The IGA-HGS strategy for solving inverse problems with direct problem described by parabolic differential equations

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Abstract

This paper presents the strategy for solving difficult inverse problems with direct problem described by parabolic differential equations. The Hierarhical Genetic Strategy (HGS) is used to solve optimization problem and utilize Isogeometric Finite Element Method (IGA-FEM) for solving direct problem. The strategy consist of proper HGS and IGA-FEM errors balance to lower computational cost while keep the high accuracy of the solution. The concept is verified mathematically.

1 Motivation

The inverse optimization problems, which have wide application in engineering, belong to very time consuming computational tasks. They require sequence of experiments, so called direct problem solutions e.g. obtained by Finite Element Method (FEM). Computational time for single experiment depends on required accuracy of the solution. Using high accuracy for solving direct problem in each iteration of inverse problem solver leads to enormous computational costs. The better strategy is to balance dynamically the accuracy of solving direct and inverse problems. Such method was presented in [3]. In this paper we extend this strategy for new class of problems - represented by parabolic differential equations. We propose to use IGA-FEM for solving direct problem and adaptive strategy called Hierarchical Genetic Strategy (HGS) to solve inverse problem. We dynamically balance accuracy of both iterations (direct and inverse) which is better approach than using maximal accuracy in all cases and gives the solution with the same accuracy. We presented a new algorithm, based on mathematical results. The relation between errors of both strategies is prooved which allows us to execute this strategy.

2 Preliminaries

As direct problem we consider a non-linear flow in homogeneous media problem described in [3]. We are looking for the pressure field u satisfying following equations:

$$\frac{\partial u}{\partial t} - \nabla(k(x,u)\nabla u) = h(x,t,u) \quad \text{in} \quad \Omega \times [0,T], \tag{2.1}$$

$$\nabla u \hat{n} = 0 \quad \text{in} \quad \partial\Omega \times [0,T] \tag{2.2}$$

$$u(x,0) = u_0(x) \quad \text{in} \quad \Omega. \tag{2.3}$$

We consider the problem in the domain $\Omega = [0,1]^3$, where T is the time of the simulation, $\kappa(x,u)$ is permeability of the medium, $h(x,t,u)$ describes induced force. We assume some initial state $u_0(x)$ and zero Neumann conditions.

Using the different methods and forward Euler scheme we obtain discretization of the variational form [3]:

$$(v, \frac{u_{t_{n+1},h(x)} - u_{t_n,h}(x)}{t_{n+1} - t_n})_\Omega + (\kappa(x,u) \bigtriangledown u_{t_n,h}, \bigtriangledown v)_\Omega = (v, h(t_n))_\Omega \tag{2.4}$$

$$\forall v \in V_h \tag{2.5}$$

$$n = 1, ..., N \tag{2.6}$$

$$u(x,0) = u_0(x) \quad \text{in} \quad \Omega \tag{2.7}$$

where $0 = t_0 < ... < t_N = T \quad t_{i+1} - t_i = t_{j+1} - t_j \quad \forall i,j = 1, ..., N$, $V_h \subset V$ is the finite dimensional subspace of the linear space of test functions $V \subset H^1(\Omega)$

The energy of the solution u we are looking for takes the form:

$$E(u(x,t)) = ||u(x,t)||^2_\Omega = \int_\Omega |u(x,t)|^2 d\Omega$$

and the energy of solution computed by IGA-FEM is defined as follows:

$$E_h(u_h(x,t)) = ||u_h(x,t)||^2_\Omega = \int_\Omega |u_h(x,t)|^2 d\Omega$$

The inverse problem under consideration is to find x^* such as

$$J(x) = \frac{h}{2}|E(u(x_0,t) - u_h(x^*,t))| = min_{h\to\infty} min_{x\in\Omega} \frac{h}{2}||E(u(x_0,t) - u_h(x,t))|$$

where x_0 is the location of pumps and sinks we are looking for.

A relative FEM error utilized by the IGA-FEM code is defined as the energy norm difference between the solutions in two next steps of IGA-FEM method

$$err_{IGA-FEM} = ||u_h(x,t) - u_{h+1}(x,t)||_E$$

From definition the relative IGA-FEM error is also form by

$$||u_{h+1}(x,t) - u_h(x,t)||_E = \sqrt{a(u_{h+1}(x,t) - u_h(x,t), u_{h+1}(x,t) - u_h(x,t))}$$

where

$$a(u,v) = -(\kappa(x,u) \bigtriangledown u, \bigtriangledown v) = \int_\Omega \kappa(x,u) \bigtriangledown u \bigtriangledown v$$

The analogical definition are also true for absolute IGA-FEM error.

