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Nordhaus-Gaddum Bounds for the Distinguishing Index

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Abstract

The distinguishing index of a graph G, denoted by $D'(G)$, is the least number of colours in an edge colouring of G not preserved by any non-trivial automorphism. We investigate the Nordhaus-Gaddum type relation:

$$2 \leq D'(G) + D'(\overline{G}) \leq \max\{\Delta(G), \Delta(\overline{G})\} + 2$$

and prove that it holds for some classes of graphs. To do this, we prove some results which might be of interest as such. In particular, we show that $D'(G) \leq 2$ if G is traceable, and $D'(G) \leq 3$ if G is either claw-free or 3-connected and planar. We also characterize all connected graphs G with $D'(G) \geq \Delta(G)$.

Keywords: edge colourings; symmetry breaking in graphs; distinguishing index; claw-free graphs, planar graph

Mathematics Subject Classifications: 05C25, 05C15

1 Introduction

We follow standard terminology and notation of graph theory (see, e.g, [8]). In this paper, we consider general, i.e., not necessarily proper, edge colourings

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of graphs. Such a colouring c of a graph G *breaks an automorphism* $\varphi \in \text{Aut}(G)$ if φ does not preserve colours of c. The *distinguishing index* $D'(G)$ of a graph G is the least number d such that G admits an edge colouring with d colours that breaks all non-trivial automorphisms (such a colouring is called a *distinguishing d-colouring*). Clearly, $D'(K_2)$ is not defined, so in this paper, a graph G is called *admissible* if neither G nor $\overline{G}$ contains K_2 as a connected component.

The definition of $D'(G)$, introduced by Kalinowski and Pilśniak in [12], was inspired by the well-known *distinguishing number* $D(G)$ which was defined for general vertex colorings by Albertson and Collins [1]. Another concept is the *distinguishing chromatic number* $\chi_D(G)$ introduced by Collins and Trenk [4] for proper vertex colourings. Both numbers, $D(G)$ and $\chi_D(G)$, have been intensively investigated by many authors in recent years.

In 1956, Nordhaus and Gaddum obtained the following lower and upper bounds for the sum of the chromatic numbers of a graph and its complement (actually, the upper bound was first proved by Zykov [18] in 1949).

Theorem 1 *[13] If G is a graph of order n with a chromatic number $\chi(G)$, then*

$$2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n + 1.$$

Since then, Nordhaus-Gaddum type bounds were obtained for many graph invariants. An exhaustive survey is given in [2]. Here, we adduce only those closely related to the topic of our paper.

In 1964, Vizing [15] considered proper edge colourings and he proved Nordhaus-Gaddum type bounds for the chromatic index of a graph.

Theorem 2 *[15] If G is a graph of order n with a chromatic index $\chi'(G)$, then*

$$n - 1 \leq \chi'(G) + \chi'(\overline{G}) \leq 2(n - 1).$$

In 2013, Collins and Trenk [5] proved Nordhaus-Gaddum type inequalities for the distinguishing chromatic number.

Theorem 3 *[5] For every graph of order n and distinguishing number $D(G)$ the following inequalities are satisfied*

$$2\sqrt{n} \leq \chi_D(G) + \chi_D(\overline{G}) \leq n + D(G).$$

Kalinowski and Pilśniak [12] also introduced a *distinguishing chromatic index* $\chi'_D(G)$ of a graph G as the least number of colours in a proper edge colouring that breaks all non-trivial automorphisms of G. They proved the following somewhat unexpected result.

Theorem 4 *[12] If G is a connected graph of order $n \geq 3$, then*

$$\chi'_D(G) \leq \Delta(G) + 1$$

except for four graphs of small orders C_4, K_4, C_6, $K_{3,3}$.

Clearly, $\chi'_D(G) \geq \chi'(G)$. Therefore, the following Nordhaus-Gaddum type inequalities for the distinguishing chromatic index can be easily derived from Theorem 2 and Theorem 4.

Theorem 5 *If G is an admissible graph of order $n \geq 7$, then*

$$n - 1 \leq \chi'_D(G) + \chi'_D(\overline{G}) \leq 2(n-1).$$

□

Collins and Trenk observed in [5] that the Nordhaus-Gaddum type relation is trivial for the distinguishing number, as $D(G) + D(\overline{G}) = 2D(G)$ since $\mathrm{Aut}(\overline{G}) = \mathrm{Aut}(G)$ and every colouring of $V(G)$ breaking all non-trivial automorphisms of G also breaks those of $\overline{G}$.

