arc-labeled with integers in the interval $[1, k]$ so that the sum of values in-coming to x is distinct from the sum of values out-going from y for every arc $(x, y) \in A$, is denoted by $\bar{\chi}^e_{\bar{L}}(D)$. This corresponds to one of possible directed versions of the well-known 1-2-3 Conjecture. Unlike in the case of other possibilities, we show that $\bar{\chi}^e_{\bar{L}}(D)$ is unbounded in the family of digraphs for which this parameter is well defined. However, if the family is restricted by excluding the digraphs with so-called lonely arcs, we prove that $\bar{\chi}^e_{\bar{L}}(D) \leq 4$, and we conjecture that $\bar{\chi}^e_{\bar{L}}(D) \leq 3$ should hold.

Keywords: edge coloring, digraph, 1-2-3 Conjecture
2000 MSC: 05C15, 05C20

1. Introduction

The origins of the problem go back to the eighties of the twentieth century and are associated with attempts to define the notion of irregularity of a graph using labels (colors) on the edges of a graph. Among those attempts, it was the irregularity strength that attracted the greatest attention. Perhaps this was due to a simple "geometric" interpretation based on the fact that although each graph of order greater than one contains at least two vertices

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of the same degree, an analogous statement is not true for multigraphs, *i.e.*, graphs in which we allow more than one edge between two (distinct) vertices.

Let $G = (V, E)$ be a graph. Given an integer k, a k-edge-coloring (labeling) of G is a function $f : E \to \{1, 2, \dots, k\}$. For $x \in V$, we put $\sigma(x) = \sum_{e \ni x} f(e)$. We say that two vertices x, y are *sum-distinguished* (the coloring f is *sum-distinguishing*) if $\sigma(x) \neq \sigma(y)$. The *irregularity strength* of G is the minimum k such that there exists a k-edge-coloring f sum-distinguishing all vertices in the graph G. The coloring f can be represented by substituting each edge e by a multiedge with multiplicity $f(e)$. The sum $\sigma(x)$ of labels around a vertex x is then equal to the degree of x in the respective multigraph.

A k-edge-coloring f of G is called *neighbor-sum-distinguishing* if $\sigma(x) \neq \sigma(y)$ whenever xy is an edge of G (we refer to it as to an *nsd-coloring* for short). Such a local variant of the irregularity strength gained great popularity in the twenty first century due to the following beautiful conjecture of Karoński, Łuczak, and Thomason [5], commonly called the **1-2-3 Conjecture** nowadays.

Conjecture 1. *If $G = (V, E)$ is a graph without isolated edges, then there is an nsd-coloring $f : E \to \{1, 2, 3\}$ of G.*

Following the notation from the survey paper by Seamone [7] we will denote the least k so that there is an nsd-k-edge-coloring of a graph G by $\chi^e_\Sigma(G)$. The 1-2-3 Conjecture thus presumes that $\chi^e_\Sigma(G) \leq 3$ for every graph G without isolated edges. The best currently known general upper bound stating that $\chi^e_\Sigma(G) \leq 5$ is due to Kalkowski, Karoński and Pfender [4]. The conjecture is verified for particular graph classes, *e.g.*, bipartite graphs, see [5].

Theorem 2. *If G is a bipartite graph without isolated edges, then $\chi^e_\Sigma(G) \leq 3$.* ■

We will focus on nsd-colorings of digraphs $D = (V, A)$, where we will use a simplified notation xy for an arc (x, y). Given a k-arc-coloring $f : A \to \{1, 2, \dots, k\}$ and a vertex $x \in V$, we discern *out-going arcs* $xy \in A$ and *in-coming arcs* $yx \in A$, and analogously the *out-sum* $\sigma^+(x) = \sum_{xy \in A} f(xy)$ and the *in-sum* $\sigma^-(x) = \sum_{yx \in A} f(yx)$ of x. Several variants of nsd-colorings of digraphs have already been considered.

The first problem of this type was introduced by Borowiecki, Grytczuk, and Pilśniak, and concerned so-called relative sums, defined for a vertex x as

$\sigma_{\pm}(x) = \sigma^{+}(x) - \sigma^{-}(x)$. The least k so that a k-arc-coloring of a given digraph $D = (V, A)$ exists with $\sigma_{\pm}(x) \neq \sigma_{\pm}(y)$ for every arc $xy \in A$ is denoted by $\chi^{e}_{\pm}(D)$. The authors proved in [3] the sharp upper bound $\chi^{e}_{\pm}(D) \leq 2$ valid for every digraph D.

