Geometric proof for normally hyperbolic invariant manifolds

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Abstract

We present a new proof of the existence of normally hyperbolic manifolds and their whiskers for maps. Our result is not perturbative. Based on the bounds on the map and its derivative, we establish the existence of the manifold within a given neighbourhood. Our proof follows from a graph transform type method and is performed in the state space of the system. We do not require the map to be invertible. From our method follows also the smoothness of the established manifolds, which depends on the smoothness of the map, as well as rate conditions, which follow from bounds on the derivative of the map. Our method is tailor made for rigorous, interval arithmetic based, computer assisted validation of the needed assumptions.

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1. Introduction

The goal of our paper is to present a geometric proof of the existence of normally hyperbolic invariant manifolds (NHIMs) for maps, in a vicinity of an approximate invariant manifold. There are four important features of our approach: 1) we do not assume that the given map is a perturbation of some other map for which we have a normally hyperbolic invariant manifold, 2) we do not require that the map is invertible, 3) the assumptions can be rigorously checked with computer assistance if our approximation of the invariant manifold is good enough 4) our method does not require high order smoothness. From our proof follows the high order smoothness of the manifolds (provided that the map is suitably smooth), but it is enough to consider C^1 bounds for the proof of their existence.

In the standard approach to the proof of various invariant manifold theorems, all considerations are done in suitable function spaces or sequences spaces. Moreover the

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existence of the invariant manifold for nearby map (or ODE) is usually assumed, see for example [4, 16, 20] and the references given there. Typically these proofs do not give any computable bounds for the size of perturbation for which the invariant manifold exists.

Our result is in similar spirit to a number of results for establishing invariant manifolds that have recently emerged, which assume that there exists a manifold that is 'approximately' invariant, and provide conditions that ensure the existence of a true invariant manifold within a given neighborhood. In [1] Bates, Lu and Zeng present such approach within a context of semiflows, which makes their method general and applicable to infinite dimensional systems and PDEs. Compared to [1] our results is more explicit. Contrary to [1], where main theorems about NHIM require that some constants are sufficiently small depending on other constants, in our main theorem we just have several explicit inequalities between pairs of constants. In [3, 11, 12, 13, 14] Calleja, Celletti, Haro, de la Llave, Figueras, Fontich and Sire provide a framework and results of establishing existence of whiskered tori with quasi periodic dynamics, which is suitable for computer assisted validation. Our approach however allows for more general dynamics. All above proofs are based on constructions in suitable function spaces.

In contrast to the above mentioned approach, in our method the whole proof is made in the phase space. This method is not entirely new. For example, a similar approach is adapted in the proof of Jones [17] in the context of slow-fast system of ODEs. Jones though considered a perturbation of a normally hyperbolic invariant manifold. In [5, 8] an approach in the same spirit as in this paper has been applied to establish existence of topologically normally hyperbolic invariant manifolds. These results are based on topological arguments and do not establish the smoothness and the foliations of the invariant manifolds. Similar approach has been applied by Berger and Bounemoura [2], where persistence and smoothness of invariant manifolds is established using geometric and topological methods. The result relies though on a perturbation of a normally hyperbolic invariant manifold.

The method in this paper is based on two types of conditions. The first are the topological conditions, which we refer to as 'covering relations'. These ensure that we have good topological alignment of the coordinates of the set within which we establish the existence of the manifold. The second type of conditions are based on the first derivative of the map and we refer to these as the 'rate conditions'. Our rate conditions are in the same spirit to those of Fenichel [9, 10]. They measure the strength of the hyperbolic contraction and expansion (within a neighborhood in which we search for our manifold), in comparison to the dynamics on the normal coordinates. The stronger the hyperbolicity is, the higher is the order of the smoothness that can be established.

Our construction of the manifolds follows from a graph transform type method. We prove that the manifolds emerge from passing to the limit of graphs in appropriate coordinates. This construction follows primarily from the covering conditions. To prove that the manifolds are Lipschitz, we show that our graphs are contained in cones (this is the approach that was taken in [8]). The novelty of this paper lies in the proof of the higher order smoothness. In our proof, this follows from establishing appropriate cone conditions for the graphs. We define higher order cones, which span around Taylor expansions of the graphs. We prove that these cones are preserved as we iterate the graphs. (Verification of this fact follows from our rate conditions.) We then show that higher order cone conditions imply higher order smoothness of the graphs, and that this smoothness is preserved as we pass to the limit.

We emphasize that in order to apply our method it is sufficient have a good guess on the position of the manifold and good estimates of the first derivative of the map. We do not require any estimates on its higher order derivatives. It is sufficient that we know that the map is appropriately smooth, and that the first derivative implies our rate conditions.

We believe that this approach is very well suited for computer assisted (rigorous, interval arithmetic based) validation of the needed assumptions. Similar approach has already been successfully applied in [5, 8] in the setting of the rotating Hénon map, in [6] to establish the center manifold in the restricted three body problem, in [7] in the setting of a driven logistic map or in [21] to establish a hyperbolic attractor in the Kuznetsov system. All these results follow from verification of cone conditions based on the estimates of the derivative. We believe that such estimates also imply rate conditions, hence the method from this paper can easily be used to establish smoothness and fibration of the manifolds. At present moment it appears that of approaches to NHIMs mentioned earlier in the introduction only the one based on the parameterization method [3, 11, 12, 13, 14] are ready for computer assisted proof. This method is however restricted by the requirement of the quasi-periodic dynamics on the invariant torus.

The paper is organized as follows. After preliminaries introducing basic notations in Sections 3 we state our main result for the case of the torus. Sections 4–10 contain the proof of our main result for the torus. In Section 11 we show to how our construction can be carried over from the torus to arbitrary compact manifold. We decided to work first with the torus rather then a general manifold, because in that case we can have a global coordinate chart and the main ideas are not mixed with the technicalities connected with different charts. In Section 12 we apply our method to the rotating Hénon map.

2. Preliminaries

2.1. Notations

For a point $p = (x, y)$ we shall use $\pi_x(p) = x$ to denote the projection of p onto the x coordinate. We use a notation $B_k(p, R)$ for a ball of radius R in $\mathbb{R}^k$, centered at p. To simplify notations we shall write $B_k(R) = B_k(0, R)$. For a set $A \subset \mathbb{R}^k$ we shall write $\overline{A}$ for closure of A and ∂A for the boundary of A. Throughout the work, the notation $\| \, \|$ will stand for the Euclidean norm, unless explicitly stated otherwise. For a set $U \subset \mathbb{R}^n$ and a continuous function (homotopy) $h : [0, 1] \times U \to \mathbb{R}^n$, for $\alpha \in [0, 1]$ we shall write $h_\alpha(x)$ for $h(\alpha, x)$.

Definition 1. *Let $f : \mathbb{R}^n \to \mathbb{R}^k$ be a C^1 function. We define the interval enclosure of the derivative Df on $U \subset \mathbb{R}^n$, as a set $[Df(U)] \subset \mathbb{R}^{k\times n}$, defined as*

$$[Df(U)] = \left\{ A = (a_{ij})_{\substack{i=1,\ldots,k \\ j=1,\ldots,n}} : a_{ij} \in \left[\inf_{x\in U} \frac{\partial f_j}{\partial x_j}(x), \sup_{x\in U} \frac{\partial f_j}{\partial x_j}(x) \right] \right\}.$$

Definition 2. *Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. Let $\|x\|$ be any norm on $\mathbb{R}^n$, then we define*

$$m(A) = \max \left\{ L \in \mathbb{R} : \|Ax\| \geq L\|x\| \text{ for all } x \in \mathbb{R}^n \right\}.$$

For an interval matrix $\mathbf{A} \subset \mathbb{R}^{k\times n}$ we set

$$m(\mathbf{A}) = \inf_{A\in\mathbf{A}} m(A).$$

2.2. Taylor formula

In this section we quickly set up the notations for the Taylor formula. Let

$$f : \mathbb{R}^n \to \mathbb{R}^m$$

The k-th derivative of f at p is a symmetric k-linear operator. On the basis it is defined as

$$D^k f(p)\,(e_{i_1}, \ldots, e_{i_k}) = \left(\frac{\partial^k f_1(p)}{\partial x_{i_1} \ldots \partial x_{i_k}}, \ldots, \frac{\partial^k f_m(p)}{\partial x_{i_1} \ldots \partial x_{i_k}} \right).$$

Using the following multi-index notation $j = (j_1, \ldots, j_n) \in \mathbb{N}^n$, $h = (h_1, \ldots, h_n) \in \mathbb{R}^n$

$$|j| = j_1 + \ldots + j_n, \qquad h^j = h_1^{j_1} h_2^{j_2} \ldots h_n^{j_n},$$
$$j! = j_1! j_1! \ldots j_n!, \qquad \frac{\partial^j g}{\partial x^j} = \frac{\partial^{|j|} g}{\partial x_1^{j_1} \ldots \partial x_n^{j_n}},$$

we can write out the value of $D^k f(p)$ on the diagonal as

$$D^k f(p)(h^{[k]}) = D^k f(p)(\underbrace{h, \ldots, h}_{k}) = \begin{pmatrix} \sum_{|j|=k} \frac{k!}{j!} h^j \frac{\partial^j f_1}{\partial x^j}(p) \\ \vdots \\ \sum_{|j|=k} \frac{k!}{j!} h^j \frac{\partial^j f_m}{\partial x^j}(p) \end{pmatrix}.$$

The above formula is convenient to formulate the multi-dimensional version of the Taylor formula:

$$f(p+h) = f(p) + T_{f,m,p}(h) + R_{f,m,p}(h),$$

where $T_{f,m,p}$ stands for the Taylor expansion of order m

$$T_{f,m,p}(h) = \sum_{k=1}^{m} \frac{D^k f(p)}{k!}(h^{[k]}),$$

and the reminder $R_{f,m,p}(h)$ can be computed in the integral form

$$R_{f,m,p}(h) = \int_0^1 \frac{(1-t)^m}{m!} D^{m+1} f(p+th) \left(h^{[m+1]}\right) dt.$$

For $f : \mathbb{R}^n \supset \mathrm{dom}(f) \to \mathbb{R}^m$ and a set $A \subset \mathrm{dom}(f)$ we define

$$\begin{aligned} \left\| D^k f(p) \right\| &= \sup \left\{ \left\| D^k f(p) \left(h^{[k]}\right) \right\| : \|h\| = 1 \right\}, \\ \|f\|_{C^m} &= \sup_{p \in \mathrm{dom}(f)} \max_{|k| \le m} \left\| \frac{\partial^k f}{\partial x^k}(p) \right\|, \\ \|f(A)\|_{C^m} &= \|f|_A\|_{C^m}. \end{aligned}$$

3. Main results

The goal of this section is to set up the structure for our NHIM, which will be diffeomorphic with a manifold Λ. To make the setup as simple as possible we will focus on the special case where Λ is a torus. This will simplify notations in many of the arguments, since we will not need to work with various local charts. We shall prepare the setup though in a way that will allow for a straightforward generalization to an arbitrary manifold without boundary. This will be done in section 11.

3.1. Definitions and setup

In the simple situation when Λ is an c-dimensional torus, we are in a convenient situation, since we have a covering

$$\varphi : \mathbb{R}^c \to \Lambda = (\mathbb{R}/\mathbb{Z})^c ,$$

which gives us the set of charts being the restriction of φ to balls B in $\mathbb{R}^c$, which are small enough so that $\varphi : B \to \Lambda$ is a homeomorphism on its image. We introduce a notation $R_\Lambda > 0$ for a radius such that $\varphi_{|B(\lambda,R_\Lambda)}$ is a homeomorphism onto its image. When Λ is a torus, we can simply take $R_\lambda = \frac{1}{2}$. Introducing the notation R_Λ here though will simplify our future discussion in section 11, where we generalize the results.

Let $R < \frac{1}{2}R_\Lambda$ and denote by D the set

$$D = \Lambda \times \overline{B}_u(R) \times \overline{B}_s(R),$$

where $\overline{B}_n(R)$ stands for a closed ball of radius R, centered at zero, in $\mathbb{R}^n$. We consider a C^{k+1} map, for $k \geq 1$,

$$f : D \to \Lambda \times \mathbb{R}^u \times \mathbb{R}^s.$$

Throughout the paper we shall use the notation $z = (\lambda, x, y)$ to denote points in D. This means that notation λ will stand for points on Λ, notation x for points in $\mathbb{R}^u$, and y for points in $\mathbb{R}^s$. We will write f as (f_λ, f_x, f_y) , where f_λ, f_x, f_y stand for projections onto Λ, $\mathbb{R}^u$ and $\mathbb{R}^s$, respectively. On $\mathbb{R}^c \times \mathbb{R}^u \times \mathbb{R}^s$ we will use the Euclidian norm.

In view of the further generalization to arbitrary manifold let us stress that our set D can be thought as a subset of the trivial vector bundle $\mathbb{T}_c \times \mathbb{R}^u \times \mathbb{R}^s$.

Definition 3. *The set of points which are in the same good chart with point $q \in D$ will be denoted by*

$$P(q) = \{z \in D \mid \|\pi_\lambda z - \pi_\lambda q\| \leq R_\Lambda/2\}.$$

Let $L \in \left(\frac{2R}{R_\Lambda}, 1\right)$, and let us define

$$\mu_{s,1} = \sup_{z\in D}\left\{\left\|\frac{\partial f_y}{\partial y}(z)\right\| + \frac{1}{L}\left\|\frac{\partial f_y}{\partial(\lambda,x)}(z)\right\|\right\},$$

$$\mu_{s,2} = \sup_{z\in D}\left\{\left\|\frac{\partial f_y}{\partial y}(z)\right\| + L\left\|\frac{\partial f_{(\lambda,x)}}{\partial y}(z)\right\|\right\},$$

$$\xi_{u,1} = \inf_{z\in D}\left\{m\left(\frac{\partial f_x}{\partial x}(z)\right) - \frac{1}{L}\left\|\frac{\partial f_x}{\partial(\lambda,y)}(z)\right\|\right\},$$

$$\xi_{u,1,P} = \inf_{z\in D} m\left[\frac{\partial f_x}{\partial x}(P(z))\right] - \frac{1}{L}\sup_{z\in D}\left\|\frac{\partial f_x}{\partial(\lambda,y)}(z)\right\|,$$

$$\xi_{u,2} = \inf_{z\in D}\left\{m\left(\frac{\partial f_x}{\partial x}(z)\right) - L\left\|\frac{\partial f_{(\lambda,y)}}{\partial x}(z)\right\|\right\},$$

$$\mu_{cs,1} = \sup_{z\in D} \left\{ \left\| \frac{\partial f_{(\lambda,y)}}{\partial (\lambda,y)}(z) \right\| + L \left\| \frac{\partial f_{(\lambda,y)}}{\partial x}(z) \right\| \right\},$$

$$\mu_{cs,2} = \sup_{z\in D} \left\{ \left\| \frac{\partial f_{(\lambda,y)}}{\partial (\lambda,y)}(z) \right\| + \frac{1}{L} \left\| \frac{\partial f_{x}}{\partial (\lambda,y)}(z) \right\| \right\},$$

$$\xi_{cu,1} = \inf_{z\in D} \left\{ m\left(\frac{\partial f_{(\lambda,x)}}{\partial (\lambda,x)}(z) \right) - L \left\| \frac{\partial f_{(\lambda,x)}}{\partial y}(z) \right\| \right\},$$

$$\xi_{cu,1,P} = \inf_{z\in D} m\left[\frac{\partial f_{(\lambda,x)}}{\partial (\lambda,x)}(P(z)) \right] - L \sup_{z\in D} \left\| \frac{\partial f_{(\lambda,x)}}{\partial y}(z) \right\|,$$

$$\xi_{cu,2} = \inf_{z\in D} \left\{ m\left(\frac{\partial f_{(\lambda,x)}}{\partial (\lambda,x)}(z) \right) - \frac{1}{L} \left\| \frac{\partial f_{y}}{\partial (\lambda,x)}(z) \right\| \right\}.$$

Remark 4. *Throughout the work, the $L \in (\frac{2R}{R_\Lambda}, 1)$ is a fixed constant. We shall later see that L is associated with Lipschitz bounds on the established manifolds (hence the choice of notation).*

The key to the naming of the constants is the following:

- $\xi_{u,\cdot}$, $\xi_{cu,\cdot}$ - the constants describing lower bound on the expansion in the unstable or center-unstable directions.
- $\mu_{s,\cdot}$, $\mu_{cs,\cdot}$ - the constants describing upper bound for contraction constant in the stable or center-stable direction.
- The number 1 or 2 as second lower index is used according to the following rule: 1, when both partial derivatives are of the same component of f, for example $f_{(\lambda,x)}$ in $\mu_{cs,1}$, while 2 is used the differentiation is done with respect to the same block of variables of various components of f.
- $\xi_{u,1}$, $\xi_{u,2}$, $\xi_{cu,1}$, $\xi_{cu,2}$ are the expansion bounds and $\mu_{s.1}$, $\mu_{s,2}$, $\mu_{cs,1}$, $\mu_{cs,2}$ are the contraction bounds, that are used for the establishing of smoothness of invariant manifolds and their fibres.
- $\xi_{u,1,P}$, $\xi_{cu,1,P}$ are more stringent bounds (i.e. $\xi_{\cdot,1,P} \leq \xi_{\cdot,1}$). They are used to ensure lower bounds on the expansion on the x and (λ, x) coordinates.

Definition 5. *We say that f satisfies rate conditions of order $k \geq 1$ if $\xi_{u,1}$, $\xi_{u,1,P}$, $\xi_{u,2}$, $\xi_{cu,1}$, $\xi_{cu,1,P}$, $\xi_{cu,2}$ are strictly positive, and for all $k \geq j \geq 1$ holds*

$$\mu_{s,1} < 1 < \xi_{u,1,P}, \tag{1}$$

$$\frac{\mu_{cs,1}}{\xi_{u,1,P}} < 1, \qquad \frac{\mu_{s,1}}{\xi_{cu,1,P}} < 1, \tag{2}$$

$$\frac{(\mu_{cs,1})^{j+1}}{\xi_{u,2}} < 1, \qquad \frac{\mu_{s,2}}{(\xi_{cu,1})^{j+1}} < 1, \tag{3}$$

$$\frac{\mu_{cs,2}}{\xi_{u,1}} < 1, \qquad \frac{\mu_{s,1}}{\xi_{cu,2}} < 1. \tag{4}$$

We say that f satisfies rate conditions of order zero, if only (1)–(2) are satisfied.

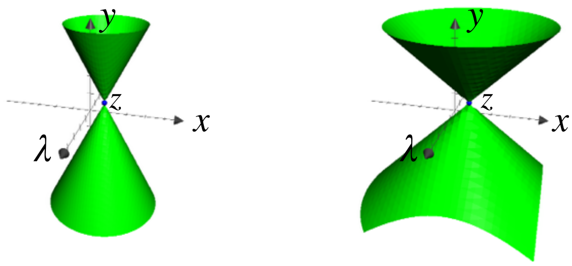

Figure 1: The stable cone $J_s(z, M)$ for $M = \frac{1}{2}$ on the left, and $M = 1$ on the right.

We introduce the following notation:

$$J_s(z, M) = \{(\lambda, x, y) : \|(\lambda, x) - \pi_{\lambda,x} z\| \leq M \|y - \pi_y z\|\}, \tag{5}$$
$$J_u(z, M) = \{(\lambda, x, y) : \|(\lambda, y) - \pi_{\lambda,y} z\| \leq M \|x - \pi_x z\|\}. \tag{6}$$

We shall refer to $J_s(z, M)$ as a stable cone of slope M at z, and to $J_u(z, M)$ as an unstable cone of slope M at z. The cones are depicted in Figures 1, 2.

Remark 6. *For any $z^* \in D$ and $z \in J_u(z^*, M)$ with $M \leq \frac{1}{L}$ we see that*

$$\|\pi_\lambda(z^* - z)\| \leq \|\pi_{(\lambda,y)}(z - z^*)\| \leq 1/L \|\pi_x (z - z^*)\| \leq 2R/L < R_\Lambda.$$

This means that

$$J_u(z^*, M) \cap D \subset \overline{B}_c(\lambda^*, R_\Lambda) \times \overline{B}_u(R) \times \overline{B}_s(R),$$

for $\lambda = \pi_\lambda z^*$. Similarly, for $M \leq \frac{1}{L}$

$$J_s(z^*, M) \cap D \subset \overline{B}_c(\lambda^*, R_\Lambda) \times \overline{B}_u(R) \times \overline{B}_s(R).$$

In other words, intersections of unstable (stable) cones with D are contained in sets on which we can use a single chart $P(z^*)$.

Definition 7. *We say that a sequence $\{z_i\}_{i=-\infty}^{0}$ is a (full) backward trajectory of a point z if $z_0 = z$, and $f(z_{i-1}) = z_i$ for all $i \leq 0$.*

Definition 8. *We define the center-stable set in D as*

$$W^{cs} = \{z : f^n(z) \in D \text{ for all } n \in \mathbb{N}\}.$$

Definition 9. *We define the center-unstable set in D as*

$$W^{cu} = \{z : \text{there is a full backward trajectory of } z \text{ in } D\}.$$

Definition 10. *We define the maximal invariant set in D as*

$$\Lambda^* = \{z : \text{there is a full trajectory of } z \text{ in } D\}.$$

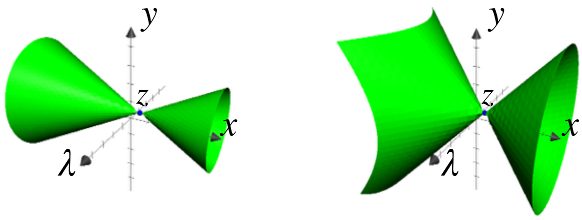

Figure 2: The stable cone $J_u(z, M)$ for $M = \frac{1}{2}$ on the left, and $M = 1$ on the right.

Definition 11. *Assume that $z \in W^{cs}$. We define the stable fiber of z as*

$$W_z^s = \{p \in D : f^n(p) \in J_s(f^n(z), 1/L) \cap D \text{ for all } n \in \mathbb{N}\}.$$

Definition 12. *Assume that $z \in W^{cu}$. We define the unstable fiber of z as*

$$\begin{aligned} W_z^u = \{ & p \in D : \exists \text{ backward trajectory } \{p_i\}_{i=-\infty}^0 \text{ of } p \text{ in } D, \\ & \text{for any such backward trajectory} \\ & \text{and any backward trajectory } \{z_i\}_{i=-\infty}^0 \text{ of } z \text{ in } D \\ & \text{holds } \; p_i \in J_u(z_i, 1/L) \cap D \}. \end{aligned}$$

The definitions of W_z^s and W_z^u are related to cones, which is a nonstandard approach, the standard one is through convergence rates. We will show that our definition implies the convergence rate as in the standard theory.

Under our assumptions it will turn out that f is injective on W^{cu}. Therefore the backward orbit in the definition of W_q^u is unique.

Definition 13. *We say that f satisfies backward cone conditions if the following condition is fulfilled:*

If $z_1, z_2, f(z_1), f(z_2) \in D$ and $f(z_1) \in J_s(f(z_2), 1/L)$ then

$$z_1 \in J_s(z_2, 1/L).$$

Remark 14. *The assumption that f satisfies backward cone conditions will turn out to be necessary in order to ensure that the established NHIM is a graph over Λ. After formulating our main Theorem 16, we follow up with Examples 21, 22, in which we demonstrate that without backward cone conditions the result cannot be obtained.*

For $\lambda \in \Lambda$ we define the following sets:

$$\begin{aligned} D_\lambda &= \overline{B}_c(\lambda, R_\Lambda) \times \overline{B}_u(R) \times \overline{B}_s(R), \\ D_\lambda^+ &= \overline{B}_c(\lambda, R_\Lambda) \times \overline{B}_u(R) \times \partial B_s(R), \\ D_\lambda^- &= \overline{B}_c(\lambda, R_\Lambda) \times \partial \overline{B}_u(R) \times B_s(R). \end{aligned}$$

Definition 15. *We say that f satisfies covering conditions if for any $z \in D$ there exists a $\lambda^* \in \Lambda$, such that the following conditions hold:*

For $U = J_u(z, 1/L) \cap D$, there exists a homotopy h

$$h : [0,1] \times U \to B_c(\lambda^*, R_\Lambda) \times \mathbb{R}^u \times \mathbb{R}^s,$$

and a linear map $A : \mathbb{R}^u \to \mathbb{R}^u$ which satisfy:

1. $h_0 = f|_U$,
2. *for any $\alpha \in [0,1]$,*

$$h_\alpha\left(U \cap D^-_{\pi_\theta z}\right) \cap D_{\lambda^*} = \emptyset, \tag{7}$$

$$h_\alpha(U) \cap D^+_{\lambda^*} = \emptyset, \tag{8}$$

3. $h_1(\lambda, x, y) = (\lambda^*, Ax, 0)$,
4. $A(\partial B_u(R)) \subset \mathbb{R}^u \setminus \overline{B}_u(R)$.

In the above definition a reasonable choice for λ^* will be $\lambda^* = \pi_\lambda f(z)$. In fact any point sufficiently close to $\pi_\lambda f(z)$ will be also good.

3.2. The main theorem

Theorem 16. *(Main result) Let $k \geq 1$ and $f : D \to \Lambda \times \mathbb{R}^u \times \mathbb{R}^s$ be a C^{k+1} map. If f satisfies covering conditions, rate conditions of order k and backward cone conditions, then W^{cs}, W^{cu} and Λ^* are C^k manifolds, which are graphs of C^k functions*

$$w^{cs} : \Lambda \times \overline{B}_s(R) \to \overline{B}_u(R),$$
$$w^{cu} : \Lambda \times \overline{B}_u(R) \to \overline{B}_s(R),$$
$$\chi : \Lambda \to \overline{B}_u(R) \times \overline{B}_s(R),$$

meaning that

$$W^{cs} = \left\{(\lambda, w^{cs}(\lambda, y), y) : \lambda \in \Lambda, y \in \overline{B}_s(R)\right\},$$
$$W^{cu} = \left\{(\lambda, x, w^{cu}(\lambda, y)) : \lambda \in \Lambda, x \in \overline{B}_u(R)\right\},$$
$$\Lambda^* = \{(\lambda, \chi(\lambda)) : \lambda \in \Lambda\}.$$

Moreover, $f_{|W^{cu}}$ is an injection, w^{cs} and w^{cu} are Lipschitz with constants L, and χ is Lipschitz with the constant $\frac{\sqrt{2}L}{\sqrt{1-L^2}}$. The manifolds W^{cs} and W^{cu} intersect transversally, and $W^{cs} \cap W^{cu} = \Lambda^*$.

The manifolds W^{cs} and W^{cu} are foliated by invariant fibers W^s_z and W^u_z. The W^s_z and W^u_z are graphs of C^k functions

$$w^s_z : \overline{B}_s(R) \to \Lambda \times \overline{B}_u(R),$$
$$w^u_z : \overline{B}_u(R) \to \Lambda \times \overline{B}_s(R),$$

meaning that

$$W^s_z = \left\{(w^s_z(y), y) : y \in \overline{B}_s(R)\right\},$$
$$W^u_z = \left\{(\pi_\lambda w^u_z(x), x, \pi_y w^u_z(x)) : x \in \overline{B}_u(R)\right\}.$$

The functions w_z^s and w_z^u are Lipschitz with constants $1/L$. Moreover,

$$\begin{aligned} W_z^s &= \{p \in D : f^n(p) \in D \text{ for all } n \geq 0, \text{ and} \\ & \quad \exists n_0, \exists C > 0 \text{ (which can depend on } p) \\ & \quad \text{s.t. for } n \geq n_0,\ f^n(p), f^n(z) \text{ are in the same chart and} \\ & \quad \|f^n(p) - f^n(z)\| \leq C\mu_{s,1}^n \}, \end{aligned}$$

and if $\{z_i\}_{i=-\infty}^0$ is the unique backward trajectory of z in D, then

$$\begin{aligned} W_z^u &= \{p \in W^{cu} : \text{such that the unique backward trajectory } \{p_i\}_{i=-\infty}^0 \\ & \quad \text{of } p \text{ in } D \text{ satisfies the following condition} \\ & \quad \exists n_0 \geq 0, \exists\ C > 0 \text{ (which can depend on } p) \\ & \quad \text{s.t. for } n \geq n_0,\ p_{-n}, z_{-n} \text{ are in the same chart and} \\ & \quad \|p_{-n} - z_{-n}\| \leq C\xi_{u,1,P}^{-n} \Big\}. \end{aligned}$$

Observe that bound on $L \in \left(\frac{2R}{R_\Lambda}, 1\right)$ gives us lower bounds for the Lipschitz constants for functions w^{cu}, w^{cs}, w^u, w^s, which is clearly an overestimate for the case when $\mathbb{T} \times \{0\} \times \{0\}$ is our NHIM. This lower bound is a consequence of choices we have made when formulating Theorem 16, as we did not want to introduce different constants for each type of cones, plus several inequalities between them. However, below we give conditions which allow to obtain better Lipschitz constants.

