

Multigenerative Grammar Systems

ALEXANDER MEDUNA¹ AND ROMAN LUKAS²

Brno University of Technology, Department of Information Systems

Faculty of Information Technology

Bozotechnova 2, Brno 61266, Czech Republic

e-mail: ¹meduna@fit.vutbr.cz, ²lukas@fit.vutbr.cz

Abstract. This paper presents new models for all recursive enumerable languages. These models are based on multigenerative grammar systems that simultaneously generate several strings in a parallel way. The components of these models are context-free grammars, working in a leftmost way. The rewritten nonterminals are determined by a finite set of nonterminal sequences.

Keywords: grammar system, context-free grammar, leftmost derivation.

1. Introduction

The formal language theory has recently intensively investigated various grammar systems (see [1, 2, 8]), which consist of several cooperating components, usually represented by grammars. Although this variant is extremely broad, all these grammar systems always make a derivation that generates a single string. In this paper, however, we introduce grammar systems that simultaneously generate several strings, which are subsequently composed into a single string by some common string operation, such as concatenation.

More precisely, for a positive integer n , an n -multigenerative grammar system discussed in this paper works with n context-free grammatical components in a leftmost way – that is, in every derivation step, each of these components rewrites the leftmost

nonterminal occurring in its current sentential form. These n leftmost derivations are controlled n -tuples of nonterminals or rules. Under a control like this, the grammar system generates n strings, out of which the strings that belong to the generated language are made by some basic operations. Specifically, these operations include union, concatenation and a selection of the string generated by the first component.

In this paper, we prove that all the multigenerative grammar systems under discussion characterize the family of recursively enumerable languages. Besides this fundamental result, we give several transformation algorithms of these multigenerative grammar systems.

2. Preliminaries

This paper assumes that the reader is familiar with the formal language theory (see [4]). For a set, Q , $card(Q)$ denotes the cardinality of Q . For an alphabet, V , V^* represents the free monoid generated by V under the operation of concatenation. The unit of V^* is denoted by ε . Set $V^+ = V^* - \{\varepsilon\}$; algebraically, V^+ is thus the free semigroup generated by V under the operation of concatenation. For every $w \in V^*$, $|w|$ denotes the length of w . Furthermore, for every $0 \leq i \leq |w|$ and $L \subseteq V^*$, we introduce the following denotation: $sym(w, i)$ denotes the i -th symbol in w .

A *context-free grammar* is a quadruple, $G = (N, T, P, S)$, where N and T are two disjoint alphabets. Symbols in N and T are referred to as *nonterminals* and *terminals*, respectively, and $S \in N$ is the *start symbol* of G . P is a finite set of *rules* of the form $A \rightarrow x$, where $A \in N$ and $x \in (N \cup T)^*$. To declare that a label r denotes the rule, this is written as $r: A \rightarrow x$. The nonterminal A is the left-hand side of r , denoted by $lhs(r)$. The string x is the right-hand side of r , denoted by $rhs(r)$. Let $u, v \in (N \cup T)^*$. For every $A \rightarrow x \in P$, write $uAv \Rightarrow uxv$. Let \Rightarrow^* denote the transitive-reflexive closure of \Rightarrow . The *language of G* , $L(G)$, is defined as

$$L(G) = \{w: S \Rightarrow^* w \text{ in } G, \text{ for some } w \in T^*\}.$$

3. Definitions

In this section, we define the fundamental notions of this paper. More specifically, we introduce two types of multigenerative grammar systems and the languages they generate in three different modes.

DEFINITION 1. An *n-multigenerative nonterminal-synchronized grammar system* (n -

MGN) is an $n+1$ tuple,

$$\mathbf{G} = (G_1, G_2, \dots, G_n, Q),$$

where $G_i = (N_i, T_i, P_i, S_i)$ is a context-free grammar for each $i = 1, \dots, n$, and Q is a finite set of n -tuples of the form (A_1, A_2, \dots, A_n) , where $A_i \hat{\mathbf{I}} N_i$ for all $i = 1, \dots, n$. Let $\mathbf{G} = (G_1, G_2, \dots, G_n, Q)$ be an n -MGN. Then, a sentential n -form of n -MGN is an n -tuple of the form $\mathbf{c} = (x_1, x_2, \dots, x_n)$, where $x_i \hat{\mathbf{I}} (N_i \tilde{\mathbf{E}} T_i)^*$ for all $i = 1, \dots, n$. Let $\mathbf{c} = (u_1 A_1 v_1, u_2 A_2 v_2, \dots, u_n A_n v_n)$ and $\mathbf{?} = (u_1 x_1 v_1, u_2 x_2 v_2, \dots, u_n x_n v_n)$ be two sentential n -forms, where $A_i \hat{\mathbf{I}} N_i$, $u_i \hat{\mathbf{I}} T_i^*$, and $v_i, x_i \hat{\mathbf{I}} (N_i \tilde{\mathbf{E}} T_i)^*$ for all $i = 1, \dots, n$. Let $A_i \otimes x_i \hat{\mathbf{I}} P_i$ for all $i = 1, \dots, n$ and $(A_1, A_2, \dots, A_n) \hat{\mathbf{I}} Q$. Then \mathbf{c} directly derives $\mathbf{?}$ in \mathbf{G} , denoted by $\mathbf{c} \mathbf{P} \mathbf{?}$. In the standard way, we generalize \mathbf{P} to $\mathbf{P}^k, k \geq 0, \mathbf{P}^+,$ and \mathbf{P}^* .

The n -language of \mathbf{G} , $n\text{-}L(\mathbf{G})$, is defined as

$$n\text{-}L(\mathbf{G}) = \{(w_1, w_2, \dots, w_n) : (S_1, S_2, \dots, S_n) \mathbf{P}^* (w_1, w_2, \dots, w_n), \\ w_i \hat{\mathbf{I}} T_i^* \text{ for all } i = 1, \dots, n\}.$$

The language generated by \mathbf{G} in the union mode, $L_{\text{union}}(\mathbf{G})$, is defined as

$$L_{\text{union}}(\mathbf{G}) = \{w : (w_1, w_2, \dots, w_n) \hat{\mathbf{I}} n\text{-}L(\mathbf{G}), w \hat{\mathbf{I}} \{w_i : i = 1, \dots, n\}\}.$$

The language generated by \mathbf{G} in the concatenation mode, $L_{\text{conc}}(\mathbf{G})$, is defined as

$$L_{\text{conc}}(\mathbf{G}) = \{w_1 w_2 \dots w_n : (w_1, w_2, \dots, w_n) \hat{\mathbf{I}} n\text{-}L(\mathbf{G})\}.$$

The language generated by \mathbf{G} in the first mode, $L_{\text{first}}(\mathbf{G})$, is defined as

$$L_{\text{first}}(\mathbf{G}) = \{w_1 : (w_1, w_2, \dots, w_n) \hat{\mathbf{I}} n\text{-}L(\mathbf{G})\}.$$

