Heteroclinic Connections between Periodic Orbits in Planar Restricted Circular Three Body Problem - Part II

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Abstract

We present a method for proving the existence of symmetric periodic, heteroclinic or homoclinic orbits in dynamical systems with the reversing symmetry. As an application we show that the Planar Restricted Circular Three Body Problem (PCR3BP) corresponding to the Sun-Jupiter-Oterma system possesses an infinite number of symmetric periodic orbits and homoclinic orbits to the Lyapunov orbits. Moreover, we show the existence of symbolic dynamics on six symbols for PCR3BP and the possibility of resonance transitions of the comet. This extends earlier results by Wilczak and Zgliczynski[12].

1 Introduction.

The Planar Restricted Circular Three Body Problem (PCR3BP) has attracted much attention of scientists. In particular, some transport properties of this system may be applied in space mission design (see [4] and references given there). The problem has been studied by Koon, Lo, Marsden and Roos in [4], where the numerical evidence of the resonance transitions for the PCR3BP for the parameter values corresponding to the Sun-Jupiter-Oterma system is presented. The rigorous proof of some facts discovered in [4] was given by Wilczak and Zgliczynski in [12] (see also Stoffer and Kirchgraber paper [8]).

In the present paper, as in [12], we restrict our attention to the following parameter values for PCR3BP $C = 3.03$, $\mu = 0.0009537$ - the parameter values for *Oterma* comet in the Sun-Jupiter system (see [4]). For this parameter values

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there are two hyperbolic periodic orbits L_1^* and L_2^*, the Liapunov orbits, around the libration points L_1 and L_2, respectively. In [12] and [8] it was proven that there exist homoclinic solutions to both L_1^* and L_2^* periodic orbits and a pair of heteroclinic connections between them in both directions.

In this paper we present the proof of the following facts:

- The PCR3BP possesses two pairs of homoclinic orbits both to L_1^* and L_2^*. These homoclinic orbits are geometrically different. Informally speaking they are close to different resonances, namely 3:2, 5:3 for the orbits homoclinic to L_1^* and 1:2, 2:3 for the orbits homoclinic to L_2^*. Moreover, it is possible for a comet to move between these four resonances in arbitrary order.

- The PCR3BP possesses an infinite number of geometrically different symmetric periodic and homoclinic orbits.

Let describe now what constitute a new element in the present paper. While the numerical evidence of three of the above mentioned homoclinic orbits is given in [4], the 2:3 homoclinic orbit appear to be a new one. The technique of the proof of the existence of these homoclinic orbits and the symbolic dynamics is the same as in [12], i.e. combines topological tools (covering relations) with rigorous numerics.

The main novelty of the present paper, when compared to [4, 12], is the existence of an infinite number of geometrically different symmetric periodic and homoclinic orbits. While the numerical evidence of the simplest symmetric homoclinic orbits is given in [4] the numerical method used there cannot yield the existence of an infinite number of them even with the help of validated numerics. In this paper we use the topological method introduced recently by Wilczak in [10] and developed later by both authors in [13]. For the purpose of this introduction we briefly describe the main points of method for symmetric periodic points. The method is based on two observations:

- to detect symmetric periodic orbits for map P (a Poincaré map) with an reversing symmetry R (in the PCR3BP case composition of a suitable reflection and the time inversion) it is enough to look for intersections of $\mathrm{Fix}(R) = \{x \mid R(x) = x\}$ with $P^k(\mathrm{Fix}(R))$. This is the Fixed Set Iteration method [5, 6] (also known as DeVogelaere method [3]). Any point from such intersection give rise to $2k$-periodic point.

- covering relations give some control of pieces of $P^k(\mathrm{Fix}(R))$, which make it possible to prove that $P^k(\mathrm{Fix}(S)) \cap \mathrm{Fix}(R)$ is nonempty for k sufficiently large and that the period of the periodic point is indeed equal to $2k$.

The paper is organized as follows. In Section 2 we recall the PCR3BP and its properties. In Section 3 the proof of the existence of a new pair of homoclinic orbits both in exterior and interior regions to the Lyapunov orbits L_1^*, L_2^* is presented. In Section 4 the symbolic dynamics on six symbols is established.

We also discuss the resonance transitions there. In Section 5 the existence of symmetric periodic and homoclinic orbits is proven.

Throughout the paper we will use the definitions and notations from [12].

2 Short description of the system.

We follow papers [4, 12] and use the notation introduced there.

Let S and J be two bodies called Sun and Jupiter, of masses $m_s = 1-\mu$ and $m_j = \mu$, $\mu \in (0,1)$, respectively. They rotate in the plane in circles counter clockwise about their common center and with angular velocity normalized as one. Choose a rotating coordinate system, so that origin is at the center of mass and the Sun and Jupiter are fixed on the x-axis at $(-\mu, 0)$ and $(1-\mu, 0)$ respectively. In this coordinate frame the equations of motion of a massless particle called the comet or the spacecraft under the gravitational action of Sun and Jupiter are (see [4] and references given there)

$$\ddot{x} - 2\dot{y} = \Omega_x(x,y), \qquad \ddot{y} + 2\dot{x} = \Omega_y(x,y), \tag{1}$$

where

$$\begin{aligned} \Omega(x,y) &= \frac{x^2+y^2}{2} + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{\mu(1-\mu)}{2} \\ r_1 = \sqrt{(x+\mu)^2+y^2}, &\qquad r_2 = \sqrt{(x-1+\mu)^2+y^2} \end{aligned}$$

Equations (1) are called the equations of the planar circular restricted three-body problem (PCR3BP). They have a first integral called the *Jacobi integral*, which is given by

$$C(x,y,\dot{x},\dot{y}) = -(\dot{x}^2+\dot{y}^2) + 2\Omega(x,y). \tag{2}$$

We consider PCR3BP on the hypersurface

$$\mathcal{M}(\mu, C) = \{(x,y,\dot{x},\dot{y}) \mid C(x,y,\dot{x},\dot{y}) = C\},$$

and we restrict our attention to the following parameter values $C = 3.03$, $\mu = 0.0009537$ - the parameter values for *Oterma* comet in the Sun-Jupiter system (see [4]).

The projection of $\mathcal{M}(\mu, C)$ onto position space is called a Hill's region and gives the region in the (x,y)-plane, where the comet is free to move. The Hill's region for the parameter considered in this paper is shown on Figure 1 in white, the forbidden region is dark. The Hill's region consists of three regions: an interior (Sun) region, an exterior region and Jupiter region.

As was mentioned in the Introduction we restrict our attention to the following parameter values $C = 3.03$, $\mu = 0.0009537$ - the parameter values for *Oterma* comet in the Sun-Jupiter system (see [4]). Since we work with fixed parameter values we usually drop the dependence of various objects defined throughout the paper on μ and C, so for example $\mathcal{M} = \mathcal{M}(\mu, C)$.