Theorem 17. *Let $M \in (0, 1/L)$ and*

$$\mu = \sup_{z \in D} \left\{ \left\| \frac{\partial f_y}{\partial y}(z) \right\| + M \left\| \frac{\partial f_y}{\partial (\lambda, x)}(z) \right\| \right\}, \tag{9}$$

$$\xi = \inf_{z \in D} m\left(\left[\frac{\partial f_{(\lambda,x)}}{\partial (\lambda, x)}(P(z)) \right] \right) - \frac{1}{M} \sup_{z \in D} \left\| \frac{\partial f_{(\lambda,x)}}{\partial y}(z) \right\|. \tag{10}$$

If assumptions of Theorem 16 hold true and also $\frac{\xi}{\mu} > 1$, then the function w_z^s from Theorem 16 is Lipschitz with constant M.

Theorem 18. *Let $M \in (0, 1/L)$ and*

$$\begin{aligned} \xi &= \inf_{z \in D} m\left(\left[\frac{\partial f_x}{\partial x}(P(z)) \right] \right) - M \sup_{z \in D} \left\| \frac{\partial f_x}{\partial (\lambda, y)}(z) \right\|, \\ \mu &= \sup_{z \in D} \left\{ \left\| \frac{\partial f_{(\lambda,y)}}{\partial (\lambda, y)}(z) \right\| + \frac{1}{M} \left\| \frac{\partial f_{(\lambda,y)}}{\partial x}(z) \right\| \right\}. \end{aligned}$$

If assumptions of Theorem 16 hold true and also $\frac{\xi}{\mu} > 1$, then the function w_z^u from Theorem 16 is Lipschitz with constant M.

Theorem 19. *Let $M \in (0, L)$ and*

$$\begin{aligned} \xi &= \inf_{z \in D} m\left[\frac{\partial f_{(\lambda,x)}}{\partial (\lambda, x)}(P(z)) \right] - M \sup_{z \in D} \left\| \frac{\partial f_{(\lambda,x)}}{\partial y}(z) \right\|, \\ \mu &= \sup_{z \in D} \left\{ \left\| \frac{\partial f_y}{\partial y}(z) \right\| + \frac{1}{M} \left\| \frac{\partial f_y}{\partial (\lambda, x)}(z) \right\| \right\}. \end{aligned}$$

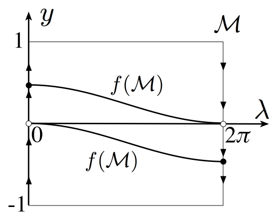

Figure 3: The Möbius strip from Example 21.

If assumptions of Theorem 16 hold true and also $\frac{\xi}{\mu} > 1$*, then the function* w^{cu} *from Theorem 16 is Lipschitz with constant* M*.*

Theorem 20. *Let* $M \in (0, L)$ *and*

$$\begin{aligned}
\xi &= \inf_{z\in D} m\left[\frac{\partial f_x}{\partial x}(P(z))\right] - \frac{1}{M}\sup_{z\in D}\left\|\frac{\partial f_x}{\partial(\lambda,y)}(z)\right\|, \\
\mu &= \sup_{z\in D}\left\{\left\|\frac{\partial f_{(\lambda,y)}}{\partial(\lambda,y)}(z)\right\| + M\left\|\frac{\partial f_{(\lambda,y)}}{\partial x}(z)\right\|\right\}
\end{aligned}$$

If assumptions of Theorem 16 hold true and also $\frac{\xi}{\mu} > 1$*, then the function* w^{cs} *from Theorem 16 is Lipschitz with constant* M*.*

3.3. Comments on the inequalities and examples

Let $J_s^c(z, M)$ and $J_u^c(z, M)$ stand for the complements of $J_s(z, M)$ and $J_u(z, M)$, respectively. We now comment about what various inequalities in Definition 5 of rate conditions mean and what they are needed for:

- $\mu_{cs,1} < \xi_{u,1,P}$: the forward invariance of $J_u(z, 1/L)$ (Corollary 34). $\xi_{u,1,P} > 1$: the expansion in $J_u(z, 1/L)$ for x - coordinate (Lemma 36). This is needed for the proof of the existence of W^{cs} (Section 7).
- $\xi_{cu,1,P} > \mu_{s,1}$: the forward invariance of $J_s^c(z, 1/L)$ (Corollary 35). $\mu_{s,1} < 1$: the contraction in y-direction in $J_s(z, 1/L)$ (Lemma 37). This is needed for the proof of the existence of W^{cu} (Section 6).
- $\frac{\mu_{s,2}}{(\xi_{cu,1})^{j+1}} < 1$, $j = 1, \ldots, k$: the C^k-smoothness of W^{cu} (Lemma 48).
- $\frac{(\mu_{cs,1})^{j+1}}{\xi_{u,2}} < 1$, $j = 1, \ldots, k$: the C^k-smoothness of W^{cs} (Lemma 52).
- $\frac{\mu_{cs,1}}{\xi_{u,1,P}} < 1$: the existence of fibers W_q^u (Lemma 57). $\frac{\mu_{cs,2}}{\xi_{u,1}} < 1$: the C^k smoothness of W_q^u (Lemma 59).
- $\frac{\mu_{s,1}}{\xi_{cu,1,P}} < 1$: the existence of fibers W_q^s (Lemma 64). $\frac{\mu_{s,1}}{\xi_{cu,2}} < 1$: the C^k smoothness of W_q^s (Lemma 66).

We now give two examples which show that in the absence of the backward cone condition, the invariant set might not be a graph over Λ.

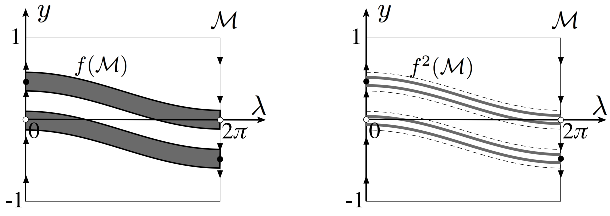

Figure 4: The Möbius strip from Example 22.

Example 21. *Consider a Möbius strip $\mathcal{M}$ depicted in Figure 3. The Möbius strip is parameterised by (λ, y), with $\lambda \in [0, 2\pi)$ and $y \in \overline{B}_s(R) = [-1, 1]$. The two vertical edges which are glued together are depicted with arrows.*

Let $\xi > 2$ be a constant. We consider a map $f : \mathcal{M} \times \overline{B}_u(R) \to \mathcal{M} \times \overline{B}_u(R)$,

$$f((\lambda, y), x) = \left(\left(2\lambda, \frac{1}{4} + \frac{1}{4}\cos\lambda\right), \xi x\right).$$

On the unstable coordinate x, f is simply a linear expansion. The stable coordinate y is the vertical coordinate on $\mathcal{M}$. The coordinates (λ, y) and x are decoupled. Intuitively, on (λ, y) the map does the following. It projects $\mathcal{M}$ into a horizontal circle, and then stretches it and wraps twice around $\mathcal{M}$ as in Figure 3. For such a map all assumptions of Theorem 16 are fulfilled, except for the backward cone conditions. We see that in the absence of the backward cone conditions, the invariant manifold can be a set which is not a graph over Λ.

Example 22. *We can modify Example 21 slightly to obtain a more interesting result. Assume that $|\mu| < \frac{1}{4}$, $\xi > 2$, and consider*

$$f((\lambda, y), x) = \left(\left(2\lambda, \frac{1}{4} + \frac{1}{4}\cos\lambda + \mu y\right), \xi x\right).$$

The difference is that instead of collapsing $\mathcal{M}$ completely, we contract in the y coordinate. Then $f(\mathcal{M})$ will be the set depicted on the left plot of Figure 4. The second iterate is shown in the right plot of Figure 4. We thus see that the invariant set has a Cantor structure.

Above examples are artificial. Similar features though can be found for instance in the Kuznetzov system (see [18],[21]), where we have a hyperbolic invariant set in $\mathbb{R}^3$, which has a Cantor set structure. By adding the assumption that f satisfies backward cone conditions we rule out such cases, and establish NHIMs that are graphs over Λ.

4. Cone evolution

In this section we introduce the notion of "higher order cones". These will be used to control the smoothness of established manifolds. The section contains auxiliary results. The construction of the manifolds is performed in Sections 6, 7 and 8.

4.1. Unstable cones

In this section we introduce the cones. We formulate the results in a setting where we have two coordinates x and y, instead of the three coordinates λ, x, y from Section 3. This is because the results are formulated in more general terms. Later, we shall apply these taking $\mathrm{x} = (\lambda, x)$ and $\mathrm{y} = y$ (or, in other instances, $\mathrm{x} = x$ and $\mathrm{y} = (\lambda, y)$) in our construction of the manifolds. Thus, the subtle change of font in x and y plays an important role.

Let

$$\mathcal{P}_m : \mathbb{R}^u \to \mathbb{R}^s, \qquad \mathcal{P}_m(0) = 0.$$

be a polynomial of degree m.

Definition 23. *We define an unstable cone of order m at z, spanned on $\mathcal{P}_m$, with a bound $M > 0$, as a set of the form*

$$J_u(z, \mathcal{P}_m, M) = \left\{ z + (\mathrm{x}, \mathrm{y} + \mathcal{P}_m(\mathrm{x})) : \quad \|\mathrm{y}\| \leq M \|\mathrm{x}\|^{m+1} \right\}. \tag{11}$$

Remark 24. *We emphasize that the index m in $J_u(z_0, \mathcal{P}_m, M)$ is important since it stands for the order m of the cone. Cones of order m are always associated with polynomials of degree m. Let us also observe that if we take a polynomial (of degree zero) $\mathcal{P}_0 = 0$, then for $\mathrm{x} = x$ and $\mathrm{y} = (\lambda, y)$ the cones defined in (6) and (11) are the same:*

$$J_u(z, \mathcal{P}_0 = 0, M) = J_u\left(z, M\right).$$

For $\delta > 0$ we define

$$J_u(z_0, \mathcal{P}_m, M, \delta) = J_u(z_0, \mathcal{P}_m, M) \cap \overline{B}(0, \delta).$$

The above defined cones are devised to control higher order derivatives of functions. The following lemmas explain this relation.

Lemma 25. *Assume that $g : \mathbb{R}^u \supset \mathrm{dom}(g) \to \mathbb{R}^s$ is a C^{m+1} function. Let $\mathrm{x}_0 \in \mathrm{dom}(g)$, $M > \|D^{m+1} g(\mathrm{x}_0)\|$. Then there exists a $\delta > 0$, such that*

$$\{(\mathrm{x}, g(\mathrm{x})) \mid \|\mathrm{x} - \mathrm{x}_0\| \leq \delta\} \subset J_u(z_0, \mathcal{P}_m, M/\left(m+1\right)!) \tag{12}$$

for $z_0 = (\mathrm{x}_0, g(\mathrm{x}_0))$ and $\mathcal{P}_m(\mathrm{x}) = T_{g,m,\mathrm{x}_0}(\mathrm{x})$.

Proof. The proof is given in Appendix B. ■

The crucial property of J_u is that Lemma 25 can be reversed to give bounds on the higher order derivatives:

Lemma 26. *Assume that $g : \mathbb{R}^u \supset \mathrm{dom}(g) \to \mathbb{R}^s$ is a C^{m+1} function. Let $\mathrm{x}_0 \in \mathrm{dom}(g)$ and assume that there exists $\delta > 0$, such that*

$$\{(\mathrm{x}, g(\mathrm{x})) \mid \|\mathrm{x} - \mathrm{x}_0\| \leq \delta\} \subset J_u(z_0, \mathcal{P}_m, M), \tag{13}$$

where $z_0 = (\mathrm{x}_0, g(\mathrm{x}_0))$ and $\mathcal{P}_m(\mathrm{x}) = T_{g,m,\mathrm{x}_0}(\mathrm{x})$. Then there exists a constant C (which depends only on m and s), such that for any $j_1, \ldots, j_{m+1} \in \{1, \ldots, u\}$

$$\left\| \frac{\partial^{m+1} g(\mathrm{x}_0)}{\partial \mathrm{x}_{i_1} \ldots \partial \mathrm{x}_{i_{m+1}}} \right\| \leq CM.$$

Proof. See Appendix C. ■

We now show that when f satisfies certain conditions, unstable cones are mapped into themselves. We start with a simple case of cones of order zero.

Theorem 27. *Let $U \subset \mathbb{R}^u \times \mathbb{R}^s$ be a convex neighborhood of zero and assume that $f : U \to \mathbb{R}^u \times \mathbb{R}^s$ is a C^1 map satisfying $f(0) = 0$. If for $M > 0$*

$$m\left[\frac{\partial f_{\mathrm{x}}}{\partial \mathrm{x}}(U)\right] - M \sup_{\mathrm{x}\in U}\left\|\frac{\partial f_{\mathrm{x}}}{\partial \mathrm{y}}(z)\right\| \geq \xi, \tag{14}$$

$$\sup_{z\in U}\left\{\left\|\frac{\partial f_{\mathrm{y}}}{\partial \mathrm{y}}(z)\right\| + \frac{1}{M}\left\|\frac{\partial f_{\mathrm{y}}}{\partial \mathrm{x}}(z)\right\|\right\} \leq \mu, \tag{15}$$

and

$$\frac{\xi}{\mu} > 1, \tag{16}$$

then

$$f(J_u(0, \mathcal{P}_0 = 0, M) \cap U) \subset \mathrm{int} J_u(0, \mathcal{R}_0 = 0, M) \cup \{0\}.$$

Proof. See Appendix D. ■

The following theorem shows that, under appropriate assumptions, cones of order m map to other cones, with the same bound M.

Theorem 28. *Let $D \subset \mathbb{R}^u \times \mathbb{R}^s$ be a convex bounded neighborhood of zero and assume that $f : D \to \mathbb{R}^u \times \mathbb{R}^s$ is a C^{m+1} map satisfying $f(0) = 0$ and $\|f(D)\|_{C^{m+1}} \leq C$. Assume that we have two polynomials $\mathcal{P}_m, \mathcal{R}_m : \mathbb{R}^u \to \mathbb{R}^s$ with coefficients bounded by C, such that*

$$\mathrm{graph}(T_{\pi_y f\circ(\mathrm{id},\mathcal{P}_m),m,0}) \subset \mathrm{graph}\left(\mathcal{R}_m\right). \tag{17}$$

If for $\xi > 0$, and $\rho < 1$

$$\begin{aligned} m\left(\tfrac{\partial f_{\mathrm{x}}}{\partial \mathrm{x}}(0) + \tfrac{\partial f_{\mathrm{x}}}{\partial \mathrm{y}}(0) D\mathcal{P}_m(0)\right) &\geq \xi, \\ \left\|\tfrac{\partial f_{\mathrm{x}}}{\partial \mathrm{y}}(0)\right\| &\leq B, \\ \left\|\tfrac{\partial f_{\mathrm{y}}}{\partial \mathrm{y}}(0) - D\mathcal{R}_m(0)\tfrac{\partial f_{\mathrm{x}}}{\partial \mathrm{y}}(0)\right\| &\leq \mu, \end{aligned} \tag{18}$$

and

$$\frac{\mu}{\xi^{m+1}} < \rho, \tag{19}$$

then there exists a constant $M^* = M^*(C, B, 1/\xi, \rho)$, such that for any $M > M^*$ there exists a $\delta = \delta(M, C, B, 1/\xi)$ such that

$$f(J_u(0, \mathcal{P}_m, M, \delta) \cap D) \subset J_u(0, \mathcal{R}_m, M).$$

Moreover, if for some $K > 0$ holds $C, B, \frac{1}{\xi} \in [0, K]$, then M^* depends only on K and ρ.

Proof. See Appendix E. ■

4.2. Stable cones

Let

$$\mathcal{Q}_m : \mathbb{R}^s \to \mathbb{R}^u, \qquad \mathcal{Q}_m(0) = 0.$$

be a polynomial of degree m.

Definition 29. *We define a stable cone of order m at z_0, spanned on $\mathcal{Q}_m$, with a bound $M > 0$, as a set of the form*

$$J_s(z_0, \mathcal{Q}_m, M) = \left\{ z_0 + (\mathrm{x} + \mathcal{Q}_m(\mathrm{y}), \mathrm{y}) : \quad \|\mathrm{x}\| \leq M \|\mathrm{y}\|^{m+1} \right\}.$$

For $\delta > 0$ we define

$$J_s(z_0, \mathcal{Q}_m, M, \delta) = J_s(z_0, \mathcal{Q}_m, M) \cap \overline{B}(0, \delta),$$

and we also denote complements of the cones as

$$\begin{aligned} J_s^c(z_0, \mathcal{Q}_m, M) &= \mathbb{R}^u \times \mathbb{R}^s \setminus J_s(z_0, \mathcal{Q}_m, M), \\ J_s^c(z_0, \mathcal{Q}_m, M, \delta) &= \overline{B}(0, \delta) \setminus J_s(z_0, \mathcal{Q}_m, M, \delta). \end{aligned}$$

Mirror results to Lemmas 25, 26 can be formulated for stable cones:

Lemma 30. *Assume that $g : \mathbb{R}^s \supset \mathrm{dom}(g) \to \mathbb{R}^u$ is a C^{m+1} function. Let $\mathrm{y}_0 \in \mathrm{dom}(g)$, $M > \|D^{m+1} g(\mathrm{y}_0)\|$. Then there exists $\delta > 0$, such that*

$$\{(g(\mathrm{y}), \mathrm{y}) \mid \|\mathrm{y} - \mathrm{y}_0\| \leq \delta\} \subset J_s(z_0, \mathcal{P}_m, M/(m+1)!)$$

for $z_0 = (g(\mathrm{y}_0), \mathrm{y}_0)$ and $\mathcal{P}_m(\mathrm{y}) = T_{g,m,\mathrm{y}_0}(\mathrm{y})$.

Lemma 31. *Assume that $g : \mathbb{R}^s \supset \mathrm{dom}(g) \to \mathbb{R}^u$ is a C^{m+1} function. Let $\mathrm{y}_0 \in \mathrm{dom}(g)$ and assume that there exists $\delta > 0$, such that*

$$\{(g(\mathrm{y}), \mathrm{y}) \mid \|\mathrm{y} - \mathrm{y}_0\| \leq \delta\} \subset J_u(z_0, \mathcal{P}_m, M),$$

where $z_0 = (g(\mathrm{y}_0), \mathrm{y}_0)$ and $\mathcal{P}_m(\mathrm{y}) = T_{g,m,\mathrm{y}_0}(\mathrm{y})$. Then there exists a constant C (which depends only on m), such that for any $j_1, \ldots, j_{m+1} \in \{1, \ldots, u\}$

$$\left\| \frac{\partial^{m+1} g(\mathrm{y}_0)}{\partial \mathrm{y}_{i_1} \ldots \partial \mathrm{y}_{i_{m+1}}} \right\| \leq CM.$$

Since proofs of Lemmas 30, 31 follow from mirror arguments to the proofs of Lemmas 25, 26, we omit their proofs.

We now give the following theorems, which are in similar spirit to Theorem 27, 28. The difference is that they concern images of **complements** of cones (and not images of the cones themselves, as is the case in Theorems 27, 28.)

Theorem 32. *Let $U \subset \mathbb{R}^u \times \mathbb{R}^s$ be a convex neighborhood of zero and assume that $f : U \to \mathbb{R}^u \times \mathbb{R}^s$ is a C^1 map satisfying $f(0) = 0$. Assume that for $M > 0$*

$$m\left(\left[\frac{\partial f_{\mathrm{x}}}{\partial \mathrm{x}}(U)\right]\right) - \frac{1}{M} \sup_{z \in U} \left\| \frac{\partial f_{\mathrm{x}}}{\partial \mathrm{y}}(z) \right\| \quad \geq \quad \xi, \tag{20}$$

$$\sup_{z \in U} \left\{ \left\| \frac{\partial f_{\mathrm{y}}}{\partial \mathrm{y}}(z) \right\| + M \left\| \frac{\partial f_{\mathrm{y}}}{\partial \mathrm{x}}(z) \right\| \right\} \quad \leq \quad \mu, \tag{21}$$

and
$$\frac{\xi}{\mu} > 1, \tag{22}$$
then
$$f(\overline{J_s^c(0, \mathcal{Q}_0 = 0, M)} \cap U) \subset J_s^c(0, \mathcal{R}_0 = 0, M) \cup \{0\}.$$

Proof. The result follows from Theorem 27. Details are given in Appendix F. ■

Theorem 33. *Let $D \subset \mathbb{R}^u \times \mathbb{R}^s$ be a convex bounded neighborhood of zero and assume that $f : D \to \mathbb{R}^u \times \mathbb{R}^s$ is a C^{m+1} map satisfying $f(0) = 0$ and $\|f(D)\|_{C^{m+1}} \leq C$. Assume that we have two polynomials $\mathcal{Q}_m, \mathcal{R}_m : \mathbb{R}^s \to \mathbb{R}^u$ with coefficients bounded by C, such that*
$$\mathrm{graph}(T_{\pi_x f \circ (\mathcal{Q}_m, \mathrm{id}), m, 0}) \subset \mathrm{graph}\left(\mathcal{R}_m\right).$$

If for $\xi > 0$, and $\rho < 1$
$$\begin{aligned} m\left(\tfrac{\partial f_{\mathrm{x}}}{\partial \mathrm{x}}(0) + D\mathcal{P}_m(0)\tfrac{\partial f_{\mathrm{y}}}{\partial \mathrm{x}}(0)\right) &\geq \xi, \\ \left\|\tfrac{\partial f_{\mathrm{y}}}{\partial \mathrm{x}}(0)\right\| &\leq B, \\ \left\|\tfrac{\partial f_{\mathrm{y}}}{\partial \mathrm{y}}(0) - \tfrac{\partial f_{\mathrm{y}}}{\partial \mathrm{x}}(0) D\mathcal{Q}_m(0)\right\| &\leq \mu, \end{aligned}$$
and
$$\frac{\mu^{m+1}}{\xi} < \rho, \tag{23}$$
then there exists a constant $M^* = M^*(C, B, 1/\xi, \rho)$, such that for any $M > M^*$ there exists $\delta = \delta(M)$ such that $\delta = \delta(M, C, B, 1/\xi)$
$$f(J_s^c(0, \mathcal{Q}_m, M, \delta) \cap U) \subset J_s^c(0, \mathcal{R}_m, M).$$
Moreover, if for some $K > 0$ holds $C, B, \frac{1}{\xi} \in [0, K]$, then M^* depends only on K and ρ.

Proof. The proof is given in Appendix G. ■

4.3. Center-stable and center-unstable cones

We now return to the setting in which we have three coordinates (λ, x, y). Recall that in these coordinates stable cones $J_s(z, M)$ and unstable cones $J_u(z, M)$ were defined using (5–6). In addition we define center-stable and center-unstable cones as
$$\begin{aligned} J_{cs}(z, M) &= \{(\lambda, x, y) : \|x - \pi_x z\| < M\|(\lambda, y) - \pi_{\lambda,y} z\|\} \cup \{z\}, \\ J_{cu}(z, M) &= \{(\lambda, x, y) : \|y - \pi_y z\| < M\|(\lambda, x) - \pi_{\lambda,x} z\|\} \cup \{z\}, \end{aligned}$$
respectively.

Observe that $J_{cs}(z, M) = J_u^c(z, 1/M) \cup \{z\}$ and $J_{cu}(z, M) = J_s^c(z, 1/M) \cup \{z\}$. We see that $J_{cs}(z, M)$ and $J_{cu}(z, M)$ as defined above are not contained in domain of single good chart. However we will always take intersections of these cones with the domain of a good chart.

As in section 3, we consider a C^{k+1} map, with $k \geq 0$,

$$f : D \to \Lambda \times \mathbb{R}^u \times \mathbb{R}^s,$$

where $D = \Lambda \times \overline{B}_u(R) \times \overline{B}_s(R)$. We rewrite some of the results from sections 4.1, 4.2 in terms of coordinates (λ, x, y), formulating them as corollaries.

Corollary 34. *If f satisfies the rate conditions of order $k = 0$ (see Definition 5) then for any $z \in D$*

$$f\left(J_u\left(z, 1/L\right) \cap D\right) \subset \mathrm{int} J_u\left(f(z), 1/L\right) \cup \{f(z)\}.$$

In alternative notation, $f\left(J_{cs}^c\left(z, L\right) \cap D\right) \subset \mathrm{int} J_{cs}^c\left(f(z), L\right) \cup \{f(z)\}$.

Proof. This follows from Theorem 27, taking coordinates $\mathrm{x} = x$, $\mathrm{y} = (\lambda, y)$ and constants $M = 1/L$, $\xi = \xi_{u,1,P}$, $\mu = \mu_{cs,1}$. The assumption (16) of Theorem 27 follows from the rate condition (2). ■

Corollary 35. *If f satisfies the rate conditions of order $k = 0$ then for any $z \in D$*

$$f\left(\overline{J_{cu}\left(z, L\right)} \cap \overline{B}_c(\pi_\lambda z, R_\Lambda) \times \overline{B}_u(R) \times \overline{B}_s(R)\right) \subset J_{cu}\left(f(z), L\right).$$

In alternative notation,

$$f(\overline{J_s^c(z, 1/L)} \cap \overline{B}_c(\pi_\lambda z, R_\Lambda) \times \overline{B}_u(R) \times \overline{B}_s(R)) \subset J_s^c(f(z), 1/L) \cup \{f(z)\}.$$

Proof. This follows from Theorem 32, taking coordinates $\mathrm{x} = (\lambda, x)$, $\mathrm{y} = y$ and constants $M = 1/L$, $\xi = \xi_{cu,1,P}$, $\mu = \mu_{s,1}$. ■

Lemma 36. *If f satisfies the rate conditions of order $k = 0$ and two points $z_1, z_2 \in D$ satisfy $z_1 \in J_u\left(z_2, 1/L\right)$, then*

$$\left\|\pi_x\left(f(z_1) - f(z_2)\right)\right\| \geq \xi_{u,1,P}\left\|\pi_x(z_1 - z_2)\right\|.$$

Proof. See Appendix H. ■

Lemma 37. *If f satisfies the rate conditions of order $k = 0$ and two points $z_1, z_2 \in D$, satisfy $z_1 \in J_s\left(z_2, 1/L\right)$ and $f(z_1) \in J_s\left(f(z_2), 1/L\right)$, then*

$$\left\|\pi_y\left(f(z_1) - f(z_2)\right)\right\| \leq \mu_{s,1}\left\|\pi_y(z_1 - z_2)\right\|.$$

Proof. See Appendix I. ■

Lemma 38. *Assume that z_1, z_2 are in the same chart. If f satisfies the rate conditions of order $k = 0$ and $z_1 \in J_{cu}\left(z_2, L\right)$, then*

$$\left\|\pi_{(\lambda,x)}\left(f(z_1) - f(z_2)\right)\right\| \geq \xi_{cu,1,P}\left\|\pi_{(\lambda,x)}\left(z_1 - z_2\right)\right\|.$$

Proof. See Appendix J. ■

Lemma 39. *Assume that z_1, z_2 and $f(z_1), f(z_2)$ are in the same charts. If f satisfies the rate conditions of order $k = 0$ and $z_1 \in J_{cs}(z_2, L)$, then*

$$\left\|\pi_{(\lambda,y)}\left(f(z_1) - f(z_2)\right)\right\| \leq \mu_{cs,1}\left\|\pi_{(\lambda,y)}\left(z_1 - z_2\right)\right\|.$$

Proof. See Appendix K. ■

5. Discs

In this section we introduce the notion of discs. These will be the building blocks for the construction of our invariant manifolds.