EXAMPLE 1. $\mathbf{G} = (G_1, G_2, Q)$, where

$$G_1 = (\{S_1, A_1\}, \{a, b, c\}, \{S_1 \rightarrow aS_1, S_1 \rightarrow aA_1, A_1 \rightarrow bA_1c, A_1 \rightarrow bc\}, S_1),$$

$$G_2 = (\{S_2, A_2\}, \{d\}, \{S_2 \rightarrow S_2A_2, S_2 \rightarrow A_2A_2 \rightarrow d\}, S_2), Q = \{(S_1, A_1), (S_2, A_2)\}$$

is a 2-multigenerative nonterminal-synchronized grammar system. Notice that $2\text{-}L(\mathbf{G}) = \{(a^n b^n c^n, d^n) : n \geq 1\}$, $L_{\text{union}}(\mathbf{G}) = \{a^n b^n c^n : n \geq 1\} \cup \{d^n : n \geq 1\}$,

$$L_{\text{conc}}(\mathbf{G}) = \{a^n b^n c^n d^n : n \geq 1\}, \text{ and } L_{\text{first}}(\mathbf{G}) = \{a^n b^n c^n : n \geq 1\}.$$

DEFINITION 2. An n -multigenerative rule-synchronized grammar system (n -MGR) is $n+1$ tuple

$$\mathbf{G} = (G_1, G_2, \dots, G_n, Q),$$

where $G_i = (N_i, T_i, P_i, S_i)$ is a context-free grammar for each $i = 1, \dots, n$, and Q is a finite set of n -tuples of the form (p_1, p_2, \dots, p_n) , where $p_i \hat{\mathbf{I}} P_i$ for all $i = 1, \dots, n$. A sentential n -form for n -MGR is defined as the sentential n -form for an n -

MGN. Let $\mathbf{G} = (G_1, G_2, \dots, G_n, Q)$ be an n -MGR.

Let $\mathbf{c} = (u_1A_1v_1, u_2A_2v_2, \dots, u_nA_nv_n)$ and $\mathbf{?} = (u_1x_1v_1, u_2x_2v_2, \dots, u_nx_nv_n)$ be two sentential n -forms, where $A_i \hat{\mathbf{I}} N_i$, $u_i \hat{\mathbf{I}} T_i^*$, and $v_i, x_i \hat{\mathbf{I}} (N_i \hat{\mathbf{E}} T_i)^*$ for all $i = 1, \dots, n$. Let $p_i: A_i \hat{\mathbf{O}} x_i \hat{\mathbf{I}} P_i$ for all $i = 1, \dots, n$ and $(p_1, p_2, \dots, p_n) \hat{\mathbf{I}} Q$. Then \mathbf{c} directly derives $\mathbf{?}$ in \mathbf{G} , denoted by $\mathbf{c} \hat{\mathbf{P}} \mathbf{?}$. An n -language for n -MGR is defined as the n -language for n -MGN, and a language generated by n -MGR in the X mode for each $X \hat{\mathbf{I}} \{\text{union, conc, first}\}$, is defined as the language generated by n -MGN in the X mode.

EXAMPLE 2. $\mathbf{G} = (G_1, G_2, Q)$, where

$$G_1 = (\{S_1, A_1\}, \{a, b, c\}, \{\mathbf{1}: S_1 \rightarrow aS_1, \mathbf{2}: S_1 \rightarrow aA_1, \mathbf{3}: A_1 \rightarrow bA_1c, \mathbf{4}: A_1 \rightarrow bc\}, S_1),$$

$$G_2 = (\{S_2\}, \{d\}, \{\mathbf{1}: S_2 \rightarrow S_2S_2, \mathbf{2}: S_2 \rightarrow S_2, \mathbf{3}: S_2 \rightarrow d\}, S_2),$$

$$Q = \{(\mathbf{1}, \mathbf{1}), (\mathbf{2}, \mathbf{2}), (\mathbf{3}, \mathbf{3}), (\mathbf{4}, \mathbf{3})\}$$

is 2-multigenerative rule-synchronized grammar system. Notice that

$$2-L(\mathbf{G}) = \{(a^n b^n c^n, d^n): n \geq 1\}, \quad L_{\text{union}}(\mathbf{G}) = \{a^n b^n c^n: n \geq 1\} \cup \{d^n: n \geq 1\},$$

$$L_{\text{conc}}(\mathbf{G}) = \{a^n b^n c^n d^n: n \geq 1\}, \quad \text{and} \quad L_{\text{first}}(\mathbf{G}) = \{a^n b^n c^n: n \geq 1\}.$$

4. Results

CLAIM 1. Let Γ be any n -MGN, let $\bar{\Gamma}$ be any n -MGR and let $n-L(\Gamma) = n-L(\bar{\Gamma})$. Then, $L_X(\Gamma) = L_X(\bar{\Gamma})$, for each $X \hat{\mathbf{I}} \{\text{union, conc, first}\}$.

PROOF:

I. We prove that

$$L_{\text{union}}(\Gamma) = L_{\text{union}}(\bar{\Gamma}): L_{\text{union}}(\Gamma) = \{w: (w_1, w_2, \dots, w_n) \in n-L(\Gamma), w \in \{w_i: i = 1, \dots, n\}\}$$

$$= \{w: (w_1, w_2, \dots, w_n) \in n-L(\bar{\Gamma}), w \in \{w_i: i = 1, \dots, n\}\} = L_{\text{union}}(\bar{\Gamma}).$$

II. We prove that

$$L_{\text{conc}}(\Gamma) = L_{\text{conc}}(\bar{\Gamma}): L_{\text{conc}}(\Gamma) = \{w_1 w_2 \dots w_n: (w_1, w_2, \dots, w_n) \in n-L(\Gamma)\}$$

$$= \{w_1 w_2 \dots w_n: (w_1, w_2, \dots, w_n) \in n-L(\bar{\Gamma})\} = L_{\text{conc}}(\bar{\Gamma}).$$

III. We prove that

$$L_{\text{first}}(\Gamma) = L_{\text{first}}(\bar{\Gamma}): L_{\text{first}}(\Gamma) = \{w_1: (w_1, w_2, \dots, w_n) \in n-L(\Gamma)\}$$

$$= \{w_1: (w_1, w_2, \dots, w_n) \in n-L(\bar{\Gamma})\} = L_{\text{first}}(\bar{\Gamma}).$$

ALGORITHM 1. Conversion of n -MGN to n -MGR

INPUT: n -MGN $\Gamma = (G_1, G_2, \dots, G_n, Q)$

OUTPUT: n -MGR $\bar{\Gamma} = (G_1, G_2, \dots, G_n, \bar{Q})$; n - $L(\Gamma) = n$ - $L(\bar{\Gamma})$

METHOD:

Let $G_i = (N_i, T_i, P_i, S_i)$ for all $i = 1, \dots, n$, then:

$$\bar{Q} := \{(A_1 \rightarrow x_1, A_2 \rightarrow x_2, \dots, A_n \rightarrow x_n) : A_i \rightarrow x_i \in P_i \text{ for all } i = 1, \dots, n, \text{ and} \\ (A_1, A_2, \dots, A_n) \in Q\}.$$

THEOREM 1. Let $\Gamma = (G_1, G_2, \dots, G_n, Q)$ be a n -MGN. With Γ as its input, Algorithm 1 halts and correctly construct a n -MGR $\bar{\Gamma} = (G_1, G_2, \dots, G_n, \bar{Q})$, such that n - $L(\Gamma) = n$ - $L(\bar{\Gamma})$, and $L_X(\Gamma) = L_X(\bar{\Gamma})$, for each $X \in \{\text{union, conc, first}\}$.