2.1 Poincaré maps.

We consider Poincaré sections: $\Theta = \{(x, y, \dot{x}, \dot{y}) \in \mathcal{M} \mid y = 0\}$, $\Theta_+ = \Theta \cap \{\dot{y} > 0\}$, $\Theta_- = \Theta \cap \{\dot{y} < 0\}$.

On $\Theta_\pm$ we can express $\dot{y}$ in terms of x and $\dot{x}$ as follows

$$\dot{y} = \pm\sqrt{2\Omega(x, 0) - \dot{x}^2 - C}$$

Hence the sections $\Theta_\pm$ can be parameterized by two coordinates $(x, \dot{x})$ and we will use this identification throughout the paper. More formally, we have the transformation $T_\pm : \mathbb{R}^2 \to \Theta_\pm$ given by the following formula

$$T_\pm(x, \dot{x}) = (x, 0, \dot{x}, \pm\sqrt{2\Omega(x, 0) - \dot{x}^2 - C}\ )$$

The domain of $T_\pm$ is given by an inequality $2\Omega(x, 0) - \dot{x}^2 - C \geq 0$.

Let $\pi_{\dot{x}} : \Theta_\pm \longrightarrow \mathbb{R}$ and $\pi_x : \Theta_\pm \longrightarrow \mathbb{R}$ denote the projection onto $\dot{x}$ and x coordinate, respectively. We have $\pi_{\dot{x}}(x_0, \dot{x}_0) = \dot{x}_0$ and $\pi_x(x_0, \dot{x}_0) = x_0$.

We will say that $(x, \dot{x}) \in \Theta_\pm$ meaning that $(x, \dot{x})$ represents two-dimensional coordinates of a point on $\Theta_\pm$. Analogously we give a meaning to the statement $M \subset \Theta_\pm$ for a set $M \subset \mathbb{R}^2$.

We define the following Poincaré maps between sections

$$\begin{aligned} P_+ &: \Theta_+ \to \Theta_+ \\ P_- &: \Theta_- \to \Theta_- \\ P_{\frac{1}{2},+} &: \Theta_+ \to \Theta_- \\ P_{\frac{1}{2},-} &: \Theta_- \to \Theta_+. \end{aligned}$$

As a rule the sign $+$ or $-$ tells that the domain of the maps $P_\pm$ or $P_{\frac{1}{2},\pm}$ is contained in $\Theta_\pm$ (the same sign). Observe that

$$P_+(x) = P_{\frac{1}{2},-} \circ P_{\frac{1}{2},+}(x), \qquad P_-(x) = P_{\frac{1}{2},+} \circ P_{\frac{1}{2},-}(x)$$

whenever $P_+(x)$ and $P_-(x)$ are defined. These identities express the following simple fact: to return to Θ_+ we need to cross Θ with negative $\dot{y}$ (this is $P_{\frac{1}{2},+}$ first and then we return to Θ with $\dot{y} > 0$ (this is $P_{\frac{1}{2},-}$).

Sometimes we will drop signs in $P_\pm$ and $P_{\frac{1}{2},\pm}$, hence $P(z) = P_+(z)$ if $z \in \Theta_+$ and $P(z) = P_-(z)$ if $z \in \Theta_-$, a similar convention will be applied to $P_{\frac{1}{2}}$.

2.2 Symmetry properties of PCR3BP

Notice that PCR3BP has the following symmetry

$$R(x, y, \dot{x}, \dot{y}, t) = (x, -y, -\dot{x}, \dot{y}, -t),$$

which expresses the following fact, if $(x(t), y(t))$ is a trajectory for PCR3BP, then $(x(-t), -y(-t))$ is also a trajectory for PCR3BP. From this it follows immediately that

$$\begin{aligned} &\text{if} \quad P_\pm(x_0, \dot{x}_0) = (x_1, \dot{x}_1) \qquad \text{then} \quad P_\pm(x_1, -\dot{x}_1) = (x_0, -\dot{x}_0), \\ &\text{if} \quad P_{\frac{1}{2},\pm}(x_0, \dot{x}_0) = (x_1, \dot{x_1}) \qquad \text{then} \quad P_{\frac{1}{2},\mp}(x_1, -\dot{x}_1) = (x_0, -\dot{x}_0). \end{aligned} \tag{3}$$

We will denote also by R the map $R : \Theta_\pm \to \Theta_\pm$ $R(x, \dot{x}) = (x, -\dot{x})$ for $(x, \dot{x}) \in \Theta_\pm$. Now Eq. (3) can be written as

$$\begin{aligned} &\text{if} \quad P_\pm(x_0) = x_1 \qquad \text{then} \quad P_\pm(R(x_1)) = R(x_0), \\ &\text{if} \quad P_{\frac{1}{2},\pm}(x_0) = x_1 \qquad \text{then} \quad P_{\frac{1}{2},\mp}(R(x_1)) = R(x_0). \end{aligned}$$

3 The existence of new homoclinic orbits.

The goal of this section is to present the proof of the existence of new homoclinic orbits with different resonances.

The notion of the resonance. We rewrite here an informal definition of the resonance from [4, Sec. 5.1]. Recall that the PCR3BP is a perturbation of the two-body problem. Hence, outside a small neighborhood of Jupiter, the trajectory of a comet follows essentially a two-body orbit around the Sun. In the heliocentric inertial frame, the orbit is nearly elliptical. The mean motion resonance of the comet with respect to Jupiter is equal to $a^{-3/2}$ where a is the semi-major axis of this elliptical orbit. Recall that the Sun-Jupiter distance is normalized to be 1 in the PCR3BP. The comet is said to be in $p : q$ resonance with Jupiter if $a^{-3/2} \approx p/q$, where p and q are small integers. In heliocentric inertial frame, the comet makes roughly p revolutions around the Sun in q Jupiter periods. Observe that this definition of the resonance make also sense for the orbits, which are non-periodic (for example orbits homoclinic to L_1^* or L_2^*), we just have to compute the semi-major axis for the piece of orbit away from Jupiter. A heuristic approach, which allows to read the resonance of an orbit from the trajectory in the rotating frame is described in Appendix.

In [12] the following theorem was proved.

Theorem 3.1. *[12, Thm.6.5,Thm.6.7] Consider PCR3BP with $C = 3.03$, $\mu = 0.0009537$. Then*

- *there exist a homoclinic orbit to the L_1^* orbit (in Sun region). This orbit is close to the $3 : 2$ resonance.*
- *there exist a homoclinic orbit to the L_2^* orbit (in exterior region). This orbit is close to the $1 : 2$ resonance.*

These orbits are presented in Fig. 1.

In this section we establish the existence of new homoclinic connections both in exterior and interior regions. The new homoclinic orbit in exterior region is close to the 2:3 resonance. As was mentioned in the Introduction this orbit has been found numerically in [4], see Fig. 5.4 the and the intersection stable and unstable manifolds of L_2^* at $L = \sqrt{a} \approx 1.26$. The other new homoclinic orbit in interior region is close to the 5:3 resonance appears to be a new one.

