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Abstract

Let $E_n = \{x_i + x_j = x_k,\ x_i \cdot x_j = x_k\colon\ i, j, k \in \{1, \ldots, n\}\}$. For each integer $n \geqslant 13$, J. Browkin defined a system $B_n \subseteq E_n$ which has exactly b_n solutions in integers $x_1, \ldots, x_n$, where $b_n \in \mathbb{N} \setminus \{0\}$ and the sequence $\{b_n\}_{n=13}^{\infty}$ rapidly tends to infinity. For each integer $n \geqslant 12$, we define a system $T_n \subseteq E_n$ which has exactly t_n solutions in integers $x_1, \ldots, x_n$, where $t_n \in \mathbb{N} \setminus \{0\}$ and $\lim\limits_{n\to\infty} \dfrac{t_n}{b_n} = \infty$.

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For a non-negative integer n, let $r_k(n)$ denote the number of representations of n as a sum of k squares of integers. Let $E_n = \{x_i + x_j = x_k,\ x_i \cdot x_j = x_k\colon\ i, j, k \in \{1, \ldots, n\}\}$. For an integer $n \geqslant 13$, let B_n denote the following system of equations ([1]):

$$\begin{cases} \forall i \in \{1, \ldots, n-13\}\ x_i \cdot x_i & = & x_{i+1} \\ x_{n-11} + x_{n-11} & = & x_1 \\ x_{n-11} \cdot x_{n-11} & = & x_1 \\ x_{n-10} \cdot x_{n-10} & = & x_{n-9} \\ x_{n-8} \cdot x_{n-8} & = & x_{n-7} \\ x_{n-6} \cdot x_{n-6} & = & x_{n-5} \\ x_{n-4} \cdot x_{n-4} & = & x_{n-3} \\ x_{n-9} + x_{n-7} & = & x_{n-2} \\ x_{n-5} + x_{n-3} & = & x_{n-1} \\ x_{n-2} + x_{n-1} & = & x_n \\ x_{n-11} + x_{n-12} & = & x_n \end{cases}$$

The system B_n is contained in E_n and B_n equivalently expresses that

$$x_1 = \ldots = x_n = 0$$

or

$$(x_{n-11} = 2) \wedge \left(x_n = 2 + 2^{2^{n-12}} = x_{n-10}^2 + x_{n-8}^2 + x_{n-6}^2 + x_{n-4}^2\right) \wedge$$

all the other variables are uniquely determined by the above conjunction and the equations of B_n.

Theorem 1. *([1]) The system B_n has exactly* $1 + r_4\left(2 + 2^{2^{n-12}}\right) = 1 + 8\sigma\left(2 + 2^{2^{n-12}}\right)$ *solutions in integers $x_1, \ldots, x_n$, where $\sigma(\cdot)$ denote the sum of positive divisors.*

For a positive integer $n \geqslant 12$, let T_n denote the following system of equations:

$$\begin{cases} \forall i \in \{1, \ldots, n-12\}\ x_i \cdot x_i &= x_{i+1} \\ x_{n-10} \cdot x_{n-10} &= x_{n-10} \\ x_{n-10} + x_{n-10} &= x_{n-9} \\ x_{n-8} + x_{n-9} &= x_1 \\ x_{n-8} \cdot x_{n-7} &= x_{n-11} \\ x_{n-10} \cdot x_{n-7} &= x_{n-7} \\ x_{n-6} \cdot x_{n-6} &= x_{n-5} \\ x_{n-4} \cdot x_{n-4} &= x_{n-3} \\ x_{n-2} \cdot x_{n-2} &= x_{n-1} \\ x_{n-5} + x_{n-3} &= x_n \\ x_{n-1} + x_n &= x_{n-11} \end{cases}$$

Theorem 2. *The system T_n is contained in E_n and T_n has exactly* $1 + \sum\limits_{k=0}^{2^{n-12}} r_3\left(\left(2 \pm 2^k\right)^{2^{n-12}}\right)$ *solutions in integers $x_1, \ldots, x_n$.*

Proof. The system T_n equivalently expresses that

$$x_1 = \ldots = x_n = 0$$

or

$$(x_{n-10} = 1) \wedge \left((x_1 - 2) \cdot x_{n-7} = x_{n-11} = x_1^{2^{n-12}} = x_{n-6}^2 + x_{n-4}^2 + x_{n-2}^2\right) \wedge$$

all the other variables are uniquely determined by the above conjunction and the equations of T_n.