Definition 40. *We say that a continuous function $b : \overline{B}_u(R) \to D$ is a horizontal disc if for any $x \in \overline{B}_u(R)$*

$$\pi_x b(x) = x \qquad \text{and} \qquad b\left(\overline{B}_u(R)\right) \subset J_u\left(b(x), 1/L\right). \tag{24}$$

Definition 41. *We say that a continuous function $b : \overline{B}_s(R) \to D$ is a vertical disc if for any $y \in \overline{B}_s(R)$*

$$\pi_y b(y) = y \qquad \text{and} \qquad b\left(\overline{B}_s(R)\right) \subset J_s\left(b(y), 1/L\right). \tag{25}$$

By Remark 6, we see that any horizontal or vertical disc can be contained in a set on which we can use a single chart. This fact will prove important in Section 11 where we reformulate our results for more general Λ.

In our former works [8, 22] the disks as defined above where said to satisfy cone conditions.

Definition 42. *We say that a continuous function $b : \Lambda \times \overline{B}_u(R) \to D$ is a center-horizontal disc if for any $(\lambda, x) \in \Lambda \times \overline{B}_u(R)$*

$$\pi_{(\lambda,x)} b(\lambda, x) = (\lambda, x)$$

and

$$b\left(\overline{B}_c(\lambda, R_\Lambda) \times \overline{B}_u(R)\right) \subset J_{cu}\left(b(\lambda, x), L\right). \tag{26}$$

Definition 43. *We say that a continuous function $b : \Lambda \times \overline{B}_s(R) \to D$ is a center-vertical disc if for any $(\lambda, y) \in \Lambda \times \overline{B}_s(R)$*

$$\pi_{(\lambda,y)} b(\lambda, y) = (\lambda, y)$$

and

$$b\left(\overline{B}_c(\lambda, R_\Lambda) \times \overline{B}_s(R)\right) \subset J_{cs}\left(b(\lambda, y), L\right). \tag{27}$$

Lemma 44. *Assume that $b : \overline{B}_u(R) \to D$ is a horizontal disc. If f satisfies the covering conditions and the rate conditions of order $l = 0$, then there exists a horizontal disc $b^* : \overline{B}_u(R) \to D$ such that $f \circ b(\overline{B}_u(R)) \cap D = b^*(\overline{B}_u(R))$. Moreover, if f and b are C^k, then so is b^*.*

Proof. The proof is given in appendix Appendix L. ■

The disc b^* from Lemma 44 is a graph transform of b. From now on we shall use the notation $\mathcal{G}_h(b)$ instead of b^*.

Lemma 45. *Assume that $b : \Lambda \times \overline{B}_u(R) \to D$ is a center-horizontal disc. If f satisfies the covering conditions, backward cone conditions and the rate conditions of order $l = 0$, then there exists a center-horizontal disc $b^* : \Lambda \times \overline{B}_u(R) \to D$ such that*

$$f \circ b(\Lambda \times \overline{B}_u(R)) \cap D = b^*(\Lambda \times \overline{B}_u(R)).$$

Moreover, if f and b are C^k, then so is b^*.

Proof. The proof is given in appendix Appendix M. ■

From now on we shall use the notation $\mathcal{G}_{ch}(b)$ instead of b^* for the disc from Lemma 45.

6. Center-unstable manifold

In this section we prove the existence and smoothness the manifold W^{cu} from Theorem 16. The proof follows from a graph transform type method, in which we take successive iterates of center-horizontal discs, and these converge to the center unstable manifold.

We start with the following lemma, which establishes the existence of W^{cu}.

Lemma 46. *Assume that f satisfies covering conditions, backward cone conditions and rate conditions of order $l \geq 0$. Let b_i be the sequence of center-horizontal discs defined as $b_0(\lambda, x) = (\lambda, x, 0)$, $b_{i+1} = \mathcal{G}_{ch}(b_i)$ for $i > 0$. Then b_i converge uniformly to a center-horizontal disc $(\lambda, x) \to (\lambda, x, w^{cu}(\lambda, x))$, where*

$$w^{cu} : \Lambda \times \overline{B}_u(R) \to \overline{B}_s(R).$$

Moreover

$$W^{cu} = \left\{(\lambda, x, w^{cu}(\lambda, x)) : \Lambda \times \overline{B}_u(R)\right\}.$$

Proof. We use a notation $\theta = (\lambda, x)$. We will show that $\pi_y b_i$ is a Cauchy sequence in the supremum norm, which converges to W^{cu}.

Let us fix $\theta \in \Lambda \times \overline{B}_u(R)$. For any $k \in \mathbb{N}$, since b_k is center-horizontal disk, there exists a finite backward orbit $\{q_i^k\}_{i=-k,\ldots,0}$, such that

$$q_0^k = b_k(\theta), \quad \pi_\theta(q_0^k) = \theta.$$

From the backward cone condition it follows that for any $i < 0$ points $\{q_i^k\}$ for $k \geq |i|$ are in the same chart and

$$q_i^{k_1} \in J_s(q_i^{k_2}, 1/L), \qquad k_1, k_2 \geq |i|.$$

Therefore we have

$$\|q_i^{k_1} - q_i^{k_2}\| \leq (1 + 1/L)\, \|\pi_y(q_i^{k_1} - q_i^{k_2})\|. \tag{28}$$

From Lemma 37 it follows that for $j \in \mathbb{Z}_- \cup \{0\}$, and $k_2 > k_1 \geq |j|$ holds

$$\begin{aligned} \|\pi_y q_j^{k_1} - \pi_y q_j^{k_2}\| &= \|\pi_y f^{k_1+j}(q_{-k_1}^{k_1}) - \pi_y f^{k_1+j}(q_{-k_1}^{k_2})\| \leq \\ &\leq (\mu_{s,1})^{k_1+j} \|\pi_y(q_{-k_1}^{k_1} - q_{-k_1}^{k_2})\| \leq 2R(\mu_{s,1})^{k_1+j}. \end{aligned} \tag{29}$$

From (29) and (28) it follows that for each $j \in \mathbb{Z}_- \cup \{0\}$ holds

$$\|q_j^{k_1} - q_j^{k_2}\| \leq (1 + 1/L) 2R(\mu_{s,1})^{k_1+j}, \qquad k_2 > k_1 \geq |j|. \tag{30}$$

Since $q_0^k = b_k(\theta)$ condition (30) establishes uniform convergence of b_k to b^*, moreover also the backward orbits form a Cauchy sequence and converge to full backward orbit of $b^*(\theta)$.

From the above it follows also that $\pi_y b_k$ converge uniformly to a continuous function $w^{cu}(\theta) = \pi_y b^*(\theta)$.

Assume now that we have a $z \in D$ that has a full backward trajectory $\{z_k\}_{k=-\infty}^0$ in D. We need to show that $z = (\theta^*, w^{cu}(\theta^*))$ for some $\theta^* \in \Lambda \times \overline{B}_u(R)$. Let $z^* = (\theta^*, w^{cu}(\theta^*))$ for $\theta^* = \pi_\theta z$. We will show that $z = z^*$. Since $\pi_\theta z = \pi_\theta z^*$,

$$z \in J_s(z^*, 1/L).$$

Then for the backward trajectory $\{z_k^*\}_{k=-\infty}^0$ of z^*, by the backward cone conditions,

$$z_k \in J_s(z_k^*, 1/L) \qquad \text{for } k = 0, -1, -2, \dots .$$

By Lemma 37, this implies that

$$\|\pi_y(z^* - z)\| \leq (\mu_{s,1})^k \|\pi_y(z_k^* - z_k)\| \leq 2(\mu_{s,1})^k .$$

Since $\mu_{s,1} < 1$, we see that $z^* = z$.

Passing to the limit in the cone condition (26) for b_k one can see that

$$b^*\left(\overline{B}_c(\lambda, R_\Lambda) \times \overline{B}_u(R)\right) \subset \overline{J_{cu}(b(\lambda, x))}.$$

Since W^{cu} is invariant under f, by Corollary 35 we obtain (26) for b^*. Thus b^* is a center-horizontal disc. ■

Lemma 47. *Assume that f is C^{k+1} and satisfies covering conditions, backward cone conditions and rate conditions of order $l \geq 0$. Let $m \leq k$. Let b_i be the sequence of center-horizontal discs defined as $b_0(\lambda, x) = (\lambda, 0, 0)$, $b_{i+1} = \mathcal{G}_{ch}(b_i)$ for $i = 0, 1, 2, \dots$. Assume that b_i are C^m and that for any i, $\|\pi_y b_i\|_{C^m} < c_m$, with c_m independent of i. If the order l of the rate conditions is greater or equal to m, then $\|\pi_y b_i\|_{C^{m+1}} < c_{m+1}$ for a constant independent of i.*

Proof. Let us fix $i \in \mathbb{N}$. Our aim will be to show that $\|\pi_y b_i\|_{C^{m+1}}$ is bounded and that the bound is independent of i. Let θ_i be any chosen point from $\Lambda \times \overline{B}_u(R)$ and let $\theta_0, \dots, \theta_i \in \Lambda \times \overline{B}_u(R)$ be a sequence such that

$$b_{l+1}(\theta_{l+1}) = f(b_l(\theta_l)),$$

for $l = 0, \dots, i-1$. Note that

$$\theta_{l+1} = \pi_\theta f(b_l(\theta_l)).$$

For $l = 0, \dots, i$ let $\mathcal{P}_m^l : \mathbb{R}^{c+u} \supset B(0, \delta) \to \mathbb{R}^s$ be a polynomial of degree m, defined as

$$\mathcal{P}_m^l = \pi_y b_l(\theta_l) + T_{\pi_y b_l, m, \theta_l}.$$

Observe that since $\|\pi_y b_l\|_{C^m} < c_m$ for c_m independent from l, the polynomials $\mathcal{P}_m^l$ have a uniform bounds for their coefficients, which is independent from l and i. Since $b_{l+1} = \mathcal{G}_{ch}(b_l)$ we also see that for $l = 0, \dots, i-1$

$$\begin{aligned} \mathrm{graph}(T_{\pi_y f \circ (\mathrm{id}, \mathcal{P}_m^l), m, 0}) &= \mathrm{graph}(T_{\pi_y f \circ b_l, m, 0}) \\ &= \mathrm{graph}(T_{\pi_y b_{l+1}, m, 0}) \\ &= \mathrm{graph}\left(\mathcal{P}_m^{l+1}\right). \end{aligned}$$

Since $\pi_y b_l$ are Lipschitz with a constant L

$$\left\|D\mathcal{P}_m^l(\theta)\right\| \leq L.$$

Let us consider coordinates $\mathrm{x} = (\lambda, x)$ $\mathrm{y} = y$ and let, $\xi = \xi_{cu,1}$, $\mu = \mu_{s,2}$ and $B = \|f\|_{C^1}$. Then, from Theorem 28, for sufficiently large M and sufficiently small δ

$$f(J_{cu}(b_l(\theta_l), \mathcal{P}_m^l, M, \delta)) \subset J_{cu}(b_{l+1}(\theta_{l+1}), \mathcal{P}_m^{l+1}, M). \tag{31}$$

Note that the choice of M and δ does not depend on l. Since b_0 is a flat disc, we have

$$b_0(B_{c+u}(\theta_0,\delta)) \subset J_{cu}(b_0(\theta_0), \mathcal{P}_m^0 = 0, M, \delta).$$

This by (31) implies that

$$b_l(B_{c+u}(\theta_l,\delta)) \subset J_{cu}(b_l(\theta_l), \mathcal{P}_m^l, M, \delta). \tag{32}$$

From (32), by Lemma 26, we obtain a uniform bound

$$\left\| \frac{\partial^{m+1}\pi_y b_i(\theta_i)}{\partial \mathrm{x}_{i_1}\dots\partial \mathrm{x}_{i_{m+1}}} \right\| \leq CM,$$

where C is independent of θ_i. This means that

$$\|\pi_y b_i\|_{C^{m+1}} \leq c_{m+1},$$

where c_{m+1} depends on CM and c_m, but is independent of i. ∎

Lemma 48. *If f satisfies the assumptions of Theorem 16, then the manifold W^{cu} is C^k.*

Proof. Let b_i be the sequence of center-horizontal discs defined as $b_0(\lambda,x) = (\lambda,0,0)$, $b_{i+1} = \mathcal{G}_{ch}(b_i)$, for $i > 0$. By Lemma 45, we know that b_i are C^{k+1}. By Lemma 46 we know that they converge uniformly to

$$W^{cu} = \{(\theta, w^{cu}(\theta)) : \theta \in \Lambda \times \overline{B}_u\}.$$

We need to show that C^k smoothness is preserved as we pass to the limit.

Since $\pi_y b_i$ are Lipschitz with a constant L, we see that $\|\pi_y b_i\|_{C^1} \leq c_1$, where c_1 is independent of i. Rate conditions of order k, imply rate conditions of order m for $m \leq k$; in particular for $m = 1$. Hence, by Lemma 47 we obtain that $\|\pi_y b_i\|_{C^2} \leq c_2$.

Applying Lemma 47 inductively we obtain that $\|\pi_y b_i\|_{C^{k+1}} \leq c_{k+1}$, with c_{k+1} independent of i. This implies that derivatives of $\pi_y b_i$ of order smaller or equal to k are uniformly bounded and uniformly equicontinuous. This by the Arzela Ascoli theorem implies that $\pi_y b_i$ and their derivatives of order smaller or equal to k converge uniformly. Thus w^{cu} is C^k, as required. ∎

Lemma 49. *If f satisfies the assumptions of Theorem 16, then $f|_{W^{cu}}$ is injective.*

Proof. If $p_1, p_2 \in D$ and $f(p_1) = f(p_2)$ then $f(p_1) \in J_s(f(p_2), 1/L)$ and by the backward cone conditions $p_1 \in J_s(p_2, 1/L)$ hence by Remark 6, p_1 and p_2 are in the same chart. This means that it is enough to show that for any $p_1, p_2 \in W^{cu}$ which are on the same chart, we can not have $p_1 \neq p_2$ and $f(p_1) = f(p_2)$.

Let $\theta_1 \in \Lambda \times \overline{B}_u(R)$, $\theta_2 \in \overline{B}_c(\pi_\lambda\theta, R_\Lambda) \times \overline{B}_u(R)$ and $\theta_1 \neq \theta_2$. By Corollary 36 it follows that

$$\|\pi_\theta f(\theta_1, w^{cu}(\theta_1)) - \pi_\theta f(\theta_2, w^{cu}(\theta_2))\| \geq \xi_{cu,1,P} \|\theta_1 - \theta_2\|.$$

This implies that $f(\theta_1, w^{cu}(\theta_1)) \neq f(\theta_2, w^{cu}(\theta_2))$, as required. ∎

Lemmas 46, 48 and 49 combined, prove the assertion about W^{cu} from Theorem 16.

We finish the section by proving Theorem 19.

Proof of Theorem 19. The result follows from showing that

$$\|\pi_y (b_i(\theta_1) - b_i(\theta_2))\| \leq M \|\pi_\theta (b_i(\theta_1) - b_i(\theta_2))\|. \tag{33}$$

By definition of b_0, (33) clearly holds for $i = 0$. To prove (33) for all $i \in \mathbb{N}$, one can inductively apply the same argument as the one from the proof of Theorem 32 (page 55). Passing with i to infinity we obtain our claim. ■

7. Center-stable manifold

The goal of this section is to establish the existence of the center stable manifold W^{cs} from Theorem 16.

We will represent W^{cs} as a limit of graphs of smooth functions. Here we take the first step in this direction. For any $i \in \mathbb{Z}_+$ and $(\lambda, y) \in \Lambda \times \overline{B}_s(R)$ we consider the following problem: Find x such that

$$\pi_x f^i(\lambda, x, y) = 0 \tag{34}$$

under the constraint

$$f^l(\lambda, x, y) \in D, \quad l = 0, 1, \ldots, i. \tag{35}$$

From Lemma 44 it follows immediately that this problem has a unique solution $x_i(\lambda, y)$ which is as smooth as f.

Lemma 50. *Let $b_i : \Lambda \times \overline{B}_s(R) \to D$ be given by $b_i(\lambda, y) = (\lambda, x_i(\lambda, y), y)$. Then b_i is a center vertical disc and the sequence b_i converges uniformly to W^{cs}. Moreover, W^{cs} is a center vertical disk in D, such that*

$$\pi_x W^{cs} \subset B_u(R). \tag{36}$$

Proof. To show that b_i is a center vertical disc, we have to prove that if $\lambda_1 \in B_c(\lambda_2, R_\Lambda)$, then $b_i(\lambda_1, y_1) \in J_{cs}(b_i(\lambda_2, y_2), L)$. We will argue by the contradiction. Assume that $b_i(\lambda_1, y_1) \notin J_{cs}(b_i(\lambda_2, y_2), L)$, which implies $b_i(\lambda_1, y_1) \in J_u(b_i(\lambda_2, y_2), 1/L)$. Then from Lemma 36, applied inductively, it follows that

$$\|\pi_x(f^i(b_i(\lambda_1, y_1)) - f^i(b_i(\lambda_2, y_2)))\| \geq \xi^i_{u,1,P} \|\pi_x (b_i(\lambda_2, y_2) - b_i(\lambda_2, y_2))\| > 0.$$

This contradicts (34). This establishes that b_i are center vertical discs.

To prove the uniform convergence of b_i we show the Cauchy condition for this sequence. Let $i, j \in \mathbb{Z}_+$. We have $b_i(\lambda, y) \in J_u(b_{i+j}(\lambda, y), 1/L)$, hence by Lemma 36

$$\|\pi_x(f^i(b_i(\lambda, y)) - f^i(b_{i+j}(\lambda, y)))\| \geq \xi^i_{u,1,P} \|\pi_x (b_i(\lambda, y) - b_{i+j}(\lambda, y))\|.$$

Since

$$\|\pi_x(f^i(b_i(\lambda, y)) - f^i(b_{i+j}(\lambda, y)))\| = \left\|\pi_x f^i(b_{i+j}(\lambda, y))\right\| \leq R$$

we obtain

$$\|\pi_x (b_i(\lambda, y) - b_{i+j}(\lambda, y))\| \leq \frac{R}{\xi^i_{u,1,P}}.$$

This, since $\xi_{u,1,P} > 1$, proves uniform convergence of b_i to some disk b. Observe that

$$\left\{b(\lambda, y) : (\lambda, y) \in \Lambda \times \overline{B}_s(R)\right\} \subset W^{cs},$$

because for each $(\lambda, y) \in \Lambda \times \overline{B}_s(R)$, $b(\lambda, y) = \lim_{i\to\infty} b_i(\lambda, y)$ and

$$f^l(b_i(\lambda, y)) \in D, \quad l = 0, \ldots, i. \tag{37}$$

Fixing l in (37) and passing to the limit with i, we obtain that for all $l \in \mathbb{Z}_+$, $f^l(b(\lambda, y)) \in D$.

We now need to show that we can not have a point $z \in W^{cs}$ such that $z \neq b\left(\pi_{(\lambda,y)}z\right)$. Since $b\left(\pi_{(\lambda,y)}z\right) \in J_u\left(z, 1/L\right)$, by Lemma 36, for all $i \geq 0$

$$\|\pi_x(f^i(b(\pi_{(\lambda,y)}z)) - f^i(z))\| \geq \xi^i_{u,1,P}\|\pi_x\left(b(\pi_{(\lambda,y)}z) - z\right)\|.$$

Since $\|\pi_x(f^i(b(\pi_{(\lambda,y)}z)) - f^i(z))\| \leq 2R$ and since $\xi_{u,1,P} > 1$, we see that $\pi_x\left(b(\pi_{(\lambda,y)}z) - z\right) = 0$, which implies that $b(\pi_{(\lambda,y)}z) = z$.

Condition (36) is an immediate consequence of (7), since if we had $z \in W^{cs}$ with $\|\pi_x z\| = R$ then $f(z) \notin D$.

We finish by showing that b is a center-vertical disc. We have already established that b_i are center-vertical discs. Passing to the limit, for any $(\lambda, y) \in \Lambda \times B_s(R)$,

$$b\left(\overline{B}_c(\lambda, R_\Lambda) \times \overline{B}_s(R)\right) \subset \overline{J_{cs}\left(b(\lambda, y), L\right)}. \tag{38}$$

The condition (27) follows from Corollary 34 by the following argument. If we had a point in $(\lambda^*, y^*) \neq (\lambda, y)$ such that

$$b\left(\lambda^*, y^*\right) \in \partial J_{cs}\left(b(\lambda, y), L\right)$$

then $b\left(\lambda^*, y^*\right) \in J^c_{cs}\left(b(\lambda, y), L\right)$ and by Corollary 34,

$$f\left(b\left(\lambda^*, y^*\right)\right) \in \mathrm{int} J^c_{cs}\left(f(b(\lambda, y)), L\right). \tag{39}$$

Since $f(W^{cu}) = W^{cu}$, (39) contradicts (38). ■

Lemma 51. *Let $m \leq k$. Let b_i be the sequence of center-horizontal discs defined in Lemma 50. Assume that b_i are C^m and that for any i, $\|\pi_x b_i\|_{C^m} < c_m$, with c_m independent of i. If f satisfies rate conditions of order m, then $\|\pi_x b_i\|_{C^{m+1}} < c_{m+1}$ for a constant independent of i.*

Proof. The proof goes along the same lines as the proof of Lemma 47. We shall write $\theta = (\lambda, y)$. Since b_i follows from the solution of problem (34)

$$f(b_{l+1}(\Lambda \times \overline{B}_s(R))) \subset b_l(\Lambda \times \overline{B}_s(R)). \tag{40}$$

Let us fix $i \in \mathbb{N}$. Our aim will be to show that $\|\pi_x b_i\|_{C^{m+1}}$ is bounded and that the bound is independent of i. Let θ_i be any chosen point from $\Lambda \times \overline{B}_s(R)$ and let $\theta_{i-1}, \ldots, \theta_0 \in \Lambda \times \overline{B}_u(R)$ be a sequence defined as

$$\theta_l = \pi_\theta f(b_{l+1}(\theta_{l+1})),$$

for $l=0,\ldots,i-1$. By (40),

$$b_l(\theta_l)=f(b_{l+1}(\theta_{l+1})).$$

For $l=0,\ldots,i$, let $\mathcal{P}_m^l:\mathbb{R}^{c+s}\supset B(0,\delta)\to\mathbb{R}^s$ be a polynomial of degree m, defined as

$$\mathcal{P}_m^l=\pi_x b_l(\theta_l)+T_{\pi_x b_l,m,\theta_l}.$$

Since $\|\pi_y b_l\|_{C^m}<c_m$ for c_m independent from l, the polynomials $\mathcal{P}_m^l$ have a uniform bounds for their coefficients, which is independent from l and i. Since b_l are center vertical discs, $\pi_x b_l$ are Lipschitz with a constant L, hence

$$\left\|D\mathcal{P}_m^l(0)\right\|\leq L.$$

Let us consider coordinates $\mathrm{x}=x$, $\mathrm{y}=(\lambda,y)$ and constants $\xi=\xi_{u,2}$, $\mu=\mu_{cs,1}$ and $B=\|f\|_{C^1}$. By (40) we see that for $k=0,\ldots,i-1$

$$\begin{aligned}\mathrm{graph}(T_{\pi_x f\circ(\mathrm{id},\mathcal{P}_m^{l+1}),m,0}) &= \mathrm{graph}(T_{\pi_x f\circ b_{l+1},m,0})\\ &\subset \mathrm{graph}(T_{\pi_x b_l,m,0})\\ &= \mathrm{graph}\left(\mathcal{P}_m^l\right).\end{aligned}$$

From Theorem 33, for sufficiently large M and sufficiently small δ

$$f(J_{cs}^c(b_{l+1}(\theta_{l+1}),\mathcal{P}_m^{l+1},M,\delta))\subset J_{cs}^c(b_l(\theta_l),\mathcal{P}_m^l,M). \tag{41}$$

Note that the choice of M and δ does not depend on l. Since $b_0(\lambda,y)=(\lambda,0,y)$ is a flat disc, we have

$$b_0(B_{c+s}(\theta_0,\delta))\subset J_{cu}(b_0(\theta_0),\mathcal{P}_m^0=0,M,\delta).$$

This by (41) implies that

$$b_l(B_{c+u}(\theta_k,\delta))\subset J_{cu}(b_l(\theta_l),\mathcal{P}_m^l,M,\delta). \tag{42}$$

From (42), by Lemma 31, we obtain a uniform bound

$$\left\|\frac{\partial^{m+1}\pi_x b_i(\theta_i)}{\partial\mathrm{y}_{i_1}\ldots\partial\mathrm{y}_{i_{m+1}}}\right\|\leq CM,$$

where C is independent of θ_i. This means that

$$\|\pi_x b_i\|_{C^{m+1}}\leq c_{m+1},$$

where c_{m+1} depends on CM and c_m, but is independent of i. ■

Lemma 52. *If f satisfies the assumptions from Theorem 16, then the manifold W^{cs} is C^k.*

Proof. The functions $\pi_x b_i$ are C^{k+1} and uniformly Lipschitz with constant L. The fact that C^k smoothness is preserved as we pass to the limit follows from Lemma 51 and mirror arguments to the proof of Lemma 48. ■

We finish the section by proving Theorem 20:

Proof of Theorem 20. We shall write $\theta = (\lambda, y)$. Our first aim is to show that for $\theta_1 \neq \theta_2$

$$\|\pi_x (b_i(\theta_1) - b_i(\theta_2))\| \leq M \|\theta_1 - \theta_2\|. \tag{43}$$

Let $z_1 = b_i(\theta_1)$, $z_2 = b_i(\theta_2)$ and suppose that (43) does not hold. Then $z_1 \in J_u(z_2, 1/M)$. From Theorem 27 (taking $\mathrm{x} = x$, $\mathrm{y} = \theta$ and $1/M$ in place of M) we see that for $l = 0, \ldots i$, $f^l(z_1) \in J_u\left(f^l(z_2), 1/M\right)$. By the same argument as the one in the proof of Lemma 36 (on page 57) we have

$$\left\|\pi_x\left(f^i(z_1) - f^i(z_2)\right)\right\| \geq \xi^i \|\pi_x(z_1 - z_2)\| > 0,$$

which contradicts the fact that by definition of b_i, $\pi_x f^i(z_1) = \pi_x f^i(z_2) = 0$. Thus we have proven (43).

The claim follows by passing with i to infinity in (43). ■

8. Normally hyperbolic manifold

In this section we establish the existence of the normally hyperbolic invariant manifold from Theorem 16. Throughout the section we assume that assumptions of Theorem 16 are satisfied.