The main aim of this paper is to investigate Nordhaus-Gaddum type inequalities for the distinguishing index of a graph. We formulate and discuss the following conjecture.

Conjecture 6 *Let G be an admissible graph of order $n \geq 7$, and let $\Delta = \max\{\Delta(G), \Delta(\overline{G})\}$. Then*

$$2 \leq D'(G) + D'(\overline{G}) \leq \Delta + 2.$$

2 Preliminary results

In the sequel, we make use of some facts proved in [12].

Proposition 7 *[12] $D'(P_n) = 2$ for every $n \geq 3$.*

Proposition 8 *[12]* $D'(C_n) = 3$ *for* $n \leq 5$, *and* $D'(C_n) = 2$ *for* $n \geq 6$.

Proposition 9 *[12]* $D'(K_n) = 3$ *if* $3 \leq n \leq 5$, *and* $D'(K_n) = 2$ *if* $n \geq 6$.

Recall that every finite tree T has either a central vertex or a central edge, which is fixed by every automorphism of T. A *symmetric tree*, denoted by $T_{h,d}$, is a tree with a central vertex v_0, all leaves at the same distance h from v_0 and all vertices that are not leaves of equal degree d. A *bisymmetric tree*, denoted by $T''_{h,d}$, is a tree with a central edge e_0, all leaves at the same distance h from the edge e_0 and all vertices which are not leaves of equal degree d.

Theorem 10 *[12] If* T *is a tree of order* $n \geq 3$, *then* $D'(T) \leq \Delta(T)$. *Moreover, equality is achieved if and only if* T *is either a symmetric or a bisymmetric tree.*

For connected graphs in general there is the following upper bound for $D'(G)$.

Theorem 11 *[12] If* G *is a connected graph of order* $n \geq 3$, *then*

$$D'(G) \leq \Delta(G)$$

unless G *is* C_3, C_4 *or* C_5.

It follows for connected graphs that $D'(G) \geq \Delta(G)$ if and only if $D'(G) = \Delta(G) + 1$ and G is a cycle of length at most 5. The equality $D'(G) = \Delta(G)$ holds for all paths, for cycles of length at least 6, for K_4, $K_{3,3}$ and for symmetric or bisymmetric trees. Now, we show that $D'(G) < \Delta(G)$ for all other connected graphs. A *palette* of a vertex is the set of colours of edges incident to it.

Theorem 12 *Let* G *be a connected graph that is neither a symmetric nor an asymmetric tree. If the maximum degree of* G *is at least* 3, *then* $D'(G) \leq \Delta(G) - 1$ *unless* G *is* K_4 *or* $K_{3,3}$.

Proof. The conclusion is true for trees due to Theorem 10. We assume that the order of a graph G is at least 7 as the claim for smaller graphs can be easily verified (we skip this to save space).

Denote $\Delta = \Delta(G)$. Consider a maximal subgraph G' of G without pendant subtrees and pendant triangles (a subgraph is pendant if it has only one vertex in common with the rest of a graph). First, we construct an edge colouring c stabilizing all vertices of G' by any automorphism preserving c. Next, we can easily colour pendant subtrees and pendant triangles with $\Delta-1$ colours, even if G' is empty.

We use a similar notation as in the proof of Theorem 11 in [12]. By $N_i(v)$ we denote the set of vertices of distance i from a vertex v. Let x be a vertex with the maximum degree of G. We colour all edges incident to x with 1. In our edge colouring c of the graph G', the vertex x will be the unique vertex of the maximum degree with the monochromatic palette $\{1\}$. Hence, it will be fixed by every automorphism φ preserving c. The neighbourhood $N_1(x)$ can be partitioned into subsets M_k, for $k = 0, 1, \ldots, \Delta - 1$, defined as

$$M_k = \{v \in N_1(x) \ : \ |N_1(v) \cap N_2(x)| = k\}.$$

Denote $M_k = \{v_1, \ldots, v_{l_k}\}$, $k = 0, 1, \ldots, \Delta-1$. Thus, $l_0+l_1+\ldots+l_{\Delta-1} = \Delta$.

We want to find a colouring of the edges of $G'[N_1(x) \cup N_2(x)]$ such that each vertex of $N_1(x) \cup N_2(x)$ is fixed by every automorphism preserving this colouring. We proceed in a number of steps.