Only just then Baudon, Bensmail, and Sopena considered the least integer k admitting a k-arc-coloring of a digraph $D = (V, A)$ such that $\sigma^{+}(x) \neq \sigma^{+}(y)$ for every $xy \in A$. We denote such k by $\chi^{e}_{+}(D)$. In [2] the authors showed that $\chi^{e}_{+}(D) \leq 3$ for every digraph D and proved that given a digraph D, the problem of determining whether $\chi^{e}_{+}(D) \leq 2$ is NP-complete. (Note that obviously we obtain the same thesis for the twin graph invariant $\chi^{e}_{-}(D)$ of the above one, where we require: $\sigma^{-}(x) \neq \sigma^{-}(y)$ for every $xy \in A$.)

The third natural variant was suggested by Łuczak [6], who proposed to study the sum-distinguishing requirement $\sigma^{+}(x) \neq \sigma^{-}(y)$ for $xy \in A$. Barme *et al.* [1] observed that the corresponding parameter $\chi^{e}_{Ł}(D)$ is not defined provided that D has an arc xy satisfying $d^{+}(x) = 1 = d^{-}(y)$, called a *lonely arc*. Nevertheless, they were able to prove the following upper bound.

Theorem 3. *If D is a digraph without lonely arcs, then $\chi^{e}_{Ł}(D) \leq 3$.* ■

The proof of Theorem 3 is based on the equivalence between the inequality $\chi^{e}_{Ł}(D) \leq k$ and the existence of an nsd-k-edge-coloring of a special (undirected) bipartite graph associated with D. Thus by the classification from the paper of Thomassen, Wu and Zhang [8], one may moreover determine $\chi^{e}_{Ł}(D)$ for any digraph D (without lonely arcs) in a polynomial time now.

In this note we study the inverse (in a way) of the problem of Łuczak above, requiring that $\sigma^{-}(x) \neq \sigma^{+}(y)$ for $xy \in A$ (which seems to be the last natural open issue in this new field). In the next section we discuss when the corresponding graph invariant $\bar{\chi}^{e}_{Ł}(D)$ is well defined, and, surprisingly, we prove that for those digraphs $\bar{\chi}^{e}_{Ł}(D)$ may be arbitrarily large. On the other hand, in Section 3 we show that $\bar{\chi}^{e}_{Ł}(D) \leq 4$ if lonely arcs are additionally forbidden. Finally, in the last section we pose a conjecture that then $\bar{\chi}^{e}_{Ł}(D) \leq 3$ should hold, and present a few rich families of digraphs supporting this new 1-2-3-Conjecture for digraphs.

2. Boundlessness of the inverse Łuczak's problem

We call a digraph $D = (V, A)$ *tractable* if for a suitable k there is a k-arc-coloring f of D such that for any arc $xy \in A$, $\sigma^{-}(x) \neq \sigma^{+}(y)$. The least such k for a tractable digraph D is denoted by $\bar{\chi}^{e}_{Ł}(D)$.

There are two obvious obstacles for tractability. Consider a k-arc-coloring f of a digraph $D = (V, A)$. For a vertex $x \in V$, we denote by $A^-(x)$ ($A^+(x)$) the set of arcs in D in-coming to x (out-going from x, respectively). An arc $xy \in A$ is called a *source-sink arc*, an *s-s arc* for short, if x is a source and y is a sink of D (*i.e.*, $d^-(x) = 0$ and $d^+(y) = 0$). Then, inevitably, $\sigma^-(x) = 0 = \sigma^+(y)$. The situation is similar if both arcs xy and yx belong to A and xy is an s-s arc in the digraph $D' = D - yx$. We then say that $\{xy, yx\}$ is a *source-sink edge* (an *s-s edge* for short). Then $A^-(x) = A^+(y) = \{yx\}$, and hence $\sigma^-(x) = f(yx) = \sigma^+(y)$. It is straightforward to see that if we forbid these two configurations in D, then $A^-(x) \neq A^+(y)$ for every arc $xy \in A$, and thus there exists a k-arc-coloring of D with $\sigma^-(x) \neq \sigma^+(y)$ for every $xy \in A$ for sufficiently large k.