Lemma 53. *For any $\lambda^* \in \Lambda$ there exists a point $p^* \in W^{cu} \cap W^{cs}$ with $\pi_\lambda p^* = \lambda^*$.*

Proof. Let $G : \overline{B}_c(\lambda^*, R_\Lambda) \times \overline{B}_u(R) \times \overline{B}_s(R) \to \overline{B}_c(\lambda^*, R_\Lambda) \times \overline{B}_u(R) \times \overline{B}_s(R)$ be defined as

$$G(\lambda, x, y) = (\lambda^*, w^{cs}(\lambda, y), w^{cu}(\lambda, x)).$$

By the Brouwer fixed point theorem, there exists a p^* such that $G(p^*) = p^*$. We see that $\pi_\lambda p^* = \lambda^*$. Let $x^* = \pi_x p^*$ and $y^* = \pi_y p^*$. Since $G(p^*) = p^*$,

$$\begin{aligned} w^{cs}(\lambda^*, y^*) &= x^*, \\ w^{cu}(\lambda^*, x^*) &= y^*, \end{aligned}$$

hence

$$p^* = (\lambda^*, x^*, w^{cu}(\lambda^*, x^*)) = (\lambda^*, w^{cs}(\lambda^*, y^*), y^*)$$

clearly lies on $W^{cu} \cap W^{cs}$. ■

Lemma 54. *Let $p \in W^{cu} \cap W^{cs}$, then W^{cu} and W^{cs} intersect transversally at p.*

Proof. The manifold W^{cu} is parameterized by $\phi_{cu} : (\lambda, x) \to (\lambda, x, w^{cu}(\lambda, x))$ and W^{cs} is parameterized by $\phi_{cs} : (\lambda, y) \to (\lambda, w^{cs}(\lambda, y), y)$. Let

$$V = \mathrm{span}\{D\phi_{cu}(p)v + D\phi_{cs}(p)w : v \in \mathbb{R}^c \times \mathbb{R}^u, w \in \mathbb{R}^c \times \mathbb{R}^s\}.$$

We need to show that

$$V = \mathbb{R}^c \times \mathbb{R}^u \times \mathbb{R}^s. \tag{44}$$

We see that V is equal to the range of the $(c + u + s) \times (c + c + u + s)$ matrix

$$A = \begin{pmatrix} \mathrm{id} & \mathrm{id} & 0 & 0 \\ 0 & \frac{\partial w_{cs}}{\partial \lambda} & \mathrm{id} & \frac{\partial w_{cs}}{\partial y} \\ \frac{\partial w_{cu}}{\partial \lambda} & 0 & \frac{\partial w_{cu}}{\partial x} & \mathrm{id} \end{pmatrix}.$$

We will show that

$$B=\begin{pmatrix} \mathrm{id} & \frac{\partial w_{cs}}{\partial y} \\ \frac{\partial w_{cu}}{\partial x} & \mathrm{id} \end{pmatrix}$$

is invertible. If for $q=(x,y)$, $Bq=0$ then

$$x-\frac{\partial w_{cs}}{\partial y}\frac{\partial w_{cu}}{\partial x}x=0,$$

which since w_{cs} and w_{cu} are Lipschitz with constant $L<1$ implies that $x=0$, and in turn that $y=0$. Since B is invertible, it is evident that the rank of A is $c+u+s$, which implies (44). ■

Lemma 55. *The $\Lambda^*=W^{cu}\cap W^{cs}$ is a C^k manifold, which is a graph over Λ of a function $\chi:\Lambda\to\overline{B}_u\times\overline{B}_s$, which is Lipschitz with a constant $\frac{\sqrt{2}L}{\sqrt{1-L^2}}$.*

Proof. From Lemma 53 it follows that for every $\lambda\in\Lambda$ the set $W^{cu}\cap W^{cs}\cap\{p\in D\mid\pi_\lambda=\lambda\}$ is nonempty. We will show that this set consists from one point $\chi(\lambda)$ and χ is a function satisfying the Lipschitz condition.

Assume that $p_1=(\lambda_1,x_1,y_1)\in W^{cs}\cap W^{cu}$ and $p_2=(\lambda_2,x_2,y_2))\in W^{cs}\cap W^{cu}$. Moreover, we assume that $\lambda_1\in B_c(\lambda_2,R_\Lambda)$ (they are in the same chart). Therefore we know that

$$p_1 \in J_{cu}(p_2,L), \tag{45}$$
$$p_1 \in J_{cs}(p_2,L). \tag{46}$$

Let $(\lambda,x,y)=p_1-p_2=(\lambda_1-\lambda_2,x_1-x_2,y_1-y_2)$. By (45–46) we obtain

$$\|y\|\leq L\|(\lambda,x)\|, \qquad \|x\|\leq L\|(\lambda,y)\|,$$

hence

$$\|y\|^2\leq L^2\left(\|\lambda\|^2+\|x\|^2\right),$$
$$\|x\|^2\leq L^2\left(\|\lambda\|^2+\|y\|^2\right).$$

From above

$$\left(\|x\|^2+\|y\|^2\right)(1-L^2)\leq 2L^2\|\lambda\|^2,$$

which gives

$$\|(x,y)\|\leq\frac{\sqrt{2}L}{\sqrt{1-L^2}}\|\lambda\|.$$

Observe that this implies that if $\lambda_1=\lambda_2$, then $p_1=p_2$. This establishes the uniqueness of the intersection of $W^{cu}\cap W^{cs}\cap\{p\in D\mid\pi_\lambda=\lambda\}$, therefore $\chi(\lambda)$ is well defined.

From the above computations it follows that

$$\|\chi(\lambda_1)-\chi(\lambda_2)\|\leq\frac{\sqrt{2}L}{\sqrt{1-L^2}}\|\lambda_1-\lambda_2\|.$$

The fact that Λ^* is a graph of a C^k function $\chi:\Lambda\to\overline{B}_u\times\overline{B}_s$ follows from (54) and the fact that W^{cu} and W^{cs} are C^k. ■

9. Unstable fibers

The goal of this section is to establish the existence of the foliation of W^{cu} into unstable fibers W_z^u for $z \in W^{cu}$. In this section $\theta = (\lambda, y)$. Throughout this section we assume that assumptions of Theorem 16 hold.

For $z \in D$ we define b_z as a horizontal disk in D by $b_z(x) = (\pi_\lambda z, x, \pi_y z)$.

By Lemma 49 we know that $f|_{W^{cu}}$ is injective, hence for any $z \in W^{cu}$ the backward trajectory is unique and equal to $\left\{ (f|_{W^{cu}})^i \right\}_{i=-\infty}^{0}$. To simplify notations we shall denote such backward trajectory by $z_i = (f|_{W^{cu}})^i$, for $i = 0, -1, \ldots$.

For $z \in W^{cu}$ consider a sequence of horizontal disks in D,

$$d_{n,z} = \mathcal{G}_h^n(b_{z_{-n}}), \quad n = 1, 2, \ldots \tag{47}$$

where G_h is the graph transform defined just after Lemma 44. Our aim will be to show that $d_{n,z}$ converge to W_z^u, as defined by Definition 12. We start with a technical lemma.

Lemma 56. *Assume that $f^j(z) \in D$ for $j = 0, 1, \ldots, n$. If $f^j(q_i) \in J_u\left(f^j(z), 1/L\right) \cap D$, for $i = 1, 2$ and $j = 0, 1, \ldots, n$ and*

$$f^n(q_1) \notin J_u(f^n(q_2), 1/L)$$

then for $j = 0, 1, \ldots, n$ holds

$$\|\pi_\theta(f^j(q_1) - f^j(q_2))\| \leq \frac{4R}{L} \left(\frac{\mu_{cs,1}}{\xi_{u,1,P}}\right)^j \frac{1}{(\xi_{u,1,P})^{n-j}},$$
$$\|f^j(q_1) - f^j(q_2)\| \leq (1 + L) \frac{4R}{L} \left(\frac{\mu_{cs,1}}{\xi_{u,1,P}}\right)^j \frac{1}{(\xi_{u,1,P})^{n-j}}.$$

Proof. Our assumption $f^j(q_i) \in J_u\left(f^j(z), 1/L\right)$, $i = 1, 2$ and $j = 0, 1, \ldots, n$ implies that $f^j(q_1), f^j(q_2), f^j(z)$ are contained in the same charts for $j = 0, 1, \ldots, n$.

By Corollary 34, $f^j(q_1) \notin J_u(f^j(q_2), 1/L)$ for $j = 0, 1, \ldots, n$, hence

$$f^j(q_1) \in J_{cs}(f^j(q_2), L).$$

By Lemma 39 this implies for $j = 1, 2, \ldots, n$

$$\|\pi_\theta(f^j(q_1) - f^j(q_2))\| \leq \mu_{cs,1} \|\pi_\theta(f^{j-1}(q_1) - f^{j-1}(q_2))\| \leq \mu_{cs,1}^j \|\pi_\theta(q_1 - q_2)\|. \tag{48}$$

We estimate $\|\pi_\theta(q_1 - q_2)\|$ using the expansion in the x-direction. By Lemma 36 we have for $i = 1, 2$

$$2R \geq \|\pi_x(f^n(q_i) - f^n(z))\| \geq \xi_{u,1,P} \|\pi_x(f^{n-1}(q_i) - f^{n-1}(z))\| \geq \xi_{u,1,P}^n \|\pi_x(q_i - z)\|,$$

hence we obtain

$$\|\pi_x(q_i - z)\| \leq \frac{2R}{\xi_{u,1,P}^n}.$$

Since $q_i \in J_u(z, 1/L)$ we get

$$\|\pi_\theta(q_i - z)\| \leq \frac{1}{L} \|\pi_x(q_i - z)\| \leq \frac{2R}{L\xi_{u,1,P}^n}.$$

From the triangle inequality we obtain

$$\|\pi_\theta(q_1-q_2)\| \le \|\pi_\theta(q_1-z)\| + \|\pi_\theta(q_2-z)\| \le \frac{4R}{L\xi_{u,1,P}^n}.$$

By combining the above inequality with (48) we obtain

$$\|\pi_\theta(f^j(q_1)-f^j(q_2))\| \le \frac{4R}{L}\left(\frac{\mu_{cs,1}}{\xi_{u,1,P}}\right)^j \frac{1}{\xi_{u,1,P}^{n-j}}.$$

Since $f^n(q_1)\notin J_u(f^n(q_2),1/L)$,

$$\|\pi_\theta\left(f^n(q_1)-f^n(q_2)\right)\| > \frac{1}{L}\left\|\pi_x\left(f^n(q_1)-f^n(q_2)\right)\right\|,$$

hence

$$\begin{aligned}\|f^j(q_1)-f^j(q_2)\| &\le \left\|\pi_\theta\left(f^j(q_1)-f^j(q_2)\right)\right\| + \left\|\pi_x\left(f^j(q_1)-f^j(q_2)\right)\right\| \\ &\le (1+L)\,\frac{4R}{L}\left(\frac{\mu_{cs,1}}{\xi_{u,1,P}}\right)^j \frac{1}{(\xi_{u,1,P})^{n-j}}\end{aligned}$$

as required. ■

Lemma 57. *Assume that $z\in W^{cu}$. For any $n\ge 0$, $d_{n,z}$ is a horizontal disc and the sequence $d_{n,z}$ converges uniformly to a horizontal disk d_z. Moreover,*

$$W_z^u = \left\{\left(\pi_\lambda w_z^u\left(x\right), x, \pi_y w_z^u\left(x\right)\right) : x\in \overline{B}_u(R)\right\},$$

where $w_z^u : \overline{B}_u(R)\to \Lambda\times\overline{B}_s(R)$ and $d_z(x) = \left(\pi_\lambda w_z^u\left(x\right), x, \pi_y w_z^u\left(x\right)\right)$.

Proof. We show first the uniform convergence. Let us fix $x\in\overline{B}_u(R)$. Our goal is to estimate $\|d_{n+j,z}(x)-d_{n,z}(x)\|$. Observe first that from the definition of the graph transform $\mathcal{G}_h$, it follows that for each $n\in\mathbb{Z}_+$ and for each $x\in\overline{B}_u(R)$ the point $d_{n,z}(x)$ has a backward orbit $\{p_i\}_{i=-n}^0$ of length $n+1$,

$$\begin{aligned}&p_0 = d_{n,z}(x), \quad f(p_i) = p_{i+1} \quad \text{for } i=-n,-n+1,\ldots,-1\\ &p_i\in J_u\left(z_i,1/L\right), \quad p_i\in d_{n+i,z_i}\left(\overline{B}_u(R)\right) \quad \text{for } i=-n,-n+1,\ldots,-1,0.\end{aligned}$$

Let n,j be positive integers. From the above observation we can find (define) q_1 and q_2 as follows. Let q_1 be such that $f^i(q_1)\in J_u\left(z_{-n+i},1/L\right)$ for $i=0,1,\ldots,n$ and $f^n(q_1)=d_{n,z}(x)$, analogously, let q_2 be such that $f^i(q_2)\in J_u\left(z_{-n+i},1/L\right)$ for $i=0,1,\ldots,n$ and $f^n(q_2)=d_{n+j,z}(x)$.

Observe that since

$$\pi_x\left(f^n(q_2)-f^n(q_1)\right) = \pi_x\left(d_{n,z}(x)-d_{n+j,z}(x)\right) = 0,$$

we have $\|\pi_\theta(f^n(q_2)-f^n(q_1))\| = \|f^n(q_2)-f^n(q_1)\|$. Assume that $f^n(q_2)\neq f^n(q_1)$, then from Lemma 56 applied to q_1, q_2 and z_{-n} it follows that

$$\|d_{n,z}(x)-d_{n+j,z}(x)\| = \|\pi_\theta(d_{n,z}(x)-d_{n+j,z}(x))\| \le \frac{4R}{L}\left(\frac{\mu_{cs,1}}{\xi_{u,1,P}}\right)^n.$$

Since by our assumptions $\frac{\mu_{cs,1}}{\xi_{u,1,P}} < 1$, we see that $d_{n,z}$ is a Cauchy sequence. Let us denote the limit by d_z.

Since $b_{z_{-n}}$ is a horizontal disc, so by Lemma 44 is $d_{n,z} = \mathcal{G}_h^n\left(b_{z_{-n}}\right)$. The properties (24) are preserved when passing to the limit, hence d_z is a horizontal disc.

We show that for all $x \in \overline{B}_u(R)$, $d_z(x) \in W^{cu}$. For this we need to construct a full backward orbit through $d_z(x)$. Let us consider backward orbits through $d_{n,z}(x)$ of length $n+1$. From Lemma 56 it follows that they converge to full backward orbit through $d_z(x)$. Therefore $d_z(x) \in W^{cu}$ for $x \in \overline{B}_u(R)$. From this reasoning it follows also that for $i < 0$

$$\mathcal{G}_h(d_{z_i}) = d_{z_{i+1}}. \tag{49}$$

We will now show that $\{d_z(x) \mid x \in \overline{B}_u(R)\} \subset W_z^u$. For any $x \in \overline{B}_u(R)$ and backward trajectory $\{p_i\}_{i=-\infty}^0$ of $d_z(x)$, from (49) it follows that $p_i \in d_{z_i}$ for $i \leq 0$. Since d_{z_i} are horizontal discs we infer that $p_i \in J_u\left(z_i, 1/L\right)$, as required.

To show that $W_z^u \subset \{d_z(x) \mid x \in \overline{B}_u(R)\}$, let us consider $p \in W^{cu}$, with a backward trajectory (note that by Lemma 49 such trajectory is unique) $\{p_i\}_{i=-\infty}^0$, $p_i = \left(f|_{W^{cu}}\right)^i(p)$, such that $p_i \in J_u\left(z_i, 1/L\right)$ for all $i < 0$. Let $x = \pi_x p$. We will show that $p = d_z(x)$. From Lemma 56 it follows that

$$\|p - d_z(x)\| = \|\pi_\theta(p - d_z(x))\| = \lim_{n\to\infty} \|\pi_\theta(p - d_{n,z}(x))\| \leq \lim_{n\to\infty} \frac{4R}{L}\left(\frac{\mu_{cs,1}}{\xi_{u,1,P}}\right)^n = 0.$$

Therefore $p = d_z(x)$.

The function w_z^u can be defined as $w_z^u(x) = \pi_\theta d_z(x)$. ■

Lemma 58. *Let $m \leq k$. Let $d_{n,z}$ be the sequence of horizontal discs defined as $d_{n,z} = \mathcal{G}_h^i(b_{z_{-n}})$. Assume that $d_{n,z}$ are C^m and that for any i, $\|\pi_\theta d_{n,z}\|_{C^m} < c_m$, with c_m independent of n. If f satisfies rate conditions of order m, then $\|\pi_\theta d_{n,z}\|_{C^{m+1}} < c_{m+1}$ for a constant independent of n.*

Proof. The proof follows from identical arguments to the proof of Lemma 47. The only difference is that when we apply Theorem 28, we choose coordinates $\mathrm{x} = x$, $\mathrm{y} = \theta = (\lambda, y)$ and constants $\xi = \xi_{1,u}$, $\mu = \mu_{cs,2}$. Note that conditions (1), (4) imply (19) for any $m \geq 0$. ■

Lemma 59. *For any $z \in W^{cu}$ the manifold W_z^u is C^k.*

Proof. The functions $\pi_\theta d_{n,z}$ are C^{k+1} and uniformly Lipschitz with constant $1/L$. The fact that C^k smoothness is preserved as we pass to the limit follows from Lemma 58 and mirror arguments to the proof of Lemma 48. ■

Lemma 60. *For any $z \in W^{cu}$. If $p \in W_z^u$, then for $n \geq 0$ $f_{|W^{cu}}^{-n}(p), f|_{W^{cu}}^{-n}(z)$ are in the same chart and*

$$\left\|\left(f_{|W^{cu}}\right)^{-n}(p) - \left(f|_{W^{cu}}\right)^{-n}(z)\right\| \leq \left(1 + \frac{1}{L}\right)\|\pi_x(p - z)\|\xi_{u,1,P}^{-n}, \qquad n \geq 0.$$

Proof. We first observe that for any $q_1, q_2 \in D$, such that $q_1 \in J_u(q_2, 1/L)$, holds

$$\|\pi_\theta (q_1 - q_2)\| \leq \frac{1}{L} \|\pi_x (q_1 - q_2)\|,$$

$$\|q_1 - q_2\| \leq \|\pi_\theta (q_1 - q_2)\| + \|\pi_x (q_1 - q_2)\| \leq \left(1 + \frac{1}{L}\right) \|\pi_x (q_1 - q_2)\|. \tag{50}$$

Let $p \in W_z^u$. By Lemma 49, backward trajectories of p and z are unique and equal to $p_{-n} = (f|_{W^{cu}})^{-n}(p)$, $z_{-n} = (f|_{W^{cu}})^{-n}(z)$, for $n \geq 0$. Note that by the definition of W_z^u, for $i = 0, \ldots, n$, $p_{-i} \in J_u(z_{-i}, 1/L)$. Let us fix $n \geq 0$. From Lemma 36 and (50), it follows that

$$\begin{aligned}
\|\pi_x(p - z)\| &= \|\pi_x (f^n(p_{-n}) - f^n(z_{-n}))\| \\
&\geq \xi_{u,1,P}^n \|\pi_x (p_{-n} - z_{-n})\| \\
&\geq \xi_{u,1,P}^n \left(1 + \frac{1}{L}\right)^{-1} \|p_{-n} - z_{-n}\|.
\end{aligned}$$

This proves that for any point in W_z^u holds

$$\|p_{-n} - z_{-n}\| \leq \xi_{u,1,P}^{-n} C,$$

for $C = \left(1 + \frac{1}{L}\right) \|\pi_x(p - z)\|$, as required. ■

Lemma 61. *For $z \in W^{cu}$ we define a set $U = U(z)$ as*

$$\begin{aligned}
U \quad = \quad & \{p \in D : \exists \text{ backward trajectory } \{p_i\}_{i=-\infty}^0 \text{ of } p \in D, \text{ and} \\
& \text{for any such } \{p_i\},\ \exists C > 0 \text{ (which may depend on } p),\ \exists n_0 \geq 0 \\
& \text{s.t. for } n \geq n_0,\ p_{-n} \text{ and } (f|_{W^{cu}})^{-n}(z) \text{ are in the same good chart} \\
& \text{and } \left\|p_{-n} - (f|_{W^{cu}})^{-n}(z)\right\| \leq C\xi_{u,1,P}^{-n}\}.
\end{aligned}$$

Then

$$W_z^u = U.$$

Proof. Observe that from Lemma 60 we obtain $W_z^u \subset U$. We will show that $U \subset W_z^u$ by contradiction.

Let $p \in U \setminus W_z^u$. Obviously $p \in W^{cu}$, hence for $i \leq 0$, by Lemma 49, its backward trajectory is uniquely defined. Let $i^* = -n_0 \leq 0$. Then for $n \leq i^*$ points p_{-n}, z_{-n} lie in the same chart and

$$\|p_{-n} - z_{-n}\| \leq C\xi_{u,1,P}^{-n}. \tag{51}$$

Since $p \notin W_z^u$ then there exists $j_0 \leq 0$ such that $p_{j_0} \notin J_u(z_{j_0}, 1/L)$. From the forward invariance of J_u's (see Cor. 34) it follows that $p_j \notin J_u(z_j, 1/L)$ for $j \leq j_0$. Hence we can find $i^{**} \leq i^* < 0$ such that $p_{i^{**}} \notin J_u(z_{i^{**}}, 1/L)$. For any $i \leq i^{**}$ holds

$$p_i \in J_{cs}(z_i, L), \quad \text{for } i \leq i^{**}$$

For $n > |i^{**}|$ from Lemma 39

$$\|\pi_\theta (f(p_{-n}) - f(z_{-n}))\| \leq \mu_{cs,1} \|\pi_\theta (p_{-n} - z_{-n})\|,$$

hence, by the same argument,

$$\begin{aligned}\left\|\pi_\theta\left(p_{i^{**}}-z_{i^{**}}\right)\right\| &= \left\|\pi_\theta\left(f^{n+i^{**}}(p_{-n})-f^{n+i^{**}}(z_{-n})\right)\right\| \\ &\leq \mu_{cs,1}^{n+i^{**}}\left\|\pi_\theta\left(p_{-n}-z_{-n}\right)\right\| \\ &= \mu_{cs,1}^{n}\mu_{cs,1}^{i^{**}}\left\|\pi_\theta\left(p_{-n}-z_{-n}\right)\right\|,\end{aligned}$$

and in turn

$$\begin{aligned}\left\|p_{-n}-z_{-n}\right\| &\geq \left\|\pi_\theta\left(p_{-n}-z_{-n}\right)\right\| \\ &\geq \mu_{cs,1}^{-n}\left(\mu_{cs,1}^{-i^{**}}\left\|\pi_\theta\left(p_{i^{**}}-z_{i^{**}}\right)\right\|\right).\end{aligned} \tag{52}$$

Since $\xi_{u,1,P}>\mu_{cs,1}$, conditions (51) and (52) contradict each other. This means that $p\in W_z^u$, as required. ■

Lemmas 57, 59, 61 combined prove the claims about W_z^u from Theorem 16. We now prove Theorem 18, which can be used to obtain tighter Lipschitz bounds on w_z^u.

Proof of Theorem 18. From Theorem 27, taking coordinates $\mathrm{x}=x$, $y=\theta$, for $q\in D$, since $\frac{\xi}{\mu}>1$,

$$f\left(J_u\left(q,M\right)\cap D\right)\subset J_u\left(f\left(q\right),M\right). \tag{53}$$

By definition of $d_{0,z}$, it follows that $d_{0,z}(x_1)\in f\left(J_u\left(d_{0,z}(x_2),M\right)\cap D\right)$, for any $x_1,x_2\in\overline{B}_u$. By (53), since $d_{n,z}=\mathcal{G}\left(d_{n-1,z}\right)$, we see that

$$d_{n,z}(x_1)\in f\left(J_u\left(d_{n,z}(x_2),M\right)\cap D\right),$$

for any $x_1,x_2\in\overline{B}_u$. Hence

$$\left\|\pi_\theta\left(d_{n,z}(x_1)-d_{n,z}(x_2)\right)\right\|\leq M\left\|\pi_x\left(d_{n,z}(x_1)-d_{n,z}(x_2)\right)\right\|=M\left\|x_1-x_2\right\|,$$

and passing with n to infinity gives

$$\left\|w_z^u(x_1)-w_z^u(x_2)\right\|\leq M\left\|x_1-x_2\right\|,$$

as required. ■

Proposition 62. *Let $z\in W^{cu}$. Then the intersection $W_z^u\cap W^{cs}$ consists of a single point and is transversal. Also the intersection $W_z^u\cap\Lambda^*$ consists of a single point.*

Proof. The proof follows from similar arguments to the proofs of Lemma 53 and Theorem 54.

First we show that W_z^u and W^{cs} intersect. By Remark 6, for any point $q\in W_z^u$ we have

$$W_z^u\subset D_{\pi_\lambda q}=\overline{B}_c\left(\pi_\lambda q,R_\Lambda\right)\times\overline{B}_u(R)\times\overline{B}_s(R).$$

Let us define the following function $G:D_{\pi_\lambda q}\to D_{\pi_\lambda q}$,

$$G\left(\lambda,x,y\right)=\left(\pi_\lambda w_z^u(x),w^{cs}\left(\lambda,y\right),\pi_y w_z^u(x)\right).$$

By the Brouwer theorem we know that there exists a $q^*=(\lambda^*,x^*,y^*)$ for which $q^*=G(q^*)$. This means that

$$W_z^u\ni\left(\pi_\lambda w_z^u(x^*),x^*,\pi_y w_z^u(x^*)\right)=q^*=\left(\lambda^*,w^{cs}\left(\lambda^*,y^*\right),y^*\right)\in W^{cs},$$

hence $q^* W_z^u \cap W^{cs}$.

Now we show that the intersection point is unique. Let $q_1, q_2 \in W_z^u \cap W^{cs}$. Since W_z^u is a vertical disc,

$$\left\|\pi_{(\lambda,y)}\left(q_1 - q_2\right)\right\| \leq 1/L \left\|\pi_x\left(q_1 - q_2\right)\right\|. \tag{54}$$

Since W^{cs} is a center-vertical disc, if $q_1 \neq q_2$ then

$$\left\|\pi_x\left(q_1 - q_2\right)\right\| < L \left\|\pi_{(\lambda,y)}\left(q_1 - q_2\right)\right\|,$$

a contradiction with (54), hence $q_1 = q_2$.

Now we prove the transversality of the intersection. This is a similar argument to the proof of Theorem 54. We first note that since W_z^u is a center-vertical disc, w_z^u is Lipschitz with a constant $\rho < 1/L$. The manifold W_z^u is parameterized by $\phi_{z,u} : x \to (\pi_\lambda w_z^u(x), x, \pi_y w_z^u(x))$ and W^{cs} is parameterized by $\phi_{cs} : (\lambda, y) \to (\lambda, w^{cs}(\lambda, y), y)$. Let

$$V = \mathrm{span}\{D\phi_{z,u}(x^*)v + D\phi_{cs}(\lambda^*, y^*)w : v \in \mathbb{R}^u, w \in \mathbb{R}^c \times \mathbb{R}^s\}.$$

We need to show that

$$V = \mathbb{R}^c \times \mathbb{R}^u \times \mathbb{R}^s. \tag{55}$$

We see that V is equal to the range of the $(c+u+s) \times (c+u+s)$ matrix

$$A = \left(\begin{array}{lll} \frac{\partial \pi_\lambda w_z^u}{\partial x} & \mathrm{id} & 0 \\ \mathrm{id} & \frac{\partial w^{cs}}{\partial \lambda} & \frac{\partial w^{cs}}{\partial y} \\ \frac{\partial \pi_y w_z^u}{\partial x} & 0 & \mathrm{id} \end{array}\right).$$

We will show that

$$B = \left(\begin{array}{ll} \frac{\partial \pi_\lambda w_z^u}{\partial x} & \mathrm{id} \\ \mathrm{id} & \frac{\partial w^{cs}}{\partial \lambda} \end{array}\right)$$

is invertible. If for $p = (\lambda, x)$, $Bp = 0$ then

$$x - \frac{\partial \pi_\lambda w_z^u}{\partial x}\frac{\partial w^{cs}}{\partial \lambda} x = 0,$$

which since w_z^u and w^{cs} are Lipschitz with constants $\rho < 1/L$ and $L < 1$, respectively, implies that $x = 0$, and in turn that $\lambda = 0$. Since B is invertible, it is evident that the rank of A is $c+u+s$, which implies (55).