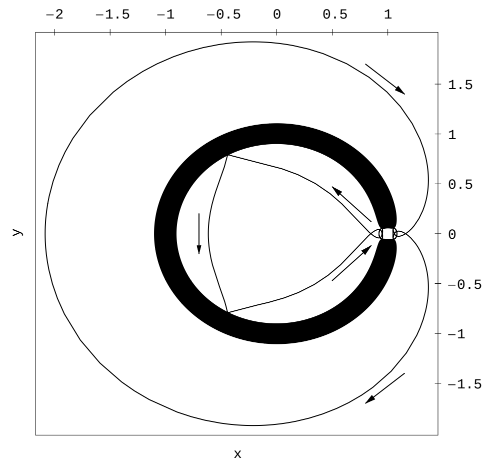

Figure 1: 3:2 homoclinic orbit to L_1^* Lyapunov orbit (interior region) and 1:2 homoclinic orbit to L_2^* Lyapunov orbit (exterior region).

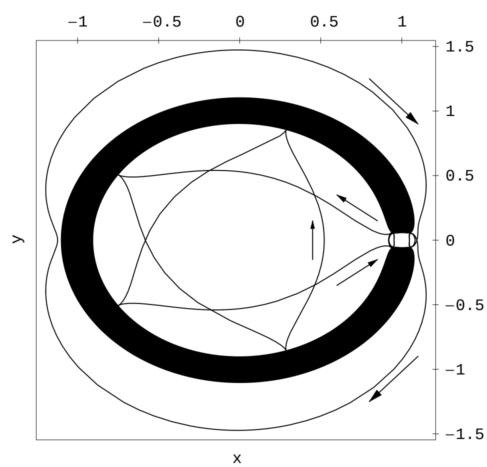

Figure 2: 5:3 homoclinic orbit to L_1^* Lyapunov orbit (interior region) and 2:3 homoclinic orbit to L_2^* Lyapunov orbit (exterior region).

3.1 The existence of the $2:3$ homoclinic orbit in the exterior region.

We define the following h-sets $G_i = t(c_i, u_i, s_i)$, for $i = 0, \ldots, 4$, where

$$\begin{aligned}
c_0 &= (-1.12327231155833984, 0), \\
c_1 &= (1.093337837571255552, -0.0251009417067904358 4), \\
c_2 &= (1.047131544421841024, -0.00105618794351394969 6), \\
c_3 &= (1.08194053721089792, -2.521361165903333888 \cdot 10^{-5}), \\
c_4 &= (1.04682616720451456, -9.169345277545603072 \cdot 10^{-7})
\end{aligned}$$

and

$$\begin{aligned}
s_0 &= (-1 \cdot 10^{-8}, 4 \cdot 10^{-7}), & u_0 &= -R(s_0), \\
s_1 &= (1 \cdot 10^{-7}, 21 \cdot 10^{-8}), & u_1 &= -R(s_1)/10, \\
s_2 &= (-1 \cdot 10^{-7}, 35 \cdot 10^{-8}), & u_2 &= -R(s_2)/10, \\
s_3 &= (-1 \cdot 10^{-7}, 23 \cdot 10^{-8}), & u_3 &= -R(s_3)/10, \\
s_4 &= (-1 \cdot 10^{-7}, 35 \cdot 10^{-8}), & u_4 &= -R(s_4)/4.
\end{aligned}$$

We assume, that $G_0, G_2, G_4 \subset \Theta_+$ and $G_1, G_3 \subset \Theta_-$. With a computer assistance we proved the following

Lemma 3.2. *The maps*

$$\begin{aligned}
P_{\frac{1}{2},+} &: G_0 \cup G_2 \cup G_4 \to \Theta_-, \\
P_{\frac{1}{2},-} &: G_1 \cup G_3 \to \Theta_+
\end{aligned}$$

are well defined and continuous. Moreover, the following covering relations hold

$$G_0 \overset{P_{1/2,+}}{\Longrightarrow} G_1 \overset{P_{1/2,-}}{\Longrightarrow} G_2 \overset{P_{1/2,+}}{\Longrightarrow} G_3 \overset{P_{1/2,-}}{\Longrightarrow} G_4 \overset{P_{1/2,+}}{\Longrightarrow} H_2^2.$$

Theorem 3.3. *For PCR3BP with $C = 3.03$ and $\mu = 0.0009537$ there exists an orbit homoclinic to L_2^* close to the $2:3$ resonance.*

Proof. From Lemma 3.2 and [12, Lemma 5.6] it follows that

$$G_0 \overset{P_{1/2,+}}{\Longrightarrow} G_1 \overset{P_{1/2,-}}{\Longrightarrow} G_2 \overset{P_{1/2,+}}{\Longrightarrow} G_3 \overset{P_{1/2,-}}{\Longrightarrow} G_4 \overset{P_{1/2,+}}{\Longrightarrow} H_2^2 \overset{P_-}{\Longrightarrow} H_2 \overset{P_-}{\Longrightarrow} H_2$$

Note that the h-set G_0 is R-symmetric by its definition. Therefore

$$\begin{aligned}
H_2 = R(H_2) \overset{P_-}{\Longleftarrow} R(H_2) \overset{P_-}{\Longleftarrow} R(H_2^2) \overset{P_{1/2,-}}{\Longleftarrow} R(G_4) \overset{P_{1/2,+}}{\Longleftarrow} R(G_3) \\
R(G_3) \overset{P_{1/2,-}}{\Longleftarrow} R(G_2) \overset{P_{1/2,+}}{\Longleftarrow} R(G_1) \overset{P_{1/2,-}}{\Longleftarrow} R(G_0) = G_0
\end{aligned}$$

Since P_- is hyperbolic on $|H_2|$ ([12, Lemma 5.5]) the assertion is a consequence of [2, Theorem 4]. $\square$

3.2 The existence of the $5:3$ homoclinic orbit in the interior region.

As in the previous section we construct a chain of covering relations in order to prove the existence of homoclinic orbit to L_1^* orbit. We define h-sets $V_i = t(c_i, u_i, s_i)$, for $i = 0, \ldots, 4$, where

$$
\begin{aligned}
c_0 &= (0.5217056203008400006, 0),\\
c_1 &= (-0.5822638014577352639, -0.2793408708392046136),\\
c_2 &= (0.919204446847046941, 0.004093829363524479834),\\
c_3 &= (0.9522506335647477061, 0.0001333182992547130779),\\
c_4 &= (0.9208022956271231241, 2.918364277340028028 \cdot 10^{-6})
\end{aligned}
$$

and

$$
\begin{aligned}
s_0 &= (-1 \cdot 10^{-7}, 2 \cdot 10^{-7}), & u_0 &= -R(s_0),\\
s_1 &= (2 \cdot 10^{-8}, 4 \cdot 10^{-7}), & u_1 &= (3 \cdot 10^{-8}, 0),\\
s_2 &= (-4 \cdot 10^{-7}, 102 \cdot 10^{-8}), & u_2 &= -R(s_2)/5,\\
s_3 &= (-1 \cdot 10^{-7}, 365 \cdot 10^{-9}), & u_3 &= -R(s_3)/10,\\
s_4 &= (-1 \cdot 10^{-7}, 25733011 \cdot 10^{-14}), & u_4 &= -R(s_4)/2.
\end{aligned}
$$