In the second case, by the polynomial identity

$$x_1^{2^{n-12}} = 2^{2^{n-12}} + (x_1 - 2) \cdot \sum_{k=0}^{2^{n-12}-1} 2^{2^{n-12}-1-k} \cdot x_1^k$$

we obtain that

$$2^{2^{n-12}} = (x_1 - 2) \cdot \left(x_{n-7} - \sum_{k=0}^{2^{n-12}-1} 2^{2^{n-12}-1-k} \cdot x_1^k\right)$$

Hence, $x_1 - 2$ divides $2^{2^{n-12}}$. Therefore, $x_1 \in \left\{2 \pm 2^k \colon k \in \left[0, 2^{n-12}\right] \cap \mathbb{Z}\right\}$. Consequently,

$$x_{n-11} = x_1^{2^{n-12}} \in \left\{\left(2 \pm 2^k\right)^{2^{n-12}} : k \in \left[0, 2^{n-12}\right] \cap \mathbb{Z}\right\}$$

Since the last five equations of T_n equivalently express that $x_{n-11} = x_{n-6}^2 + x_{n-4}^2 + x_{n-2}^2$, the proof is complete. □

The following lemma is a consequence of Siegel's theorem ([4]).

Lemma 1. *([2, p. 119], [3, p. 271]) For every* $\varepsilon \in (0, \infty)$ *there exists* $c(\varepsilon) \in (0, \infty)$ *such that* $r_3\left(4^s \cdot m\right) \geqslant c(\varepsilon) \cdot m^{\frac{1}{2} - \varepsilon}$ *for every non-negative integer s and every positive integer* $m \notin \{4k \colon k \in \mathbb{Z}\} \cup \{8k + 7 \colon k \in \mathbb{Z}\}$.

Let b_n denote the number of integer solutions of B_n, and let t_n denote the number of integer solutions of T_n.

Theorem 3. $\lim\limits_{n\to\infty} \dfrac{t_n}{b_n} = \infty$.

Proof. Let an integer n is greater than 12. By Theorem 2,

$$t_n > r_3\left(\left(2 + 2^{2^{n-12}}\right)^{2^{n-12}}\right) = r_3\left(4^{2^{n-13}} \cdot \left(1 + 2^{2^{n-12}} - 1\right)^{2^{n-12}}\right)$$

By Lemma 1, for each $\varepsilon \in (0, \infty)$ there exists $c(\varepsilon) \in (0, \infty)$ such that

$$r_3\left(4^{2^{n-13}} \cdot \left(1 + 2^{2^{n-12}} - 1\right)^{2^{n-12}}\right) \geqslant c(\varepsilon) \cdot \left(\left(1 + 2^{2^{n-12}} - 1\right)^{2^{n-12}}\right)^{\frac{1}{2} - \varepsilon}$$

for every integer $n > 12$. We take $\varepsilon = \dfrac{1}{4}$. Since $2^{n-12} - 1 \geqslant 2^{n-13}$, we get

$$c(\varepsilon) \cdot \left(\left(1 + 2^{2^{n-12}} - 1\right)^{2^{n-12}}\right)^{\frac{1}{2} - \varepsilon} > c\left(\frac{1}{4}\right) \cdot \left(2^{2^{n-13}}\right)^{2^{n-12} \cdot \frac{1}{4}} = c\left(\frac{1}{4}\right) \cdot 2^{2^{2n-27}}$$

Since $1 < \sigma\left(2 + 2^{2^{n-12}}\right)$ and $2 + 2^{2^{n-12}} < 2^{2^{n-11}}$, Theorem 1 gives:

$$b_n = 1 + 8\sigma\left(2 + 2^{2^{n-12}}\right) < 9\sigma\left(2 + 2^{2^{n-12}}\right) \leqslant 9 \cdot \sum_{k=1}^{2 + 2^{2^{n-12}}} k < 9 \cdot \sum_{k=1}^{2^{2^{n-11}}} k < 9 \cdot 2^{2^{n-11}} \cdot 2^{2^{n-11}} =$$

$$9 \cdot 2^{2^{n-10}} < 2^{2^{n-9}}$$

Therefore, $\dfrac{t_n}{b_n} > \dfrac{c\left(\frac{1}{4}\right) \cdot 2^{2^{2n-27}}}{2^{2^{n-9}}}$. This quotient tends to infinity when n tends to infinity, which completes the proof. ◻

The following *Mathematica* code first computes decimal approximations of $\frac{t_n}{b_n}$ for all integers $n \in [13, 19]$.

```
In[1]:= Clear[b, n, t, k]
        b[n_] := 1 + SquaresR[4, 2 + 2^(2^(n - 12))]
        t[n_] := 1 + Sum[SquaresR[3, (2 + 2^k)^(2^(n - 12))] +
             SquaresR[3, (2 - 2^k)^(2^(n - 12))], {k, 0, 2^(n - 12)}]
        Table[N[t[n] / b[n]], {n, 13, 19}]
        Print[N[SquaresR[3, (2 + (2^256))^256] / b[20]]]
Out[4]= {0.824742, 5.85304, 1.28489 × 10^6, 2.48193 × 10^31,
         5.35896 × 10^139, 4.706882258096019 × 10^587, 1.954869785930356 × 10^2408}

        2.751429528430034 × 10^9748
```

The output results show that $t_n > b_n$ for every integer $n \in [14, 19]$. By Theorem 2,

$$\frac{t_{20}}{b_{20}} > \frac{r_3\left(\left(2 + 2^{2^{20-12}}\right)^{2^{20-12}}\right)}{b_{20}} = \frac{r_3\left(\left(2 + 2^{256}\right)^{256}\right)}{b_{20}}$$

The last command of the code finds the decimal approximation of the last quotient. Hence, $\frac{t_{20}}{b_{20}} > 2.75 \cdot 10^{9748}$. It seems that $t_n > b_n$ for every integer $n \geqslant 21$, although this remains unproven.

