A Limit Theorem for the System of Leaky Buckets

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Abstract. The sequence of systems of m leaky buckets and one multiplexer is considered. It turns out that the states of the buffers for tokens and the state of the buffer of the multiplexer satisfy differential equations in the limit as the size of the cells of the data and the value of the tokens tend to zero. The results are compared with the equations proposed by N.U. Ahmed and K.L. Teo.

Keywords: leaky bucket scheme, token buffer, multiplexer, jumping system, continuous system, approximation.

1. Introduction

One of the most important problems in the traffic management of communication networks is to prevent the network from becoming a bottleneck. One of the methods to do this is the so-called leaky bucket scheme (see, for example, [3]).

The basic idea behind this approach is that a cell, before entering the network, must obtain a token from the token buffer (see Fig. 1). An arriving cell will consume one token and immediately depart from the leaky bucket if there is at least one token available in the token buffer. If there is no token available in the token buffer, the cell will be dropped. In the classical leaky bucket scheme it is assumed that tokens are generated at a constant

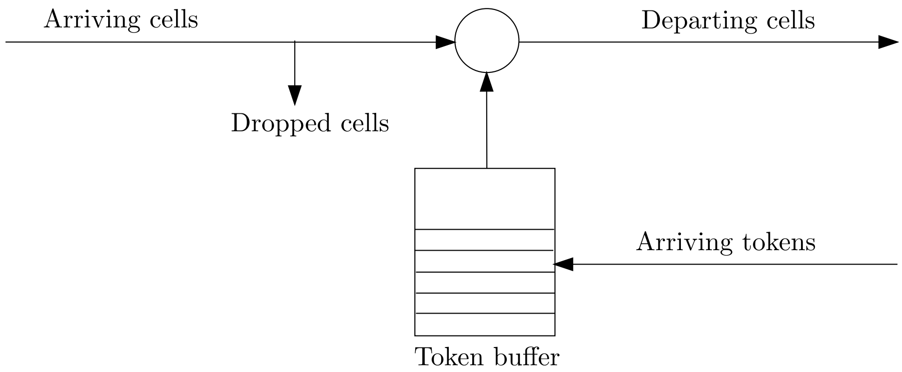

Fig. 1. Classical leaky bucket scheme

rate. There is an upper bound on the number of tokens that can be waiting in the buffer and tokens arriving at a time when the token buffer is full are discarded. The size of the token buffer imposes an upper bound on the burst length and determines the number of cells that can be transmitted, controlling the burst length. The maximum number of cells that can exit the leaky bucket is greater than the buffer, since, while cells arrive and consume tokens, new tokens are generated and placed at the buffer. There are some versions of the leaky bucket scheme (see, for example, [3, 5]).

The stream of cells can be homogenized to the continuous data flux if the cell is sufficiently small with respect to the total amount of data to be transmitted. This condition can be frequently accepted e.g. by the digital voice and video transmission.

It turns out that the flow of the data treated as the fluid and regulated by the leaky bucket scheme can be described by the differential equations, namely, by some type of stochastic differential equations. An attempt to this approach is given by N.U. Ahmed and K.L. Teo [1]. However, the equations given by Ahmed and Teo are introduced using rather an intuitive argument and some arguments indicate that the equations should be modified.

Our purpose is to work out these equations in a rigorous way. We obtain them as a limit of a sequence of difference equations for the discrete leaky bucket scheme as the size of cells and the value of tokens tend to zero.

In Section 2 we introduce basic notions and formulate main results.