Since $W_z^u \subset W^{cu}$ we see that $q^* \in W_z^u \cap W^{cs} \subset W^{cu} \cap W^{cs} = \Lambda^*$, hence $q^* \in W_z^u \cap \Lambda^*$. The fact that the intersection point is unique follows from the fact that $\Lambda^* \subset W^{cu}$ and already established uniqueness of the intersection point $W_z^u \cap W^{cs}$. ■

10. Stable fibers

The goal of this section is to establish the existence of the foliation of W^{cs} into the stable fibers W_z^s for $z \in W^{cs}$. In this section $\theta = (\lambda, x)$. Throughout the section we assume that the assumptions of Theorem 16 hold.

Let us fix a point $z \in W^{cs}$. Let $y \in \overline{B}_s$ and consider the following problem: Find θ such that

$$\pi_\theta\left(f^n\left(\theta, y\right) - f^n\left(z\right)\right) = 0,$$

under the constraint

$$f^i(\theta, y) \in D \qquad \text{for } i = 1, \ldots, n.$$

By taking $b_y(\theta) = (\theta, y)$ and observing that $f^i(\theta, y) = f^i(b_y(\theta))$, from Lemma 45 it follows immediately that this problem has a unique solution $\theta(y)$ which is as smooth as f.

We define

$$d_{n,z}(y) = (\theta(y), y). \tag{56}$$

Our objective will be to prove that $d_{n,z}(y)$ are vertical disks converging uniformly to W_z^s as n tends to infinity. First we prove a technical lemma.

Lemma 63. *Assume that $f^j(z) \in D$ for $j = 0, 1, \ldots, n$. If $f^j(q_i) \in J_s\left(f^j(z), 1/L\right) \cap D$ for $i = 1, 2$, $j = 1, \ldots, n$ and if*

$$q_1 \notin J_s(q_2, 1/L)$$

then

$$\begin{aligned} \|\pi_\theta(q_1 - q_2)\| &\leq \frac{4R}{L}\left(\frac{\mu_{s,1}}{\xi_{cu,1,P}}\right)^n, \\ \|q_1 - q_2\| &\leq (1 + L)\frac{4R}{L}\left(\frac{\mu_{s,1}}{\xi_{cu,1,P}}\right)^n. \end{aligned}$$

Proof. Our assumption $f^j(q_i) \in J_s\left(f^j(z), 1/L\right)$, $i = 1, 2$ and $j = 0, 1, \ldots, n$ implies that $f^j(q_1), f^j(q_2), f^j(z)$ are contained in the same charts for $j = 0, 1, \ldots, n$.

By Corollary 35, $f^j(q_1) \notin J_s(f^j(q_2), 1/L)$ for $j = 0, 1, \ldots, n$, hence $f^j(q_1) \in J_{cu}(f^j(q_2), L)$. By Lemma 38 this implies

$$\|\pi_\theta(f^n(q_1) - f^n(q_2))\| \geq \xi_{cu,1,P}\|\pi_\theta(f^{n-1}(q_1) - f^{n-1}(q_2))\| \geq \ldots \geq \xi_{cu,1,P}^n\|\pi_\theta(q_1 - q_2)\|. \tag{57}$$

We estimate $\|\pi_\theta(f^n(q_1) - f^n(q_2))\|$ using the contraction in the y-direction. By Lemma 37, for $i = 1, 2$,

$$\begin{aligned} \|\pi_y(f^n(q_i) - f^n(z))\| &\leq \mu_{s,1}\left\|\pi_y\left(f^{n-1}(q_i) - f^{n-1}(z)\right)\right\| \\ &\leq \mu_{s,1}^n\|\pi_y(q_i - z)\| \leq \mu_{s,1}^n 2R. \end{aligned}$$

Since $f^n(q_i) \in J_s(f^n(z), 1/L)$,

$$\|\pi_\theta(f^n(q_i) - f^n(z))\| \leq \frac{1}{L}\|\pi_y(f^n(q_i) - f^n(z))\| \leq \mu_{s,1}^n\frac{2R}{L}$$

which by the triangle inequality implies

$$\begin{aligned} \|\pi_\theta(f^n(q_1) - f^n(q_2))\| &\leq \|\pi_\theta(f^n(q_1) - f^n(z))\| + \|\pi_\theta(f^n(q_2) - f^n(z))\| \\ &\leq \mu_{s,1}^n\frac{4R}{L}. \end{aligned}$$

Combining the above with (57),

$$\|\pi_\theta(q_1 - q_2)\| \leq \frac{4R}{L}\left(\frac{\mu_{s,1}}{\xi_{cu,1,P}}\right)^n.$$

Since $q_1 \in J_{cu}(q_2, L)$, then

$$\|\pi_y (q_1 - q_2)\| \leq L \|\pi_\theta (q_1 - q_2)\|,$$

hence

$$\begin{aligned} \|q_1 - q_2\| &\leq \|\pi_y (q_1 - q_2)\| + \|\pi_\theta (q_1 - q_2)\| \\ &\leq (1 + L) \frac{4R}{L} \left(\frac{\mu_{s,1}}{\xi_{cu,1,P}} \right)^n, \end{aligned}$$

as required. ■

Lemma 64. *For any $n \geq 0$, $d_{n,z}$ is a vertical disc and the sequence $d_{n,z}$ converges uniformly to horizontal disk d_z. Moreover, $W_z^s = \left\{ (w_z^s(y), y) : y \in \overline{B}_s(R) \right\}$, where $w_z^s : \overline{B}_s(R) \to \Lambda \times \overline{B}_u(R)$ and $d_z(y) = (w_z^s(y), y)$*

Proof. Let $y_1, y_2 \in \overline{B}_s(R)$. By construction,

$$\pi_{(\lambda,x)} f^n (d_{n,z}(y_1)) = \pi_{(\lambda,x)} f^n (z) = \pi_{(\lambda,x)} f^n (d_{n,z}(y_2)), \tag{58}$$

and $f^i (d_{n,z}(y_1)), f^i (d_{n,z}(y_2)) \in D$ for $i = 0, \ldots, n$. Since (58) implies that

$$f^n (d_{n,z}(y_1)) \in J_s (f^n (d_{n,z}(y_2)), 1/L),$$

by the backward cone condition,

$$d_{n,z}(y_1) \in J_s (d_{n,z}(y_2), 1/L),$$

which means that $d_{n,z}$ is a vertical disc. Also, for any $y \in \overline{B}_s(R)$, since $f^n (d_{n,z}(y)) \in J_s (f^n (z), 1/L)$, by the backward cone condition,

$$f^j (d_{n,z}(y)) \in J_s \left(f^j (z), 1/L \right) \quad \text{for } j = 0, \ldots, n. \tag{59}$$

Observe that since

$$\pi_y (d_{n,z}(y) - d_{n+j,z}(y)) = 0, \tag{60}$$

we have $\|\pi_\theta(d_{n,z}(y) - d_{n+j,z}(y))\| = \|d_{n,z}(y) - d_{n+j,z}(y)\|$. Assume that $d_{n,z}(y) \neq d_{n+j,z}(y)$. By (60) we see that $d_{n,z}(y) \notin J_s (d_{n+j,z}(y), 1/L)$. From Lemma 63 applied to $q_1 = d_{n,z}(y)$, $q_2 = d_{n+j,z}(y)$ and z it follows that

$$\|d_{n,z}(y) - d_{n+j,z}(y)\| = \|\pi_\theta(d_{n,z}(y) - d_{n+j,z}(y))\| \leq \frac{4R}{L} \left(\frac{\mu_{s,1}}{\xi_{cu,1,P}} \right)^n. \tag{61}$$

Note that if $d_{n,z}(y) = d_{n+j,z}(y)$, then (61) also holds. Since by our assumptions $\frac{\mu_{s,1}}{\xi_{cu,1,P}} < 1$, therefore $d_{n,z}$ is a Cauchy sequence in supremum norm. Let us denote the limit by d_z.

The $d_{n,z}$ are vertical discs. The properties (25) are preserved when passing to the limit, hence d_z is a vertical disc.

We show that for all $y \in \overline{B}_s(R)$, $d_z(y) \in W^{cs}$. By construction, for any $i \geq 0$ and $n \geq i$, $f^i (d_{n,z}(y)) \in D$. Passing to the limit with n to infinity gives $f^i (d_z(y)) \in D$, as required.

By (59), passing to the limit with n to infinity, we see that for any $j \geq 0$

$$f^j(d_{n,z}(y)) \in J_s\left(f^j(z), 1/L\right),$$

hence $\{d_z(y) \mid y \in \overline{B}_s(R)\} \subset W_z^s$.

To show that $W_z^s \subset \{d_z(y) \mid y \in \overline{B}_s(R)\}$, let us consider $p \in W^{cs}$, such that $f^j(p) \in J_s\left(f^j(z), 1/L\right)$ for all $j \geq 0$. Let $y = \pi_y p$. We will show that $p = d_z(y)$. From Lemma 63, (taking $q_1 = p$, and $q_2 = d_z(y)$,) it follows that

$$\|p - d_z(y)\| = \|\pi_\theta(p - d_z(y))\| \leq \frac{4R}{L}\left(\frac{\mu_{s,1}}{\xi_{cu,1,P}}\right)^n \to 0, \quad n \to \infty.$$

Therefore $p = d_z(y)$.

The function w_z^s can be defined as $w_z^s(y) = \pi_\theta d_z(y)$. ∎

Lemma 65. *Let $m \leq k$. Let $d_{n,z}$ be the sequence of vertical discs defined in (56). Assume that $d_{n,z}$ are C^m and that for any n, $\|\pi_\theta d_{n,z}\|_{C^m} < c_m$, with c_m independent of n. If f satisfies rate conditions of order m, then $\|\pi_\theta d_{n,z}\|_{C^{m+1}} < c_{m+1}$ for a constant independent of n.*

Proof. The proof follows from identical arguments to the proof of Lemma 51. The noticeable difference is that when we apply Theorem 33, we should choose coordinates $\mathrm{x} = \theta = (\lambda, x)$, $\mathrm{y} = y$ and constants $\xi = \xi_{cu,2}$, $\mu = \mu_{s,1}$. Note that conditions (1), (4) imply (23) for any $m \geq 0$. ∎

Lemma 66. *For any $z \in W^{cs}$ the manifold W_z^s is C^k.*

Proof. The functions $\pi_\theta d_{n,z}$ are C^{k+1} and uniformly Lipschitz with constant $1/L$. The fact that C^k smoothness is preserved as we pass to the limit follows from Lemma 65 and mirror arguments to the proof of Lemma 48. ∎

Lemma 67. *Let $z \in W^{cs}$. If $p \in W_z^s$, then for $n \geq 0$ $f^n(p), f^n(z)$ are in the same chart and*

$$\|f^n(p) - f^n(z)\| \leq \left(1 + \frac{1}{L}\right)\|\pi_y(p - z)\|\mu_{s,1}^n, \qquad n \geq 0.$$

Proof. We first observe that for any $q_1, q_2 \in D$, such that $q_1 \in J_s(q_2, 1/L)$, holds

$$\|\pi_\theta(q_1 - q_2)\| \leq \frac{1}{L}\|\pi_y(q_1 - q_2)\|,$$

hence

$$\|q_1 - q_2\| \leq \|\pi_\theta(q_1 - q_2)\| + \|\pi_y(q_1 - q_2)\| \leq \left(1 + \frac{1}{L}\right)\|\pi_y(q_1 - q_2)\|.$$

Let $p \in W_z^s$, which means that $f^i(p) \in J_s\left(f^i(z), 1/L\right) \cap D$, for all $i \geq 0$. From Lemma 37 it follows for $n > 0$ that

$$\|\pi_y(f^n(p) - f^n(z))\| \leq (\mu_{s,1})^n \|\pi_y(p - z)\|.$$

Therefore

$$\begin{aligned}\|f^n(p) - f^n(z)\| &\leq \left(1 + \frac{1}{L}\right) \|\pi_y (f^n(p) - f^n(z))\| \\ &\leq \mu_{s,1}^n \left(1 + \frac{1}{L}\right) \|\pi_y (p - z)\|,\end{aligned}$$

as required. ■

Lemma 68. *For any $z \in W^{cs}$, let us define a set $U = U(z)$ as*

$$\begin{aligned} U &= \{p \in D : f^n(p) \in D \text{ for all } n \geq 0, \text{ and} \\ &\quad \exists n_0 \geq 0, \exists C > 0 \text{ (which may depend on } p, z) \\ &\quad \text{s.t. for } n \geq n_0, \ f^n(p) \text{ and } f^n(z) \text{ are in the same good chart and} \\ &\quad \|f^n(p) - f^n(z)\| \leq C\mu_{s,1}^n\}. \end{aligned}$$

Then $W_z^s = U$.

Proof. From Lemma 67 it follows that $W_z^s \subset U$. It remains to prove that $U \subset W_z^s$.

For the proof by the contradiction let us consider $p \in U \setminus W_z^s$. Observe that from the backward cone condition (Definition 13), since $p \notin W_z^s$, it follows that for $i \geq n_0$ holds $f^i(p) \notin J_s\left(f^i(z), 1/L\right)$. Hence $f^i(p) \in J_{cu}(f^i(z), L)$ for any $i \geq n_0$ (this makes sense because $f^i(p)$ and $f^i(z)$ are in the same good chart for $i \geq n_0$.) Hence from Lemma 38 it follows that

$$\left\|\pi_\theta \left(f^{n_0+i}(p) - f^{n_0+i}(z)\right)\right\| \geq \xi_{cu,1,P}^i \|\pi_\theta (f^{n_0}(p) - f^{n_0}(z))\|,$$

and thus for any $n > n_0$

$$\|\pi_\theta (f^n(p) - f^n(z))\| \geq \xi_{cu,1,P}^n \left(\xi_{cu,1,P}^{-n_0} \|\pi_\theta (f^{n_0}(p) - f^{n_0}(z))\|\right). \tag{62}$$

By our assumption

$$\|f^n(p) - f^n(z)\| \leq C\mu_{s,1}^n, \quad n \geq n_0. \tag{63}$$

Since $\mu_{s,1} < \xi_{cu,1,P}$, conditions (63) and (62) contradict each other. This means that $p \in W_z^s$, as required. ■

Lemmas 64, 66, 68 combined prove the claims about W_z^u from Theorem 16. We now prove Theorem 17, which can be used to obtain tighter Lipschitz bounds on w_z^s.

Proof of Theorem 17. Observe that by definition of $d_{n,z}$, for any $y_1, y_2 \in \overline{B}_s(R)$

$$\pi_{(\lambda,x)} (f^n (d_{n,z} (y_1))) = \pi_{(\lambda,x)} (f^n (z)) = \pi_{(\lambda,x)} f^n (d_{n,z} (y_2)),$$

hence for $y_1 \neq y_2$

$$f^n (d_{n,z} (y_2)) \notin J_s^c (f^n (d_{n,z} (y_1)), M). \tag{64}$$

By Theorem 32, taking $\mathrm{x} = x$ and $\mathrm{y} = (\lambda, y)$, for $i = 1, \ldots, n$,

$$f(J_s^c(f^{i-1} (d_{n,z} (y_1)), M) \cap D) \subset J_s^c(f^i (d_{n,z} (y_1)), M). \tag{65}$$

This means that

$$d_{n,z}(y_2) \notin J_s^c(d_{n,z}(y_1), M). \tag{66}$$

(Since otherwise from (65) and (64) we would get a contradiction.) By definition of $d_{n,z}$,

$$\pi_y d_{n,z}(y_i) = y_i \qquad \text{for } i = 1, 2,$$

which combined with (66) gives

$$\left\|\pi_{(\lambda,x)}(d_{n,z}(y_1) - d_{n,z}(y_2))\right\| \leq M \left\|\pi_y(d_{n,z}(y_1) - d_y(y_2))\right\| = M \left\|y_1 - y_2\right\|,$$

as required. ■

Proposition 69. *Let $z \in W^{cs}$. Then the intersection $W_z^s \cap W^{cu}$ consists of a single point and is transversal. The intersection $W_z^s \cap \Lambda^*$ consists of a single point.*

Proof. The result follows from similar arguments to the proof of Proposition 62. ■

11. Invariant manifolds for vector bundles

The previous discussion was focused on the setting where Λ was a torus. We now generalize the result for Λ which are compact manifolds without boundaries.

11.1. Vector bundles

We start by recalling the definition of the vector bundle [15].

Definition 70. *Let B, E be topological spaces. Let $p : E \to B$ be a continuous map. A* vector bundle chart *on (p, E, B) with* domain *U and* dimension *n is a homeomorphism $\varphi : p^{-1}(U) \to U \times \mathbb{R}^n$, where $U \subset B$ is open and such that*

$$\pi_1 \circ \varphi(z) = p(z), \qquad \text{for } z \in p^{-1}(U).$$

We will denote such bundle chart by a pair (φ, U).

For each $\lambda \in U$ we define the homeomorphism φ_λ to be the composition

$$\varphi_\lambda : p^{-1}(\lambda) \xrightarrow{\varphi} \{\lambda\} \times \mathbb{R}^n \to \mathbb{R}^n.$$

A vector bundle atlas *Φ on (p, E, B) is a family of vector bundle charts on (p, E, B) with the values in the same $\mathbb{R}^n$, whose domains cover B and such that whenever (φ, U) and (ψ, V) are in Φ and $\lambda \in U \cap V$, the homeomorphism $\psi_\lambda \varphi_\lambda^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is linear. The map*

$$U \cap V \ni \lambda \mapsto \psi_\lambda \varphi_\lambda^{-1} \in GL(n)$$

is continuous for all pairs of charts in Φ.

A maximal vector bundle atlas Φ is a vector bundle structure *on (p, E, B). We then call $\gamma = (p, E, B, \Phi)$ a* vector bundle *having* (fibre) dimension n, projection p, total space E *and* base space B.

The fibre *over $\lambda \in B$ is the space $p^{-1}(\lambda) = \gamma_\lambda = E_\lambda$. γ_λ has the vector space structure.*

If the E, B are C^r manifolds and all maps appearing in the above definition are C^r, then we will say that the bundle (p, E, B, Φ) is a C^r-bundle.

One can introduce the notion of subbundles, morphisms etc. (see [15] and references given there). The fibers can have a structure: for example a scalar product, a norm, which depend continuously on the base point.

Definition 71. *We say that the vector bundle γ is a Banach vector bundle with fiber being the Banach space $(F, \|\cdot\|)$, if for each $\lambda \in B$ the fiber γ_λ is a Banach space with norm $\|\cdot\|_\lambda$ such that for each bundle chart (φ, U) the map $\varphi_\lambda : E_\lambda \to F$ is an isometry ($\|\varphi_\lambda(v)\| = \|v\|_\lambda$).*

For vector bundles γ_1, γ_2 over the same base space one can define $\gamma = \gamma_1 \oplus \gamma_2$ by setting $\gamma_\lambda = \gamma_{1,\lambda} \oplus \gamma_{2,\lambda}$. In the following, points in $\gamma_1 \oplus \gamma_2$ will be denoted by a triple (λ, v_1, v_2), where $\lambda \in B$, $v_1 \in \gamma_{1,\lambda}$ and $v_2 \in \gamma_{2,\lambda}$. If γ_1 and γ_2 are both Banach bundles, then $\gamma_1 \oplus \gamma_2$ is also a Banach vector bundle with the norm on $\gamma_{1,\lambda} \oplus \gamma_{2,\lambda}$ defined by $\|(v_1, v_2)\|_\lambda = \sqrt{\|v_1\|_\lambda^2 + \|v_2\|_\lambda^2}$. We will also always assume that the atlas on bundle $\gamma_1 \oplus \gamma_2$ respects this structure, namely if (η, U) is a bundle chart for $\gamma_1 \oplus \gamma_2$, then its restriction (obtained through projection) to γ_i is also a bundle chart for γ_i for $i = 1, 2$.

11.2. Formulation of the result

Assume that, we have Banach vector bundles γ_u, γ_s, $\gamma = \gamma_u \oplus \gamma_s$. Let u and s be the fiber dimension of γ_u and γ_s, respectively. Let the base space for γ, denoted by Λ, be a C^k compact manifold without boundary of dimension c. We consider $D \subset \gamma_u \oplus \gamma_s$ defined as:

$$D = \{(\lambda, v_1, v_2) \in \gamma_u \oplus \gamma_s \mid \lambda \in \Lambda, v_1 \in \gamma_{u,\lambda}, v_2 \in \gamma_{s,\lambda}, \quad \|v_1\| \leq R, \quad \|v_2\| \leq R\}.$$

Consider a finite open covering $\{\mathcal{U}_i\}$ of Λ and an atlas $\{(\eta_i, \mathcal{U}_i)\}$, where

$$\eta_i : \mathcal{U}_i \to \eta_i(\mathcal{U}_i) \subset \mathbb{R}^c$$

are charts. We assume that there exists a $R_\Lambda > 0$ such that for any $\lambda \in \Lambda$ there exists an i such that

$$\overline{B}_c(\eta_i(\lambda), R_\Lambda) \subset \eta_i(\mathcal{U}_i). \tag{67}$$

Also, we assume that for any η_i there exists a λ such that (67) holds true.

Definition 72. *We refer to a chart $(\eta_i, \mathcal{U}_i)$ satisfying (67) as a good chart for λ.*

We assume that for each $(\eta_i, \mathcal{U}_i)$ we have a vector bundle chart for γ of the form $\varphi_i = (p, \varphi_i^u, \varphi_i^s) : p^{-1}(\mathcal{U}_i) \to \mathcal{U}_i \times \mathbb{R}^s \times \mathbb{R}^u$. We define maps

$$\tilde{\eta}_i : p^{-1}(\mathcal{U}_i) \to \mathbb{R}^c \times \mathbb{R}^u \times \mathbb{R}^s,$$

as

$$\tilde{\eta}_i(\lambda, v_1, v_2) = \left(\eta_i(\lambda), \varphi_{i,\lambda}^u(v_1), \varphi_{i,\lambda}^s(v_2)\right),$$

and sets

$$\widetilde{\mathcal{U}}_i = p^{-1}(\mathcal{U}_i) \cap D,$$
$$D_i = \tilde{\eta}_i\left(\widetilde{\mathcal{U}}_i\right) = \eta_i(\mathcal{U}_i) \times B_u(R) \times B_s(R).$$

Definition 73. *We say that $\tilde{\eta}_i$ is a good chart for $z \in \gamma_u \oplus \gamma_s$ if η_i is a good chart for $p(z) \in \Lambda$.*

We use a notation $z = (\lambda, v_1, v_2)$ for points in $\gamma_u \oplus \gamma_s$ and $(\theta, x, y) \in \mathbb{R}^c \times \mathbb{R}^u \times \mathbb{R}^s$ to make the distinction between those on the bundle and those in local coordinates.

We fix a constant $L \in \mathbb{R}$ satisfying

$$L \in \left(\frac{2R}{R_\Lambda}, 1 \right).$$

Remark 74. *Using mirror arguments to those in Remark 6 we see that for any $M \leq \frac{1}{L}$ and any good chart $\tilde{\eta}_i$ around $z \in D$, holds*

$$J_s(\tilde{\eta}_i(z), M) \cap D_i \subset \overline{B}_c(\pi_\theta \tilde{\eta}_i(z), R_\Lambda) \times \overline{B}_u(R) \times \overline{B}_s(R),$$
$$J_u(\tilde{\eta}_i(z), M) \cap D_i \subset \overline{B}_c(\pi_\theta \tilde{\eta}_i(z), R_\Lambda) \times \overline{B}_u(R) \times \overline{B}_s(R).$$

This is important for us, since it is one of the reasons why the proof presented in previous sections will also work for the current setting. For instance, one of the founding blocks of the proof was the Lemma 44, which states that images of horizontal discs are horizontal discs. Horizontal discs are contained in cones, and here we see that the relevant fragments of the cones will lie in local coordinates. This will allow us to consider conditions defined locally.

We consider a map

$$f : D \to \gamma_u \oplus \gamma_s.$$

For any $z \in D$, a good chart $\tilde{\eta}_i$ around z and a good chart $\tilde{\eta}_j$ around $f(z)$ we can define (locally around $\tilde{\eta}_i(z)$)

$$f_{ji} := \tilde{\eta}_j \circ f \circ \tilde{\eta}_i^{-1}.$$

For a chart η_i we define sets $D_i^+, D_i^- \subset \mathbb{R}^c \times \mathbb{R}^u \times \mathbb{R}^s$ as

$$D_i^+ = \eta_i(\mathcal{U}_i) \times \overline{B}_u(R) \times \partial B_s(R),$$
$$D_i^- = \eta_i(\mathcal{U}_i) \times \partial \overline{B}_u(R) \times B_s(R).$$

Definition 75. *We say that f satisfies covering conditions if for any $z \in D$ the following conditions hold:*

For any good chart $\tilde{\eta}_i$ around z, there exists a good chart $\tilde{\eta}_j$ around $f(z)$ such that the set

$$U = J_u(\tilde{\eta}_i(z), 1/L) \cap D_i$$

is contained in the domain of f_{ji}. Additionally, for $\theta^* = \pi_\theta \tilde{\eta}_j(f(z))$, there exists a homotopy

$$h : [0,1] \times U \to B_c(\theta^*, R_\Lambda) \times \mathbb{R}^u \times \mathbb{R}^s,$$

and a linear map $A : \mathbb{R}^u \to \mathbb{R}^u$ which satisfy:

1. $h_0 = f_{ji}|_U$,

2. *for any* $\alpha \in [0,1]$,

$$h_\alpha \left(U \cap D_i^-\right) \cap D_j = \emptyset, \tag{68}$$
$$h_\alpha \left(U\right) \cap D_j^+ = \emptyset, \tag{69}$$

3. $h_1(\theta, x, y) = (\theta^*, Ax, 0)$,
4. $A(\partial B_u(R)) \subset \mathbb{R}^u \setminus \overline{B}_u(R)$.

Definition 76. *For* $z \in D$, *we refer to* $(\tilde{\eta}_j, \tilde{\eta}_i)$ *which satisfy the conditions of Definition 75 as a good charts pair for* z.

We assume that for any $z \in D$ and any good chart η_j for z there exists an i such that $(\tilde{\eta}_j, \tilde{\eta}_i)$ is a good charts pair.

We use a notation

$$\mathfrak{C}(z) = \{(j,i) : (\tilde{\eta}_j, \tilde{\eta}_i) \text{ is a good charts pair for } z\},$$

$$\mathfrak{C} = \bigcup_{z \in D} \mathfrak{C}(z).$$

We now define the constants

$$\xi_{u,1,P} = \inf_{(j,i)\in\mathfrak{C}} m\left[\frac{\partial (f_{ji})_x}{\partial x}(\mathrm{dom}\,(f_{ji}))\right] - \frac{1}{L}\sup_{(j,i)\in\mathfrak{C}, z\in \mathrm{dom}(f_{ji})}\left\|\frac{\partial (f_{ji})_x}{\partial(\lambda, y)}(z)\right\|,$$
$$\xi_{cu,1,P} = \inf_{(j,i)\in\mathfrak{C}} m\left[\frac{\partial (f_{ji})_{(\lambda,x)}}{\partial(\lambda, x)}(\mathrm{dom}\,(f_{ji}))\right] - L\sup_{(j,i)\in\mathfrak{C}, z\in \mathrm{dom}(f_{ji})}\left\|\frac{\partial (f_{ji})_{(\lambda,x)}}{\partial y}(z)\right\|.$$

Similarly, we define constants which are analogues of $\mu_{\iota,\kappa}$, $\xi_{\iota,\kappa}$, (for $\iota \in \{u, s, cu, cs\}$ and $\kappa \in \{1,2\}$) from section 3, by changing the conditions under the sup and inf, from "$z \in D$" to "$(j,i) \in \mathfrak{C}, z \in \mathrm{dom}\,(f_{ji})$".