We assume, that $V_0, V_2, V_4 \subset \Theta_+$ and $V_1, V_3 \subset \Theta_-$. With a computer assistance we proved the following

Lemma 3.4. *The maps*

$$
\begin{aligned}
P_{\frac{1}{2},+} &: V_0 \cup V_2 \cup V_4 \to \Theta_-,\\
P_{\frac{1}{2},-} &: V_1 \cup V_3 \to \Theta_+
\end{aligned}
$$

are well defined and continuous. Moreover, we have the following chain of covering relations

$$V_0 \overset{P_{1/2,+}}{\Longrightarrow} V_1 \overset{P_{1/2,-}}{\Longrightarrow} V_2 \overset{P_{1/2,+}}{\Longrightarrow} V_3 \overset{P_{1/2,-}}{\Longrightarrow} V_4 \overset{P_+}{\Longrightarrow} H_1^2.$$

Theorem 3.5. *For PCR3BP with $C = 3.03$ and $\mu = 0.0009537$ there exists an orbit homoclinic to L_1^* close to the $5:3$ resonance.*

Proof. Since the sets H_1 and H_1^2 are R-symmetric [12, Lemma 5.6] and [12, Corollary 3.14] imply that

$$H_1^2 = R(H_1^2) \overset{P_+}{\Longleftarrow} R(H_1) = H_1 \overset{P_+}{\Longrightarrow} H_1.$$

After combining the above with Lemma 3.4 we obtain

$$V_0 \overset{P_{1/2,+}}{\Longrightarrow} V_1 \overset{P_{1/2,-}}{\Longrightarrow} V_2 \overset{P_{1/2,+}}{\Longrightarrow} V_3 \overset{P_{1/2,-}}{\Longrightarrow} V_4 \overset{P_+}{\Longrightarrow} H_1^2 \overset{P_+}{\Longleftarrow} H_1 \overset{P_+}{\Longrightarrow} H_1.$$

Note that the h-set V_0 is R-symmetric by its definition. Therefore

$$H_1 \overset{P_+}{\Longrightarrow} H_1^2 \overset{P_+}{\Longleftarrow} R(V_4) \overset{P_{1/2,+}}{\Longleftarrow} R(V_3) \overset{P_{1/2,-}}{\Longleftarrow} R(V_2) \overset{P_{1/2,+}}{\Longleftarrow} R(V_1) \overset{P_{1/2,-}}{\Longleftarrow} R(V_0) = V_0$$

Since P_+ is hyperbolic on $|H_1|$ ([12, Lemma 5.5]) the assertion is a consequence of [2, Theorem 4]. □

4 Symbolic dynamics on six symbols and resonance transitions.

As a consequence of theorems proved in [12] and in the previous section we obtain the existence of symbolic dynamics on six symbols. Let L_1, L_2 denote the Lyapunov orbits regions (see [4]) , S and I denote two parts of the Sun region corresponding to suitable vicinities of two homoclinic orbits to L_1^*. Let X and E denote two parts the exterior region corresponding to suitable vicinities of two homoclinic orbits to L_2^* orbit. Schematically this situation is shown in Fig. 3

In [12] the symbolic dynamics on four symbols, i.e. $\{L_1, L_2, X, S\}$ was established. The new homoclinic orbits allow us to include more symbols in it. We state this result more precisely. Let $\alpha, \beta \in \{L_1, L_2, X, E, I, S\}$ be such that

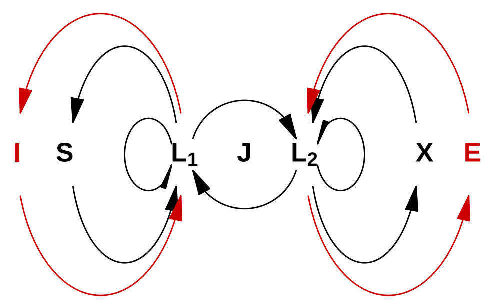

Figure 3: The graph of symbolic dynamics on six symbols.

there is an arrow from α to β on the graph presented in Fig. 3. We define the function

$$f_{\beta,\alpha} = \begin{cases} P_+, & \text{if } (\alpha,\beta) = (L_1, L_1) \\ P_-, & \text{if } (\alpha,\beta) = (L_2, L_2) \\ P_- \circ (P_{1/2,+} \circ P_{1/2,-})^4 \circ P_{1/2,+} \circ P_+, & \text{if } (\alpha,\beta) = (L_1, L_2) \\ P_+ \circ P_{1/2,-} \circ (P_{1/2,+} \circ P_{1/2,-})^4 \circ P_-, & \text{if } (\alpha,\beta) = (L_2, L_1) \\ P_+ \circ (P_{1/2,-} \circ P_{1/2,+})^2 \circ P_{1/2,-} & \text{if } (\alpha,\beta) = (S, L_1) \\ P_+ \circ P_{1/2,-} \circ (P_{1/2,+} \circ P_{1/2,-})^2 & \text{if } (\alpha,\beta) = (L_1, S) \\ P_-^2 \circ P_{1/2,+} \circ (P_{1/2,-} \circ P_{1/2,+})^2 & \text{if } (\alpha,\beta) = (X, L_2) \\ (P_{\frac{1}{2},-} \circ P_{1/2,+})^2 \circ P_{1/2,-} \circ P_-^2 & \text{if } (\alpha,\beta) = (L_2, X) \\ P_- \circ (P_{1/2,+} \circ P_{1/2,-})^2 \circ P_{1/2,+}, & \text{if } (\alpha,\beta) = (E, L_2) \\ P_{1/2,-} \circ (P_{1/2,+} \circ P_{1/2,-})^2 \circ P_-, & \text{if } (\alpha,\beta) = (L_2, E) \\ P_+^2 \circ (P_{1/2,-} \circ P_{1/2,+})^2, & \text{if } (\alpha,\beta) = (I, L_1) \\ (P_{1/2,-} \circ P_{1/2,+})^2 \circ P_+^2, & \text{if } (\alpha,\beta) = (L_1, I) \end{cases} \tag{4}$$

For each symbol $\alpha \in \{L_1, L_2, X, E, I, S\}$ we define the h-set Q_α, where $Q_{L_1} = H_1$, $Q_{L_2} = H_2$, $Q_S = E_0$, $Q_X = F_0$, $Q_I = V_0$, $Q_E = G_0$.