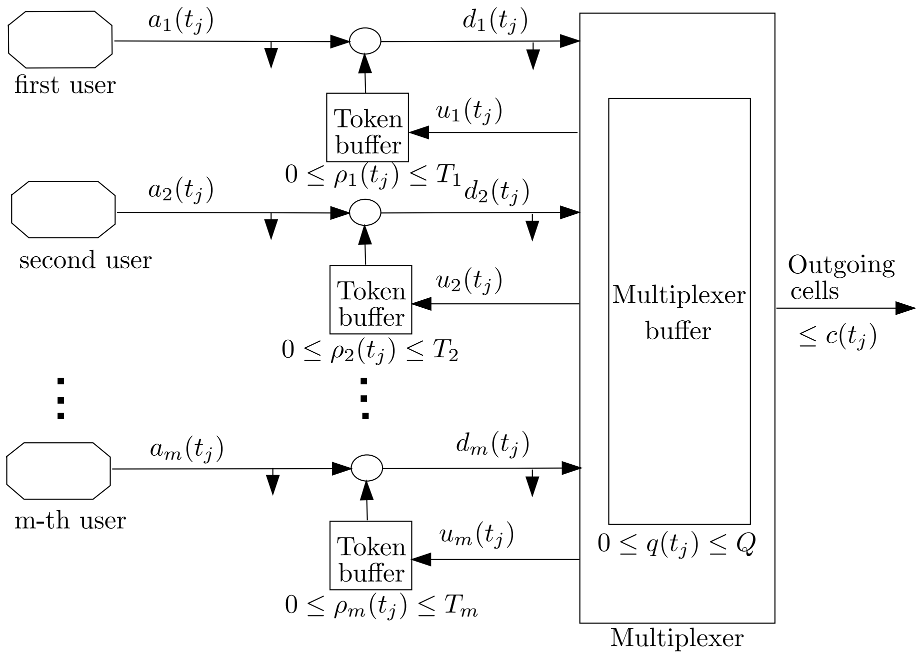

Fig. 2. System of m leaky buckets and one multiplexer

2. Basic notions and main results

In this paper we consider one node (a system of m leaky buckets and one multiplexer) of a communication network. The function of such node is to regulate the flow of the incoming cells before they are launched into the outgoing channel.

Let us consider m users linked to a multiplexer. Moreover, let us assume that each user is controlled by the leaky bucket scheme, that is each user has its token buffer (see Fig. 2).

In the classical leaky bucket scheme the multiplexer sends to each user one token after each unit of time. The tokens are stored in the token buffers of each user. A user can send a cell of data to the multiplexer if the token buffer is not empty and each cell sent by the user causes a reduction of the state of the token buffer by one token. Obviously any user cannot store more tokens than the size of the buffer. It imposes an upper bound on the number cells sent by users in a short interval of time.

To simplify computations as well as to provide wider applicability we generalize this system of leaky buckets. We will assume that the tokens can be sent in any portions and the size of this portion can change from time to time. Also the intervals of time between consecutive moments of arriving of the tokens can change from time to time. The same we will assume about the size of cells and the capacity of the multiplexer to send out cells in a moment of time. The changes of the above three values are independent of each other. Moreover, we will assume that the cell of the size a sent by a user will diminish the state of the token buffer by the same number a.

We introduce the following notations.

- T_r, $r = 1, 2, \ldots, m$, denotes the size of the token buffer of the r-th user.
- $Q > 0$ denotes the size of the multiplexer buffer.
- t_j, $0 < t_j < T$, $j = 1, 2, \ldots, k$, denote the moment when a user sends cells or the multiplexer sends tokens to any user or the multiplexer sends out cells.
- $a_r(t_j) \geq 0$, $r = 1, 2 \ldots, m$, denotes the size of the cell of data sent by the r-th user at time t_j, $j = 1, 2, \ldots, k$.
- $u_r(t_j) \geq 0$, $r = 1, 2 \ldots, m$, denotes the size of the portion of tokens sent by the multiplexer at time t_j, $j = 1, 2, \ldots, k$ to the r-th user.
- $c(t_j) \geq 0$ denotes the amount of data that the multiplexer is able to send out at time t_j, $j = 1, 2, \ldots, k$.
- $\rho_r(t)$, $0 \leq \rho_r(t) \leq T_r$, $r = 1, 2 \ldots, m$, denotes the state of the r-th token buffer at time $t \in [0, T]$.
- $q(t)$, $0 \leq q(t) \leq Q$, $r = 1, 2 \ldots, m$, denotes the state of the multiplexer buffer at time $t \in [0, T]$.

We remind once more that $a_r(t_j)$ and $u_r(t_j)$ need not to be natural numbers.