Proposition 4. *A digraph D is tractable if and only if D has neither s-s arcs nor s-s edges.* ∎

The three parameters χ^e_+, χ^e_- and $\chi^e_{\L}$ fulfill a correspondingly formulated 1-2-3-Conjecture. Is it the case for the parameter $\bar{\chi}^e_{\L}$, too? The digraph D_4 drawn in Figure 1, gives us a negative answer to this question.

First, observe that D_4 has neither an s-s arc nor an s-s edge. Consider an arc-coloring f of D_4 such that $\sigma^-(x) \neq \sigma^+(y)$ whenever xy is an arc of D_4. Let $f(x_1x_2) = a$, $f(x_3x_4) = b$, $f(x_5x_6) = c$, $f(x_7x_8) = d$. The digraph D_4 satisfies $A^+(x_{2i-1}) = \{x_{2i-1}x_{2i}\} = A^-(x_{2i})$, $i = 1, 2, 3, 4$. Moreover, for any i, j with $1 \leq i < j \leq 4$, the arc $x_{2i}x_{2j-1}$ belongs to D_4, and hence

$$f(x_{2i-1}x_{2i}) = \sigma^-(x_{2i}) \neq \sigma^+(x_{2j-1}) = f(x_{2j-1}x_{2j}).$$

Therefore, the colors a, b, c, d of the dashed arcs $x_{2i-1}x_{2i}$, $i = 1, 2, 3, 4$, are pairwise distinct, and so $\bar{\chi}^e_{\L}(D_4) \geq 4$.

Proposition 5. *For any integer $k \geq 2$ there is a digraph D_k with $\bar{\chi}^e_{\L}(D_k) \geq k$.*

Proof. Consider a digraph D_k with the vertex set $\{x_1, x_2, \ldots, x_{2k}\}$ and the arc set $\bigcup_{i=1}^k \left(\{x_{2i-1}x_{2i}\} \cup \bigcup_{j=i+1}^k \{x_{2i}x_{2j-1}\}\right)$. Suppose that an l-arc-coloring $f : E(D_k) \to \{1, 2, \ldots, l\}$ satisfies $\sigma^-(x_i) \neq \sigma^+(x_j)$ whenever $x_ix_j \in E(D_k)$. It is easy to see proceeding as above that then necessarily $l \geq k$. ∎

Corollary 6. *The parameter $\bar{\chi}^e_{\L}$ is not bounded from above by an absolute constant.* ∎

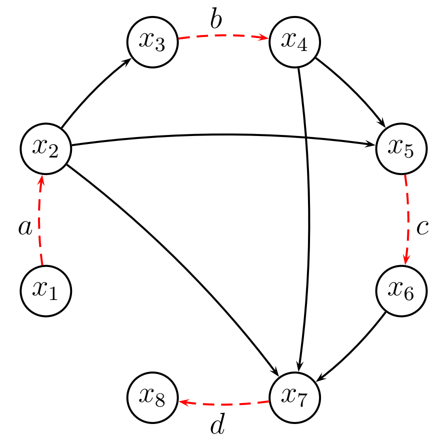

Figure 1: A digraph D_4 where the color 4 is needed

3. Graphs without lonely arcs

Let us observe that in the digraph D_k from Proposition 5, the arcs $x_{2i-1}x_{2i}$, which necessitate the use of a large number of colors, are lonely arcs. Having this in mind, it seems natural to ask whether, if a digraph does not contain such arcs, it is possible to color its arcs in the desired way using only colors 1, 2, 3. The question remains as yet unanswered. However, we are able to show that positive integers up to four are enough in this case. Note that forbidding lonely arcs in a digraph D forbids s-s edges in D, too, and so guarantees the tractability of D.

Theorem 7. *If D is a digraph without s-s arcs and without lonely arcs, then* $\bar{\chi}^e_{\not{L}}(D) \leq 4$.

To prove Theorem 7 we adapt the concept of so-called associated bipartite graphs used in [1]. Let $D = (V, A)$ be a digraph of order n with $V = \{v_1, v_2, \ldots, v_n\}$. The *associated bipartite graph of* D is the undirected bipartite graph $B(D) = (X, Y, E)$ of order $2n$ with $X = \{x_1, x_2, \ldots, x_n\}$, $Y = \{y_1, y_2, \ldots, y_n\}$, and the edge set defined as follows: $x_iy_j \in E \Leftrightarrow v_iv_j \in A$, $1 \leq i, j \leq n$ (note that here x_iy_j is a shortened form for $\{x_i, y_j\}$).