Definition 4.1. *The bi-infinite sequence $(\alpha_i)_{i\in\mathbb{Z}}$ is called* admissible *if for every $i \in \mathbb{Z}$ there is an arrow from α_i to α_{i+1} on the graph presented in Fig. 3.*

The finite sequence $(\alpha_0, \alpha_1, \ldots, \alpha_n)$ is called admissible *if for every $i = 0, 1, \ldots, n-1$ there is an arrow from α_i to α_{i+1} on the graph presented in Fig. 3.*

Let Γ be the set of all admissible sequences $(\alpha_i)_{i\in\mathbb{Z}} \in \{L_1, L_2, X, E, I, S\}^{\mathbb{Z}}$.

Theorem 4.2. *For every $(\alpha_i)_{i\in\mathbb{Z}} \in \Gamma$ there exists a sequence $(x_i)_{i\in\mathbb{Z}}$ satisfying*

1. *$x_i \in |Q_{\alpha_i}|$ for $i \in \mathbb{Z}$,*
2. *$f_{\alpha_{i+1},\alpha_i}(x_i) = x_{i+1}$, for $i \in \mathbb{Z}$.*

Moreover, we have
periodic orbits: *if the sequence $(\alpha_i)_{i\in\mathbb{Z}}$ is periodic with the principal period k then the trajectory $(x_i)_{i\in\mathbb{Z}}$ may be chosen so that $x_k = x_0$, hence its trajectory is periodic*
homo- and heteroclinic orbits: *if the sequence $(\alpha_i)_{i\in\mathbb{Z}}$ is such that $\alpha_k = L_{i_-}$ for $k \leq k_-$ and $\alpha_k = L_{i_+}$ for $k \geq k_+$, where $i_-, i_+ \in \{1, 2\}$ then*

$$\lim_{k\to-\infty} x_k = L^*_{i_-}, \qquad \lim_{k\to\infty} x_k = L^*_{i_+}.$$

Proof. The same as [12, Theorem 7.1]. □

4.1 Resonance transitions.

Theorem 4.2 implies the possibility for a comet to move between various resonances. If we interpret staying close to L_1^* or L_2^* periodic orbits as the 1 : 1

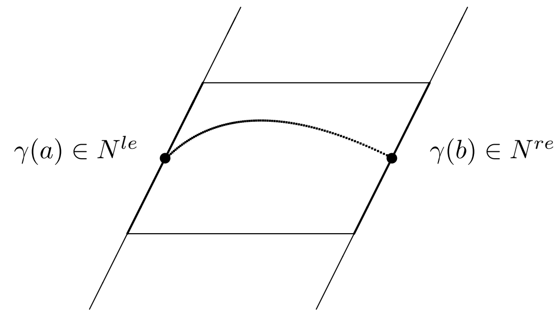

Figure 4: An h-set N and a horizontal curve γ in N.

resonance, then Theorem 4.2 says that the comet can travel between exterior and Sun regions in both directions and can move between $5:3$, $3:2$, $1:2$, $2:3$ and $1:1$ resonances in an arbitrary order.

5 Symmetric periodic and homoclinic orbits.

In Section 2.2 the symmetry property of PCR3BP and the associated Poincaré maps are described. In this section we give the proof of the existence of an infinite number of symmetric periodic and homoclinic orbits.

Definition 5.1. *Let $I \ni t \to u(t) \in \mathbb{R}^4$ be a solution of PCR3BP, where I is the maximal interval of the existence of the solution. An orbit $t \to u(t)$ is called R-*symmetric *iff*

$$\mathrm{Image}(u) = \{u(t) \mid t \in I\} = \{R(u(t)) \mid t \in I\} = R(\mathrm{Image}(u)).$$

In this section we apply the method for finding symmetric periodic, homo and heteroclinic orbits first introduced in [10, 11] for the planar case and later developed in [13] in multidimensional situation. We recall here the basic definitions.

Definition 5.2. *Let N be a h-set with one unstable and one stable direction and let $\gamma : [a,b] \to \mathbb{R}^2$ be a continuous curve. We say that γ is a horizontal curve in N if the following conditions hold:*

1. $\gamma((a,b)) \subset \mathrm{int}(|N|)$
2. *either* $\gamma(a) \in N^{le}$ *and* $\gamma(b) \in N^{re}$,
   or $\gamma(a) \in N^{re}$ *and* $\gamma(b) \in N^{le}$.

The geometry of this concept is shown in Fig. 4.

Definition 5.3. *Let N be a h-set with one unstable and one stable direction and let $\gamma : [a,b] \to \mathbb{R}^2$ be a continuous curve. We say that γ is a vertical curve in N if the following conditions hold:*

1. $\gamma((a,b)) \subset \mathrm{int}(|N|)$

2. *either* $\gamma(a) \in N^{te}$ *and* $\gamma(b) \in N^{be}$,
   or $\gamma(a) \in N^{te}$ *and* $\gamma(b) \in N^{be}$.

The following theorem is a special case of [13, Thm.3].

Theorem 5.4. *Assume $N_0, N_1, \ldots, N_k$ are h-sets with one unstable and one stable direction and*

$$N_0 \overset{f_0}{\Longleftrightarrow} N_1 \overset{f_1}{\Longleftrightarrow} \cdots \overset{f_{k-1}}{\Longleftrightarrow} N_k.$$

If $\gamma : [a,b] \to \mathbb{R}^2$ is a horizontal curve in N_0 and $\bar{\gamma} : [\bar{a},\bar{b}] \to \mathbb{R}^2$ is a vertical curve in N_k then there exists $t_0 \in (a,b)$ such that

$$(f_m \circ \cdots \circ f_0 \circ \gamma)(t_0) \in \mathrm{int}(|N_{m+1}|), \tag{5}$$

for $m = 0, \ldots, k-1$ and

$$(f_{k-1} \circ \cdots \circ f_0 \circ \gamma)(t_0) \in \bar{\gamma}((\bar{a},\bar{b})). \tag{6}$$

Theorem 5.4 was first proven in [10] for a planar case and direct covering relations. The generalization to a higher dimension with one unstable direction and the direct (forward) covering is presented in [11]. The proof of a general situation (i.e. direct and backward covering in multidimensional case) requires more sophisticated techniques and is presented in [13].

5.1 Symmetric periodic orbits.

In this section we will use Theorem 5.4 in order to prove the existence of an infinite number of geometrically different symmetric periodic orbits.

Before we state the main result in this section we introduce some notation. Let $(\alpha,\beta) \in \{L_1, L_2, X, E, I, S\}^2$ be an admissible sequence of symbols. Let the maps $f_{\beta,\alpha}$ be defined as in (4).