Obviously the states of the token buffers and the multiplexer buffer are changed at most at the moments t_j, $j = 1, 2, \ldots, k$ and $\rho_r(t)$, $r = 1, 2, \ldots, m$, and $q(t)$ are constant between these moments. We will also assume that $\rho_r(t)$, $r = 1, 2, \ldots, m$, and $q(t)$ are right-hand side continuous.

If tokens are sent by the multiplexer at the same time when a user send cells, the user can use these tokens or not. According to the way of the utilization of tokens we will consider two types of a schedule that describes the behaviour of the system.

If we consider that the user can use tokens sent by the multiplexer at the same time when the user sends cells (the first type of a schedule), then the state of the r-th token buffer is described by the equation

$$\rho_r(t_j) = \rho_r(t_{j-1}) + b\{-\rho_r(t_{j-1}), u_r(t_j) - a_r(t_j), T_r - \rho_r(t_{j-1})\}, \qquad (1)$$

where

$$b\{d, x, u\} = \max\{d, \min\{x, u\}\}.$$

This means that at time t_j the number of tokens in the buffer cannot decrease more than $\rho_r(t_{j-1})$, cannot increase more than $T_r - \rho_r(t_{j-1})$ and changes exactly $u_r(t_j) - u_r(t_j)$ if $-\rho_r(t_{j-1}) \leq u_r(t_j) - a_r(t_j) \leq T_r - \rho_r(t_{j-1})$. It is not excluded that $a_r(t_j) = 0$ or $u_r(t_j) = 0$.

The number of units of information that reaches the multiplexer from the r-th user at time t_j, $j = 1, 2, \ldots, k$, is described by the equation

$$d_r(t_j) = \min\{a_r(t_j), \rho_r(t_{j-1}) + u_r(t_j)\}. \tag{2}$$

If we consider that the user cannot use tokens sent by the multiplexer at the same time when the user sends cells (the second type of a schedule), then the state of the r-th token buffer is described by the equation

$$\rho_r(t_j) = \min\{\max\{0, \rho_r(t_{j-1}) - a_r(t_j)\} + u_r(t_j), T_r\}. \tag{3}$$

In this case the number of units of information that reaches the multiplexer from the r-th user at time t_j, $j = 1, 2, \ldots, m$, is described by the equation

$$d_r(t_j) = \min\{a_r(t_j), \rho_r(t_{j-1})\}. \tag{4}$$

Let $d(t_j)$ denote the number of units of information that reaches the multiplexer from all the users at time t_j, $j = 1, 2, \ldots, k$. In both cases we obviously have

$$d(t_j) = \sum_{r=1}^{m} d_r(t_j). \tag{5}$$

It is likely to be that too many cells reach the multiplexer from a few users at the same time and the multiplexer has to reject some data. Then a problem arises from whom users to accept cells and from which to reject, but it is not essential for our further considerations now.

The state of the multiplexer buffer is described by the equation

$$q(t_j) = q(t_{j-1}) + b\{-q(t_{j-1}), d(t_j) - c(t_j), Q - q(t_{j-1})\}. \tag{6}$$

We will assume that the multiplexer can send out data and store incoming data at the same moment of time even if the buffer is full. Technically, it is possible.

Let us put

$$A_r(t) = \sum_{t_j \leq t} a_r(t_j). \tag{7}$$

$$U_r(t) = \sum_{t_j \leq t} u_r(t_j), \tag{8}$$

$$C(t) = \sum_{t_j \le t} c(t_j) \tag{9}$$

for every $t \in [0, T]$.

It is not difficult to see that $A_r(t)$, $U_r(t)$, $r = 1, 2, \ldots, m$, and $C(t)$ are nonnegative nondecreasing jumping functions, left-hand side discontinuous at most at points t_j, $j = 1, 2, \ldots, k$.

It is easy to see that a system of m leaky bucket, one multiplexer and its behaviour on the interval $[0, T]$ is unambiguously characterized by the sequence

$$(T_r, Q, A_r(t), U_r(t), C(t), r = 1, 2, \ldots, m, \tau),$$

where $A_r(t)$, $U_r(t)$, $r = 1, 2, \ldots$ and $C(t)$ are nondecreasing jumping functions on the interval $[0, T]$, $A_r(0) = 0$, $U_r(0) = 0$, $r = 1, 2, \ldots, m$, and $C(0) = 0$, τ denotes the way of the utilization of tokens.