There is a one-to-one correspondence between the arcs of D and the edges of $B(D)$. It is easy to see that the arcs out-going from v_i correspond to the edges incident with x_i, and the arcs in-coming to v_j correspond to the edges incident with y_j. In particular, the arc v_iv_j is lonely (in D) if and only if the edge x_iy_j is isolated (in $B(D)$). Let us observe that, in an obvious way, an arc-coloring of D induces an edge-coloring of $B(D)$, and *vice versa*. Moreover, for the arc-coloring of D and the edge-coloring of $B(D)$ inducing each other, we have $\sigma^+(v_i) = \sigma(x_i)$ and $\sigma^-(v_i) = \sigma(y_i)$.

In the following lemma we use the group $\mathbb{Z}_4 = \{\mathbb{0}, \mathbb{1}, \mathbb{2}, \mathbb{3}\}$, where $\mathbb{i} \in \mathbb{Z}_4$ is the set of integers congruent to i modulo 4, $i = 0, 1, 2, 3$.

Lemma 8. *Let $G = (X, Y, E)$ be a bipartite graph without isolated vertices and edges. Then there exists a mapping $f : E \to \mathbb{Z}_4$ such that the mapping $\sigma : X \cup Y \to \mathbb{Z}_4$, defined by $\sigma(u) = \sum_{uv \in E} f(uv)$, satisfies $\sigma(x) \in \{\mathbb{2}, \mathbb{3}\}$ for each $x \in X$ and $\sigma(y) \in \{\mathbb{0}, \mathbb{1}\}$ for each $y \in Y$.*

Proof. We define a required edge coloring of G componentwise. For that purpose let k be the number of components of G, and let $G_l = (X_l, Y_l, E_l)$, $l \in \{1, 2, \ldots, k\}$, be the lth component of G, where $X_l \subseteq X$ and $Y_l \subseteq Y$; notice that $|E_l| \geq 2$. Suppose that $X_l = \{x_0, x_1, \ldots, x_p\}$, with $d = d(x_0) \geq d(x_i)$ for $i = 1, 2, \ldots, p$, and let $y_1, y_2, \ldots, y_d$ be the neighbors of x_0.

If $d \geq 2$, we determine values of f for edges belonging to E_l in several stages. In the stage 0 we put on each edge in E_l the temporary value $\mathbb{0}$.

In the stage $j \in \{1, 2, \ldots, p\}$ we choose an arbitrary path P in G_l joining x_0 with x_j, and we add to temporary values of the edges of P alternately $\mathbb{1}$ and $\mathbb{3}$. Since $\mathbb{1} + \mathbb{3} = \mathbb{0}$, and $\mathbb{0}$ is the identity element in $\mathbb{Z}_4$, temporary sum values do not change for inner vertices of P, hence after finishing the stage j we have temporary sum values $\sigma(x_0) = j\mathbb{1}$, $\sigma(x_i) = \mathbb{3}$ for $i = 1, 2, \ldots, j$, and $\sigma(u) = \mathbb{0}$ for all remaining vertices $u \in X_l \cup Y_l$.

Consider the situation after finishing the stage p, when $\sigma(x_0) = p\mathbb{1} = \mathbb{q}$ with $p \equiv q \pmod 4$ and $q \in \{0, 1, 2, 3\}$. If $q \in \{2, 3\}$, we are done.

If $q \in \{0, 1\}$, in the stage $p + 1$ we add $\mathbb{1}$ to the temporary value of the edge $x_0 y_i$ for each i satisfying $1 \leq i \leq 2 - q$ to finish with $\sigma(y_i) = \mathbb{1}$ and $\sigma(x_0) = \mathbb{2}$.

In the case $d = 1$ we have $G_l \cong K_{1,p+1}$ with $p \geq 1$, and $E_l = \{x_i y_1 : i = 0, 1, \ldots, p\}$. Colors of f for the edges in E_l are then defined as follows (and it is straightforward to check that the mapping σ, derived from f, has the required property for all vertices in $X_l \cup Y_l$):

If p is odd, then $f(x_i y_1) = \mathbb{2}$ for $i = 0, 1, \ldots, p$.

If $p = 2$, then $f(x_0 y_1) = f(x_1 y_1) = f(x_2 y_1) = \mathbb{3}$.