Notation: By $Q_\alpha \overset{f_{\beta,\alpha}}{\Longleftrightarrow} Q_\beta$ we will denote the chain of covering relations associated with the sequence (α,β), i.e.

$$Q_\alpha \overset{P_1}{\Longleftrightarrow} V_1 \overset{P_2}{\Longleftrightarrow} V_2 \overset{P_3}{\Longleftrightarrow} \cdots \overset{P_{k-1}}{\Longleftrightarrow} V_{k-1} \overset{P_k}{\Longleftrightarrow} Q_\beta,$$

where $f_{\beta,\alpha} = P_k \circ \ldots \circ P_1$ and V_i, $i = 1, \ldots, k-1$ are suitable h-sets.

Definition 5.5. *Let $f : X \to X$. By $\mathrm{Fix}(f)$ we will denote the set of fixed points of f, i.e.*

$$\mathrm{Fix}(f) = \{y \in X \mid f(y) = y\}.$$

Theorem 5.6. *Let $\phi : \mathbb{R} \times \mathbb{R}^4 \not\to \mathbb{R}^4$ denotes the local flow induced by the PCR3BP with $C = 3.03$ and $\mu = 0.0009537$. Assume $(\alpha_0, \alpha_1, \ldots, \alpha_n) \in$*

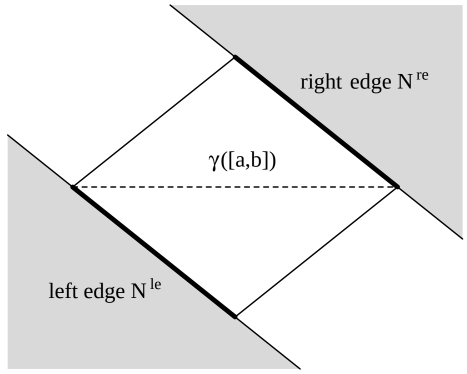

Figure 5: A symmetric h-set. The γ curve is both horizontal and vertical in N.

$\{S, I, X, E, L_1, L_2\}^n$, $n > 0$ *is admissible sequence of symbols. Then there exists a point* $x_0 \in |Q_{\alpha_0}|$ *such that*

$$\begin{aligned}(f_{\alpha_m,\alpha_{m-1}} \circ \cdots \circ f_{\alpha_1,\alpha_0})(x_0) &\in |Q_{\alpha_m}|,\\ (f^{-1}_{\alpha_{m-1},\alpha_m} \circ \cdots \circ f^{-1}_{\alpha_0,\alpha_1})(x_0) &\in |Q_{\alpha_m}|,\end{aligned} \tag{7}$$

for $m = 1, \ldots, n$, *i.e., the trajectory of* x_0 *is coded by the periodic sequence of symbols*

$$(\alpha_n, \ldots, \alpha_1, \alpha_0, \alpha_1, \ldots, \alpha_n). \tag{8}$$

Moreover, x_0 *is periodic and its orbit is* R*-symmetric.*

Proof. From the definitions of the h-sets used in the proof of homo- and heteroclinic chains it follows that the sets $H_1, H_2, V_0, G_0, E_0, F_0$ are R-symmetric. Therefore $\mathrm{Fix}(R)$ may be parameterized both as a horizontal and as a vertical curve in each of these sets (see Fig 5). Let $\gamma : [a, b] \to |Q_{\alpha_0}|$ be the horizontal curve in Q_{α_0} and $\bar{\gamma} : [\bar{a}, \bar{b}] \to |Q_{\alpha_n}|$ be the vertical curve in Q_{α_n}, such that $\gamma([a, b]) \cup \bar{\gamma}([\bar{a}, \bar{b}]) \subset \mathrm{Fix}(R)$. Now, Theorem 5.4 applied to the sequence

$$Q_{\alpha_0} \overset{f_{\alpha_1,\alpha_0}}{\Longleftrightarrow} Q_{\alpha_1} \overset{f_{\alpha_2,\alpha_1}}{\Longleftrightarrow} Q_{\alpha_2} \overset{f_{\alpha_3,\alpha_2}}{\Longleftrightarrow} \cdots \overset{f_{\alpha_n,\alpha_{n-1}}}{\Longleftrightarrow} Q_{\alpha_n}$$

implies that there exists a point $x_0 = \gamma(t_0) \in |Q_{\alpha_0}| \cap \mathrm{Fix}(R)$ such that

$$\begin{aligned}(f_{\alpha_m,\alpha_{m-1}} \circ \cdots \circ f_{\alpha_1,\alpha_0})(x_0) \in |Q_{\alpha_m}|, \quad \text{for} \quad m = 1, \ldots, n,\\ (f_{\alpha_n,\alpha_{n-1}} \circ \cdots \circ f_{\alpha_1,\alpha_0})(x_0) \in \bar{\gamma}((\bar{a}, \bar{b})) \subset \mathrm{Fix}(R).\end{aligned}$$

From the definition of $f_{\beta,\alpha}$ (see Eq.(4)) as a composition of suitable Poincaré maps it follows that there exists $T > 0$ such that

$$\phi(T, x_0) = (f_{\alpha_n,\alpha_{n-1}} \circ \cdots \circ f_{\alpha_1,\alpha_0})(x_0) \in \mathrm{Fix}(R).$$

Since R is the reversing symmetry of ϕ we obtain

$$\phi(T, x_0) = R(\phi(T, x_0)) = \phi(-T, R(x_0)) = \phi(-T, x_0)$$

which proves x_0 is periodic and its orbit is R-symmetric.

There remains to prove that the trajectory of x_0 is coded by the sequence (8), i.e. (7) is satisfied. We formulate this as a separate lemma. $\square$

Lemma 5.7. *Assume* $(\alpha_0, \ldots, \alpha_n)$ *is an admissible sequence of symbols. If* $x \in \mathrm{dom}(f_{\alpha_n,\alpha_{n-1}} \circ \cdots \circ f_{\alpha_1.\alpha_0})$ *then* $R(x) \in \mathrm{dom}(f^{-1}_{\alpha_{n-1},\alpha_n} \circ \cdots \circ f^{-1}_{\alpha_0,\alpha_1})$ *and*

$$(R \circ f_{\alpha_m,\alpha_{m-1}} \circ \cdots \circ f_{\alpha_1,\alpha_0})(x) = (f^{-1}_{\alpha_{m-1},\alpha_m} \circ \cdots \circ f^{-1}_{\alpha_0,\alpha_1} \circ R)(x),$$

for $m = 1, \ldots, n$. *Moreover, if* $x = R(x)$ *then*

$$(f^{-1}_{\alpha_{m-1},\alpha_m} \circ \cdots \circ f^{-1}_{\alpha_0,\alpha_1})(x) \in |Q_{\alpha_m}|$$

for $m = 1, \ldots, n$.