PROOF:

CLAIM A: Let $(S_1, S_2, \dots, S_n) \Rightarrow^m (y_1, y_2, \dots, y_n)$ in Γ , where $m \geq 0$,

$y_i \in (N_i \cup T_i)^*$ for all $i = 1, \dots, n$. Then, $(S_1, S_2, \dots, S_n) \Rightarrow^m (y_1, y_2, \dots, y_n)$ in $\bar{\Gamma}$.

PROOF: This claim is proved by induction:

Basic: Let $m = 0$. Then $(S_1, S_2, \dots, S_n) \Rightarrow^0 (S_1, S_2, \dots, S_n)$ in Γ . Observe that

$(S_1, S_2, \dots, S_n) \Rightarrow^0 (S_1, S_2, \dots, S_n)$ in $\bar{\Gamma}$, so the claim holds.

Induction hypothesis: Assume that Claim A holds for all m -step derivations, where $m = 0, \dots, k$, for some $k \geq 0$.

Induction step: Consider $(S_1, S_2, \dots, S_n) \Rightarrow^{k+1} (y_1, y_2, \dots, y_n)$ in Γ . Then, there exists a sentential n -form $(u_1 A_1 v_1, u_2 A_2 v_2, \dots, u_n A_n v_n)$, where $u_i \in T_i^*$, $A_i \in N_i$, $v_i \in (N_i \cup T_i)^*$ such that $(S_1, S_2, \dots, S_n) \Rightarrow^k (u_1 A_1 v_1, u_2 A_2 v_2, \dots, u_n A_n v_n) \Rightarrow (u_1 x_1 v_1, u_2 x_2 v_2, \dots, u_n x_n v_n)$ in Γ , where $u_i x_i v_i = y_i$ for all $i = 1, \dots, n$.

I. $(S_1, S_2, \dots, S_n) \Rightarrow^k (u_1 A_1 v_1, u_2 A_2 v_2, \dots, u_n A_n v_n)$ in Γ implies $(S_1, S_2, \dots, S_n) \Rightarrow^k (u_1 A_1 v_1, u_2 A_2 v_2, \dots, u_n A_n v_n)$ in $\bar{\Gamma}$, by the induction hypothesis.

II. Let $(u_1 A_1 v_1, u_2 A_2 v_2, \dots, u_n A_n v_n) \Rightarrow (u_1 x_1 v_1, u_2 x_2 v_2, \dots, u_n x_n v_n)$ in Γ . Then, there holds in Γ $(A_1, A_2, \dots, A_n) \in Q$ and $A_i \rightarrow x_i \in P_i$ for all $i = 1, \dots, n$.

Algorithm 1 implies that $(A_1 \rightarrow x_1, A_2 \rightarrow x_2, \dots, A_n \rightarrow x_n) \in \bar{Q}$ in $\bar{\Gamma}$, so

$(u_1 A_1 v_1, u_2 A_2 v_2, \dots, u_n A_n v_n) \Rightarrow (u_1 x_1 v_1, u_2 x_2 v_2, \dots, u_n x_n v_n)$ in $\bar{\Gamma}$.

I. and II. imply that $(S_1, S_2, \dots, S_n) \Rightarrow^{k+1} (u_1 x_1 v_1, u_2 x_2 v_2, \dots, u_n x_n v_n)$ in $\bar{\Gamma}$.

CLAIM B: Let $(S_1, S_2, \dots, S_n) \Rightarrow^m (y_1, y_2, \dots, y_n)$ in $\bar{\Gamma}$, where $m \geq 0$, $y_i \in (N_i \cup T_i)^*$ for all $i = 1, \dots, n$. Then, $(S_1, S_2, \dots, S_n) \Rightarrow^m (y_1, y_2, \dots, y_n)$ in Γ .

PROOF: This claim is proved by induction:

Basic: Let $m = 0$. Then $(S_1, S_2, \dots, S_n) \Rightarrow^0 (S_1, S_2, \dots, S_n)$ in $\overline{\Gamma}$. Observe that

$(S_1, S_2, \dots, S_n) \Rightarrow^0 (S_1, S_2, \dots, S_n)$ in Γ , so the claim holds.

Induction hypothesis: Assume that Claim B holds for all m -step derivations, where $m = 0, \dots, k$, for some $k \geq 0$.

Induction step: Consider $(S_1, S_2, \dots, S_n) \Rightarrow^{k+1} (y_1, y_2, \dots, y_n)$ in $\overline{\Gamma}$. Then, there exists a sentential n -form $(u_1A_1v_1, u_2A_2v_2, \dots, u_nA_nv_n)$, where $u_i \in T_i^*$, $A_i \in N_i$,

$v_i \in (N_i \cup T_i)^*$ such that $(S_1, S_2, \dots, S_n) \Rightarrow^k (u_1A_1v_1, u_2A_2v_2, \dots, u_nA_nv_n) \Rightarrow (u_1x_1v_1, u_2x_2v_2, \dots, u_nx_nv_n)$ in $\overline{\Gamma}$, where $u_ix_iv_i = y_i$ for all $i = 1, \dots, n$.

I. $(S_1, S_2, \dots, S_n) \Rightarrow^k (u_1A_1v_1, u_2A_2v_2, \dots, u_nA_nv_n)$ in $\overline{\Gamma}$ implies $(S_1, S_2, \dots, S_n) \Rightarrow^k (u_1A_1v_1, u_2A_2v_2, \dots, u_nA_nv_n)$ in Γ , by the induction hypothesis.

II. Let $(u_1A_1v_1, u_2A_2v_2, \dots, u_nA_nv_n) \Rightarrow (u_1x_1v_1, u_2x_2v_2, \dots, u_nx_nv_n)$ in $\overline{\Gamma}$. Then, there exists $(A_1 \rightarrow x_1, A_2 \rightarrow x_2, \dots, A_n \rightarrow x_n) \in \overline{Q}$ in $\overline{\Gamma}$. Algorithm 1 implies that $(A_1, A_2, \dots, A_n) \in Q$ and $A_i \rightarrow x_i \in P_i$ for all $i = 1, \dots, n$, so $(u_1A_1v_1, u_2A_2v_2, \dots, u_nA_nv_n) \Rightarrow (u_1x_1v_1, u_2x_2v_2, \dots, u_nx_nv_n)$ in Γ .