Step M_0. Observe that, by our choice of G', a subgraph $G'[M_0]$ of G' induced by the vertices of the set M_0 contains neither isolated vertices nor isolated edges. Moreover $\Delta(G'[M_0]) \leq \Delta - 1$ and we want to colour edges of $G'[M_0]$ with $\Delta - 1$ colours. This is possible by Theorem 11 unless $G'[M_0]$ either is a small cycle of length at most 5 or it is disconnected. If $l_0 = \Delta$ and $G'[M_0] \in \{C_3, C_4, C_5\}$, then $G \in \{K_4, K_5, K_6\}$, respectively. A distinguishing colouring is given by Theorem 9, and it uses Δ colours for K_4. If $l_0 < \Delta$, we can use a third colour for small cycles since then $\Delta \geq 4$.

If $G'[M_0]$ is disconnected then $\Delta \geq 6$ and we have to distinguish all isomorphic components. Denote such a component by G_1. Suppose that $tG_1 \subseteq G'[M_0]$, for some $t > 1$. Recall that $|G_1| \geq 3$, so $t \leq \frac{\Delta}{3}$. We can choose distinct sets of colours for every component since

$$\binom{\Delta-1}{\frac{\Delta}{t}} \geq \binom{\Delta-1}{3} \geq \frac{\Delta}{3} \geq t,$$

where $\frac{\Delta}{t} - 1$ is an upper bound for the maximum degree of G_1. Thus, each vertex of M_0 is fixed.

Step M_1. For every $i = 1, \ldots, l_1$, we colour every edge $v_i u$, where $u \in N_2(x)$, with a distinct colour from $\{1, \ldots, \Delta - 1\}$. This is impossible only if $l_1 = \Delta$. Then we choose two vertices a and b in $G'[M_1]$ such that its neighbours a' and b', respectively, in $N_2(x)$ have distinct neighbourhoods in $N_2(x)$ or in $N_3(x)$. Then we colour with 1 one edge incident with b' (but neither $a'b'$ nor bb'). It is impossible only if $|N_2(x)| = 1$. However, it is easy to find a distinguishing colouring also in this case. Next, we colour all the remaining edges incident to $v_i \in M_1$ with 1, and all the remaining edges in $N_2(x)$ with 2. Thus, each vertex of M_1 is fixed.

Step M_2. For every $i = 1, \ldots, l_2$, we colour the edges v_iu_1, v_iu_2 where $\{u_1, u_2\} \subseteq N_2(x)$, with two distinct colour sets from among $\binom{\Delta-1}{2}$ sets. This is impossible only in three cases:

a) if $l_2 = \Delta = 3$. Then we choose two vertices a and b in $G'[M_2]$ such that $N(a) \cap N(b) \cap N_2(x) = \{y\}$. We colour the edges aa' and cc' with 1 (also if $c' = y$) and the edges ay, bb', by, cc" with 2. If such a choice of vertices a and b is impossible then either

– $N(a) \cap N(b) \cap N(c) \cap N_2(x) = \{y, z\}$, and then G is isomorphic to $K_{3,3}$; or

– $N(a) \cap N(b) \cap N_2(x) = \{y, z\}$ and $N(a) \cap N(c) \cap N_2(x) = \emptyset$, and then we colour an edge by with 1 and edges ay, az, bz with 2 , and two edges incident with a vertex c with 1 and 2, or

– for every two vertices a, b of $G'[M_2]$, the set $N(a) \cap N(b) \cap N_2(x)$ is empty. There exists an i such that $N_i(x)$ contains vertices a' in the subtree T_a and b' in the subtree T_b such that $a'b' \in E(G')$ since G' does not have pendant subtrees and triangles. Similarly, there exists a j such that $N_j(x)$ contains vertices a'' in the subtree T_a and c'' in the subtree T_c such that $a''c'' \in E(G')$. Then we colour these two edges $a'b'$, $a''c''$ with 1, and all remaining edges of $G'[N_i(x)]$ and $G'[N_j(x)]$ with 2. Moreover, let a_1 be a vertex of $G'[N_2(x)]$ which is on the path $a - a'$, let b_1 be a vertex of $G'[N_2(x)]$ which is on the path $b - b'$, and let c_1 be a vertex of $G'[N_2(x)]$ which is on the path $c - c''$. If a_1 is on the path $a - a''$, then we colour the edges aa_1, bb_2 and cc_1 with 2, and the edges aa_2, bb_1 and cc_2 with 1. If a_1 is not on the path $a - a''$, then we colour the edges aa_2, bb_1 and cc_1 with 2, and the edges aa_1, bb_2 and cc_2 with 1.