Proof. One observes that if (α, β) is admissible, then (β, α) is admissible, too. Moreover, from the definition of $f_{\beta,\alpha}$ (Eq. (4)) it follows that $R \circ f_{\beta,\alpha} = f^{-1}_{\alpha,\beta} \circ R$. Let $x \in \mathrm{dom}(f_{\alpha_k,\alpha_{k-1}} \circ \cdots \circ f_{\alpha_1.\alpha_0})$. Then

$$\begin{aligned} (R \circ f_{\alpha_m,\alpha_{m-1}} \circ \cdots \circ f_{\alpha_1,\alpha_0})(x) &= \\ (f^{-1}_{\alpha_{m-1},\alpha_m} \circ R \circ f_{\alpha_{m-1},\alpha_{m-2}} \circ \cdots \circ f_{\alpha_1,\alpha_0})(x) &= \\ \cdots = (f^{-1}_{\alpha_{m-1},\alpha_m} \circ \cdots \circ f^{-1}_{\alpha_0,\alpha_1} \circ R)(x), \end{aligned}$$

for $m = 1, \ldots, n$. If in addition $x = R(x)$ then $x \in \mathrm{dom}(f^{-1}_{\alpha_{m-1},\alpha_m} \circ \cdots \circ f^{-1}_{\alpha_0,\alpha_1})$ and

$$(f^{-1}_{\alpha_{m-1},\alpha_m} \circ \cdots \circ f^{-1}_{\alpha_0,\alpha_1})(x) \in R(|Q_{\alpha_m}|) = |Q_{\alpha_m}|.$$

$\square$

Remark 5.8. *Theorem 5.6 implies that there exist infinitely many geometrically different symmetric periodic orbits. This follows immediately from the fact that there exists an infinite number of admissible chains satisfying the assumptions of Theorem 5.6.*

5.2 Symmetric homoclinic orbits.

In this section we apply Theorem 5.4 in order to prove the existence of infinitely many geometrically different symmetric homoclinic orbits to L_1^* and L_2^* Lyapunov orbits.

The following theorem shows how to use the method of covering relations in order to prove the existence of symmetric homoclinic or heteroclinic orbits. Later we will apply it to Poincaré maps for PCR3BP.

Theorem 5.9. *Let* $N_0, N_1, \ldots, N_k$ *be h-sets, such that*

$$N_0 \overset{f_0}{\Longleftrightarrow} N_1 \overset{f_1}{\Longleftrightarrow} \cdots \overset{f_{k-1}}{\Longleftrightarrow} N_k \overset{f_k}{\Longleftrightarrow} N_k$$

and let $\gamma : [a,b] \to \mathbb{R}^2$ in N_0 be a horizontal curve in N_0. If f_k is hyperbolic (see [2, Def. 1]) on N_k, then there exists a point $x_0 \in \gamma((a,b))$ such that

$$(f_m \circ \cdots \circ f_0)(x_0) \in \mathrm{int}(|N_{m+1}|), \quad \textit{for } m = 0,1,\ldots,k-1,$$
$$(f_k^n \circ f_{k-1} \circ \cdots \circ f_0)(x_0) \in \mathrm{int}(|N_k|), \quad \textit{for } n > 0.$$

Moreover,

$$\lim_{n\to\infty} (f_k^n \circ f_{k-1} \circ \cdots \circ f_0)(x_0) = x_*,$$

where x_* is a unique fixed point of f_k in $|N_k|$.

Proof. From Theorem 5.4 it follows that for every $n > 0$ there exists $t_n \in [a,b]$ such that

$$(f_m \circ \cdots \circ f_0)(\gamma(t_n)) \in \mathrm{int}(|N_{m+1}|), \quad \text{for } m = 0,1,\ldots,k-1,$$
$$(f_k^n \circ f_{k-1} \circ \cdots \circ f_0)(\gamma(t_n)) \in \mathrm{int}(|N_k|).$$

Since $\gamma([a,b])$ is compact we can find $t_* \in [a,b]$ such that

$$(f_m \circ \cdots \circ f_0)(\gamma(t_*)) \in \mathrm{int}(|N_{m+1}|), \quad \text{for } m = 0,1,\ldots,k-1,$$
$$(f_k^n \circ f_{k-1} \circ \cdots \circ f_0)(\gamma(t_*)) \in \mathrm{int}(|N_k|), \quad \text{for } n > 0.$$

Since neither $f(\gamma(a)) \notin N_1$ nor $f(\gamma(b)) \notin N_1$ we get $t_* \in (a,b)$. Now, f_k is hyperbolic on N_k. Therefore by Theorem 3 in [2],

$$\lim_{n\to\infty} (f_k^n \circ f_{k-1} \circ \cdots \circ f_0)(x_0) = x_*,$$

where $x_0 := \gamma(t_*)$. □

Now we can state the basic result in this section.

Theorem 5.10. *Assume $(\alpha_0, \alpha_1, \ldots, \alpha_n)$ is an admissible nonconstant chain of symbols $\{S, I, X, E, L_1, L_2\}$, such that $\alpha_n \in \{L_1, L_2\}$. Then there exists a symmetric homoclinic orbit associated with the sequence of symbols*

$$(\ldots, \alpha_n, \alpha_n, \alpha_{n-1}, \ldots, \alpha_1, \alpha_0, \alpha_1, \ldots, \alpha_{n-1}, \alpha_n, \alpha_n, \ldots). \tag{9}$$

Proof. Let $\gamma : [a,b] \to |Q_{\alpha_0}|$ be a horizontal curve in Q_{α_0} such that $\gamma([a,b]) \subset \mathrm{Fix}(R)$. From Lemma 5.5 in [12] it follows that P_+ is hyperbolic on $|H_1| = Q_{L_1}$ and P_- is hyperbolic on $|H_2| = Q_{L_2}$. Since $\alpha_n \in \{L_1, L_2\}$ Theorem 5.9 there exists $x_0 \in \gamma((a,b))$ such that

$$(f_{\alpha_m,\alpha_{m-1}} \circ \cdots \circ f_{\alpha_1,\alpha_0})(x_0) \in \mathrm{int}(|Q_{\alpha_m}|), \quad \text{for } m = 1,\ldots,n,$$
$$(f^k_{\alpha_n,\alpha_n} \circ f_{\alpha_n,\alpha_{n-1}} \circ \cdots \circ f_{\alpha_1,\alpha_0})(x_0) \in \mathrm{int}(|Q_{\alpha_n}|), \quad \text{for } k > 0,$$
$$\lim_{k\to\infty} (f^k_{\alpha_n,\alpha_n} \circ f_{\alpha_n,\alpha_{n-1}} \circ \cdots \circ f_{\alpha_1,\alpha_0})(x_0) = L,$$

where $L = L_1^*$ or $L = L_2^*$ is a unique fixed point in $|Q_{\alpha_n}|$. Since $x_0 = R(x_0)$ Lemma 5.7 implies that

$$\begin{aligned}
(f^{-1}_{\alpha_{m-1},\alpha_m} \circ \cdots \circ f^{-1}_{\alpha_0,\alpha_1})(x_0) &\in \mathrm{int}(|Q_{\alpha_m}|), \quad \text{for } m = 1, \ldots, n, \\
(f^{-k}_{\alpha_n,\alpha_n} \circ f^{-1}_{\alpha_{n-1},\alpha_n} \circ \cdots \circ f^{-1}_{\alpha_0,\alpha_1})(x_0) &\in \mathrm{int}(|Q_{\alpha_n}|), \quad \text{for } k > 0, \\
\lim_{k\to\infty} (f^{-k}_{\alpha_n,\alpha_n} \circ f^{-1}_{\alpha_{n-1},\alpha_n} \circ \cdots \circ f^{-1}_{\alpha_0,\alpha_1})(x_0) &= R(L) = L
\end{aligned}$$

which proves that the trajectory of x_0 is a symmetric homoclinic orbit coded by the sequence of symbols (9). $\square$