I. and II. imply that $(S_1, S_2, \dots, S_n) \Rightarrow^{k+1} (u_1x_1v_1, u_2x_2v_2, \dots, u_nx_nv_n)$ in Γ .

Consider Claim A with $m \geq 0$, $y_i \in T_i^*$ for all $i = 1, \dots, n$. At this point, if $(S_1, S_2, \dots, S_n) \Rightarrow^* (y_1, y_2, \dots, y_n)$ in Γ , then $(S_1, S_2, \dots, S_n) \Rightarrow^* (y_1, y_2, \dots, y_n)$ in $\overline{\Gamma}$. Hence, $n-L(\Gamma) \subseteq n-L(\overline{\Gamma})$. Consider Claim B with $m \geq 0$, $y_i \in T_i^*$ for all $i = 1, \dots, n$. At this point, if $(S_1, S_2, \dots, S_n) \Rightarrow^* (y_1, y_2, \dots, y_n)$ in $\overline{\Gamma}$, then $(S_1, S_2, \dots, S_n) \Rightarrow^* (y_1, y_2, \dots, y_n)$ in Γ . Hence, $n-L(\overline{\Gamma}) \subseteq n-L(\Gamma)$. $n-L(\Gamma) \subseteq n-L(\overline{\Gamma})$ and $n-L(\overline{\Gamma}) \subseteq n-L(\Gamma)$ imply $n-L(\Gamma) = n-L(\overline{\Gamma})$. This equality implies by the Claim 1 $L_X(\Gamma) = L_X(\overline{\Gamma})$, for each $X \in \{\text{union, conc, first}\}$.

ALGORITHM 2. Conversion of n-MGR to n-MGN

INPUT: n-MGR $\Gamma = (G_1, G_2, \dots, G_n, Q)$

OUTPUT: n-MGN $\overline{\Gamma} = (\overline{G}_1, \overline{G}_2, \dots, \overline{G}_n, \overline{Q})$; $n-L(\Gamma) = n-L(\overline{\Gamma})$

METHOD:

Let $G_i = (N_i, T_i, P_i, S_i)$ for all $i = 1, \dots, n$, then:

$\overline{G}_i = (\overline{N}_i, T_i, \overline{P}_i, S_i)$ for all $i = 1, \dots, n$, where:

$$\overline{N}_i := \{\langle A, x \rangle : A \rightarrow x \in P_i\} \cup \{S_i\},$$

$$\overline{P}_i := \{\langle A, x \rangle \rightarrow y : A \rightarrow x \in P_i, y \in \mathbf{t}_i(x)\} \cup \{S_i \rightarrow y : y \in \mathbf{t}_i(S_i)\},$$

where \mathbf{t}_i is a substitution from $N_i \cup T_i$ to $\overline{N}_i \cup T_i$ defined as:

$\mathbf{t}_i(a) = \{a\}$ for all $a \in T_i$; $\mathbf{t}_i(A) = \{ \langle A, x \rangle : A \rightarrow x \in P_i \}$ for all $A \in N_i$.

$$\overline{Q} := \{ \langle \langle A_1, x_1 \rangle, \langle A_2, x_2 \rangle, \dots, \langle A_n, x_n \rangle \rangle : (A_1 \rightarrow x_1, A_2 \rightarrow x_2, \dots, A_n \rightarrow x_n) \in Q \} \cup \{ (S_1, S_2, \dots, S_n) \}.$$

THEOREM 2. Let $\Gamma = (G_1, G_2, \dots, G_n, Q)$ be a n -MGR. With Γ as its input, Algorithm 2 halts and correctly construct a n -MGN $\overline{\Gamma} = (\overline{G}_1, \overline{G}_2, \dots, \overline{G}_n, \overline{Q})$, such that $L_X(\Gamma) = L_X(\overline{\Gamma})$, where $X \in \{\text{union, conc, first}\}$.

PROOF:

CLAIM A: Let $(S_1, S_2, \dots, S_n) \Rightarrow^m (z_1, z_2, \dots, z_n)$ in Γ , where $m \geq 0$, $z_i \in (N_i \cup T_i)^*$ for all $i = 1, \dots, n$. Then, $(S_1, S_2, \dots, S_n) \Rightarrow^{m+1} (\overline{z}_1, \overline{z}_2, \dots, \overline{z}_n)$ in $\overline{\Gamma}$ for any $\overline{z}_i \in \mathbf{t}_i(z_i)$.

PROOF: This claim is proved by induction:

Basic: Let $m = 0$. Then, $(S_1, S_2, \dots, S_n) \Rightarrow^0 (S_1, S_2, \dots, S_n)$ in Γ . Observe that $(S_1, S_2, \dots, S_n) \Rightarrow^1 (\overline{z}_1, \overline{z}_2, \dots, \overline{z}_n)$ in $\overline{\Gamma}$ for any $\overline{z}_i \in \mathbf{t}_i(S_i)$, because Algorithm 2 implies $(S_1, S_2, \dots, S_n) \in \overline{Q}$ and $S_i \rightarrow \overline{z}_i \in \overline{P}_i$, for any $\overline{z}_i \in \mathbf{t}_i(S_i)$ for all $i = 1, \dots, n$, so the claim holds.

Induction hypothesis: Assume that Claim A holds for all m -step derivations, where $m = 0, \dots, k$, for some $k \geq 0$.

Induction step: Consider $(S_1, S_2, \dots, S_n) \Rightarrow^{k+1} (y_1, y_2, \dots, y_n)$ in Γ . Then, there exists a sentential n -form $(u_1 A_1 v_1, u_2 A_2 v_2, \dots, u_n A_n v_n)$ in Γ , where $u_i \in T_i^*$,

$A_i \in N_i$, $v_i \in (N_i \cup T_i)^*$ such that $(S_1, S_2, \dots, S_n) \Rightarrow^k (u_1 A_1 v_1, u_2 A_2 v_2, \dots, u_n A_n v_n) \Rightarrow (u_1 x_1 v_1, u_2 x_2 v_2, \dots, u_n x_n v_n)$, where $u_i x_i v_i = y_i$ for all $i = 1, \dots, n$.

I. $(S_1, S_2, \dots, S_n) \Rightarrow^k (u_1 A_1 v_1, u_2 A_2 v_2, \dots, u_n A_n v_n)$ in Γ implies $(S_1, S_2, \dots, S_n) \Rightarrow^{k+1} (\overline{w}_1, \overline{w}_2, \dots, \overline{w}_n)$ in $\overline{\Gamma}$ for any $\overline{w}_i \in \mathbf{t}_i(u_i A_i v_i)$ for all $i = 1, \dots, n$, by the induction hypothesis.

II. Let $(u_1 A_1 v_1, u_2 A_2 v_2, \dots, u_n A_n v_n) \Rightarrow (u_1 x_1 v_1, u_2 x_2 v_2, \dots, u_n x_n v_n)$ in Γ . Then, there holds $(A_1 \rightarrow x_1, A_2 \rightarrow x_2, \dots, A_n \rightarrow x_n) \in Q$ in Γ . Algorithm 2 implies that $\langle \langle A_1, x_1 \rangle, \langle A_2, x_2 \rangle, \dots, \langle A_n, x_n \rangle \rangle \in \overline{Q}$ in $\overline{\Gamma}$ and $\langle A_i, x_i \rangle \rightarrow \overline{y}_i \in \overline{P}_i$ for any $\overline{y}_i \in \mathbf{t}_i(x_i)$ for all $i = 1, \dots, n$. Let \overline{w}_i be any sentential form of the form $\overline{u}_i \langle A_i, x_i \rangle \overline{v}_i$ for all $i = 1, \dots, n$, where $\overline{u}_i \in \mathbf{t}_i(u_i)$, $\overline{v}_i \in \mathbf{t}_i(v_i)$, and observe that by definitions of substitution \mathbf{t}_i holds $\langle A_i, x_i \rangle \in \mathbf{t}_i(A_i)$.