b) if $l_2 = \Delta = 4$. Then we choose two vertices a and b in $G'[M_2]$ such that $N(a) \cap N_2(x) \neq N(b) \cap N_2(x)$ and $N(a) \cap N(b) \cap N_2(x) \neq \emptyset$. We colour with 2 and 3 the edges incident with a and with 2 both edges incident with b. It is impossible only if $G'[M_2] \cup N(G'[M_2]) \cap N_2(x) \subseteq K_{3,4}$ (then two colours suffice to fix all seven vertices, by Theorem 14, as $K_{3,4}$ is traceable), or if for every a and b in $G'[M_2]$, the set $N(a) \cap N(b) \cap N_2(x)$ is empty (then two vertices of $G'[M_2]$ obtain the same pair of colours and we can distinguish them in next levels recursively).

c) if $l_2 = \Delta - 1$ and $\Delta = 3$. Let a and b be the two vertices in $G'[M_2]$. If $N(a) \cap N(b) \cap N_2(x) \neq \emptyset$, then we colour with 1 and 2 the two edges incident to a and both edges incident to b with 2. If the set $N(a) \cap N(b) \cap N_2(x)$ is empty, then there exists an i such that $N_i(x)$ contains vertices a' in the subtree T_a and b' in the subtree T_b such that $a'b' \in E(G')$ because G' does not have pendant subtrees and triangles. Then we colour the edge $a'b'$ with 1 and all remaining edges of $G'[N_i(x)]$ with 2. Let a_1 be a vertex of $G'[N_2(x)]$ which is on the path $a - a'$, and let b_1 be a vertex of $G'[N_2(x)]$ which is on the path $b - b'$. Then we colour the edges aa_1, bb_2 with 1, and the edges aa_2, bb_1 with 2.

Next, we colour all the remaining edges incident to $v_i \in M_2$ with 2 and all the remaining edges in $N_2(x)$ with 2. Thus, each vertex of M_2 is fixed.

Step M_j, for $j \geq 3$. For every $i = 1, \ldots, l_j$, we colour the edges $v_i u$, where $u \in N_2(x)$, with distinct sets of j colours from $\binom{\Delta-1}{j}$ sets. It is always possible whenever $\binom{\Delta-1}{j} \geq l_j$. This inequality does not hold only in two cases.

- If $j = \Delta - 2$ and $l_j = \Delta$, then we define a colouring with $\Delta - 1$ colours like in Step M_2 b).

- If $j = \Delta - 1$ and $l_j \geq 2$, then we can use multisets of colours (without a monochromatic set $\{1\}$) for colouring edges incident with $v \in M_j$ and we define a colouring with $\Delta - 1$ colours like in Step M_2 a) and c), but it is more technical and complicated.

Clearly, each vertex of $N_1(x) \cup N_2(x)$ is fixed by every automorphism preserving the colouring c.

Then for $v_j \in N_j(x)$, $j \geq 2$, we colour all edges $v_j u$, $u \in N_{j+1}(x)$, with distinct colours from $\{1, \ldots, \Delta - 1\}$ and the remaining edges incident to v_j with 2.

Then we recursively colour the edges incident to consecutive spheres $N_j(x)$ in the same way as previously. It is easily seen that it is always possible. Hence, all vertices of G' are fixed by any automorphism φ preserving our colouring c.

It is not difficult to observe that x is the unique vertex of the maximum degree with the monochromatic palette $\{1\}$. □

3 Some classes of graphs

We say that a graph G is *almost spanned* by a subgraph H if $G-v$ is spanned by H for some $v \in V(G)$. The following observation will play a crucial role in this section.

Lemma 13 *If a graph G is spanned or almost spanned by a subgraph H, then*

$$D'(G) \leq D'(H) + 1.$$

Proof. We colour the edges of H with colours $1, \ldots, D'(H)$, and all other edges of G with an additional colour 0. If φ is an automorphism of G preserving this colouring, then $\varphi(x) = x$, for each $x \in V(H)$. Moreover, if H is a spanning subgraph of $G - v$, then also $\varphi(v) = v$. Therefore, φ is the identity. □