If p is even, $p \geq 4$, then $f(x_0 y_1) = f(x_1 y_1) = f(x_2 y_1) = \mathbb{3}$ and $f(x_i y_1) = \mathbb{2}$ for $i = 3, 4, \ldots, p$.

This completes the proof of the lemma. ∎

Proof of Theorem 7 Let B be the associated bipartite graph for the digraph $D = (V, A)$, and let $G = (X, Y, E)$ be created from B by excluding all its isolated vertices. The absence of lonely arcs in D causes the absence of isolated edges in G. Therefore, by Lemma 9, there is a coloring $f : E \to \mathbb{Z}_4$ such that $\sigma(x) \in \{\mathbb{2}, \mathbb{3}\}$ for each $x \in X$ and $\sigma(y) \in \{\mathbb{0}, \mathbb{1}\}$ for each $y \in Y$.

Consider the mapping $\tilde{f} : A \to \{1, 2, 3, 4\}$ defined so that if $x_i y_j \in E$ (with $x_i \in X$ and $y_j \in Y$), then $\tilde{f}(v_i v_j) \in f(x_i y_j)$; this is well-defined since the congruence class $f(x_i y_j) \in \{\mathbb{0}, \mathbb{1}, \mathbb{2}, \mathbb{3}\}$ has a unique representative in the set $\{1, 2, 3, 4\}$. Let $\tilde{\sigma}^-$ be the in-sum function and $\tilde{\sigma}^+$ the out-sum function that correspond to $\tilde{f}$. To show that $\tilde{f}$ distinguishes vertices $v_i, v_j \in V$ with $v_i v_j \in A$ we first note that $d^-(v_i) + d^+(v_j) > 0$ (otherwise $v_i v_j$ would be an s-s arc in D), and then we reason as follows:

If $d^-(v_i) = 0$, then $d^+(v_j) > 0$, and so $\tilde{\sigma}^-(v_i) = 0 < \tilde{\sigma}^+(v_j)$.

If $d^+(v_j) = 0$, then $d^-(v_i) > 0$, hence $\tilde{\sigma}^-(v_i) > 0 = \tilde{\sigma}^+(v_j)$.

If $d^-(v_i) > 0$ and $d^+(v_j) > 0$, from the definition of the mapping $\tilde{f}$ it is clear that $\tilde{\sigma}^-(v_i) \in \sigma(y_i) \in \{\mathbb{0}, \mathbb{1}\}$ and $\tilde{\sigma}^+(v_j) \in \sigma(x_j) \in \{\mathbb{2}, \mathbb{3}\}$, which immediately yields $\tilde{\sigma}^-(v_i) \neq \tilde{\sigma}^+(v_j)$. ■

4. The conjecture

Note that in the proof of Theorem 8 we have distinguished adjacent vertices of a digraph D in a stronger way than necessary. Indeed, if $v_i v_j$ is an arc of D, then the in-sum for v_i is not only distinct from the out-sum for v_j, but those sums even belong to distinct congruence classes modulo 4. This is why we believe that the following conjecture holds true.

Conjecture 9. *If D is a digraph without s-s arcs and lonely arcs, then $\bar{\chi}^e_{\Sigma}(D) \leq 3$.*

A *symmetric* digraph $D = (V, A)$ is such that $xy \in A \Rightarrow yx \in A$. If a k-arc-coloring $f : A \to \{1, 2, \ldots, k\}$ of a symmetric digraph D satisfies $xy \in A \Rightarrow \sigma^+(x) \neq \sigma^-(y)$, then it satisfies $yx \in A \Rightarrow \sigma^-(y) \neq \sigma^+(x)$, too, and *vice versa*. As a symmetric digraph cannot contain s-s arcs, by Theorem 3 we obtain the following proposition supporting Conjecture 9.

Proposition 10. *If D is a symmetric digraph without lonely arcs, then $\bar{\chi}^e_{\Sigma}(D) = \chi^e_{\Sigma}(D) \leq 3$.* ■

Moreover, a connected symmetric digraph D whose underlying graph is a cycle of an odd length $2l + 1$, satisfies $\bar{\chi}^e_{\Sigma}(D) = \chi^e_{\Sigma}(D) = \chi^e_{\Sigma}(B(D)) =$

$\chi^e_\Sigma(C_{4l+2}) = 3$. Thus the upper bound in Conjecture 9 cannot be reduced. In order to further support the plausibility of its thesis we additionally prove it for a special class of digraphs. We say a component C of a bipartite graph (X, Y, E) is an X-*star* if C is a star with $|V(C) \cap X| = 1$; similarly is defined a Y-*star*.