Then, $(\overline{u}_1 \langle A_1, x_1 \rangle \overline{v}_1, \overline{u}_2 \langle A_2, x_2 \rangle \overline{v}_2, \dots, \overline{u}_n \langle A_n, x_n \rangle \overline{v}_n) \Rightarrow (\overline{u}_1 \overline{y}_1 \overline{v}_1, \overline{u}_2 \overline{y}_2 \overline{v}_2, \dots, \overline{u}_n \overline{y}_n \overline{v}_n)$ in $\overline{\Gamma}$, where $\overline{u}_i \overline{y}_i \overline{v}_i$ is any sentential form, $\overline{u}_i \overline{y}_i \overline{v}_i \in \mathbf{t}_i(u_i x_i v_i)$ for all $i = 1, \dots, n$.

I. and II. imply that

$$(S_1, S_2, \dots, S_n) \Rightarrow^{k+1} (\bar{u}_1 \langle A_1, x_1 \rangle \bar{v}_1, \bar{u}_2 \langle A_2, x_2 \rangle \bar{v}_2, \dots, \bar{u}_n \langle A_n, x_n \rangle \bar{v}_n) \Rightarrow (\bar{u}_1 \bar{y}_1 \bar{v}_1, \bar{u}_2 \bar{y}_2 \bar{v}_2, \dots, \bar{u}_n \bar{y}_n \bar{v}_n) \text{ in } \bar{\Gamma} \text{ for any } \bar{u}_i \bar{y}_i \bar{v}_i \in \mathbf{t}_i(u_i x_i v_i) \text{ for all } i = 1, \dots, n.$$

Let $\bar{z}_i = \bar{u}_i \bar{y}_i \bar{v}_i$ and $z_i = u_i x_i v_i$ for all $i = 1, \dots, n$. Then, $(S_1, S_2, \dots, S_n) \Rightarrow^{k+2} (\bar{z}_1, \bar{z}_2, \dots, \bar{z}_n)$ in $\bar{\Gamma}$ for any $\bar{z}_i \in \mathbf{t}_i(z_i)$.

CLAIM B: Let $(S_1, S_2, \dots, S_n) \Rightarrow^m (\bar{z}_1, \bar{z}_2, \dots, \bar{z}_n)$ in $\bar{\Gamma}$, where $m \geq 1$, $\bar{z}_i \in (\bar{N}_i \cup T_i)^*$ for all $i = 1, \dots, n$. Then, $(S_1, S_2, \dots, S_n) \Rightarrow^{m-1} (z_1, z_2, \dots, z_n)$ in Γ , where $z_i \in \mathbf{t}_i^{-1}(\bar{z}_i)$ for all $i = 1, \dots, n$.

PROOF: This claim is proved by induction:

Basic: Let $m = 1$. Then, $(S_1, S_2, \dots, S_n) \Rightarrow^1 (\bar{z}_1, \bar{z}_2, \dots, \bar{z}_n)$ in $\bar{\Gamma}$ implies $S_i \rightarrow \bar{z}_i \in \bar{P}_i$, for all $i = 1, \dots, n$. Algorithm 2 implies that $\bar{z}_i \in \mathbf{t}_i(S_i)$ for all $i = 1, \dots, n$, so $(S_1, S_2, \dots, S_n) \Rightarrow^0 (S_1, S_2, \dots, S_n)$ in Γ and holds $S_i \in \mathbf{t}_i^{-1}(\bar{z}_i)$ for all $i = 1, \dots, n$.

Induction hypothesis: Assume that Claim B holds for all m -step derivations, where $m = 1, \dots, k$, for some $k \geq 1$.

Induction step: Consider $(S_1, S_2, \dots, S_n) \Rightarrow^{k+1} (\bar{y}_1, \bar{y}_2, \dots, \bar{y}_n)$ in $\bar{\Gamma}$. Then, there exists a sentential n -form $(\bar{u}_1 \langle A_1, x_1 \rangle \bar{v}_1, \bar{u}_2 \langle A_2, x_2 \rangle \bar{v}_2, \dots, \bar{u}_n \langle A_n, x_n \rangle \bar{v}_n)$ in $\bar{\Gamma}$, where $\bar{u}_i \in T_i^*$, $\langle A_i, x_i \rangle \in \bar{N}_i$, $\bar{v}_i \in (\bar{N}_i \cup T_i)^*$ such that: $(S_1, S_2, \dots, S_n) \Rightarrow^k (\bar{u}_1 \langle A_1, x_1 \rangle \bar{v}_1, \bar{u}_2 \langle A_2, x_2 \rangle \bar{v}_2, \dots, \bar{u}_n \langle A_n, x_n \rangle \bar{v}_n) \Rightarrow (\bar{u}_1 \bar{x}_1 \bar{v}_1, \bar{u}_2 \bar{x}_2 \bar{v}_2, \dots, \bar{u}_n \bar{x}_n \bar{v}_n)$, where $\bar{u}_i \bar{x}_i \bar{v}_i = \bar{y}_i$ for all $i = 1, \dots, n$.

I. $(S_1, S_2, \dots, S_n) \Rightarrow^k (\bar{u}_1 \langle A_1, x_1 \rangle \bar{v}_1, \bar{u}_2 \langle A_2, x_2 \rangle \bar{v}_2, \dots, \bar{u}_n \langle A_n, x_n \rangle \bar{v}_n)$ in $\bar{\Gamma}$ implies $(S_1, S_2, \dots, S_n) \Rightarrow^{k-1} (w_1, w_2, \dots, w_n)$ in Γ , where $w_i \in \mathbf{t}_i^{-1}(\bar{u}_i \langle A_i, x_i \rangle \bar{v}_i)$ for all $i = 1, \dots, n$, by the induction hypothesis.