Definition 77. *We say that* f *satisfies cone conditions and rate conditions of order* $k \geq 1$ *if the inequalities from Definition 5 are satisfied.*

Definition 78. *We say that* f *satisfies backward cone conditions if for any* $z \in D$ *and a good charts pair* $(\tilde{\eta}_j, \tilde{\eta}_i)$ *for* z, *the following condition is fulfilled:*

If $z' \in D$, $f(z') \in \widetilde{\mathcal{U}}_j$ *and* $\tilde{\eta}_j(f(z')) \in J_s(\tilde{\eta}_j(f(z)), 1/L)$ *then* $z' \in \widetilde{\mathcal{U}}_i$ *and*

$$\tilde{\eta}_i(z') \in J_s(\tilde{\eta}_i(z), 1/L).$$

Definition 79. *For* $z \in D$, *we refer to* $\left(\tilde{\eta}_{i_n}, \tilde{\eta}_{i_{n-1}}, \ldots, \tilde{\eta}_{i_0}\right)$ *as a good charts sequence for* z, *if* $\left(\tilde{\eta}_{i_{k+1}}, \tilde{\eta}_{i_k}\right)$ *is a good charts pair for* $f^k(z)$.

For a good chart sequence $\left(\tilde{\eta}_{i_n}, \tilde{\eta}_{i_{n-1}}, \ldots, \tilde{\eta}_{i_0}\right)$ we use a notation

$$f_{i_n,\ldots,i_0} = f_{i_n i_{n-1}} \circ \ldots \circ f_{i_2 i_1} \circ f_{i_1 i_0}.$$

We can now formulate our result.

Theorem 80. *If f is C^{k+1} and satisfies covering conditions, rate conditions of order k and backward cone conditions, then W^{cs}, W^{cu} and Λ^* are C^k manifolds. In local coordinates given by a chart $\tilde{\eta}_i$, the manifolds are graphs of C^k functions*

$$\begin{aligned} w_i^{cs} &: \eta_i(\mathcal{U}_i) \times \overline{B}_s(R) \to \overline{B}_u(R), \\ w_i^{cu} &: \eta_i(\mathcal{U}_i) \times \overline{B}_u(R) \to \overline{B}_s(R), \\ \chi_i &: \eta_i(\mathcal{U}_i) \to \overline{B}_u(R) \times \overline{B}_s(R), \end{aligned}$$

meaning that

$$\begin{aligned} \tilde{\eta}_i\left(W^{cs} \cap \widetilde{\mathcal{U}}_i\right) &= \left\{(\theta, w_i^{cs}(\theta, y), y) : \theta \in \eta_i(\mathcal{U}_i), y \in \overline{B}_s(R)\right\}, \\ \tilde{\eta}_i\left(W^{cu} \cap \widetilde{\mathcal{U}}_i\right) &= \left\{(\theta, x, w_i^{cu}(\theta, y)) : \theta \in \eta_i(\mathcal{U}_i), x \in \overline{B}_u(R)\right\}, \\ \tilde{\eta}_i\left(\Lambda^* \cap \widetilde{\mathcal{U}}_i\right) &= \left\{(\theta, \chi(\theta)) : \theta \in \eta_i(\mathcal{U}_i)\right\}. \end{aligned}$$

Moreover, $f_{|W^{cu}}$ is an injection.

For any $z \in W^{cs}$ and any good chart $\tilde{\eta}_i$ around z,

$$\tilde{\eta}_i\left(W^{cs} \cap \widetilde{\mathcal{U}}_i\right) \subset J_{cs}(\tilde{\eta}_i(z), L).$$

For any $z \in W^{cu}$ and any good chart $\tilde{\eta}_i$ around z,

$$\tilde{\eta}_i\left(W^{cu} \cap \widetilde{\mathcal{U}}_i\right) \subset J_{cu}(\tilde{\eta}_i(z), L),$$

Also, for any $z \in \Lambda^*$ and any good chart $\tilde{\eta}_i$ around z, for $M = \frac{\sqrt{2}L}{\sqrt{1-L^2}}$,

$$\tilde{\eta}_i\left(\Lambda^* \cap \widetilde{\mathcal{U}}_i\right) \subset \left\{(\theta, x, y) : \left\|(x, y) - \pi_{(x,y)}\tilde{\eta}_i(z)\right\| \leq M \left\|\theta - \pi_\theta \tilde{\eta}_i(z)\right\|\right\}.$$

The manifolds W^{cs} and W^{cu} intersect transversally, and $W^{cs} \cap W^{cu} = \Lambda^*$.

The manifolds W^{cs} and W^{cu} are foliated by invariant fibers W_z^s and W_z^u, which in local coordinates given by any good chart $\tilde{\eta}_i$ around z are graphs of C^k functions

$$\begin{aligned} w_{z,i}^s &: \overline{B}_s(R) \to \eta_i(\mathcal{U}_i) \times \overline{B}_u(R), \\ w_{z,i}^u &: \overline{B}_u(R) \to \eta_i(\mathcal{U}_i) \times \overline{B}_s(R), \end{aligned}$$

The functions $w_{z,i}^s$ and $w_{z,i}^u$ are Lipschitz with constants $1/L$. Moreover,

$$\begin{aligned} W_z^s = \{ & p \in D : f^n(p) \in D \text{ for all } n \geq 0, \text{ and} \\ & \exists\, C > 0 \text{ (which can depend on } p) \\ & \text{s.t. for } n \geq 0, \text{ and any good charts sequence } (\tilde{\eta}_{i_n}, \ldots, \tilde{\eta}_{i_0}) \text{ for } z \\ & \|f_{i_n,\ldots,i_0}(p) - f_{i_n,\ldots,i_0}(z)\| \leq C\mu_{s,1}^n \}, \end{aligned}$$

and

$$\begin{aligned} W_z^u = \{ & p \in W^{cu} : \text{the unique backward trajectory } \{p_i\}_{i=-\infty}^0 \text{ of } p \text{ in } D, \text{ and for} \\ & \text{any such } \{p_i\}, \text{ and the unique backward trajectory } \{z_i\}_{i=-\infty}^0 \text{ of } z \text{ in } D \\ & \exists\, C > 0 \text{ (which can depend on } p) \\ & \text{s.t. for } n \geq 0, \text{ and any good charts sequence } \left(\tilde{\eta}_{i_0}, \ldots, \tilde{\eta}_{i_{-n}}\right) \text{ for } z_{-n} \\ & \left\|\tilde{\eta}_{i_{-n}}(p_{-n}) - \tilde{\eta}_{i_{-n}}(z_{-n})\right\| \leq C\xi_{u,1,P}^{-n} \end{aligned}$$

11.3. Outline of the proof

The proof of the theorem follows from the same arguments as the proof of Theorem 16. The only difference is that instead of investigating compositions f^n, we consider good chart sequences $(\tilde{\eta}_{i_n}, \ldots, \tilde{\eta}_{i_0})$ and local maps $f_{i_n,\ldots,i_0}$.

We shall now focus on the needed changes to perform the construction. We first go over the construction of the center-unstable manifold (see section 6). The construction of the center-unstable manifold is based on propagation of horizontal discs. In our context we modify the definition of the horizontal disc as follows:

Definition 81. *We say that a set $b \subset D$ is a horizontal disc if for any $z \in b$ there exists a good chart $\tilde{\eta}_i$ around z such that $b \subset \widetilde{\mathcal{U}}_i$ and a continuous function $b_i : \overline{B}_u(R) \to D_i$*

$$\begin{aligned} \tilde{\eta}_i(b) &= b_i\left(\overline{B}_u(R)\right), \\ \pi_x b_i(x) &= x, \\ b_i\left(\overline{B}_u(R)\right) &\subset J_u\left(b_i(x), 1/L\right) \qquad \text{for any } x \in \overline{B}_u\left(R\right). \end{aligned} \tag{70}$$

We say that b is C^k if b_i are C^k.

With such definition we have a mirror result to Lemma 44 . This is done in Lemma 83.

Remark 82. *Lemmas 83, 86 are the core of the construction of both W^{cu} and W^{cs}. For this reason we go into some degree of detail outlining its proof, pointing out differences in approach when working with local maps.*

Lemma 83. *Assume that $b \subset D$ is a horizontal disc. If f satisfies covering conditions and rate conditions of order $k \geq 0$, then there exists a horizontal disc $b^* \subset D$ such that $f(b) \cap D = b^*$. Moreover if b and f are C^k then so is b^*.*

Proof. The proof is a mirror argument to the proof of Lemma 44. We therefore restrict our attention to setting up the local maps needed for the construction.

Let let us fix $z \in b$. Let $\tilde{\eta}_i$ be a good chart around z, for which conditions (70) hold. Let $\tilde{\eta}_j$ be the good chart around $f(z_0)$ from Definition 75. Note that $(\tilde{\eta}_j, \tilde{\eta}_i)$ is a good charts pair. Let $\theta^* = \pi_\theta \tilde{\eta}_j\left(f(z)\right)$. Existence and smoothness of $b_j^* : \overline{B}_u\left(R\right) \to D_j$ such that

$$f_{ji} \circ b_i\left(\overline{B}_u\left(R\right)\right) \cap D_i = b_j^*\left(\overline{B}_u\left(R\right)\right)$$

follows from a mirror construction to the one from the proof of Lemma 44. We can define

$$b^* = \tilde{\eta}_j^{-1} \circ b_j^*\left(\overline{B}_u\left(R\right)\right).$$

By construction, b_j^* satisfies (70).

Let us now take any $\hat{z} \in b^*$. We need to prove that we have a good chart $\tilde{\eta}_{\hat{\jmath}}$ for $\hat{z}$, for which conditions from Definition 81 hold. By our construction $\hat{z} = f(\hat{z}_0)$, for some $\hat{z}_0 = \hat{z}_0\left(\hat{z}\right) \in b$. Let $\tilde{\eta}_{\hat{\imath}}$ be the good chart around $\hat{z}_0$ for which conditions 81 hold for b. Let $\tilde{\eta}_{\hat{\jmath}}$ be the good chart around $\hat{z} = f(\hat{z}_0)$ from Definition 75. Once again, from the same construction as in the proof of Lemma 44 follows the existence and smoothness of $b_{\hat{\jmath}}^*$. ■

Remark 84. *From the proof we see that for $z \in b$ such that $f(z) \in b^* \subset D$ and for a good charts pair $(\tilde{\eta}_j, \tilde{\eta}_i)$ for z we can construct b_j satisfying (81). The chart $\tilde{\eta}_j$ is a good chart for $f(z)$.*

Definition 85. *Let $D_u = \pi_{\gamma_u} D \subset \gamma_u$. We say that a function $b : D_u \to D$ is a center-horizontal disc if for any $(\lambda, v_1) \in D_u$*

$$\pi_{\gamma_u} b(\lambda, v_1) = (\lambda, v_1),$$

and for any $z \in b(\gamma_u)$ and any good chart $\tilde{\eta}_i$ around z

$$\tilde{\eta}_i \circ b(D_u) \cap D_i \subset J_{cu}(\tilde{\eta}_i(z), L). \tag{71}$$

Lemma 86. *Assume that b is a center-horizontal disc. If f satisfies covering conditions, backward cone conditions and rate conditions of order $l \geq 0$, then there exists a center-horizontal disc b^* such that*

$$f(b(D_u)) \cap D = b^*(D_u).$$

Moreover, if f and b are C^k, then so is b^*.

Proof. The proof goes along the same lines as the proof of Lemma 45. We will outline the differences concerning the choices of local maps.

We start by showing that

$$f \circ b(D_u) \cap D \neq \emptyset. \tag{72}$$

To this end, we consider $\lambda \in \Lambda$ and define $b^\lambda = \gamma_\lambda \cap b(D_u)$. By mirror arguments to the ones from the proof of Lemma 45 it follows that b^λ is a horizontal disc. By Lemma 83 $f(b^\lambda) \cap D \neq \emptyset$, which implies (72).

Using the same arguments as those from the proof of Lemma 45 it follows that $\pi_{\gamma_u} \circ f \circ b : D_u \to \gamma_u$ is an injective open map. By (72) $\pi_{\gamma_u} \circ f \circ b(D_u) \cap D_u \neq \emptyset$. If $(\lambda, v_1) \in \partial D_u$ then $\|v_1\| = R$. Let $z = (\lambda, v_1, v_2) = b(\lambda, v_1)$. Let $(\tilde{\eta}_j, \tilde{\eta}_i)$ be a good charts pair for z and $U = J_u(\tilde{\eta}_i(z), 1/L)$. Using the same argument as in the proof of Lemma 45 it follows that $\pi_\theta f_{ji}\left(D_i^- \cap U\right) \cap D_j = \emptyset$. Thus $\pi_{\gamma_u} \circ f \circ b(\partial D_u) \cap D_u = \emptyset$. This means that

$$\pi_{\gamma_u} \circ f \circ b(D_u) \cap D_u = D_u,$$

hence for any $(\lambda^*, v_1^*) \in D_u$ there exists a $(\lambda, v_1) \in D_u$ such that $\pi_{\gamma_u} \circ f \circ b(\lambda, v_1) = (\lambda^*, v_1^*)$. We can define $b^*(\lambda^*, v_1^*) = f \circ b(\lambda, v_1)$. All the desired properties of b^* follow from mirror arguments to the proof of Lemma 45. ■

For a center-horizontal disc b we use the notation $\mathcal{G}(b)$ for the center-horizontal disc b^* from Lemma 86.

Lemma 87. *Let $b_0 : D_u \to D$ be defined as $b_0(\lambda, v_1) = (\lambda, v_1, 0)$. If assumptions of Theorem 80 are satisfied, then $\mathcal{G}^k(b_0)$ converges to W^{cu} as k tends to infinity.*

Proof. Let us fix $(\lambda, v_1) \in D_u = \pi_{\gamma_u} D$. Let $k_2 \geq k_1$ and let us define $q_0^{k_1} = \mathcal{G}^{k_1} b(\lambda, v_1)$ and $q_0^{k_2} = \mathcal{G}^{k_2} b(\lambda, v_2)$. By definition of $\mathcal{G}$, there exist backward trajectories $\{q_i^{k_1}\}_{i=-k_1}^0$, $\{q_i^{k_2}\}_{i=-k_2}^0$

$$f\left(q_i^{k_1}\right) = q_{i+1}^{k_1}, \qquad f\left(q_i^{k_1}\right) = q_{i+1}^{k_1}.$$

Since $\pi_{\gamma_u}\mathcal{G}^{k_i}b(\lambda, v_1) = (\lambda, v_1)$, we see that $q_0^{k_1} - q_0^{k_2} = \pi_{v_2}\left(q_0^{k_1} - q_0^{k_2}\right)$ hence we can compute

$$\left\|q_0^{k_1} - q_0^{k_2}\right\| = \left\|\pi_{v_2}\left(q_0^{k_1} - q_0^{k_2}\right)\right\|,$$

and the norm is independent of the considered chart. Let us take a good chart sequence $\left(\tilde{\eta}_{i_{k_1}}, \tilde{\eta}_{i_{k_1}-1}, \ldots, \tilde{\eta}_{i_0}\right)$ for $q_{-k_1}^{k_1}$. Since f satisfies backward cone conditions, $\left(\tilde{\eta}_{i_{k_1}}, \tilde{\eta}_{i_{k_1}-1}, \ldots, \tilde{\eta}_{i_0}\right)$ is also a good sequence for $q_{-k_1}^{k_2}$. From mirror computations to the ones from the proof of Lemma 46 (see (30))

$$\begin{aligned} \left\|\mathcal{G}^{k_1}b(\lambda, v_1) - \mathcal{G}^{k_2}b(\lambda, v_1)\right\| &= \|q_0^{k_1} - q_0^{k_2}\| \\ &\leq (1 + 1/L)2R(\mu_{s,1})^{k_1}. \end{aligned}$$

We note that the estimate is independent of the choice of the good chart sequence. Thus we obtain uniform convergence of $\mathcal{G}^k b(\lambda, v_1)$.

The proof of the fact that $\mathcal{G}^k b(\lambda, v_1)$ converges to W^{cu} follows from mirror arguments to the ones in the proof of Lemma 46. ■

Lemma 87 establishes the existence of W^{cu}. The proof of its smoothness follows from arguments identical to the proof of the smoothness when Λ was a torus (Lemma 48). All the arguments in the proof are local, and can be performed using local maps passing through good chart sequences.

We now move to outlining the method for the proof of the existence of W^{cs}. First we give two definitions.

Definition 88. *We say that $b \subset D$ is a vertical disc if for any $z \in b$ there exists a good chart $\tilde{\eta}_i$ around z such that $b \subset \widetilde{\mathcal{U}}_i$ and a continuous function $b_i : \overline{B}_s(R) \to D_i$*

$$\begin{aligned} \tilde{\eta}_i(b) &= b_i\left(\overline{B}_s(R)\right), \\ \pi_y b_i(y) &= y, \\ b_i\left(\overline{B}_s(R)\right) &\subset J_s\left(b_i(y), 1/L\right) \qquad \text{for any } y \in \overline{B}_s(R). \end{aligned}$$

We say that b is C^k if b_i are C^k.

Definition 89. *Let $D_s = \pi_{\gamma_s} D \subset \gamma_s$. We say that a continuous function $b : D_s \to D$ is a center-vertical disc if for any $(\lambda, v_2) \in D_s$*

$$\pi_{\gamma_s} b(\lambda, v_2) = (\lambda, v_2)$$

and for any $z \in b(\gamma_s)$ and any good chart $\tilde{\eta}_i$ around z

$$\tilde{\eta}_i \circ b(D_u) \cap D_i \subset J_{cs}(\tilde{\eta}_i(z), L).$$

The construction of W^{cs} is analogous to the one from section 7: For any $i \in \mathbb{Z}_+$ and $(\lambda, v_2) \in D_s$ we consider the following problem: Find x such that

$$\pi_{v_1} f^i(\lambda, v_1, v_2) = 0$$

under the constraint

$$f^l(\lambda, v_1, v_2) \in D, \quad l = 0, 1, \ldots, i.$$

From Lemma 83 it follows that this problem has a unique solution $v_{1,i}(\lambda, v_2)$ which is as smooth as f.

We consider $b_i : D_s \to D$ given by $b_i(\lambda, v_2) = (\lambda, v_{1,i}(\lambda, v_2), v_2)$. Then, using mirror arguments to the proof of Lemma 50, b_i is a center vertical disc and the sequence b_i converges uniformly to W^{cs}. The proof of the smoothness of W^{cs} follows from mirror arguments to the proof of Lemma 52.

Intersection of W^{cu} and W^{cs} gives the center manifold Λ^*.

Vertical and horizontal discs are contained in local charts. Thus the arguments for the existence of W^s_z and W^u_z follow from identical arguments as those from sections 9, 10. The only difference is that instead of working with compositions of f, we work with compositions of local maps passing through good chart sequences.

12. Numerical example

We consider a one dimensional torus (circle) Λ and the rotating Hénon map $F_\varepsilon : \Lambda \times \mathbb{R}^2 \to \Lambda \times \mathbb{R}^2$,

$$F_\varepsilon(\lambda, q_1, q_2) = (\theta + c + \varepsilon q_1 \cos(2\pi\lambda)), 1 + q_2 - aq_1^2 + \varepsilon \cos(2\pi\lambda), bq_1). \tag{73}$$

We take $a = 0.68$, $b = 0.1$ and an arbitrary constant $c \in \mathbb{R}$. We investigate the existence and smoothness of the NHIM and its associated stable/unstable manifolds for a range of parameters $\boldsymbol{\varepsilon} = [\varepsilon_1, \varepsilon_2]$.

We consider the maps (73) in local coordinates (λ, x, y) given by the linear change

$$(\lambda, q_1, q_2) = C(\lambda, x, y) + (0, q_1^*, q_2^*),$$

where

$$\begin{aligned} q_1^* &= \frac{-(1-b) - \sqrt{(1-b)^2 + 4a}}{2a} \approx -2.043\,3, \\ q_2^* &= bq_1^* \approx -0.204\,33. \end{aligned}$$

and

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -0.3553203857 \\ 0 & 0.03553203857 & 1 \end{pmatrix}.$$

Thus, in local coordinates $p = (\lambda, x, y)$, we consider the family of maps

$$f_\varepsilon(p) = F_\varepsilon(Cp + (0, q_1^*, q_2^*)) - (0, q_1^*, q_2^*).$$

The choice of (q_1^*, q_2^*) is dictated by the fact that this is a hyperbolic fixed point for the Hénon map (with $\varepsilon = 0$). The matrix C diagonalizes (roughly) the linear part of F into a Jordan normal form.

For a fixed interval $\boldsymbol{\varepsilon} = [\varepsilon_1, \varepsilon_2]$, we consider the set $D_{\boldsymbol{\varepsilon}} = \Lambda \times \overline{B}_{u=1}(R) \times \overline{B}_{s=1}(R)$, with $R = \varepsilon_2$. Below we take two examples of $\boldsymbol{\varepsilon} = [0, 0.0001]$ and $\boldsymbol{\varepsilon} = [0.009, 0.01]$. The bounds for $[Df_{\varepsilon}(D_{\varepsilon})]$ for these two intervals are:

$$\left[Df_{[0,0.0001]}\left(D_{[0,0.0001]}\right)\right] = \begin{pmatrix} 1^{+0.00129}_{-0.00129} & 0^{+0.000101}_{-0.000100} & 0^{+0.000036}_{-0.000036} \\ 0^{+0.000621}_{-0.000621} & 2.814^{55}_{17} & 0^{+0.000065}_{-0.000065} \\ 0^{+0.000023}_{-0.000023} & 0^{+0.000007}_{-0.000007} & -0.0355^{35}_{29} \end{pmatrix}, \tag{74}$$

$$\left[Df_{[0.009,0.01]}\left(D_{[0.009,0.01]}\right)\right] = \begin{pmatrix} 1^{+0.12924}_{-0.12924} & 0^{+0.010001}_{-0.010001} & 0^{+0.003554}_{-0.003554} \\ 0^{+0.062049}_{-0.062049} & 2.^{83257}_{79615} & 0^{+0.006468}_{-0.006468} \\ 0^{+0.002205}_{-0.002205} & 0^{+0.000647}_{-0.000647} & -0.035^{762}_{302} \end{pmatrix}. \tag{75}$$

Above, by convention, $1^{+0.00129}_{-0.00129}$ stands for the interval $[1-0.00129, 1+0.00129]$ and 2.814^{55}_{17} stands for $[2.81417, 2.81455]$. We choose $L = 1 - \frac{1}{100}$, and in Table 1 display coefficients that were computed based on (74) and (75).

	$\varepsilon = [0, 0.0001]$	$\varepsilon = [0.009, 0.01]$
$\xi_{u,1}$	2.81352	2.7303
$\xi_{u,1,P}$	2.81352	2.7303
$\xi_{u,2}$	2.81408	2.78624
$\xi_{cu,1}$	0.997718	0.748463
$\xi_{cu,2}$	0.997766	0.753236
$\xi_{cu,1,P}$	0.997718	0.748463
$\mu_{s,1}$	0.0355597	0.0382945
$\mu_{s,2}$	0.0356074	0.0430675
$\mu_{cs,1}$	1.0014	1.14097
$\mu_{cs,2}$	1.00196	1.19691

Table 1: Coefficients for the rate conditions computed from (74) and (75).

In a similar fashion one can compute the coefficients for other intervals, and based on these compute the order of the rate conditions. In Table 2 we show a sequence of intervals spanning from $\varepsilon = 0$ to $\varepsilon = \frac{1}{100}$, together with the established order.

ε	order	ε	order	ε	order
$[0, 0.0001]$	737	$[0.0005, 0.001]$	73	$[0.005, 0.006]$	11
$[0.0001, 0.0002]$	368	$[0.001, 0.002]$	36	$[0.006, 0.007]$	9
$[0.0002, 0.0003]$	245	$[0.002, 0.003]$	24	$[0.007, 0.008]$	8
$[0.0003, 0.0004]$	184	$[0.003, 0.004]$	17	$[0.008, 0.009]$	7
$[0.0004, 0.0005]$	147	$[0.004, 0.005]$	14	$[0.009, 0.01]$	6

Table 2: Rate conditions order for various parameters.

To establish the covering condition we have numerically verified that $\pi_y f_{\varepsilon}(D_{\varepsilon}) \subset \mathrm{int}\pi_y D_{\varepsilon}$ and that for $D_{\varepsilon}^{-,\mathrm{left}} = \Lambda \times \{-R\} \times \overline{B}_s(R)$ and $D_{\varepsilon}^{-,\mathrm{right}} = \Lambda \times \{R\} \times \overline{B}_s(R)$ holds

$$\pi_x f_{\varepsilon}\left(D_{\varepsilon}^{-,\mathrm{left}}\right) < -R \qquad \text{and} \qquad \pi_x f_{\varepsilon}\left(D_{\varepsilon}^{-,\mathrm{right}}\right) > R.$$

Now we show how we verified the backward cone conditions. Since $\lambda \in \mathbb{R} \bmod 2\pi$, we can take $R_\Lambda = 1$. If $p_1 \in J_s(p_2, 1/L)$, then

$$\|\pi_\lambda (p_1 - p_2)\| \leq \frac{1}{L} \|\pi_y (p_1 - p_2)\| \leq \frac{1}{L} 2\varepsilon_2.$$

Let $U = \left[-\frac{2}{L}\varepsilon_2, \frac{2}{L}\varepsilon_2\right] \times \overline{B}_u(R) \times \overline{B}_s(R)$, then $p_1 - p_2 \in U$ and

$$\left\|\pi_\lambda \left(f^{-1}(p_1) - f^{-1}(p_2)\right)\right\| \leq \max\left[\left\|\pi_\lambda \left(Df(D)\right)^{-1} U\right\|\right].$$

We verify numerically that $\max\left[\left\|\pi_\lambda \left(Df(D)\right)^{-1} U\right\|\right] < R_\Lambda$. This means that

$$f^{-1}(p_1) \in \overline{B}_c(\pi_\lambda f^{-1}(p_2), R_\Lambda) \times \overline{B}_u(R) \times \overline{B}_s(R),$$

and the backward cone condition for $z_1 = f^{-1}(p_1)$, $z_2 = f^{-1}(p_2)$ follows from Corollary 35.

Remark 90. *The smoothness established in Table 2 is not optimal. The example serves only to demonstrate that our method is applicable. We choose a single change of coordinates and use global estimates on the derivative of the map. With a more careful choice of changes to local coordinates and by a local treatment of the estimates on the derivatives one could obtain better results.*

All computations were performed using the CAPD[5] package.

Appendix A. An auxiliary lemma

Lemma 91. *Let $U \subset \mathbb{R}^u \times \mathbb{R}^s$ be a convex bounded neighborhood of zero and assume that $f : U \to \mathbb{R}^s$ is a C^{m+1} map satisfying $f(0) = 0$ and*

$$\|f(U)\|_{C^{m+1}} \leq c, \tag{A.1}$$

$$\frac{\partial^l f}{\partial x^l}(0) = 0, \qquad \text{for } |l| \leq m. \tag{A.2}$$

Then

$$f(x, y) = \frac{\partial f}{\partial y} y + g_2(\mathrm{x}, \mathrm{y}),$$

where

$$g_2(x, y) \leq C(\|y\|^2 + \|x\|\|y\| + \|x\|^{m+1}),$$

with C depending on c, the diameter of U and m.