Traceable graphs

Theorem 14 *If G is a traceable graph of order $n \geq 7$, then $D'(G) \leq 2$.*

Proof. Let $P_n = v_1 v_2 \ldots v_n$ be a Hamiltonian path of G. If $G = P_n$ then the conclusion follows from Proposition 7. If G is isomorphic to $P_n + v_1v_3$, them we colour the edge v_1v_3 with 1, and all other edges with 2 breaking all nontrivial automorphisms of G. Then suppose that G contains an edge v_iv_j distinct from v_1v_3 with $i < j - 1$. Without loss of generality we may assume that $i - 1 \leq n - j$. It is easy to see that at least one of the graphs $P_n + v_iv_j - v_{j-1}v_j$, $P_n + v_iv_j - v_{j-1}$ or $P_n + v_iv_j - v_n$ is an asymmetric spanning or almost spanning subgraph of G for any $n \geq 7$. The conclusion follows from Lemma 13. □

The assumption $n \geq 7$ is substantial in the above theorem since $D'(K_{3,3}) = 3$.

Claw-free graphs

A $K_{1,3}$-free graph, called also *a claw-free graph*, is a graph containing no copy of $K_{1,3}$ as an induced subgraph. Claw-free graphs have numerous applications, e.g., in operations research and scheduling theory. For a survey of claw-free graphs and their applications consult [6].

A *k-tree* of a connected graph is its spanning tree with the maximum degree k. Win [17] investigated spanning trees in 1-tough graphs and proved the following result.

Theorem 15 *[17] A 2-connected claw-free graph has a 3-tree.*

We use this result to give an upper bound for the distinguishing number of claw-free graphs.

Theorem 16 *If G is a connected claw-free graph, then $D'(G) \leq 3$.*

Proof. Assume first that G is 2-connected. Let T be a 3-tree of G. By Theorem 10 and Theorem 15, we have $D'(T) \leq 2$ if T is neither symmetric nor bisymmetric tree. Hence, $D'(G) \leq 3$ by Lemma13.

Let T be a symmetric tree $T_{h,3}$. Denote a central vertex of T by x and its neighbour by a, b, c. Since G is a claw-free graph, there exists in G at least one edge, say bc, in the neighbourhood of x in T. Define a subgraph $\widetilde{T} = T + ab$. We colour bc, xa and xb with 1, and xc with 2. Thus all vertices a, b, c, x are fixed by every nontrivial automorphisms of $\widetilde{T}$. We now colour the remaining edges in $\widetilde{T}$ starting from the edges incident to a, b, c in such way that two uncoloured adjacent edges obtain two different colours 1 and 2. This colouring breaks all non-trivial automorphisms of $\widetilde{T}$. Hence, $D'(G) \leq 3$ by Lemma 13.

Let T be a bisymmetric tree $T''_{h,3}$. Denote a central edge by xy and its neighbours by a, b, c, d. We colour xy, xa and yc with 1, and xb and yd with 2. Since G is a claw-free graph, there exist in G either at least one of edges by, cx or both ab and cd. We define a subgraph $\widetilde{T}$ obtained from the tree T by adding either one of the edges by, cx or both ab and cd. In the first case we colour by or cx with 1, in the second case we colour ab with 1 and cd with 2. Now all vertices a, b, c, d, x, y are fixed by every nontrivial automorphism of $\widetilde{T}$. We then colour the remaining edges of $\widetilde{T}$ as above, and we obtain the claim.

If a graph G is not 2-connected, then its graph of blocks and cut-vertices is a path, since G is claw-free. We colour every block according to the rules described above. Then to break all non-trivial automorphisms of G, it is enough to ensure that two terminal blocks has no isomorphic colourings. This is possible by exchanging 1 and 2 in a colouring of edges in a neighbourhood of a centrum of a spanning tree of G. $\square$

Planar graphs

First, recall that by a famous Theorem of Tutte [14], every 4-connected planar graph is hamiltonian. Hence its distinguishing index is at most 2, by Theorem 14. A similar result as for claw-free graphs we obtain for 3-connected planar graphs. In the proof, we use the following result of Barnette about spanning trees of such graphs.

Theorem 17 *[3] Every* 3*-connected planar graph has a* 3*-tree.*

Using a similar method as in the proof of Theorem 16, we obtain the following.

Theorem 18 *If G is* 3*-connected planar graph, then* $D'(G) \leq 3$.

Proof. Let T be a 3-tree of G. It follows from Theorem 10 that $D'(T) \leq 2$ and hence, $D'(G) \leq 3$ by Lemma 13, if T is neither a symmetric nor a bisymmetric tree.