II. Let $(\bar{u}_1 \langle A_1, x_1 \rangle \bar{v}_1, \bar{u}_2 \langle A_2, x_2 \rangle \bar{v}_2, \dots, \bar{u}_n \langle A_n, x_n \rangle \bar{v}_n) \Rightarrow (\bar{u}_1 \bar{x}_1 \bar{v}_1, \bar{u}_2 \bar{x}_2 \bar{v}_2, \dots, \bar{u}_n \bar{x}_n \bar{v}_n)$ in $\bar{\Gamma}$. Then, there holds in $\bar{\Gamma}$: $(\langle A_1, x_1 \rangle, \langle A_2, x_2 \rangle, \dots, \langle A_n, x_n \rangle) \in \bar{Q}$ and $\langle A_i, x_i \rangle \rightarrow \bar{x}_i \in \bar{P}_i$ for all $i = 1, \dots, n$. Algorithm 2 implies that

$(A_1 \rightarrow x_1, A_2 \rightarrow x_2, \dots, A_n \rightarrow x_n) \in Q$ and $A_i \rightarrow x_i \in P_i$, where $\bar{x}_i \in \mathbf{t}_i(x_i)$, so $x_i \in \mathbf{t}_i^{-1}(\bar{x}_i)$. For all $i = 1, \dots, n$, we can write w_i as $w_i = u_i A_i v_i$, where $u_i \in \mathbf{t}_i^{-1}(\bar{u}_i)$, $v_i \in \mathbf{t}_i^{-1}(\bar{v}_i)$ and observe that by definitions of substitution for \mathbf{t}_i holds:

$A_i \in \mathbf{t}_i^{-1}(\langle A_i, x_i \rangle)$. Then, $(u_1 A_1 v_1, u_2 A_2 v_2, \dots, u_n A_n v_n) \Rightarrow (u_1 x_1 v_1, u_2 x_2 v_2, \dots, u_n x_n v_n)$ in Γ , where $u_i \in \mathbf{t}_i^{-1}(\bar{u}_i)$, $v_i \in \mathbf{t}_i^{-1}(\bar{v}_i)$ and $x_i \in \mathbf{t}_i^{-1}(\bar{x}_i)$, for all $i = 1, \dots, n$, it means that $u_i x_i v_i \in$

$\mathbf{t}_i^{-1}(\bar{u}_i \bar{x}_i \bar{v}_i)$ for all $i = 1, \dots, n$.

I. and II. imply that $(S_1, S_2, \dots, S_n) \Rightarrow^{k+1} (u_1 A_1 v_1, u_2 A_2 v_2, \dots, u_n A_n v_n) \Rightarrow (u_1 x_1 v_1, u_2 x_2 v_2, \dots, u_n x_n v_n)$ in Γ , where $u_i x_i v_i \in \mathbf{t}_i^{-1}(\bar{u}_i \bar{x}_i \bar{v}_i)$ for all $i = 1, \dots, n$. Let $\bar{z}_i = \bar{u}_i \bar{y}_i \bar{v}_i$ and $z_i = u_i x_i v_i$ for all $i = 1, \dots, n$. Then, $(S_1, S_2, \dots, S_n) \Rightarrow^{k+2} (z_1, z_2, \dots, z_n)$ in Γ for all $z_i \in \mathbf{t}_i^{-1}(\bar{z}_i)$.

Consider Claim A with $m \geq 0$, $z_i \in T_i^*$ for all $i = 1, \dots, n$. At this point, if $(S_1, S_2, \dots, S_n) \Rightarrow^* (z_1, z_2, \dots, z_n)$ in Γ , then $(S_1, S_2, \dots, S_n) \Rightarrow^* (\bar{z}_1, \bar{z}_2, \dots, \bar{z}_n)$ in $\bar{\Gamma}$, where $\bar{z}_i \in \mathbf{t}_i(z_i)$. Because $\mathbf{t}_i(a_i) = \{a_i\}$ for all $a_i \in T_i^*$, then for all $i = 1, \dots, n$ hold: $\bar{z}_i = z_i$. Hence $(S_1, S_2, \dots, S_n) \Rightarrow^* (z_1, z_2, \dots, z_n)$ in $\bar{\Gamma}$, thus $n-L(\Gamma) \subseteq n-L(\bar{\Gamma})$. Consider Claim B with $m \geq 0$, $\bar{z}_i \in T_i^*$ for all $i = 1, \dots, n$. At this point, if $(S_1, S_2, \dots, S_n) \Rightarrow^* (\bar{z}_1, \bar{z}_2, \dots, \bar{z}_n)$ in $\bar{\Gamma}$, then $(S_1, S_2, \dots, S_n) \Rightarrow^* (z_1, z_2, \dots, z_n)$ in $\bar{\Gamma}$, where $z_i \in \mathbf{t}_i^{-1}(\bar{z}_i)$. Because $\mathbf{t}_i(a_i) = \{a_i\}$ for all $a_i \in T_i^*$, then for all $i = 1, \dots, n$ hold: $z_i = \bar{z}_i$. Hence $(S_1, S_2, \dots, S_n) \Rightarrow^* (\bar{z}_1, \bar{z}_2, \dots, \bar{z}_n)$ in Γ , thus $n-L(\bar{\Gamma}) \subseteq n-L(\Gamma)$. $n-L(\Gamma) \subseteq n-L(\bar{\Gamma})$ and $n-L(\bar{\Gamma}) \subseteq n-L(\Gamma)$ imply $n-L(\Gamma) = n-L(\bar{\Gamma})$. This equality implies by the Claim 1 $L_X(\Gamma) = L_X(\bar{\Gamma})$, for each $X \in \{\text{union, conc, first}\}$.

COROLLARY 1. *The class of languages generated by n-MGN in the X mode, where $X \in \{\text{union, conc, first}\}$ is equivalent to the class of languages generated by n-MGR in the X mode.*

PROOF: This corollary follows from Algorithm 1 and Algorithm 2.

THEOREM 3. *For every recursive enumerable language L over an alphabet T there exists a 2-MGR, $\Gamma = ((\bar{N}_1, T, \bar{P}_1, S_1), (\bar{N}_2, T, \bar{P}_2, S_2), Q)$ such that:*

- 1) $\{w : (S_1, S_2) \Rightarrow^* (w, w)\} = L$,
- 2) $\{w_1 w_2 : (S_1, S_2) \Rightarrow^* (w_1, w_2), w_1, w_2 \in T^*, w_1 \neq w_2\} = \emptyset$.

PROOF: Recall that for every recursive enumerable language L over an alphabet T there exist two context-free grammars $G_1 = (N_1, \bar{T}, P_1, S_1)$, $G_2 = (N_2, \bar{T}, P_2, S_2)$ and homomorphism h : from \bar{T} to T^* such that $L = \{h(x) : x \in L(G_1) \cap L(G_2)\}$ (see Theorem 10.3.1 in [3]). Furthermore, for every context-free grammar, there exists an equivalent context-free grammar in Greibach normal form (see Section 5.1.4.2 in [4]). Hence, without loss of generality, we can assume that G_1 and G_2 are in Greibach normal form. Consider an 2-MGR $\Gamma = (G_1, G_2, Q)$, where:

$G_1 = (\bar{N}_1, T, \bar{P}_1, S_1)$, where $\bar{N}_1 = N_1 \cup \{\bar{a} : a \in \bar{T}\}$,

$\bar{P}_1 = \{A \rightarrow \bar{a}x : A \rightarrow ax \in P_1, a \in \bar{T}, x \in N_1^*\} \cup \{\bar{a} \rightarrow h(a) : a \in \bar{T}\}$