Definition 1. *The tuple*

$$S = (T_r, Q, A_r(t), U_r(t), C(t), r = 1, 2, \ldots, m, \tau),$$

where $T_r > 0$, $r = 1, 2, \ldots, m$, $Q > 0$, and $A_r(t)$, $U_r(t)$, $r = 1, 2, \ldots$ and $C(t)$ are nondecreasing jumping functions on the interval $[0, T]$, $A_r(0) = 0$, $U_r(0) = 0$, $r = 1, 2, \ldots, m$, and $C(0) = 0$, τ denotes the way of the utilization of tokens, we will call a jumping system of m leaky buckets and one multiplexer.

Let $\mathcal{S}$ denote the set of all jumping systems. In the space $\mathcal{S}$ we define the distance by the following formula

$$\begin{aligned} &\operatorname{dist}(S_1, S_2) = \\ &\quad \max_{r=1,2\ldots,m} |T_r^1 - T_r^2| + |Q^1 - Q^2| + \\ &\quad \max_{r=1,2\ldots,m} \operatorname{dist}(A_r^1, A_r^2) + \max_{r=1,2\ldots,m} \operatorname{dist}(U_r^1, U_r^2) + \operatorname{dist}(C^1, C^2), \end{aligned} \tag{10}$$

where

$$\operatorname{dist}(f_1, f_2) = \sup_{t \in [0,T]} |f_1(t) - f_2(t)|,$$

and f_1, f_2 are bounded functions on the interval $[0, T]$.

It is easy to see that $(\mathcal{S}, \operatorname{dist})$ is a metric space.

We have the following approximation theorem

Theorem 1. *For every $\varepsilon > 0$, for every $M > 0$, for every $L > 0$, there exists $\delta > 0$, such that for each*

$$S_1 = (T_r^1, Q^1, A_r^1(t), U_r^1(t), C^1(t), r = 1, 2, \ldots, m, \tau) \in \mathcal{S},$$

$$S_2 = (T_r^2, Q^2, A_r^2(t), U_r^2(t), C^2(t), r = 1, 2, \ldots, m, \tau) \in \mathcal{S}$$

satisfying the following three conditions

(a)

$$dist(S_1, S_2) \leq \delta,$$

(b)

$$A_r^i(T) \leq M,\ U_r^i(T) \leq M,\ C^i(T) \leq M,$$

$$T_r^i \geq L,\ Q^i \geq L,$$

$$r = 1, 2, \ldots, m,\ i = 1, 2,$$

(c) S_1 *and* S_2 *have the same type of schedule* τ,

we have

$$\sup_{t \in [0,T]} (\rho_r^1(t), \rho_r^2(t)) \leq \varepsilon,\ r = 1, 2, \ldots, m, \tag{11}$$

$$\sup_{t \in [0,T]} (q^1(t), q^2(t)) \leq \varepsilon, \tag{12}$$

whenever

$$|\rho_r^1(0) - \rho_r^2(0)| \leq \delta,\ r = 1, 2, \ldots, m,$$

$$|q^1(0) - q^2(0)| \leq \delta.$$

For the proof of the theorem see [2].

Now we introduce the notion of a continuous system of leaky buckets and a multiplexer, that is we define the system in the case when the data is treated as a fluid.

Let $T_r > 0$, $r = 1, 2, \ldots, m$, and $Q > 0$, similarly as in the jumping systems case, denote the size of the token buffer of the r-th user and the size of the buffer of the multiplexer, respectively.

Let $a_r(t), u_r(t) \in L^1(0,T)$, $r = 1, 2, \ldots, m$, and $c(t) \in L^1(0,T)$ be non-negative functions.

Let us put

$$A_r(t) = \int_0^t a_r(x)\, dx,$$

$$U_r(t) = \int_0^t u_r(x)\, dx,$$

$$C(t) = \int_0^t c(x)\, dx,$$

for every $t \in [0,T]$. Since $a_r(t)$, $u_r(t)$, $r = 1,2,\ldots,m$, and $c(t)$ are measurable and integrable, A_r, U_r, $r = 1,2,\ldots,m$, and C are absolutely continuous.