Objective function error is defined as the difference between energy of solution found by IGA-FEM method and the energy of exact solution.

$$e_h(x,t) = |E(u_{h+1}(x,t)) - E(u(x_0,t))|$$

3 IGA-FEM and objective function error relation

Energy of difference between solutions of two next steps in IGA-FEM method is limited by the relative error of this strategy.

Lemma 3.1.

$$|E(u_{h+1}(x,t) - u_h(x,t))| \leq P||u_{h+1}(x,t) - u_h(x,t)||_E^2$$

Proof.

$$|E(u_{h+1}(x,t) - u_h(x,t))| = |\int_\Omega (u_{h+1}(x,t) - u_h(x,t))^2 d\Omega| \leq$$

from Poincare inequality

$$\leq |P \int_\Omega \bigtriangledown (u_{h+1}(x,t) - u_h(x,t))^2 d\Omega| \leq$$

from definition

$$\leq P||u_{h+1}(x,t) - u_h(x,t)||_E^2$$

□

Energy of difference between approximated solution obtain by IGA-FEM method and exact solution is limited by absolute IGA-FEM error of this strategy.

Lemma 3.2.

$$|E(u_h(x,t) - u(x,t))| \leq Q||u_h(x,t) - u(x,t)||_E^2$$

Proof.

$$|E(u_h(x,t) - u(x,t))| = |\int_\Omega (u_h(x,t) - u(x,t))^2 d\Omega| \leq$$

from Poincare inequality

$$\leq |P \int_\Omega \bigtriangledown (u_h(x,t) - u(x,t))^2 d\Omega| \leq$$

from definition

$$\leq P||u_h(x,t) - u(x,t)||_E^2$$

□

We obtain the following dependency between the inverse and direct problems errors: the objective function error is limited by the relative IGA-FEM error plus absolute IGA-FEM error plus the accuracy of HGS solution.

Theorem 3.3.

$$e_{h+1}(x,t) \leq P||u_{h+1}(x,t) - u_h(x,t)||_E^2 + Q||u_h(x,t) - u(x,t)||_E^2 + \alpha(x - x_0)$$

Proof.

$$e_{h+1}(x,t) = |E(u_{h+1}(x,t) - E(u(x_0,t))| =$$

$$|E(u_{h+1}(x,t) - u_h(x,t) + u_h(x,t) - u(x,t) + u(x,t)) - E(u(x_0,t))| =$$

$$= \int_\Omega (u_{h+1}(x,t) - u_h(x,t) + u_h(x,t) - u(x,t) + u(x,t))^2 - \int_\Omega u(x_0,t)^2 \leq$$

$$\leq \int_\Omega (u_{h+1}(x,t) - u_h(x,t))^2 + \int_\Omega (u_h(x,t) - u(x,t))^2 + \int_\Omega u(x,t)^2 - \int u(x_0,t)^2 =$$

$$|E(u_{h+1}(x,t) - u_h(x,t))| + |E(u_h(x,t) - u(x,t))| + |E(u(x,t) - E(u(x_0,t))|$$

assuming Lipshitz continuous of functional E in respect to parameter x and plugging 3.1 3.2 we obtain

$$e_{h+1}(x,t) \leq P||u_{h+1}(x,t) - u_h(x,t)||_E^2 + Q||u_h(x,t) - u(x,t)||_E^2 + \alpha(x - x_0)$$

□

Let's observe that if the objective function error is much smaller than the FEM error, increasing number of elements in FEM mesh is needed in order to improve accuracy of computation. This allows us to construct algorithm in which the objective function error and FEM error are balanced. We perform decreasing error of direct problem solution while it is greater then assumed Ratio connected with δ_j.

4 IGA-HGS - inverse problem optimization algorithm description

We present new strategy for solving ill-posed global optimization inverse problems where direct problem is described by parabolic differential equations. It couples isogeometric finite element method (IGA-FEM) for solving direct problem and Hierarchical Genetic Strategy (HGS) for solving optimization one. To obtain computing time reduction we balance the accuracy of both algorithms.

The Hierarchical Genetic Strategy–HGS introduced by Kolodziej and Schaefer [1] enables effective solving of global optimization problems using a variable, dynamically adapting accuracy.

It produces a tree-structured, dynamically changing set of dependent entities. The number of tree levels is restricted by m. Each branch is governed by a separate instance of Simple Genetic Algorithm (SGA). All entities work asynchronously and are synchronized by the message-passing mechanism if necessary.

There is different accuracy of searching on each level of the HGS tree. The branches of higher order perform more chaotic and less accurate search, finding only the promising regions in the domain. The deeper branches operate with higher accuracy and search only in the area appointed by their parents branch.