Let then T be a symmetric tree $T_{h,3}$. Denote a central vertex by x, and by T_a, T_b and T_c the connected components of $T - x$ which are trees rooted at the neighbours a, b, c of a vertex x, respectively. Since G is 3-connected, there exist an edge e between T_a and T_b in G. Consider a spanning subgraph $\widetilde{T} = T + e$. Then we colour xa and xc with 1, and xb with 2, and extend this colouring as in the proof of Theorem 16 to a colouring of $\widetilde{T}$ breaking by all non-trivial automorphisms of $\widetilde{T}$ (the colour of e is irrelevant). Consequently, $D'(G) \leq 3$ by Lemma 13.

If T is a bisymmetric tree $T''_{h,3}$ with and a central edge xy, then we can add to T one edge in a subtree of $T - xy$ rooted at x, and such a graph can be easily distinguished by two colours. Again, our claim follows from Lemma 13. $\square$

2-connected graphs

For a 2-connected planar graph G, the distinguishing index may attain $1+\left\lceil\sqrt{\Delta(G)}\right\rceil$ as it is shown by the complete bipartite graph $K_{2,q}$ with $q=r^2$ for a positive integer r. In this case, $D'(K_{2,q})=r+1$ as it follows from the result obtained independently by Fisher and Isaak [7] and by Imrich, Jerebic and Klavžar [11]. They proved exactly the following theorem.

Theorem 19 *[7], [11] Let p, q, d be integers such that $d\geq 2$ and $(d-1)^p < q\leq d^p$. Then*

$$D'(K_{p,q})=\begin{cases} d, & \text{if } \ q\leq d^p-\lceil\log_d p\rceil-1,\\ d+1, & \text{if } \ q\geq d^p-\lceil\log_d p\rceil+1.\end{cases}$$

If $q=d^p-\lceil\log_d p\rceil$ then the distinguishing index$D'(K_{p,q})$ is either d or $d+1$ and can be computed recursively in $O(\log^*(q))$ time.

In the next section, we make use of the following immediate corollary.

Corollary 20 *If $p\leq q$, then $D'(K_{p,q})\leq\lceil\sqrt[p]{q}\rceil+1$.* □

Moreover, we prove a useful property of distinguishing 2-colourings of complete bipartite graphs.

Proposition 21 *If $D'(K_{p,q})\leq 2$, then there exists a distinguishing edge 2-colouring such that the edges in one of colours induce a spanning or an almost spanning asymmetric subgraph of $K_{p,q}$.*

Proof. Let P and Q be the two sets of bipartition of $K_{p,q}$, and assume $p\leq q$. If $p=q$, then there exists a spanning asymmetric tree of $K_{p,p}$ (see [12]).

If $p<q$, then to prove the claim it suffices to show the existence of a distinguishing colouring with red and blue, such that at most one vertex in $K_{p,q}$ has no red incident edge. Suppose then that there exist two vertices v and w in P (or both in Q) without any red incident edge. Then a transposition of v and w is a non-trivial automorphism preserving the colouring, a contradiction. Now, let v be a vertex in P without any red incident edge. It is not difficult to observe that even if every vertex in P has a distinct number of incident red edges, then we have $q-p+1$ free numbers of possible red

incident edges. We choose a number i and we colour red i edges between v and edges in Q with largest number of red incident edges. It is not difficult to observe that such a colouring is preserved only by the identity. So we have a spanning or almost spanning red subgraph of $K_{p,q}$. □

Corollary 22 *If a graph G is spanned by $K_{p,q}$ and $D'(K_{p,q}) \leq 2$, then $D'(G) \leq 2$.* □

In general, for 2-connected graphs we conjecture that the complete bipartite graph K_{2,r^2} is the worst case.

Conjecture 23 *If G is a 2-connected graph, then*

$$D'(G) \leq 1 + \left\lceil \sqrt{\Delta(G)} \right\rceil .$$

4 Nordhaus-Gaddum inequalities for D'

In this section, we discuss Conjecture 6, formulated at the end of Introduction, stating that

$$2 \leq D'(G) + D'(\overline{G}) \leq \Delta + 2$$

for every admissible graph G of order $n \geq 7$, where $\Delta = \max\{\Delta(G), \Delta(\overline{G})\}$.

The left-hand inequality is obvious. Indeed, if a graph G is asymmetric, then so is $\overline{G}$. Thus we are only interested in the right-hand inequality $D'(G) + D'(\overline{G}) \leq \Delta + 2$. Note also that at least one of the graphs G and $\overline{G}$ is connected.