Remark 5.11. *Theorem 5.10 implies that there exist infinitely many symmetric homoclinic orbits which are geometrically different. This follows immediately from the fact that there exists an infinite number of admissible chains satisfying the assumptions of Theorem 5.10.*

6 Technical data.

The computer assisted proofs of Lemma 3.2 and Lemma 3.4 will be not discussed here. All ideas involved in such proof were presented in [12]. The C++ sources containing the rigorous numerical proof of Lemmas from [12], Lemma 3.2 and Lemma 3.4 is available at [9].

The program uses the interval arithmetic and set algebra package developed at Jagiellonian University by CAPD group [1].

The whole proof took 14 minutes on the Pentium IV 2.4GHz processor (the gcc-3.3.1 compiler, PLD linux distribution). The reader should note, that the computation time reported here is considerably shorter than the one from [12] (40 minutes on 1.1GHz) despite the fact that here we have more conditions to check. This is a result of improved numerical algorithms and faster computer. For comparison purposes we had also run our program on 1.1GHz machine and the resulting computation time was 34 minutes. The gain (6 minutes) was mainly due to various optimizations in the algorithms. The main one was the use of *Evaluation 5* instead of *Evaluation 3* in C^1-Lohner algorithm.

7 Appendix. Reading resonances from the trajectory in rotating frame.

We describe the heuristic approach, which allows to read the resonance from the inspection of the trajectory in the rotating coordinate frame.

We assume that Jupiter and the comet move in the heliocentric inertial frame in the counterclockwise direction and the distance comet-Sun has well discernible maxima or minima along the trajectory. This means that an approximate ellipse on which the comet is moving has nonzero eccentricity.

Let R denote the resonance. Let T be an approximate period of the comet in the heliocentric frame. Let us recall that the period of the Jupiter is equal to 1. Hence

$$R = \frac{1}{T} \tag{10}$$

Then in the heliocentric inertial frame the average angular velocity of the comet is $\frac{2\pi}{T}$ and that of the Jupiter is equal to 2π.

For an approximate periodic trajectory of a comet in the rotating frame we introduce the following notation

- θ is the number of full turns around the Sun during the whole period. This number is positive for trajectories in the interior region and negative in the exterior region.
- M - the number of maxima (or minima) of the distance between the Sun and the comet.

Since the distance Sun-comet reaches the maximum (or minimum) only when the comet is at the aphelion (or perihelion), hence consecutive maxima (minima) occur with the period T. In the rotating frame the difference between the angular variables of the comet and Jupiter is equal to $2\pi\theta/M$. Observe that this difference is the same in both reference frames, the inertial one and the rotating one. Hence

$$\begin{aligned}
\frac{2\pi\theta}{M} &= \left(\frac{2\pi}{T} - 2\pi\right) T \\
\frac{\theta}{M} &= 1 - T \\
T &= \frac{M-\theta}{M}.
\end{aligned}$$

Hence finally

$$R = \frac{M}{M-\theta}. \tag{11}$$

Let us apply (11) to Figures 1 and 2. For interior homoclinics we count the maxima and for exterior homoclinic we count the minima. We have

- the interior homoclinic orbit in Figure 1: $\theta = 1$, $M = 3$. Hence $R = \frac{3}{2} = 3:2$.
- the interior homoclinic orbit in Figure 2: $\theta = 2$, $M = 5$. Hence $R = \frac{5}{3} = 5:3$.
- the exterior homoclinic orbit in Figure 1: $\theta = -1$, $M = 1$. Hence $R = \frac{1}{2} = 1:2$.
- the exterior homoclinic orbit in Figure 2: $\theta = -1$, $M = 2$. Hence $R = \frac{2}{3} = 2:3$.

References

[1] CAPD - Computer Assisted Proofs in Dynamics, a package for rigorous numeric, `http://capd.wsb-nlu.edu.pl`.

[2] Z. Galias and P. Zgliczynski, *Abundance of homoclinic and heteroclinic orbits and rigorous bounds for the topological entropy for the Henon map*, Nonlinearity, 14 (2001), 909–932

[3] R. DeVogelaere, *On the structure of symmetric periodic solutions of conservative systems*, in: Contribution to the theory of nonlinear oscillations, vol. 4, Princeton, Princeton University Press, 1958

[4] W. S. Koon, M. W. Lo, J. E. Marsden and S. D. Ross, *Heteroclinic Connections between Periodic Orbits and Resonance Transitions in Celestial Mechanics*, Chaos, 10(2000), no. 2, 427–469.

[5] J.S.W. Lamb, *Reversing symmetries in dynamical systems*, J. Phys. A:Math. Gen. 25, 925–937 , 1992

[6] J.S.W. Lamb, *Reversing symmetries in dynamical systems*, PhD Thesis, Amsterdam University, 1994

[7] J. Moser, *On the generalization of a theorem of Liapunov*, Comm. Pure Appl. Math., 11 (1958), 257–271

[8] D. Stoffer and U. Kirchgraber, *Possible chaotic motion of comets in the Sun Jupiter system - an efficient computer-assisted approach*, Nonlinearity, 17 (2004) 281-300.

[9] D. Wilczak, `http://www.wsb-nlu.edu.pl/~dwilczak.`

[10] D. Wilczak, *Chaos in the Kuramoto–Sivashinsky equations – a computer assisted proof*, J. Diff. Eq, Vol. 194, 433-459 (2003).

[11] D. Wilczak, *Symmetric heteroclinic connections in the Michelson system – a computer assisted proof*, to appear in SIAM J. App. Dyn Sys.

[12] D. Wilczak and P. Zgliczyński, *Heteroclinic Connections between Periodic Orbits in Planar Restricted Circular Three Body Problem - A Computer Assisted Proof*, Commun. Math. Phys. 234, 37-75 (2003).

[13] D. Wilczak and P. Zgliczyński, *Topological method for symmetric periodic orbits for maps with a reversing symmetry*, submitted, available at `http://www.wsb-nlu.edu.pl/~dwilczak.`