After fixed number of genetic epochs K, called the metaepoch, each branch except leaves sprouts the new child-branch in promising region it found. This new branch is activated surrounding the best-fitted individual from the parental entity . One of methods to acquire different accuracy on each level of tree is using binary genotypes of different lenght. An unique branch of the first order called root performs search with lower accuracy utilizing the shortest genotypes. The branches of level m called leaves utilizing longest genotypes and performing most accurate search.

There is a coherency in the search between entities of different orders thanks to the special kind of hierarchical encoding that forms the sequence of nested grids. You can find that construction in [2]. At the begining the mesh for branches of order m is established. Next the mesh for every jth level is defined recursively by selecting some nodes from mesh of $j+1$th branches.The maximum diameter δ_j determined the search accuracy on jth level of HGS tree. It is clear that $\delta_1 > ... > \delta_m$

HGS also implements two additional mechanisms to reduce the redundancy of the search process. The first one, called *conditional sprouting* disables sprouting new entities in the region already occupied or explored by another entity of the same order sprouted by the same parent. The second mechanism, called *branch reduction*, reduces the branches of the same order that perform the search in the common landscape region or in the regions already explored.

There are two types of stop conditions. First is local and apply for every single branch stopping it if detects the lack of evolution progress of the current entity P. Second, global is send to all branches. It checks if the all branches found local extremes.

Let observe that two first elements on the right side of formula 3.3 are convergent. To lower the computational costs of algorithm the regresion of direct problem error (IGA-FEM error) and the inverse problem error (HGS error) should be close. According to this we choose the step h in IGA-HGS algorithm such that ratio of IGA-FEM relative error and accuracy of HGS solution is smaller than assumption constant connected with α. The main idea of IGA-HGS algorithm is to decrease the computational cost by dynamically balancing the accuracy of both HGS and IGA-FEM algorithms. We use a hierarchical strategy where we increase the accuracy of the IGA-FEM solver while restricting the search to the local neighborhood of the best individuals. Using this strategy allows us to reduce computational cost in two ways. Firstly, it decreases number of objective functions calls by using adaptation in HGS. Secondly, using different mesh sizes in IGA-FEM solver causes lowering costs of fitness function evaluations

For sufficient diversity of the search in the root the large size of population, high mutation and crossing-over rates are used. Smaller values of this parameters in leaves reduces the enormous computational costs of local searches.

For evaluating individuals in branches of order j we perform computation on IGA-FEM mesh

size equal to N_j elements, with number of time iteration t_j and time step dt_j . Let us established some initial parameters N_1, t_1 and dt_1 used for fitness function computations in root.

When we increase the mesh size by factor 2^k, so

$$N_j = N_1 * 2^{k(j-1)}$$

for $j = 1, ..., m$ according Courant-Friedrichs-Lewy condition we have to increase number of time steps by factor 2^{2k}, so

$$t_j = t_{j-1} * 2^{2k} = t_1 * 2^{2k(j-1)}$$

for $j = 1, ..., m$. Of course period of time for simulations in IGA-FEM computations should be the same in all IGA-HGS branches. To maintain this following equation have to be fulfilled $t_j * dt_j = t_k * dt_k$ for $j, k = 1, ..., m$ [4]

Based on above, dt_j is decreased by factor 2^{2k}.

$$dt_j = \frac{dt_{j-1}}{2^{2k}} = \frac{dt_1}{2^{2k(j-1)}}$$

for $j = 1, ..., m$.

Note, that IGA-HGS algorithm is defined in the correct way because of search coherency obtained by of nested grids for entities of differents levels. All individuals occurring in jth level branch are also possible in $j + 1$ level ones. The sketch of strategy called IGA-HGS for one entity is presented in Algorithm 1

Algorithm 1: IGA-HGS algorithm for branch of j^{th} order with varying accuracy of the fitness function computation

```
1  if j = 1 then
2      initialize root population P^0;
3  repeat
4      l ← 0 ;
5      if global stop condition then
6          STOP;
7      for all i ∈ P^l do
8          solve direct problem for phenothype of i using IGA-FEM with proper parameters:
9          while error_FEM > Ratio * δ_j do
10             N_j = N_j * 2,
11             t_j = t_j * 2,
12             dt_j = dt_j/2
13             computeerror_{IGA_F EM}
14         compute the fitness function f_j(i) value;
15     if branch stop condition then
16         STOP;
17     perform selection according to the fitness function value f_j;
18     perform genetic operations;
19     if NOT l mod K then
20         distinguish the best fitted individual from P^t;
21         if NOT prefix comparison AND j ≤ m then
22             sprout;
23     l ← l + 1;
24 until false;
```

5 Conclusion

The proposed strategy could be applied to solving difficullt inverse problems with direct problem described by parabolic differential equation. Through proper balancing errors of direct and inverse problems we keep the high accuracy of the solution while lowering computational costs. The correctness of the strategy is mathematically verified.

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