Theorem 11. *Let D be a digraph without s-s arcs and lonely arcs and let $B(D) = (X, Y, E)$. If $B(D)$ has no X-star components or $B(D)$ has no Y-star components, then $\bar{\chi}^e_L(D) \le 3$.*

Proof. Suppose first that $B(D)$ has no X-star components and let $G = (X', Y', E)$ be the graph created by excluding all isolated vertices from $B(D)$. Proceeding analogously as in the proof of Lemma 8 we prove that there is a mapping $f : E \to \mathbb{Z}_3 = \{\mathbb{0}, \mathbb{1}, \mathbb{2}\}$ such that $\sigma(x) \in \{\mathbb{1}, \mathbb{2}\}$ for each $x \in X'$ and $\sigma(y) = \mathbb{0}$ for each $y \in Y'$ (in this case $\mathbb{i} \in \mathbb{Z}_3$ is the set of integers congruent to i modulo 3, $i = 0, 1, 2$).

Let k be the number of components of G and let $G_l = (X_l, Y_l, E_l)$, $l \in \{1, 2, \dots, k\}$, be the lth component of G, where $X_l \subseteq X'$ and $Y_l \subseteq Y'$. From our assumptions it follows that the set $X = \{x_1, x_2, \dots, x_p\}$ satisfies $p \ge 2$; let $q = \lfloor \frac{p}{2} \rfloor$.

In the stage 0 we assign $\mathbb{0}$ as the temporary value of f to each edge of E_l.

In the stage $j \in \{1, 2, \dots, q\}$ we choose an arbitrary path P in G_l joining x_{2j-1} to x_{2j}, and we add to temporary values of the edges of P alternately $\mathbb{1}$ and $\mathbb{2}$. If p is even, we are done. If p is odd, in the stage $q + 1$ we proceed similarly as above with a path in G_l joining x_1 to x_p.

The mapping f is then used to define the mapping $\tilde{f} : A \to \{1, 2, 3\}$ similarly as in the proof of Theorem 7. Since $\tilde{f}$ distinguishes adjacent vertices of D in the required way, we have $\bar{\chi}^e_L(D) \le 3$.

If $B(D)$ has no Y-star components, we proceed the same way as above, this time however assuring that $\sigma(y) \in \{\mathbb{1}, \mathbb{2}\}$ for each $y \in Y'$ and $\sigma(x) = \mathbb{0}$ for each $x \in X'$. ■

Corollary 12. *If T is an n-vertex tournament, $n \ge 3$, then $\bar{\chi}^e_L(T) \le 3$.*

Proof. Let $V(T) = \{v_1, v_2, \dots, v_n\}$. By the way of contradiction we prove that $B(T)$ has no X-stars. Indeed, otherwise we may suppose without loss of generality that an X-star C of $B(T)$ satisfies $V(C) \cap X = \{x_1\}$ and $E(C) \supseteq \{x_1y_2, x_1y_3\}$. Since $d(y_2) = 1 = d^-(v_2)$, $v_1v_2 \in E(T)$ and T is a tournament, we have $d^+(v_2) = n - 2 = d(x_2)$, $v_2v_1 \notin E(T)$, $v_2v_3 \in E(T)$,

$x_2y_3 \in E(B(T))$ and $d(y_3) \geq 2$, a contradiction. Thus, by Theorem 11, $\bar{\chi}^e_Ł(T) \leq 3$. ■

Another wide family of examples may also be derived from the result of Thomassen, Wu, and Zhang [8], who succeeded to determine $\chi^e_\Sigma(G)$ for any bipartite graph G without isolated edges, and in particular proved that $\chi^e_\Sigma(G) = 2$ if $\delta(G) \geq 3$. Consequently, any digraph D with $\chi^e_Ł(D) = 3$ supports Conjecture 9, too. Indeed, if $\chi^e_Ł(D) = 3 = \chi^e_\Sigma(B(D))$, then by [8] it follows that $\delta(B(D)) = 2$, *i.e.* $B(D) = (X, Y, E)$ has neither X-star nor Y-star components, and hence, by Theorem 11, $\bar{\chi}^e_Ł(D) \leq 3$.

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