$G_2 = (\bar{N}_2, T, \bar{P}_2, S_2)$, where $\bar{N}_2 = N_2 \cup \{\bar{a} : a \in \bar{T}\}$,

$$\bar{P}_2 = \{A \rightarrow \bar{a}x : A \rightarrow ax \in P_2, a \in \bar{T}, x \in N_2^*\} \cup \{\bar{a} \rightarrow h(a) : a \in \bar{T}\}$$

$$Q = \{(A_1 \rightarrow \bar{a}x_1, A_2 \rightarrow \bar{a}x_2) : A_1 \rightarrow \bar{a}x_1 \in \bar{P}_1, A_2 \rightarrow \bar{a}x_2 \in \bar{P}_2, a \in \bar{T}\} \cup$$

$$\{(\bar{a} \rightarrow h(a), \bar{a} \rightarrow h(a)) : a \in \bar{T}\}$$

Now, we prove that for a 2-MGR $\Gamma = (G_1, G_2, Q)$ holds:

$$1) \{w : (S_1, S_2) \Rightarrow^* (w, w)\} = L.$$

$$2) \{w_1 w_2 : (S_1, S_2) \Rightarrow^* (w_1, w_2), w_1, w_2 \in T^*, w_1 \neq w_2\} = \emptyset.$$

$$\text{CLAIM A: } \{w : (S_1, S_2) \Rightarrow^* (w, w)\} = L:$$

PROOF: I. We prove that $L \subseteq \{w : (S_1, S_2) \Rightarrow^* (w, w)\}$:

Let $w \in L$ be any string. Then, there exists a string $a_1 a_2 \dots a_n \in \bar{T}^*$ such that:

- 1) $a_1 a_2 \dots a_n \in L(G_1)$,
- 2) $a_1 a_2 \dots a_n \in L(G_2)$,
- 3) $h(a_1 a_2 \dots a_n) = w$.

It means that there exist following derivations in G_1 and G_2 :

1. $S_1 \Rightarrow a_1 x_1 [p_1] \Rightarrow a_1 a_2 x_2 [p_2] \Rightarrow \dots \Rightarrow a_1 a_2 \dots a_n [p_n]$,
2. $S_2 \Rightarrow a_1 y_1 [r_1] \Rightarrow a_1 a_2 y_2 [r_2] \Rightarrow \dots \Rightarrow a_1 a_2 \dots a_n [r_n]$,

where $a_i \in \bar{T}$, $x_i \in N_1^*$, $y_i \in N_2^*$, $p_i \in P_1$, $r_i \in P_2$ for all $i = 1, \dots, n$.

Observe that for all $i = 1, \dots, n$ holds: $\text{sym}(\text{rhs}(p_i), 1) = \text{sym}(\text{rhs}(r_i), 1) = a_i$.

A construction of a set Q ,

$$Q = \{(A_1 \rightarrow \bar{a}x_1, A_2 \rightarrow \bar{a}x_2) : A_1 \rightarrow \bar{a}x_1 \in \bar{P}_1, A_2 \rightarrow \bar{a}x_2 \in \bar{P}_2, a \in \bar{T}\} \cup$$

$$\{(\bar{a} \rightarrow h(a), \bar{a} \rightarrow h(a)) : a \in \bar{T}\},$$

implies:

- 1) Let $p_i : A_i \rightarrow a_i u_i \in \bar{P}_1$, $r_i : B_i \rightarrow a_i v_i \in \bar{P}_2$. Then $(A_i \rightarrow \bar{a}_i u_i, B_i \rightarrow \bar{a}_i v_i) \in Q$ for all $i = 1, \dots, n$,
- 2) $(\bar{a}_i \rightarrow h(a_i), \bar{a}_i \rightarrow h(a_i)) \in Q$ for all $i = 1, \dots, n$.

Then, there exists a sequence of derivation in 2-MGR Γ :

$$\begin{aligned} (S_1, S_2) &\Rightarrow (\bar{a}_1 x_1, \bar{a}_1 y_1) \Rightarrow (h(a_1) x_1, h(a_1) y_1) \\ &\Rightarrow (h(a_1) \bar{a}_2 x_2, h(a_1) \bar{a}_2 y_2) \Rightarrow (h(a_1) h(a_2) x_2, h(a_1) h(a_2) y_2) \\ &\Rightarrow \dots \end{aligned}$$

$$\begin{aligned} &\Rightarrow (h(a_1)h(a_2)..h(a_n), h(a_1)h(a_2)..h(a_n)) = \\ &\quad (h(a_1a_2..a_n), h(a_1a_2..a_n)) = (w, w). \end{aligned}$$

It means that $(S_1, S_2) \Rightarrow^* (w, w)$ in Γ .

II. We prove, that $\{w: (S_1, S_2) \Rightarrow^* (w, w)\} \subseteq L$:

Let $(S_1, S_2) \Rightarrow^* (w, w)$ in Γ . Then, there exists a sequence of derivation in 2-MGR Γ :

$$\begin{aligned} (S_1, S_2) &\Rightarrow (\bar{a}_1x_1, \bar{a}_1y_1) \Rightarrow (h(a_1)x_1, h(a_1)y_1) \\ &\Rightarrow (h(a_1)\bar{a}_2x_2, h(a_1)\bar{a}_2y_2) \Rightarrow (h(a_1)h(a_2)x_2, h(a_1)h(a_2)y_2) \\ &\Rightarrow \dots \\ &\Rightarrow (h(a_1)h(a_2)..h(a_n), h(a_1)h(a_2)..h(a_n)) = \\ &\quad (h(a_1a_2..a_n), h(a_1a_2..a_n)) = (w, w). \end{aligned}$$

Analogically as in part I., we can prove that there exist derivations in G_1 and G_2 of the forms:

1. $S_1 \Rightarrow a_1x_1 [p_1] \Rightarrow a_1a_2x_2 [p_2] \Rightarrow \dots \Rightarrow a_1a_2..a_n [p_n]$,
2. $S_2 \Rightarrow a_1y_1 [r_1] \Rightarrow a_1a_2y_2 [r_2] \Rightarrow \dots \Rightarrow a_1a_2..a_n [r_n]$.

It means that $a_1a_2..a_n \in L(G_1)$, $a_1a_2..a_n \in L(G_2)$ and $h(a_1a_2..a_n) = w$, and so $w \in L$.

CLAIM B: $\{w_1w_2: (S_1, S_2) \Rightarrow^* (w_1, w_2), w_1, w_2 \in T^*, w_1 \neq w_2\} = \emptyset$.

PROOF (BY CONTRADICTION): Let $\{w_1w_2: (S_1, S_2) \Rightarrow^* (w_1, w_2), w_1, w_2 \in T^*, w_1 \neq w_2\} \neq \emptyset$. Then, there exist two different strings $w_1 = h(a_1)h(a_2)..h(a_n)$, $w_2 = h(b_1)h(b_2)..h(b_n)$, such that: $(S_1, S_2) \Rightarrow^* (w_1, w_2)$.

I. Assume that for all $i = 1, \dots, n$ holds: $a_i = b_i$. Then $w_1 = h(a_1)h(a_2)..h(a_n) = h(b_1)h(b_2)..h(b_n) = w_2$. But w_1 and w_2 are different strings. That is a contradiction.