The bound $\Delta + 2$ cannot be improved. To see this, consider a star $K_{1,n-1}$ of any order $n \geq 7$. As $\overline{K_{1,n-1}}$ is a disjoint union of a complete graph K_{n-1} and an isolated vertex, it follows from Proposition 9 that $D'(\overline{K_{1,n-1}}) = 2$. Therefore, $D'(K_{1,n-1}) + D'(\overline{K_{1,n-1}}) = n - 1 + 2 = \Delta + 2$.

If T is a tree, then $\Delta(T)$ can be much smaller than $\Delta = \Delta(\overline{T}) = n - 1$. However, the following holds.

Proposition 24 *If T is a tree of order $n \geq 7$, then*

$$D'(T) + D'(\overline{T}) \leq \Delta(T) + 2.$$

Proof. As it was shown above, the conclusion holds for stars. If T is not a star, then $D'(\overline{T}) \leq 2$ by Lemma 13. Indeed, as it was proved by Hedetniemi et al. in [9], every two trees distinct from a star can be packed into K_n. Thus, the complement $\overline{T}$ contains a spanning asymmetric tree. By Theorem 10, we have the inequality $D'(T) + D'(\overline{T}) \leq \Delta(T) + 2$. □

This fact emboldened us to formulate the following stronger conjecture.

Conjecture 25 *Every connected admissible graph G of order $n \geq 7$ satisfies the inequality*

$$D'(G) + D'(\overline{G}) \leq \Delta(G) + 2.$$

Now we show that Conjecture 6 holds not only for trees, but also for some other classes of graphs. To do this we use the following fact.

Theorem 26 *Let G be a connected admissible graph of order $n \geq 7$. If either G or $\overline{G}$ has the distinguishing index at most 3, then*

$$D'(G) + D'(\overline{G}) \leq \Delta + 2,$$

where $\Delta = \max\{\Delta(G), \Delta(\overline{G})\}$.

Proof.

Case A. Let $D'(\overline{G}) \leq 3$.

Then $D'(G) \leq \Delta(G) - 1$, and if $\overline{G}$ is connected, then our claim holds. Assume now that $\overline{G}$ is disconnected. Then G is spanned by $K_{p,q}$ with $p \leq q$ and $\Delta = q$. Suppose that a graph $\overline{G}$ has t isomorphic components. If we had a distinct set of three colours for every component, then $D'(\overline{G}) \leq \lceil \sqrt[3]{6t} \rceil$. We then consider two cases:

- If $q \leq 2^p - \lceil \log_2 p \rceil - 1$, then $D'(G) = 2$ by Theorem 19 and Theorem 22. Moreover, we then have at most $\frac{n}{3}$ connected components of $\overline{G}$, so $D'(\overline{G}) \leq \lceil \sqrt[3]{2n} \rceil$. And we can easily check that

$$\lceil \sqrt[3]{2n} \rceil + 2 \leq \frac{n}{2} + 2$$

for every $n \geq 4$.

- If $q \geq 2^p - \lceil \log_2 p \rceil - 1$, then there exists a big connected component (of order q) in $\overline{G}$ and we can assume that $t \leq \frac{p}{3}$ remaining components are isomorphic ($p \geq 6$). In this case, by assumptions we have $p \leq \lceil \log_2(q+1) \rceil$, therefore

$$D'(\overline{G}) \leq \lceil \sqrt[3]{6t} \rceil \leq \sqrt[3]{2\lceil \log_2(q+1) \rceil}.$$

On the other hand we have $D'(G) \leq \lceil \sqrt[p]{q} \rceil + 2$ by Theorem 20 and Theorem 13. Then it is not difficult to check that

$$\sqrt[3]{2\lceil \log_2(q+1) \rceil} + \lceil \sqrt[p]{q} \rceil + 2 \leq q + 2$$

what finishes the proof.

Case B. Let $D'(G) \leq 3$.

If $\overline{G}$ is connected, then we obtain our claim by Theorem 12. Assume now, that $\overline{G}$ has $t \geq 2$ connected components. Then $\Delta \geq \frac{n}{2}$ and, in the worst case, all connected components of $\overline{G}$ are isomorphic. Observe that the maximal degree of every component is at most $\frac{n}{t} - 1$. If we assign one unique colour to every component, then we need at most $\frac{n}{t} - 1 + (t-1)$ colours to distinguish $\overline{G}$. Hence, if

$$\frac{n}{t} + t \leq \frac{n}{2} - 1,$$

then $D'(\overline{G}) \leq \Delta - 1$, and our claim is true. The above inequality holds unless $t = 2$.