Let us define the height of a rational number $\frac{p}{q}$ by $\max(|p|, |q|)$ provided $\frac{p}{q}$ is written in lowest terms. Let us define the height of a rational tuple $(x_1, \ldots, x_n)$ as the maximum of n and the heights of the numbers $x_1, \ldots, x_n$.

For an integer $n \geqslant 4$, let S_n denote the following system of equations:

$$\begin{cases} \forall i \in \{1, \ldots, n-4\} \; x_i \cdot x_i &= x_{i+1} \\ x_{n-2} + 1 &= x_1 \\ x_{n-1} + 1 &= x_{n-2} \\ x_{n-1} \cdot x_n &= x_{n-3} \end{cases}$$

Theorem 4. *([5]) The system S_n is soluble in positive integers and has only finitely many integer solutions. Each integer solution $(x_1, \ldots, x_n)$ satisfies $|x_1|, \ldots, |x_n| \leqslant \left(2 + 2^{2^{n-4}}\right)^{2^{n-4}}$. The following equalities*

$$\begin{aligned} \forall i \in \{1, \ldots, n-3\} \; x_i &= \left(2 + 2^{2^{n-4}}\right)^{2^{i-1}} \\ x_{n-2} &= 1 + 2^{2^{n-4}} \\ x_{n-1} &= 2^{2^{n-4}} \\ x_n &= \left(1 + 2^{2^{n-4}} - 1\right)^{2^{n-4}} \end{aligned}$$

define the unique integer solution whose height is maximal.

Conjecture. *If an integer n is sufficiently large and a system*

$$U \subseteq \{x_i + 1 = x_k,\ x_i \cdot x_j = x_k :\ i, j, k \in \{1, \ldots, n\}\}$$

has only finitely many solutions in positive integers $x_1, \ldots, x_n$*, then each such solution* $(x_1, \ldots, x_n)$ *satisfies* $x_1, \ldots, x_n \leqslant \left(2 + 2^{2^{n-4}}\right)^{2^{n-4}}$.

A bit stronger version of the Conjecture appeared in [5]. The Conjecture implies that there is an algorithm which takes as input a Diophantine equation, returns an integer, and this integer is greater than the heights of integer (non-negative integer, positive integer, rational) solutions, if the solution set is finite ([5]).

Let us pose the following two questions:

Question 1. *Is there an algorithm which takes as input a Diophantine equation, returns an integer, and this integer is greater than the heights of integer solutions, if the solution set is finite?*

Question 2. *Is there an algorithm which takes as input a Diophantine equation, returns an integer, and this integer is greater than the number of integer solutions, if the solution set is finite?*

Obviously, a positive answer to Question 1 implies a positive answer to Question 2.

Lemma 2. *Let d denote the maximal height of an integer solution of a Diophantine equation* $D(x_1, \ldots, x_n) = 0$ *whose solution set in integers is non-empty and finite. We claim that the number of integer solutions to the equation*

$$D^2(x_1, \ldots, x_n) + \left(n^2 + x_1^2 + \ldots + x_n^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2 - v_1^2 - v_2^2 - v_3^2 - v_4^2\right)^2 = 0$$

is finite and greater than d.

Proof. There exists an integer tuple $(a_1, \ldots, a_n)$ such that $D(a_1, \ldots, a_n) = 0$ and $\max(n, |a_1|, \ldots, |a_n|) = d$. The equation $n^2 + a_1^2 + \ldots + a_n^2 = x + y$ has $n^2 + a_1^2 + \ldots + a_n^2 + 1$ solutions in non-negative integers x and y. Since $n^2 + a_1^2 + \ldots + a_n^2 + 1 \geqslant d^2 + 1 > d$, the claim follows from Lagrange's four-square theorem. □

Theorem 5. *A positive answer to Question 2 implies a positive answer to Question 1.*

Proof. In order to compute an upper bound on the heights of integer solutions to a Diophantine equation $D(x_1, \ldots, x_n) = 0$ with a finite number of integer solutions, we compute an upper bound on the number of integer solutions to the equation

$$D^2(x_1, \ldots, x_n) + \left(n^2 + x_1^2 + \ldots + x_n^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2 - v_1^2 - v_2^2 - v_3^2 - v_4^2\right)^2 = 0$$

By Lemma 2, this number is greater than the heights of integer solutions to $D(x_1, \ldots, x_n) = 0$. □

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