The functions $A_r(t)$, $U_r(t)$, $r = 1,2,\ldots,m$, and $C(t)$ we can interpret as the amount of data sent in the interval $[0,t]$ of time and functions $a_r(t)$, $u_r(t)$, $r = 1,2,\ldots,m$, and $c(t)$ can be treated as the rate of transmission.

Definition 2. *The tuple*

$$(T_r, Q, A_r(t), U_r(t), C(t), r = 1,2,\ldots,m),$$

where $T_r > 0$, $r = 1,2,\ldots,m$, $Q > 0$, and $A_r(t)$, $U_r(t)$, $r = 1,2,\ldots,m$, $C(t)$ are nondecreasing absolutely continuous functions on the interval $[0,T]$, $A_r(0) = 0$, $U_r(0) = 0$, $r = 1,2,\ldots,m$, and $C(0) = 0$, we will call a continuous system of m leaky buckets and one multiplexer.

Similarly as in the jumping case the functions $A_r(t)$, $U_r(t)$, $r = 1,2,\ldots,m$, $C(t)$ and the constants T_r, $r = 1,2,\ldots,m$, and Q also unambiguously characterize the system of m leaky buckets and a multiplexer.

Denote by $\widetilde{\mathcal{S}}$ the set of all continuous systems. In the space $\widetilde{\mathcal{S}}$ we can introduce distance by the same formula as in the space $\mathcal{S}$, namely by (10). Thus $\widetilde{\mathcal{S}}$ is a metric space and $\mathcal{S} \cup \widetilde{\mathcal{S}}$ is a metric space also.

Let us put

$$[\varphi] = \begin{cases} 1 & \text{if } \varphi \text{ is true} \\ 0 & \text{if } \varphi \text{ is false.} \end{cases}$$

We have the following limit theorem.

Theorem 2. *Let us consider a continuous system*

$$S = (T_r, Q, A_r(t), U_r(t), C(t), r = 1,2,\ldots,m) \in \widetilde{\mathcal{S}}$$

such that $A_r(t)$, $U_r(t)$, $r = 1,2,\ldots,m$, and $C(t)$ are nondecreasing Lipschitzean functions on the interval $[0,T]$, $A_r(0) = 0$, $U_r(0) = 0$, $r = 1,2,\ldots,$ m, and $C(0) = 0$.
Then,

(a) for every $\rho_{r,0} \in [0,T_r]$, $r = 1,2,\ldots,m$, and for every $q_o \in [0,Q]$, there exist unique functions $\rho_r(t)$, $r = 1,2,\ldots,m$, and $q(t)$ such that for every sequence $\{S_n\}$ of jumping systems convergent to S in the metric space $\mathcal{S} \cup \widetilde{\mathcal{S}}$, we have

$$\lim_{n\to\infty} \rho_r^n = \rho_r$$

$$\lim_{n\to\infty} q_r^n = q,$$

whenever $\rho_r^n(0)$ is convergent to $\rho_{r,0}$ and q_0^n is convergent to q_0, *(b)* $\rho_r(t)$*,* $r = 1, 2, \ldots, m$*, and* $q(t)$ *satisfy the following differential equation in the Caratheodory sense*

$$\rho_r'(t) = \begin{cases} 0 & \text{if } \rho_r(t) = 0 \\ u_r(t) - a_r(t) & \text{if } 0 < \rho_r(t) < T_r \\ 0 & \text{if } \rho_r(t) = T_r \end{cases} \tag{13}$$

for $r = 1, 2, \ldots, m$*,*

$$q'(t) = \begin{cases} 0 & \text{if } q(t) = 0 \\ d(t) - c(t) & \text{if } 0 < q(t) < Q, \\ 0 & \text{if } q(t) = Q \end{cases} \tag{14}$$

where

$$d(t) = \sum_{r=1}^{m} \{a_r(t)[\rho_r(t) > 0] + u_r(t)[\rho_r(t) = 0]\},$$

(c) for almost all t *such that* $\rho_r(t) = 0$ *we have*

$$u_r(t) \leq a_r(t),$$

for almost all t *such that* $\rho_r(t) = T$ *we have*

$$u_r(t) \geq a_r(t),$$

for almost all t *such that* $q(t) = 0$ *we have*

$$d(t) \leq c(t)$$

and at last, for almost all t *such that* $q(t) = Q$

$$d(t) \geq c(t).$$

For the definition of differential equations in the Caratheodory sense see, for example, [1, 4].