If there exist two isomorphic connected components in $\overline{G}$, then $D'(G) \leq 2$ due to Corollary 22 since G is spanned by $K_{\frac{n}{2},\frac{n}{2}}$. Then $D'(\overline{G}) \leq \frac{n}{2}$, and finally $D'(G) + D'(\overline{G}) \leq \frac{n}{2} + 2$. □

We now can formulate some consequences of Theorem 26 and suitable results proved in Section 3.

Corollary 27 *Let G be a connected admissible graph of order $n \geq 7$. If G satisfies at least one of the following conditions:*

- *traceable graphs,*
- *claw-free graphs,*

- *triangle-free graphs,*
- 3-*connected planar graphs,*

then

$$D'(G) + D'(\overline{G}) \leq \Delta + 2,$$

where $\Delta = \max\{\Delta(G), \Delta(\overline{G})\}$.

Proof. It suffices to apply Theorem 26 together with Theorem 14, Theorem 16 and Theorem 18, respectively. Observe also that if the girth of a graph G is at least 4, i.e., G is triangle-free, then its complement $\overline{G}$ is claw-free. □

Finally, it has to be noted that there exist graphs of order less than 7 such that the right-hand inequality in Conjecture 6 is not satisfied. For example, for the graph $K_{3,3}$ we have $D'(K_{3,3}) = D'(\overline{K_{3,3}}) = 3$ and $\Delta = 3$, hence $D'(K_{3,3}) + D'(\overline{K_{3,3}}) = \Delta + 4$. Also, $D'(C_5) + D'(\overline{C_5}) = 3 + 3 = \Delta + 4$, and $D'(K_{1,i}) + D'(\overline{K_{1,i}}) = \Delta + 3$ for $i = 3, 4, 5$.

References

[1] M. O. Albertson and K. L. Collins, Symmetry breaking in graphs, Electron. J. Combin. 3 (1996), R18.

[2] M. Aouchiche and P. Hansen, A survey of Nordhaus-Gaddum type relations, Discrete Appl. Math. 161 (2013), 466–546.

[3] D. W. Barnette, Trees in polihedral graphs, Can. J. Math. 18 (1966), 731–736.

[4] K. L. Collins and A. N. Trenk, The distinguishing chromatic number, Electron. J. Combin. 13 (2006,) R16.

[5] K. L. Collins and A. N. Trenk, Nordhaus-Gaddum Theorem for the Distinguishing Chromatic Number, Electron. J. Combin. 20 (2013), P46.

[6] R. Faudree, E. Flandrin and Z. Ryjaček, Claw-free graphs – A survey, Discrete Math. 164 (1997), 87-147.

[7] M. J. Fisher and G. Isaak, Distinguishing colorings of Cartesian products of complete graphs, Discrete Math. 308 (2008), 2240-2246.

[8] R. Hammack, W. Imrich and S. Klavžar, Handbook of Product Graphs (Second Edition), Taylor & Francis Group (2011).

[9] S. M. Hedetniemi, S. T. Hedetniemi and P. J. Slater, A note on packing two trees into K_n, Ars Combin. 11 (1981), 149–153.

[10] W. Imrich, R. Kalinowski, F. Lehner and M. Pilśniak, Endomorphism Breaking in Graphs, Electron. J. Combin. 21 (2014), P1.16.

[11] W. Imrich, J. Jerebic and S. Klavžar, The distinguishing number of Cartesian products of complete graphs, European J. Combin. 29 (2008), 922–929.

[12] R. Kalinowski and M. Pilśniak, Distinguishing graphs by edge colourings, European J. Combin. 45 (2015), 124–131.

[13] E. A. Nordhaus and J. W. Gaddum, On Complementary Graphs, Amer. Math. Monthly 63 (1956), 175-177.

[14] W. T. Tutte, A theorem on planar graphs, Trans. Amer. Math. Soc. 82 (1956), 99–116.

[15] V. G. Vizing, The chromatic class of multigraphs, Kibernetika 1 (1965), 29-39.

[16] H. Whitney, Non-separable and planar graphs, Trans. Amer. Math. Soc. 34 (1932), 339-362,

[17] S. Win, On a Connection Between the Existence of k-Trees and Toughness of Graphs, Graphs and Combin. 5 (1989), 201-205,

[18] A. A. Zykov, On Some Properties of Linear Complexes, Math. Sbornik. NS, 24 (1949), 163-188.