[5]Computer Assisted Proofs in Dynamics: http://capd.ii.uj.edu.pl/

Proof. Let us consider the Taylor formula with the integral remainder of order $(m+1)$ (here the convexity is used). We group the second or higher order terms in this expansion in three groups. The first group contains only the terms independent of x. The second group contains both x and y. The sums in both groups can be can be bounded by $C_1\|y\|^2$ and $C_2\|x\|\|y\|$, respectively, where constants C_1 and C_2 depend on c, the diameter of U and m. The last group contains a single term coming from the reminder

$$\frac{1}{m!}\int_0^1 D^{m+1}f(t(x,y))(x^{[m+1]})dt,$$

which bounded by $\frac{c}{(m+1)!}\|x\|^{m+1}$. ■

Appendix B. Proof of Lemma 25

Proof. For sufficiently small δ, if $\|\mathrm{x}-\mathrm{x}_0\| \leq \delta$ then $M > \|D^{m+1}g(\mathrm{x})\|$, hence

$$\begin{aligned}
& \|g(\mathrm{x}) - g(\mathrm{x}_0) - \mathcal{P}_m(\mathrm{x}-\mathrm{x}_0)\| \\
= & \|R_{m+1,p}(\mathrm{x}-\mathrm{x}_0)\| \\
= & \left\| \int_0^1 \frac{(1-t)^m}{m!} D^{m+1} g(\mathrm{x}_0 + th) \left((\mathrm{x}-\mathrm{x}_0)^{[m+1]} \right) dt \right\| \\
\leq & \int_0^1 \frac{(1-t)^m}{m!} M \left\|\mathrm{x}-\mathrm{x}_0\right\|^{m+1} dt \\
= & \frac{M}{(m+1)!} \left\|\mathrm{x}-\mathrm{x}_0\right\|^{m+1}
\end{aligned}$$

Therefore for $\|\mathrm{x}-\mathrm{x}_0\| \leq \delta$ we have

$$(\mathrm{x}, g(\mathrm{x})) = (\mathrm{x}_0, g(\mathrm{x}_0)) + (\mathrm{x}-\mathrm{x}_0, \mathcal{P}_m(\mathrm{x}) + \mathrm{y}),$$

where $\mathrm{y} = g(\mathrm{x}) - g(\mathrm{x}_0) - \mathcal{P}_m(\mathrm{x})$ satisfies $\|\mathrm{y}\| \leq \frac{M}{(m+1)!}\|\mathrm{x}-\mathrm{x}_0\|^{m+1}$. Hence (12) is satisfied. ■

Appendix C. Proof of Lemma 26

The proof of Lemma 26 is based on the following result.

Lemma 92. *Let $\|\cdot\|$ be an euclidean norm on $\mathbb{R}^n$. Let $B : \mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$ be k-linear symmetric form. Assume that $M > 0$ is such that for all $h \in \mathbb{R}^n$ holds*

$$|B(h^{[k]})| \leq M\|h\|^k.$$

Let $\{e_i\}_{i=1,\ldots,n} \in \mathbb{R}^n$ be an orthonormal basis.

Then there exists $c = k^k$ such that for all $(i_1, i_2, \ldots, i_k) \in \{1, \ldots, n\}^k$

$$|B(e_{i_1}, e_{i_2}, \ldots, e_{i_k})| \leq cM.$$

Proof. We now introduce some notations. For any set Z by $\#Z$ we will denote its number of elements. To deal with symmetric multiindices we define a set $S_{n,k} \subset \{1,\dots,n\}^k$ by

$$S_{n,k} = \{i \in \{1,\dots,n\}^k \mid i_m \leq i_{m+1}, \quad m = 1,\dots,k-1\}.$$

For any $i \in \{1,\dots,n\}^k$ by $z = S(i)$ we denote a unique element in $S_{n,k}$, such that for each $j \in \{1,\dots,n\}$ holds $\#\{m \mid i_m = j\} = \#\{m \mid z_m = j\}$. Hence $S(i)$ is an 'ordered' i. For $i \in S_{n,k}$ we define a multiplicity of i, denoted by $m(i)$,

$$m(i) = \#\{S^{-1}(i)\}.$$

For $i \in S_{n,k}$ and $j \in \{1,\dots,n\}$ we define a multiplicity of j in i by

$$m(j,i) = \#\{m \mid i_m = j\}.$$

It is easy to see that

$$m(i) = \frac{k!}{\Pi_{j=1,\dots,n}(m(j,i)!)}.$$

For $i \in \{1,\dots,n\}^k$ we write $x^i = x_{i_1} x_{i_2} \dots x_{i_k}$.

Let us denote by $\mathcal{D}$ the diagonal of B, i.e. $\mathcal{D} : \mathbb{R}^n \to \mathbb{R}$, $\mathcal{D}(h) = B(h^{[k]})$. Let us consider the following polynomial of degree k of n variables $x_1,\dots,x_n$

$$\begin{aligned} P(x_1,\dots,x_n) &= \mathcal{D}\left(\sum_{i=1}^n x_i e_i\right) \\ &= \sum_{i \in \{1,\dots,n\}^k} x_{i_1} x_{i_2} \dots x_{i_k} B(e_{i_1}, e_{i_2}, \dots, e_{i_k}) \\ &= \sum_{l \in S_{n,k}} m(l) x_{l_1} x_{l_2} \dots x_{l_k} B(e_{l_1}, e_{l_2}, \dots, e_{l_k}). \end{aligned}$$

Now our task can is reduced to the following one: given bounds on $P(x_1,\dots,x_n)$ can we produce bounds for its coefficients.

First of all we will develop a formula for each coefficient. To shorten some expressions let us denote coefficients of P by p_l, that is,

$$P(x_1,\dots,x_n) = \sum_{l \in S_{n,k}} p_l x^l, \quad p_l = m(l) B(e_{l_1}, e_{l_2}, \dots, e_{l_k}). \tag{C.1}$$

Each coefficient p_l can be computed by finite differences as follows.

For any polynomial $W(x_1,\dots,x_n)$ and $i = 1,\dots,n$ we define a finite difference operator Δ_i as

$$(\Delta_i W)(x_1,\dots,x_n) := W(x_1,\dots,x_i + 1/2,\dots,x_n) - W(x_1,\dots,x_i - 1/2,\dots,x_n).$$

It is easy to see that $\Delta_i W$ is a polynomial, whose degree with respect to variable x_i decreases by 1 (if it is nonzero). It is easy to check that $\Delta_i \Delta_j = \Delta_j \Delta_i$. For $l \in \{1,\dots,n\}^a$ we set

$$\Delta^l = \Delta_{l_1} \Delta_{l_2} \dots \Delta_{l_s}.$$

We shall use the fact that for any polynomial $W(x_1,\dots,x_n)=\sum_l w_l x^l$, any k and $l\in S_{n,k}$ we have

$$(\Delta^l W)(0,\dots,0)=(\Pi_{i=1,\dots,n}(m(i,l)!))\,w_l.$$

Observe that for polynomial P given by (C.1) and $l\in S_{n,k}$ $\Delta^l P$ is a constant polynomial. Therefore from the above formula we obtain

$$p_l=(\Pi_{i=1,\dots,n}m(i,l))^{-1}\,\Delta^l P.$$

Now we are ready to estimate p_l. We set $(x_1,\dots,x_n)=0$. Observe that $\Delta^l P$ will involve 2^k terms of the form $\pm P(j_1,\dots,j_n)$, where $j_r\in\{-k/2,\dots,k/2\}$ and $\sum_{r=1}^n|j_r|\le k/2$.

Hence

$$|p_l|\le|\Delta^l P|\le 2^k\max_{\|x\|\le k/2}|P(x_1,\dots,x_n)|\le Mk^k$$

Therefore

$$|B(e_{l_1},\dots,e_{l_k})|\le\frac{Mk^k}{m(l)}\le Mk^k.$$

■

We are now ready to prove Lemma 26:

Proof. Using the Taylor formula

$$\|R_{m+1,\mathrm{x}_0}(h)\|=\|g(\mathrm{x}_0+h)-g(\mathrm{x}_0)-\mathcal{P}_m(h)\|\le M\,\|h\|^{m+1}. \tag{C.2}$$

Let S^u denotes the sphere of radius 1 in $\mathbb{R}^u$. Let $e\in S^u$ and let $h=\eta e$ for $\eta\in[0,1]$. Then

$$\begin{aligned}
R_{m+1,\mathrm{x}_0}(h) &= \int_0^1\frac{(1-t)^m}{m!}D^{m+1}g(\mathrm{x}_0+th)\left(h^{[m+1]}\right)dt\\
&= \eta^{m+1}\int_0^1\frac{(1-t)^m}{m!}D^{m+1}g(\mathrm{x}_0+t\eta e)\left(e^{[m+1]}\right)dt\\
&= \eta^{m+1}\int_0^1\frac{(1-t)^m}{m!}D^{m+1}g(\mathrm{x}_0)\left(e^{[m+1]}\right)dt+\eta^{m+1}\varepsilon(\mathrm{x}_0,e,\eta)\\
&= \eta^{m+1}\frac{D^{m+1}g(\mathrm{x}_0)}{(m+1)!}\left(e^{[m+1]}\right)+\eta^{m+1}\varepsilon(\mathrm{x}_0,e,\eta),
\end{aligned}\tag{C.3}$$

where

$$\varepsilon(\mathrm{x}_0,e,\eta)=\int_0^1\frac{(1-t)^m}{m!}\left(D^{m+1}g(\mathrm{x}_0+t\eta e)\left(e^{[m+1]}\right)-D^{m+1}g(\mathrm{x}_0)\left(e^{[m+1]}\right)\right)dt$$

Since $D^{m+1}g$ is continuous, $\varepsilon(\mathrm{x}_0,e,\eta)\to 0$ as $\eta\to 0$. Combining (C.2) and (C.3) we obtain

$$\eta^{m+1}\left\|\frac{D^{m+1}g(\mathrm{x}_0)}{(m+1)!}\left(e^{[m+1]}\right)\right\|-\eta^{m+1}\|\varepsilon(\mathrm{x}_0,e,\eta)\|\le M\eta^{m+1}.$$

Dividing by η^{m+1} and passing with η to zero gives

$$\left\|\frac{D^{m+1}g(\mathrm{x}_0)}{(m+1)!}\left(e^{[m+1]}\right)\right\|\le M.$$

This by Lemma 92 gives

$$\begin{aligned}\left\|\frac{\partial^{m+1}g(\mathrm{x}_0)}{\partial \mathrm{x}_{i_1}\ldots\partial \mathrm{x}_{i_{m+1}}}\right\| &\leq \left|\frac{\partial^{m+1}g_1(\mathrm{x}_0)}{\partial \mathrm{x}_{i_1}\ldots\partial \mathrm{x}_{i_{m+1}}}\right| + \ldots + \left|\frac{\partial^{m+1}g_s(\mathrm{x}_0)}{\partial \mathrm{x}_{i_1}\ldots\partial \mathrm{x}_{i_{m+1}}}\right| \\ &\leq s\,(m+1)!cM,\end{aligned}$$

which concludes our proof. ∎

Appendix D. Proof of Theorem 27

Proof. If $(\mathrm{x},\mathrm{y}) \in J_u(0,\mathcal{P}_0,M)$ then $\|\mathrm{y}\| \leq M\|\mathrm{x}\|$. Since

$$f(\mathrm{x},\mathrm{y}) = f(0) + \int_0^1 Df(t(\mathrm{x},\mathrm{y}))dt(\mathrm{x},\mathrm{y}) \in [Df(U)]\,(\mathrm{x},\mathrm{y})$$

by (14) we obtain

$$\begin{aligned}\|\pi_{\mathrm{x}} f(\mathrm{x},\mathrm{y})\| &\geq m\left(\left[\frac{\partial f_{\mathrm{x}}}{\partial \mathrm{x}}(U)\right]\right)\|\mathrm{x}\| - \sup_{z\in U}\left\|\frac{\partial f_{\mathrm{x}}}{\partial \mathrm{y}}(z)\right\|\|\mathrm{y}\| \\ &\geq \left(m\left(\left[\frac{\partial f_{\mathrm{x}}}{\partial \mathrm{x}}(U)\right]\right) - M\sup_{z\in U}\left\|\frac{\partial f_{\mathrm{x}}}{\partial \mathrm{y}}(z)\right\|\right)\|\mathrm{x}\| \geq \xi\|\mathrm{x}\|.\end{aligned}$$

Using (15) in the last inequality, we have

$$\begin{aligned}\|\pi_{\mathrm{y}} f(\mathrm{x},\mathrm{y})\| &= \left\|\int_0^1 \frac{\partial f_{\mathrm{y}}}{\partial \mathrm{x}}(t(\mathrm{x},\mathrm{y}))\mathrm{x} + \frac{\partial f_{\mathrm{y}}}{\partial \mathrm{y}}(t(\mathrm{x},\mathrm{y})))\mathrm{y}dt\right\| \\ &\leq \int_0^1 \left\|\frac{\partial f_{\mathrm{y}}(t(\mathrm{x},\mathrm{y}))}{\partial \mathrm{x}}\right\|\|\mathrm{x}\| + \left\|\frac{\partial f_{\mathrm{y}}(t(\mathrm{x},\mathrm{y}))}{\partial \mathrm{y}}\right\|\|\mathrm{y}\|dt \\ &\leq M\int_0^1\left(\frac{1}{M}\left\|\frac{\partial f_{\mathrm{y}}(t(\mathrm{x},\mathrm{y}))}{\partial \mathrm{x}}\right\|\|\mathrm{x}\| + \left\|\frac{\partial f_{\mathrm{y}}(t(\mathrm{x},\mathrm{y}))}{\partial \mathrm{y}}\right\|\|\mathrm{x}\|\right)dt \\ &\leq M\mu\|\mathrm{x}\|.\end{aligned}$$

From the above estimates and (16), if $(\mathrm{x},\mathrm{y}) \neq 0$ we obtain

$$\frac{\|\pi_{\mathrm{x}} f(\mathrm{x},\mathrm{y})\|}{\|\pi_{\mathrm{y}} f(\mathrm{x},\mathrm{y})\|} \geq \frac{\xi\|\mathrm{x}\|}{M\mu\|\mathrm{x}\|} > \frac{1}{M}.$$

This implies

$$\|\pi_{\mathrm{y}} f(\mathrm{x},\mathrm{y})\| < M\|\pi_{\mathrm{x}} f(\mathrm{x},\mathrm{y})\|,$$

hence, $f(\mathrm{x},\mathrm{y}) \in \mathrm{int}J_u(0,\mathcal{R}_0 = 0, M)$, as required. ∎

Appendix E. Proof of Theorem 28

We start by proving the theorem with an additional assumption that $\mathcal{P}_m = \mathcal{Q}_m = 0$, and that $\frac{\partial^l f_{\mathrm{y}}}{\partial \mathrm{x}^l}(0) = 0$, for $|l| \leq m$. We formulate this as a lemma:

Lemma 93. *Let $U \subset \mathbb{R}^u \times \mathbb{R}^s$ be a convex bounded neighborhood of zero and assume that $f : U \to \mathbb{R}^u \times \mathbb{R}^s$ is a C^{m+1} map satisfying $f(0) = 0$ and*

$$\begin{aligned} \|f(U)\|_{C^{m+1}} &\leq c, && \text{(E.1)} \\ \frac{\partial^l f_{\mathrm{y}}}{\partial \mathrm{x}^l}(0) &= 0, \qquad \text{for } |l| \leq m. && \text{(E.2)} \end{aligned}$$

If for $\xi > 0$, and $\rho < 1$

$$\begin{aligned} m\left(\tfrac{\partial f_{\mathrm{x}}}{\partial \mathrm{x}}(0)\right) &\geq \xi, \\ \left\|\tfrac{\partial f_{\mathrm{x}}}{\partial \mathrm{y}}(0)\right\| &\leq B, && \text{(E.3)} \\ \left\|\tfrac{\partial f_{\mathrm{y}}}{\partial \mathrm{y}}(0)\right\| &\leq \mu, \end{aligned}$$

and

$$\frac{\mu}{\xi^{m+1}} < \rho, \tag{E.4}$$

then there exists a constant $M^* = M^*(c, B, 1/\xi)$, such that for any $M > M^*$ there exists $\delta = \delta(M, c, B, 1/\xi)$ such that

$$f(J_u(0, \mathcal{P}_m = 0, M, \delta) \cap U) \subset J_u(0, \mathcal{P}_m = 0, M).$$

Moreover, if for some $K > 0$ holds $c, B, \frac{1}{\xi} \in [0, K]$, then M^* depends only on K and ρ.

Proof. Let us introduce the following notations

$$D_{11} = \frac{\partial f_{\mathrm{x}}}{\partial \mathrm{x}}(0), \quad D_{12} = \frac{\partial f_{\mathrm{x}}}{\partial \mathrm{y}}(0), \quad D_{22} = \frac{\partial f_{\mathrm{y}}}{\partial \mathrm{y}}(0),$$

then since $\frac{\partial^l f_{\mathrm{y}}}{\partial \mathrm{x}^l}(0) = 0$ for $|l| \leq m$

$$f(\mathrm{x}, \mathrm{y}) = (D_{11}\mathrm{x} + D_{12}\mathrm{y} + g_1(\mathrm{x}, \mathrm{y}), D_{22}\mathrm{y} + g_2(\mathrm{x}, \mathrm{y})),$$

where by the Taylor formula and Lemma 91

$$\begin{aligned} g_1(\mathrm{x}, \mathrm{y}) &\leq C(\|\mathrm{x}\|^2 + \|\mathrm{y}\|^2), \\ g_2(\mathrm{x}, \mathrm{y}) &\leq C\left(\|\mathrm{y}\|^2 + \|\mathrm{x}\|\,\|\mathrm{y}\| + \|\mathrm{x}\|^{m+1}\right), \end{aligned}$$

for $(x, y) \in U$, $\|(x, y)\| \leq 1$ with C depending on c.

Let $(\mathrm{x}, \mathrm{y}) \in J_u(0, 0, M) \cap U$. Then $\|\mathrm{y}\| \leq M\|\mathrm{x}\|^{m+1}$. Let $(\mathrm{x}_1, \mathrm{y}_1) = f(\mathrm{x}, \mathrm{y})$. We have

$$\begin{aligned} \|\mathrm{x}_1\| &\geq m(D_{11})\|\mathrm{x}\| - \|D_{12}\| \cdot \|\mathrm{y}\| - C(\|\mathrm{x}\|^2 + \|\mathrm{y}\|^2) \\ &\geq \xi\|\mathrm{x}\| - BM\|\mathrm{x}\|^{m+1} - C\|\mathrm{x}\|^2(1 + M^2\|\mathrm{x}\|^{2m}) \end{aligned}$$

It is apparent that there exists $\delta = \delta(M, c, B, 1/\xi) > 0$, such that if $\|x\| \leq \delta$, then x_1 is positive. Observe that δ is decreasing with respect to all of its arguments.

We now compute

$$\begin{aligned} \|\mathrm{y}_1\| &\leq \|D_{22}\|\|\mathrm{y}\| + C\left(\|\mathrm{y}\|^2 + \|\mathrm{x}\|\|\mathrm{y}\| + \|\mathrm{x}\|^{m+1}\right) \\ &\leq M\|\mathrm{x}\|^{m+1}\left(\mu + C\left(M\|\mathrm{x}\|^{m+1} + \|\mathrm{x}\| + \frac{1}{M}\right)\right). \end{aligned}$$

By further decreasing δ if necessary we obtain for $\|\mathrm{x}\| \leq \delta$ the following inequalities

$$\|\mathrm{x}\|^m \leq M^{-2}, \qquad \|\mathrm{x}\|^{m+1} \leq M^{-2}, \qquad \|\mathrm{x}\|^{2m} \leq M^{-2}$$

hence by (E.4), for sufficiently large M

$$\begin{aligned}
\frac{\|\mathrm{y}_1\|}{\|\mathrm{x}_1\|^{m+1}} &\leq \frac{M\|\mathrm{x}\|^{m+1}\left(\mu + C\left(M\|\mathrm{x}\|^{m+1} + \|\mathrm{x}\| + \frac{1}{M}\right)\right)}{\left(\xi\|\mathrm{x}\| - BM\|\mathrm{x}\|^{m+1} - C\|\mathrm{x}\|^2(1 + M^2\|\mathrm{x}\|^{2m})\right)^{m+1}} \\
&= M\frac{\mu + C\left(M\|\mathrm{x}\|^{m+1} + \|\mathrm{x}\| + \frac{1}{M}\right)}{\xi^{m+1}\left(1 - \frac{1}{\xi}BM\|\mathrm{x}\|^m - \frac{1}{\xi}C\|\mathrm{x}\|(1 + M^2\|\mathrm{x}\|^{2m})\right)^{m+1}} \\
&\leq M\frac{\mu + \frac{1}{M}C\left(2 + \frac{1}{M}\right)}{\xi^{m+1}\left(1 - \frac{1}{M}\frac{1}{\xi}(B + 2C)\right)^{m+1}} \\
&\leq M\frac{\rho + \frac{1}{M\xi^{m+1}}C\left(2 + \frac{1}{M}\right)}{\left(1 - \frac{1}{M}\frac{1}{\xi}(B + 2C)\right)^{m+1}} \\
&\leq M.
\end{aligned}$$

The choice of the size of M depends on C, B, ρ and $\frac{1}{\xi}$. Since $\frac{\|\mathrm{y}_1\|}{\|\mathrm{x}_1\|^{m+1}} \leq M$ we have shown that $(\mathrm{x}_1, \mathrm{y}_1) \in J_u(0, 0, M)$, as required. ■

We are now ready to prove Theorem 28:

Proof. We would like to change the coordinates around zero, so that the map f in these coordinates will satisfy the assumptions of Lemma 93.

Let $(\mathrm{x}_0, \mathrm{y}_0)$ be the new coordinates in the neighborhood of zero given by $(\mathrm{x}, \mathrm{y}) = \Phi_{z_0 \to z}(\mathrm{x}_0, \mathrm{y}_0)$

$$\begin{aligned}
\mathrm{x} &= \mathrm{x}_0, \\
\mathrm{y} &= \mathrm{y}_0 + \mathcal{P}_m(\mathrm{x}_0),
\end{aligned}$$

We denote the inverse transformation as $\Phi_{z \to z_0} = \Phi_{z_0 \to z}^{-1}$.

Analogously, let us also consider coordinates $(\mathrm{x}_1, \mathrm{y}_1)$ given by $\Phi_{z_1 \to z}(\mathrm{x}_1, \mathrm{y}_1) = (\mathrm{x}, \mathrm{y})$

$$\begin{aligned}
\mathrm{x} &= \mathrm{x}_1, \\
\mathrm{y} &= \mathrm{y}_1 + \mathcal{R}_m(\mathrm{x}_1).
\end{aligned}$$

and denote $\Phi_{z \to z_1} = \Phi_{z_1 \to z}^{-1}$.

Observe that both inverse transformations $\Phi_{z_0 \to z}^{-1}$ and $\Phi_{z_1 \to z}^{-1}$ are polynomial:

$$\begin{aligned}
\Phi_{z_0 \to z}^{-1}(\mathrm{x}, \mathrm{y}) &= (\mathrm{x}, \mathrm{y} - \mathcal{P}_m(\mathrm{x})) \\
\Phi_{z_1 \to z}^{-1}(\mathrm{x}, \mathrm{y}) &= (\mathrm{x}, \mathrm{y} - \mathcal{R}_m(\mathrm{x}))
\end{aligned}$$

and satisfy he same bound on the coefficients.

Now let $\tilde{f}(\mathrm{x}_0, \mathrm{y}_0) = \Phi_{z \to z_1}(f(\Phi_{z_0 \to z}(\mathrm{x}_0, \mathrm{y}_0)))$, i.e. we express f in new coordinates.

Observe that in coordinates $(\mathrm{x}_0,\mathrm{y}_0)$ the set $J_u(0,\mathcal{P}_m,M)$ is just $J_u(0,0,M)$, i.e. $\Phi_{z\to z_0}(J_u(0,\mathcal{P}_m,M))=J_u(0,0,M)$. Analogously, $\Phi_{z\to z_1}(J_u(0,\mathcal{R}_m,M))=J_u(0,0,M)$.

Now we compute the derivative of $\tilde{f}$. We have

$$
\begin{aligned}
D\tilde{f}(0) &= \left[\begin{array}{cc} I & 0\\ -D\mathcal{R}_m(0) & I\end{array}\right]\cdot\left[\begin{array}{cc} Df_{11} & Df_{12}\\ Df_{21} & Df_{22}\end{array}\right]\cdot\left[\begin{array}{cc} I & 0\\ D\mathcal{P}_m(0) & I\end{array}\right]\\
&= \left[\begin{array}{cc} D\tilde{f}_{11} & D\tilde{f}_{12}\\ D\tilde{f}_{21} & D\tilde{f}_{22}\end{array}\right],
\end{aligned}
$$

hence

$$
\begin{aligned}
D\tilde{f}_{11} &= Df_{11}+Df_{12}D\mathcal{P}(0),\\
D\tilde{f}_{12} &= Df_{12},\\
D\tilde{f}_{22} &= Df_{22}-D\mathcal{R}(0)Df_{12}.
\end{aligned}
$$

By (18–19) we see that assumptions (E.3–E.4) of Lemma 93 are satisfied.

We now show that assumption (E.1) from Lemma 93 is satisfied with a common constant c for <u>all</u> polynomials $\mathcal{P}_m$ and $\mathcal{R}_m$ satisfying our assumptions. The fact that $\|\tilde{f}(D)\|_{C^{m+1}}$ is bounded follows from the fact that $\|f(D)\|_{C^{m+1}}\leq C$ and since $\Phi_{z_0\to z}$, $\Phi_{z\to z_1}$ are polynomial changes of coordinates. We assumed that the coefficients of $\mathcal{P}$, $\mathcal{R}$ are bounded by C, hence $\|\tilde{f}(D)\|_{C^{m+1}}$ can be bounded by a constant dependent only on C, m and the size of the set D.

What remains is to verify that condition (E.2) holds for $\tilde{f}$. From the definition of $\Phi_{z\to z_1}$ we see that

$$
\pi_{\mathrm{y}}\circ\Phi_{z\to z_1}\circ(\mathrm{x},\mathcal{R}_m(\mathrm{x}))=0. \tag{E.5}
$$

By (17), for any x

$$
T_{f\circ(\mathrm{id},\mathcal{P}_m),m,0}(\mathrm{x})=(\mathrm{x}',\mathcal{R}_m(\mathrm{x}'))
$$

for some $\mathrm{x}'\in\mathbb{R}^u$, hence from (E.5) it follows that

$$
\pi_{\mathrm{y}}\circ\Phi_{z\to z_1}\circ T_{f\circ(\mathrm{id},\mathcal{P}_m),m,0}(\mathrm{x})=0.
$$

The Taylor expansion of $\pi_{\mathrm{y}}\circ\Phi_{z\to z_1}\circ T_{f\circ(\mathrm{id},\mathcal{P}_m),m,0}$ up to order m is equal to the Taylor expansion of $\pi_{\mathrm{y}}\circ\Phi_{z\to z_1}\circ f\circ(\mathrm{id},\mathcal{P}_m)$ up to order m. This means that

$$
T_{\pi_{\mathrm{y}}\circ\Phi_{z\to z_1}\circ f\circ(\mathrm{id},\mathcal{P}_m),m,0}(\mathrm{x})=0.
$$

Since $(\mathrm{id},\mathcal{P}_m)(\mathrm{x})=\Phi_{z_0\to z}(\mathrm{x},0)$, above implies that

$$
T_{\pi_{\mathrm{y}}\circ\tilde{f}\circ(\mathrm{id},0),m,0}(\mathrm{x})=0,
$$

hence

$$
\frac{\partial^l\tilde{f}_{\mathrm{y}}}{\partial\mathrm{x}^l}(0)=0.
$$

We have thus shown (E.2), which concludes the proof. ■

Appendix F. Proof of Theorem 32

Proof. We first observe that

$$J_s^c(0, \mathcal{Q}_0 = 0, M) = \{\|\mathrm{x}\| > M\,\|\mathrm{y}\|\} = \mathrm{int}J_u\left(0, 0, \frac{1}{M}\right).$$

Conditions (20–22) imply that assumptions of Theorem 27 are satisfied (for $1/M$ in place of M). This means that

$$\begin{aligned} f\left(\overline{J_s^c(0, \mathcal{Q}_0 = 0, M)} \cap U\right) &= f\left(J_u\left(0, 0, \frac{1}{M}\right) \cap U\right) \\ &\subset \mathrm{int}J_u\left(0, 0, \frac{1}{M}\right) \cup \{0\} \\ &= J_s^c(0, \mathcal{R}_0 = 0, M) \cup \{0\}, \end{aligned}$$

as required. ■

Appendix G. Proof of Theorem 33

The proof goes along the same lines as the proof of Theorem 28. There are some differences though in the needed estimates.