II. Assume that there exists $k \leq n$, such that $a_k \neq b_k$: Then, there exists a sequence of derivations in Γ of the form:

$$\begin{aligned} (S_1, S_2) &\Rightarrow (\bar{a}_1x_1, \bar{a}_1y_1) \Rightarrow (h(a_1)x_1, h(a_1)y_1) \\ &\Rightarrow (h(a_1)\bar{a}_2x_2, h(a_1)\bar{a}_2y_2) \Rightarrow (h(a_1)h(a_2)x_2, h(a_1)h(a_2)y_2) \\ &\Rightarrow \dots \\ &\Rightarrow (h(a_1)h(a_2)..h(a_{k-1})x_{k-1}, h(a_1)h(a_2)..h(a_{k-1})y_{k-1}). \end{aligned}$$

Then, there must exist a derivation $(x_{k-1}, y_{k-1}) \Rightarrow (\bar{a}_kx_k, \bar{b}_ky_k)$, where $\bar{a}_k \neq \bar{b}_k$. Because $Q = \{(A_1 \rightarrow \bar{a}_kx_k, A_2 \rightarrow \bar{a}_ky_k): A_1 \rightarrow \bar{a}_kx_k \in \bar{P}_1, A_2 \rightarrow \bar{a}_ky_k \in \bar{P}_2, a \in \bar{T}\} \cup \{(\bar{a} \rightarrow h(a), \bar{a} \rightarrow h(a)): a \in \bar{T}\}$, there must be used a pair of rules (p, r) such that $\text{sym}(\text{rhs}(p), 1) = \text{sym}(\text{rhs}(r), 1)$. Next derivation must be of the form:

$(x_{k-1}, y_{k-1}) \Rightarrow (\bar{a}_k x_k, \bar{b}_k y_k)$, where $\bar{a}_k = \bar{b}_k$. That is a contradiction.

THEOREM 4. *For every recursive enumerable language L over an alphabet T there exist a 2-MGR, $\Gamma = (G_1, G_2, Q)$ such that: $L_{union}(\Gamma) = L$.*

PROOF: By the Theorem 3 holds: For every recursive enumerable language L over an alphabet T there exist a 2-MGR, $\bar{\Gamma} = ((N_1, T, P_1, S_1), (N_2, T, P_2, S_2), Q)$ such that: $L = \{w: (S_1, S_2) \Rightarrow^* (w, w)\}$ and

$$\{w_1 w_2: (S_1, S_2) \Rightarrow^* (w_1, w_2), w_1, w_2 \in T^*, w_1 \neq w_2\} = \emptyset.$$

Let $\Gamma = \bar{\Gamma}$. Then,

$$\begin{aligned} L_{union}(\Gamma) &= \{w: (S_1, S_2) \Rightarrow^* (w_1, w_2), w_i \in T^* \text{ for } i = 1, 2, w \in \{w_i: i = 1, 2\}\} = \\ &= \{w: (S_1, S_2) \Rightarrow^* (w, w), w \in T^*\} \cup \{w: (S_1, S_2) \Rightarrow^* (w_1, w_2), w_i \in T^* \text{ for } i = 1, 2, \\ &w \in \{w_i: i = 1, 2\}, w_1 \neq w_2\} = \{w: (S_1, S_2) \Rightarrow^* (w, w), w \in T^*\} \cup \emptyset = \{w: (S_1, S_2) \Rightarrow^* (w, \\ &w), w \in T^*\} = L. \end{aligned}$$

THEOREM 5. *For every recursive enumerable language L over an alphabet T there exist a 2-MGR, $\Gamma = (G_1, G_2, Q)$ such that: $L_{first}(\Gamma) = L$.*

PROOF: By the Theorem 3 holds: For every recursive enumerable language L over an alphabet T there exist a 2-MGR, $\bar{\Gamma} = ((N_1, T, P_1, S_1), (N_2, T, P_2, S_2), Q)$ such that: $L = \{w: (S_1, S_2) \Rightarrow^* (w, w)\}$ and

$$\{w_1 w_2: (S_1, S_2) \Rightarrow^* (w_1, w_2), w_1, w_2 \in T^*, w_1 \neq w_2\} = \emptyset.$$

$$\begin{aligned} \text{Let } \Gamma = \bar{\Gamma}. \text{ Then, } L_{first}(\Gamma) &= \{w_1: (S_1, S_2) \Rightarrow^* (w_1, w_2), w_i \in T^* \text{ for } i = 1, 2\} = \\ &= \{w: (S_1, S_2) \Rightarrow^* (w, w), w \in T^*\} \cup \{w_1: (S_1, S_2) \Rightarrow^* (w_1, w_2), w_i \in T^* \text{ for } i = 1, 2, \\ &w_1 \neq w_2\} = \{w: (S_1, S_2) \Rightarrow^* (w, w), w \in T^*\} \cup \emptyset = \{w: (S_1, S_2) \Rightarrow^* (w, w), w \in T^*\} = L. \end{aligned}$$

THEOREM 6. *For every recursive enumerable language L over an alphabet T there exist a 2-MGR, $\Gamma = (G_1, G_2, Q)$ such that: $L_{conc}(\Gamma) = L$.*

PROOF: By the Theorem 3 holds: For every recursive enumerable language L over an alphabet T there exist a 2-MGR, $\bar{\Gamma} = ((N_1, T, P_1, S_1), (N_2, T, P_2, S_2), Q)$ such that: $L = \{w: (S_1, S_2) \Rightarrow^* (w, w)\}$ and $\{w_1 w_2: (S_1, S_2) \Rightarrow^* (w_1, w_2), w_1, w_2 \in T^*, w_1 \neq w_2\} = \emptyset$.

Let $G_1 = (N_1, T, P_1, S_1)$, $G_2 = (N_2, \emptyset, \bar{P}_2, S_2)$, where $\bar{P}_2 = \{A \rightarrow g(x): A \rightarrow x \in P_2\}$, where g is a homomorphism from $(N_2 \cup T)$ to N_2 defined as: for all $X \in N_2$: $g(X) = X$, for all $X \in T$: $g(X) = \varepsilon$. We prove that $L_{conc}(\Gamma) = L$.

I. We prove that $L \subseteq L_{conc}(\Gamma)$: Let $w \in L$. Then, there exists a sequence of derivation in $\bar{\Gamma}$ of the form: $(S_1, S_2) \Rightarrow^* (w, w)$, thus, there exist a sequence of derivations in Γ of the form

$(S_1, S_2) \Rightarrow^* (w, g(w))$. Because $g(a) = \varepsilon$ for all $a \in T$, then $g(w) = \varepsilon$ for all $w \in T^*$. Thus, there exists a sequence of derivations $(S_1, S_2) \Rightarrow^* (w, \varepsilon)$ in Γ . Hence, $w\varepsilon = w \in L_{conc}(\Gamma)$.

II. We prove that $L_{conc}(\Gamma) \subseteq L$: Let $w \in L_{conc}(\Gamma)$. Then, there exists a sequence of derivations $(S_1, S_2) \Rightarrow^* (w, \varepsilon)$ in Γ , because G_2 derives only the empty string. $g(x) = \varepsilon$ for all $x \in T^*$, so there exists a sequence of derivation in $\bar{\Gamma}$ of the form: $(S_1, S_2) \Rightarrow^* (w, x)$, where x is any string. Theorem 3 implies that $x = w$, thus: $(S_1, S_2) \Rightarrow^* (w, w)$. Thus, $w \in L$.

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6. References

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