For the proof of the theorem see [2].

3. Remarks

Remark 1. In fact the functions $A_r(t)$, $U_r(t)$, $r = 1, 2, \ldots, m$, and $C(t)$ in Theorem 2 can be stochastic processes defined on a probability space Ω and on the interval [0,T] of time, namely

$$A_r : \Omega \times [0, T] \mapsto \mathbf{R},\ r = 1, 2, \ldots, m,$$

$$U_r : \Omega \times [0,T] \mapsto \mathbf{R},\ r = 1,2,\ldots,m,$$

$$C : \Omega \times [0,T] \mapsto \mathbf{R}.$$

Without loss of applicability we can assume that for every fixed $\omega \in \Omega$ functions of t $A_r(\omega,\cdot)$, $U_r(\omega,\cdot)$, $r = 1,2,\ldots,m$, and $C(\omega,\cdot)$ are lipschitzean. We can do it because the rate of transmission of data cannot be infinite. Under such assumptions $\rho_r(t)$ and $q(t)$ are also stochastic processes defined on the same space. For every fixed $\omega \in \Omega$ $\rho_r(\omega,\cdot)$ and $q(\omega,\cdot)$ are lipschitzean and satisfy equations (13), (14). Thus these equations can be treated as some kind of stochastic differential equations. These equations give us distributions of random variables $\rho_r(t)$ and $q(t)$ if we know distributions of $A_r(t)$,$U_r(t)$, $r = 1,2,\ldots,m$, and C(t). Problem of finding these distributions in the general situation is open and this will be the subject of further investigations.

REMARK 2. The equations proposed by N.U. Ahmed and K.L. Teo [1] have the form

$$\rho_r'(t) = u_r(t)[\rho_r(t) < T_r] - a_r(t)[\rho_r(t) > 0],\ r = 1,2,\ldots,m, \tag{15}$$

$$q'(t) = -c(t)[q(t) > 0] + \left(\sum_{r=1}^{m} a_r(t)[\rho_r(t) > 0]\right)[q(t) < Q], \tag{16}$$

that is

$$\rho_r'(t) = \begin{cases} u_r(t) & \text{if } \rho_r(t) = 0 \\ u_r(t) - a_r(t) & \text{if } 0 < \rho_r(t) < T_r \\ -a_r(t) & \text{if } \rho_r(t) = T_r \end{cases}$$

for $r = 1,2,\ldots,m$, and

$$q'(t) = \begin{cases} \sum_{r=1}^{m} a_r(t)[\rho_r(t) > 0] & \text{if } q(t) = 0 \\ -c(t) + \sum_{r=1}^{m} a_r(t)[\rho_r(t) > 0] & \text{if } 0 < q(t) < Q \\ -c(t) & \text{if } q(t) = Q. \end{cases}$$

The essential difference between equation (13), (14) and (15), (16) is in the value $\rho_r'(t)$ and $q'(t)$ if $\rho_r(t) = 0$ or $\rho_r(t) = T_r$ and $q(t) = 0$ or $q(t) = Q$ respectively. It seems, the difference is because N.U. Ahmed and K.L. Teo do not take into account that in the continuous case $\rho_r(t) = 0$ does not mean as in the discrete case that the buffer is empty and the transmission is possible. Maybe the quality of transmission is low and the lower it is the lesser is the fraction $u_r(t)/a_r(t)$.

Remark 3. In the classical leaky bucket scheme it is assumed that tokens are generated at a constant rate, one token after every unit of time. In the continuous case, it corresponds to the function $u(t) = \text{const}$. In practice $a_r(t)$ is piecewise constant. It is apparent if the video or voice is transmitted. In this case $A_r(t)$ and $C(t)$ are piecewise linear and the equations (13), (14) describe the behaviour of the system of m leaky buckets and a multiplexer in a satisfactory way.

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5. References

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