Similarly to the proof of Theorem 28, we start by proving the theorem with an additional assumption that $\mathcal{Q}_m = \mathcal{R}_m = 0$, and that $\frac{\partial^l f_{\mathrm{x}}}{\partial \mathrm{y}^l}(0) = 0$, for $|l| \leq m$. We formulate this as a lemma:

Lemma 94. *Let $U \subset \mathbb{R}^u \times \mathbb{R}^s$ be a convex bounded neighborhood of zero and assume that $f : U \to \mathbb{R}^u \times \mathbb{R}^s$ is a C^{m+1} map satisfying $f(0) = 0$ and*

$$\|f(U)\|_{C^{m+1}} \leq c, \tag{G.1}$$

$$\frac{\partial^l f_{\mathrm{x}}}{\partial \mathrm{y}^l}(0) = 0, \qquad \text{for } |l| \leq m. \tag{G.2}$$

If for $\xi > 0$, and $\rho < 1$

$$\begin{aligned} m\left(\tfrac{\partial f_{\mathrm{x}}}{\partial \mathrm{x}}(0)\right) &\geq \xi, \\ \left\|\tfrac{\partial f_{\mathrm{y}}}{\partial \mathrm{x}}(0)\right\| &\leq B, \\ \left\|\tfrac{\partial f_{\mathrm{y}}}{\partial \mathrm{y}}(0)\right\| &\leq \mu, \end{aligned} \tag{G.3}$$

and

$$\frac{\mu^{m+1}}{\xi} < \rho, \tag{G.4}$$

then there exists a constant $M^* = M^*(c, B, 1/\xi, \rho)$, such that for any $M > M^*$ there exists $\delta = \delta(M, c, B, 1/\xi)$ such that

$$f(J_s^c(0, \mathcal{P}_m = 0, M, \delta) \cap U) \subset J_s^c(0, \mathcal{P}_m = 0, M).$$

Moreover, if for some $K > 0$ holds $c, B, \frac{1}{\xi} \in [0, K]$, then M^* depends only on K and ρ.

Proof. Let us introduce the following notations

$$D_{11} = \frac{\partial f_{\mathrm{x}}}{\partial \mathrm{x}}(0), \quad D_{21} = \frac{\partial f_{\mathrm{y}}}{\partial \mathrm{x}}(0), \quad D_{22} = \frac{\partial f_{\mathrm{y}}}{\partial \mathrm{y}}(0),$$

then since $\frac{\partial^l f_{\mathrm{x}}}{\partial \mathrm{y}^l}(0) = 0$ for $|l| \le m$

$$f(\mathrm{x}, \mathrm{y}) = (D_{11}\mathrm{x} + g_1(\mathrm{x}, \mathrm{y}), D_{21}\mathrm{x} + D_{22}\mathrm{y} + g_2(\mathrm{x}, \mathrm{y})),$$

where by the Taylor formula and Lemma 91

$$\begin{aligned} \|g_1(\mathrm{x}, \mathrm{y})\| &\le C\left(\|\mathrm{x}\|^2 + \|\mathrm{x}\|\,\|\mathrm{y}\| + \|\mathrm{y}\|^{m+1}\right), \\ \|g_2(\mathrm{x}, \mathrm{y})\| &\le C(\|\mathrm{x}\|^2 + \|\mathrm{y}\|^2), \end{aligned}$$

for $(x, y) \in U \cap B(0, 1)$ with C depending on c.

Let $(\mathrm{x}, \mathrm{y}) \in J_s^c(0, 0, M) \cap U$. Then $\|\mathrm{x}\| > M\|\mathrm{y}\|^{m+1}$. Let $(\mathrm{x}_1, \mathrm{y}_1) = f(\mathrm{x}, \mathrm{y})$. From (G.3) we have

$$\begin{aligned} \|\mathrm{x}_1\| &\ge m(D_{11})\|\mathrm{x}\| - C\left(\|\mathrm{x}\|^2 + \|\mathrm{x}\|\,\|\mathrm{y}\| + \|\mathrm{y}\|^{m+1}\right) \\ &\ge \|\mathrm{x}\|\left(\xi - C\left(\|\mathrm{x}\| + \left(\frac{1}{M}\|\mathrm{x}\|\right)^{\frac{1}{m+1}} + \frac{1}{M}\right)\right). \end{aligned} \tag{G.5}$$

It is apparent that taking M sufficiently large and sufficiently small $\|\mathrm{x}\|$, the lower bound for $\|\mathrm{x}_1\|$ is positive.

We now compute

$$\|\mathrm{y}_1\| \le \|D_{21}\|\,\|\mathrm{x}\| + \|D_{22}\|\|\mathrm{y}\| + C(\|\mathrm{x}\|^2 + \|\mathrm{y}\|^2). \tag{G.6}$$

Taking $\|\mathrm{x}\| \le M^{-m}$ we see that $\left(\frac{1}{M}\|\mathrm{x}\|\right)^{\frac{1}{m+1}} \le M^{-1}$, hence for $(\mathrm{x}, \mathrm{y}) \in J_s^c(0, 0, M)$,

$$\begin{aligned} \|\mathrm{y}\| &< \left(\frac{1}{M}\|\mathrm{x}\|\right)^{\frac{1}{m+1}}, \\ \|\mathrm{y}\|^2 &< M^{-1}\left(\frac{1}{M}\|\mathrm{x}\|\right)^{\frac{1}{m+1}}. \end{aligned}$$

If $M > 1$ and $\|\mathrm{x}\| \le M^{-4m}$ then

$$\begin{aligned} \|\mathrm{x}\| &= \|\mathrm{x}\|^{\frac{1}{m+1}}\|\mathrm{x}\|^{\frac{m}{m+1}} \\ &\le \|\mathrm{x}\|^{\frac{1}{m+1}}\left(M^{-4m}\right)^{\frac{m}{m+1}} \\ &= \frac{1}{M}\left(\frac{1}{M}\|\mathrm{x}\|\right)^{\frac{1}{m+1}} M^{\frac{m+2-4m^2}{m+1}} \\ &< \frac{1}{M}\left(\frac{1}{M}\|\mathrm{x}\|\right)^{\frac{1}{m+1}}, \end{aligned}$$

and

$$\|\mathrm{x}\|^2 \leq M^{-1}\left(\frac{1}{M}\|\mathrm{x}\|\right)^{\frac{1}{m+1}}.$$

This by (G.6) and (G.3) means that for $M>1$ and $\|\mathrm{x}\| \leq M^{-4m}$

$$\begin{aligned}
\|\mathrm{y}_1\| &\leq \left(\frac{1}{M}\|\mathrm{x}\|\right)^{\frac{1}{m+1}}\left[\|D_{21}\| M^{-1}+\|D_{22}\|+2CM^{-1})\right] \\
&\leq \left(\frac{1}{M}\|\mathrm{x}\|\right)^{\frac{1}{m+1}}\left[\mu+\frac{1}{M}(B+2C))\right],
\end{aligned}$$

which combined with (G.5) gives

$$\begin{aligned}
\frac{\|\mathrm{y}_1\|^{m+1}}{\|\mathrm{x}_1\|} &\leq \frac{\frac{1}{M}\|\mathrm{x}\|\left[\mu+\frac{1}{M}(B+2C))\right]^{m+1}}{\|\mathrm{x}\|\left(\xi-C\left(\|\mathrm{x}\|+\left(\frac{1}{M}\|\mathrm{x}\|\right)^{\frac{1}{m+1}}+\frac{1}{M}\right)\right)} \\
&\leq \frac{\frac{1}{M}\left[\mu+\frac{1}{M}(B+2C))\right]^{m+1}}{\xi-3C/M} \\
&= \frac{1}{M}\frac{\left(\frac{\mu}{\xi^{1/m+1}}+\frac{B+2C}{M\xi^{1/m+1}}\right)^{m+1}}{\left(1-\frac{3C}{\xi M}\right)} \\
&< \frac{1}{M}\frac{\left(\rho^{1/m+1}+\frac{B+2C}{M\xi^{1/m+1}}\right)^{m+1}}{\left(1-\frac{3C}{\xi M}\right)}
\end{aligned}$$

Since $\rho<1$, taking sufficiently large M (the choice of M depends on C, B, $1/\xi$ and ρ), we see that $\frac{\|\mathrm{y}_1\|^{m+1}}{\|\mathrm{x}_1\|}<\frac{1}{M}$, hence $(\mathrm{x}_1,\mathrm{y}_1)\in J_s^c(0,0,M)$, as required. ■

We are now ready to give the proof of Theorem 33.

Proof. The claim of Theorem 33 follows from Lemma 94, by considering f in suitable local coordinates. The proof follows from a mirror argument to the proof of Theorem 28, with the only difference that we need to swap the roles of the coordinates x and y. ■

Appendix H. Proof of Lemma 36

Proof. By Remark 6, z_1 and z_2 are contained in the same chart.

Since $z_1 \in J_u(z_2, 1/L)$

$$\left\|\pi_{(\lambda,y)}(z_1-z_2)\right\| \leq \frac{1}{L}\|\pi_x(z_1-z_2)\|.$$

We have

$$f(z_1)-f(z_2)=\int_0^1 Df(z_2+t(z_1-z_2))dt(z_1-z_2),$$

hence

$$
\begin{aligned}
& \left\| \pi_x \left(f(z_1) - f(z_2) \right) \right\| \\
\geq\ & m \left(\frac{\partial f_x}{\partial x} (P(z_1)) \right) \left\| \pi_x \left(z_1 - z_2 \right) \right\| - \sup_{z \in D} \left\| \frac{\partial f_x}{\partial (\lambda, y)} z) \right\| \left\| \pi_{(\lambda, y)} \left(z_1 - z_2 \right) \right\| \\
\geq\ & \inf_{z \in D} m \left(\frac{\partial f_x}{\partial x} (P(z)) \right) \left\| \pi_x \left(z_1 - z_2 \right) \right\| - \frac{1}{L} \sup_{z \in D} \left\| \frac{\partial f_x}{\partial \left(\lambda, y \right)} \left(z \right) \right\| \left\| \pi_x \left(z_1 - z_2 \right) \right\| \\
\geq\ & \xi_{u,1,P} \| \pi_x (z_1 - z_2) \|,
\end{aligned}
$$

■

Appendix I. Proof of Lemma 37

Proof. By Remark 6, z_1 and z_2 are contained in the same chart. Since $z_1 \in J_s \left(z_2, 1/L \right)$,

$$
\left\| \pi_\theta \left(z_1 - z_2 \right) \right\| < 1/L \left\| \pi_y \left(z_1 - z_2 \right) \right\|.
$$

We have

$$
f(z_1) - f(z_2) = \int_0^1 Df(z_2 + t(z_1 - z_2)))dt(z_1 - z_2).
$$

This implies that

$$
\begin{aligned}
& \| \pi_y f(z_1) - \pi_y f(z_2) \| \\
=\ & \left\| \int_0^1 \frac{\partial f_y}{\partial y} (z_2 + t(z_1 - z_2)) \pi_y (z_1 - z_2) + \frac{\partial f_y}{\partial \theta} (z_2 + t(z_1 - z_2)) \pi_\theta (z_1 - z_2) dt \right\| \\
\leq\ & \int_0^1 \left\| \frac{\partial f_y}{\partial y} (z_2 + t(z_1 - z_2)) \right\| \| \pi_y (z_1 - z_2) \| + \left\| \frac{\partial f_y}{\partial \theta} (z_2 + t(z_1 - z_2)) \right\| \| \pi_\theta (z_1 - z_2) \| \, dt \\
\leq\ & \sup_{z \in D} \left(\left\| \frac{\partial f_y}{\partial y} (z) \right\| + \frac{1}{L} \left\| \frac{\partial f_y}{\partial \theta} (z) \right\| \right) \| \pi_y (z_1 - z_2) \| \\
\leq\ & \mu_{s,1} \left\| \pi_y (z_1 - z_2) \right\|,
\end{aligned}
$$

as required. ■

Appendix J. Proof of Lemma 38

Proof. Since $z_1 \in J_{cu} \left(z_2, L \right)$,

$$
\left\| \pi_y \left(z_1 - z_2 \right) \right\| \leq L \left\| \pi_{(\lambda, x)} \left(z_1 - z_2 \right) \right\|.
$$

We have

$$
f(z_1) - f(z_2) = \int_0^1 Df(z_2 + t(z_1 - z_2))dt(z_1 - z_2).
$$

This gives

$$
\begin{aligned}
& \left\|\pi_{(\lambda,x)}\left(f(z_1)-f(z_2)\right)\right\| \\
\geq\ & m\left(\frac{\partial f_{(\lambda,x)}}{\partial\left(\lambda,x\right)}(P(z_1))\right)\|\pi_{\lambda,x}(z_1-z_2)\|-\sup_{z\in D}\left\|\frac{\partial f_{(\lambda,x)}}{\partial y}(z)\right\|\|\pi_y(z_1-z_2)\| \\
\geq\ & \left(\inf_{z\in D} m\left(\frac{\partial f_{(\lambda,x)}}{\partial(\lambda,x)}(P(z))\right)-L\sup_{z\in D}\left\|\frac{\partial f_{(\lambda,x)}}{\partial y}(z)\right\|\right)\left\|\pi_{(\lambda,x)}\left(z_1-z_2\right)\right\| \\
\geq\ & \xi_{cu,1,P}\left\|\pi_{(\lambda,x)}\left(z_1-z_2\right)\right\|,
\end{aligned}
$$

■

Appendix K. Proof of Lemma 39

Proof. Since $z_1\in J_{cs}\left(z_2,L\right)$,

$$\left\|\pi_x\left(z_1-z_2\right)\right\|\leq L\left\|\pi_{(\lambda,y)}\left(z_1-z_2\right)\right\|.$$

We have

$$f(z_1)-f(z_2)=\int_0^1 Df(z_2+t(z_1-z_2)))dt(z_1-z_2).$$

hence

$$
\begin{aligned}
& \left\|\pi_{(\lambda,y)}\left(f(z_1)-f(z_2)\right)\right\| \\
=\ & \left\|\int_0^1 Df_{(\lambda,y)}(z_2+t(z_1-z_2)))(z_1-z_2)dt\right\| \\
\leq\ & \int_0^1\left\|\frac{\partial f_{(\lambda,y)}}{\partial\left(\lambda,y\right)}(z_2+t(z_1-z_2))\right\|\left\|\pi_{(\lambda,y)}\left(z_1-z_2\right)\right\| \\
& +\left\|\frac{\partial f_{(\lambda,y)}}{\partial x}(z_2+t(z_1-z_2))\right\|\left\|\pi_x\left(z_1-z_2\right)\right\|dt \\
\leq\ & \sup_{z\in D}\left(\left\|\frac{\partial f_{(\lambda,y)}}{\partial\left(\lambda,y\right)}(z)\right\|+L\left\|\frac{\partial f_{(\lambda,y)}}{\partial x}(z)\right\|\right)\left\|\pi_{(\lambda,y)}\left(z_1-z_2\right)\right\| \\
\leq\ & \mu_{cs,1}\left\|\pi_{(\lambda,y)}\left(z_1-z_2\right)\right\|,
\end{aligned}
$$

as required. ■

Appendix L. Proof of Lemma 44

Proof. Let $z=b(0)$ and $\lambda^*\in\Lambda$ be the point from Definition 15 for z. Note that since b is a horizontal disc, $b(\overline{B}_u(R))\subset J_u\left(z,1/L\right)$. This also means that

$$\left\|\pi_{(\lambda,y)}\left(b\left(x_1\right)-b\left(x_2\right)\right)\right\|\leq\frac{1}{L}\left\|\pi_x\left(b\left(x_1\right)-b\left(x_2\right)\right)\right\|=\frac{1}{L}\left\|x_1-x_2\right\|. \tag{L.1}$$

From Definition 15 follows that

$$f(b(\overline{B}_u(R)))\subset B_c(\lambda^*,R_\Lambda)\times\mathbb{R}^u\times\mathbb{R}^s,$$

hence

$$f(b(\overline{B}_u(R))) \cap D \subset D_{\lambda^*}.$$

Observe that by Remark 6, h_t maps the disk b in a set contained in a single chart.

We start by showing that for any $\hat{x} \in \overline{B}_u(R)$ there exists $x = x(\hat{x})$ such that

$$\pi_x f(b(x)) = \hat{x}. \tag{L.2}$$

and then disk b^* will be defined by $b^*(\hat{x}) = f(b(x(\hat{x})))$.

Let us fix $\hat{x} \in \overline{B}_u(R)$ and consider a function

$$F : \overline{B}_u(R) \to \mathbb{R}^u$$

defined as

$$F(x) = \pi_x f(b(x)) - \hat{x}.$$

Our objective is to show that there exists a unique x such that $F(x) = 0$.

Let h_α be the homotopy from Definition 15. Let us define a homotopy

$$H : [0,1] \times \overline{B}_u(R) \to \mathbb{R}^u$$

$$H_\alpha(x) = \pi_x h_\alpha(b(x)) - \hat{x}.$$

Note that $H_0 = F$. We will start by showing that

$$\forall \alpha \in [0,1] \quad \forall x \in \partial B_u(R) \qquad H_\alpha(q) \neq 0. \tag{L.3}$$

To prove (L.3) let us take $x \in \partial B_u(R)$. Since $b(x) \in J_u(z, 1/L) \cap D^-_{\pi_\lambda(z)}$, by condition (7) from Definition 15 $h_\alpha(b(x)) \notin D_{\lambda^*}$, which means that $h_\alpha(b(x)) \neq \hat{x}$, implying $H_\alpha(q) \neq 0$.

Let $U \subset \mathbb{R}^n$ be a set and $q \in \mathbb{R}^n$ be a point. We use the notation $\deg(F, U, q)$ for the Brouwer degree of F with respect to the set D at q . From condition (L.3) by the homotopy property of the Brouwer degree (see [19]), we obtain

$$\deg(F, B_u(R), 0) = \deg(H_\alpha, B_u(R), 0) = \deg(H_1, B_u(R), 0). \tag{L.4}$$

Our next step is to show that $\deg(H_1, B_u(R), 0) \neq 0$. Since $h_1(x) = Ax$ we see that

$$H_1(x) = (Ax, 0) - \hat{x}.$$

By point 4. from Definition 15 it follows that $\det(A) \neq 0$ and $A^{-1}\hat{x} \in B_u(R)$. Therefore equation $H_1(q) = 0$ has a unique solution in $B_u(R)$ and by the degree property for affine maps

$$\deg(H_1, B_u(R), 0) = \operatorname{sgn} \det A = \pm 1$$

By (L.4), this gives

$$\deg(F, B_u(R), 0) = \deg(H_1, B_u(R), 0) \neq 0.$$

This means that there exists an $x \in B_u(R)$ such that $F(x) = 0$. This finishes the proof of (L.2).

We now define the candidate for $b^*(\hat{x})$ as

$$b^*(\hat{x}) = f \circ b(x(\hat{x})). \tag{L.5}$$

By construction, $\pi_x b^*(\hat{x}) = \hat{x}$. We need to show that $b^*(\hat{x})$ is well defined (meaning that the choice of $x(\hat{x})$ is unique), and that it is a horizontal disc.

Let $x_1 \neq x_2$. By Lemma 36 we have

$$\|\pi_x(f \circ b(x_1) - f \circ b(x_2))\| \geq \xi_{u,1,P} \|\pi_x(b(x_1) - b(x_2))\| = \xi_{u,1,P} \|x_1 - x_2\| \neq 0. \quad \text{(L.6)}$$

Hence, $b^*(\hat{x})$ is well defined.

Observe that (L.6) can be rewritten as

$$\|\hat{x}_1 - \hat{x}_2\| = \|\pi_x(f \circ b(x(\hat{x_1})) - f \circ b(x(\hat{x_2})))\| \geq \xi_{u,1,P} \|x(\hat{x}_1) - x(\hat{x}_2)\|.$$

Therefore $x(\hat{x})$ is Lipschitz, hence b^* is continuous.

We will now show that for any $\hat{x} \in \overline{B}_u(R)$ we have $b^*\left(\overline{B}_u\right) \subset J_u\left(b^*(\hat{x}), 1/L\right)$. By Corollary 34

$$f\left(J_u\left(b(x), 1/L\right) \cap D\right) \subset J_u\left(f \circ b(x), 1/L\right).$$

Since for any x we have $b(\overline{B}_u) \subset J_u\left(b(x), 1/L\right)$, we obtain

$$f \circ b(\overline{B}_u) \subset J_u\left(f \circ b(x), 1/L\right),$$

which by the definition of b^* from (L.5) implies

$$b^*(\overline{B}_u) \subset J_u\left(b^*(\hat{x}), 1/L\right),$$

as required.

We now need to show that if f, b are C^k, for $k \geq 1$, then so is b^*. Let us introduce the notation

$$\begin{aligned} g &: \overline{B}_u(R) \to \mathbb{R}^u, \\ g(x) &= \pi_x f \circ b(x). \end{aligned}$$

We can rewrite the definition of b^* using g as

$$b^*\left(x^*\right) = f \circ b \circ g^{-1}(x^*).$$

To show that b^* is C^k, it is sufficient for g^{-1} to be C^k. From (L.1) we see that $\pi_{(\lambda,y)} b$ is Lipschitz with the constant $1/L$, hence

$$\begin{aligned} m(Dg(x)) &= m(D\pi_x f \circ b(x)) \\ &= m\left(\frac{\partial f_x}{\partial x}(b(x)) + \frac{\partial f_x}{\partial(\lambda, y)}(b(x)) \frac{\partial \pi_{(\lambda,y)} b}{\partial(\lambda, y)}(x)\right) \\ &\geq m\left(\frac{\partial f_x}{\partial x}(b(x))\right) - \left\|\frac{\partial f_x}{\partial(\lambda, y)}(b(x)) \frac{\partial \pi_{(\lambda,y)} b}{\partial(\lambda, y)}(x)\right\| \\ &\geq m\left(\frac{\partial f_x}{\partial x}(b(x))\right) - \frac{1}{L}\left\|\frac{\partial f_x}{\partial(\lambda, y)}(b(x))\right\| \geq \xi_{u,1} > 0. \end{aligned}$$

and by the inverse function theorem g^{-1} is C^k; as required. ∎

Appendix M. Proof of Lemma 45

Proof. First, we will prove that

$$\big(f \circ b(\Lambda \times \overline{B}_u(R))\big) \cap D \neq \emptyset. \tag{M.1}$$

For any $\lambda \in \Lambda$ let us consider $b^{\lambda} : \overline{B}_u(R) \to D$ given by $b^{\lambda}(x) = b(\lambda, x)$. We will argue that b^{λ} is a horizontal disc. We first observe that

$$\pi_x b^{\lambda}(x) = \pi_x b(\lambda, x) = \pi_x \pi_{(\lambda,x)} b(\lambda, x) = \pi_x(\lambda, x) = x.$$

We need to show that

$$b^{\lambda}(\overline{B}_u(R)) \subset J_u\left(b^{\lambda}(x), 1/L\right). \tag{M.2}$$

Since b is a center-horizontal disc, by definition, for any $x_1, x_2 \in \overline{B}_u(R)$,

$$b(\lambda, x_2) \in J_u\left(b\left(\lambda, x_2\right), L\right),$$

hence

$$\|\pi_y b(\lambda, x_1) - \pi_y b(\lambda, x_2)\| < L\left\|\pi_{(\lambda,x)} b(\lambda, x_1) - \pi_{(\lambda,x)} b(\lambda, x_2)\right\|.$$

Since $\pi_{(\lambda,x)} b(\lambda, x_i) = (\lambda, x_i)$, this gives (remember that $L < 1$)

$$\begin{aligned}
\left\|\pi_{(\lambda,y)} b^{\lambda}(x_1) - \pi_{(\lambda,y)} b^{\lambda}(x_2)\right\| &= \left\|\pi_{(\lambda,y)} b(\lambda, x_1) - \pi_{(\lambda,y)} b(\lambda, x_2)\right\| \\
&= \|(\lambda, \pi_y b(\lambda, x_1)) - (\lambda, \pi_y b(\lambda, x_2))\| \\
&= \|\pi_y b(\lambda, x_1) - \pi_y b(\lambda, x_2)\| \\
&< L\left\|\pi_{(\lambda,x)} b(\lambda, x_1) - \pi_{(\lambda,x)} b(\lambda, x_2)\right\| \\
&= L\|x_1 - x_2\| \\
&= L\left\|\pi_x b^{\lambda}(x_1) - \pi_x b^{\lambda}(x_2)\right\| \\
&\leq 1/L\left\|\pi_x b^{\lambda}(x_1) - \pi_x b^{\lambda}(x_2)\right\|,
\end{aligned}$$

which implies (M.2). We have thus shown that b^{λ} is a horizontal disc.

From Lemma 44 it follows that $f \circ b^{\lambda}(\overline{B}_u(R)) \cap D$ is a horizontal disk in D, in particular this implies (M.1).

In the remainder of the proof we will use notation $\theta = (\lambda, x)$.

We will now show that $\pi_\theta f \circ b$ is an open map, in fact it is continuous and locally injective.

Let us fix θ_1 and let us take U, an convex open neighborhood contained in a single chart and such that $f(b(U))$ is contained in a single chart. From Lemma 38 it follows that

$$\|\pi_\theta f \circ b(\theta_1) - \pi_\theta f \circ b(\theta_2)\| \geq \xi_{cu,1,P}\|\theta_1 - \theta_2\|.$$

Therefore $\pi_\theta f \circ b : U \to \Lambda \times \mathbb{R}^u$ is continuous and injective, hence by the Brouwer open map theorem we know $\pi_\theta f \circ b(U)$ is an open set. This means that $\pi_\theta f \circ b$ is an open map, and therefore $\pi_\theta f \circ b(\Lambda \times B_u(R))$ is an open set.

From the covering relation (Definition 15) we know that the points $b(\theta)$ for $\theta \in \Lambda \times \partial\overline{B}_u(R)$ are mapped by f out of the set D

$$\pi_\theta f \circ b(\Lambda \times B_u(R)) \cap (\Lambda \times \overline{B}_u(R)) = \pi_\theta f \circ b(\Lambda \times \overline{B}_u(R)) \cap (\Lambda \times \overline{B}_u(R)).$$

Therefore the set $\pi_\theta f \circ b(\Lambda \times B_u(R)) \cap (\Lambda \times \overline{B}_u(R))$ is both open and closed in $\Lambda \times \overline{B}_u(R)$ and since it is also nonempty and $\Lambda \times \overline{B}_u(R)$ is connected, we infer that

$$\pi_\theta f \circ b(\Lambda \times B_u(R)) \cap (\Lambda \times \overline{B}_u(R)) = \Lambda \times \overline{B}_u(R). \tag{M.3}$$

We need to show that the map $\pi_\theta f \circ b$ is an injection on $(\pi_\theta f \circ b)^{-1}(\Lambda \times \overline{B}_u(R))$. This is a direct consequence of the backward cone condition (see Definition 13). To show this, assume that there exists $\theta_1 \neq \theta_2$ in $\Lambda \times \overline{B}_u(R)$ such that

$$\pi_\theta f(b(\theta_1)) = \pi_\theta f(b(\theta_2)).$$

Then

$$f(b(\theta_1)) \in J_s\left(f(b(\theta_2)), 1/L\right),$$

therefore the backward cone condition implies that

$$b(\theta_1) \in J_s\left(b(\theta_2), 1/L\right),$$

which contradicts condition (26) required of center-horizontal disks.

We have shown (M.3), which means that for any $\theta^* \in \Lambda \times \overline{B}_u(R)$ there exists an θ such that

$$\pi_\theta f \circ b(\theta) = \theta^*.$$

Such θ is unique due to the fact that $\pi_\theta f \circ b$ is injective. We can therefore define

$$b^*(\theta^*) = f \circ b(\theta).$$

From the construction of b^* it follows that $\pi_\theta b^*(\theta^*) = \theta^*$. Condition (26) is a consequence of backward cone conditions, and follows from a mirror argument to the one used for the proof of injectivity of $\pi_\theta f \circ b$, which was done in the preceding paragraph.

What is left is to show that if f, b are C^k, for $k \geq 1$, then so is b^*. This follows from mirror arguments to the proof of C^k smoothness in Lemma 44. ■

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