Calculus of Invariant Manifolds - topological approach

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Introduction, notation

0.1 Introduction

In these notes to describe the geometric (topological) methods to rigorously establish the existence of some invariant objects in dynamical systems like: fixed points, periodic points and their stable and unstable manifolds, the normally hyperbolic hyperbolic manifold. The tools we present have been developed with the intention to be applicable in the context of computer assisted proofs, where constructive proofs and explicit bounds on the objects under investigation are needed. This in most cases requires a serious reformulation of the classical approach used in the dynamical systems theory, which an eye toward very robust assumptions.

Despite the fact that, the main motivation of the body of the work contained in these notes are ODE, where in most cases the rigorous analytical investigation is apparently outside of the human capabilities, we restrict ourself here to maps, i.e. dynamical systems with a discrete time. The results presented here are applicable to the Poincaré return maps for ODEs. Also, we have made a conscious effort to do not use the inverse maps. The motivation comes from the consideration of systems with have an attractor and for which a rigorous backward integration of orbits might be very difficult or the backward orbit may blow up in finite time. This allows to apply our results also to the case of non-invertible maps.

We focus here on abstract methods and theorems, we will not discuss the issue of rigorous numerics, which is as important in the context of computer assisted proofs in dynamics as the abstract mathematical theorems. But from the point of mathematical depth this issue is rather trivial.

The mathematical objects appearing in these notes can be divided in two classes: topological and differential. In the topological class the reader will meet h-sets, covering relations, the local Brouwer degree, topological disks etc. These objects are used to prove the 'global' results - the existence of the fixed point or of nonempty invariant set, the nonempty intersection of some topological disks etc.

The second class of objects, the differential objects, may be described in one word as the cones, or the cone fields. Yet, contrary to the classical approach, where the cones are considered in the tangent space, we consider them in the phasespace using the finite differences. Obviously, this makes our approach very coordinate dependent, but there is no way to avoid this dependence on the coordinates when one want explicit bounds in a concrete example. These tools are used, just in the classical theory, to establish finer properties of invariant object like their uniqueness, being a manifold etc.

In contrast to the most common approaches to the question of existence of (un)stable manifolds in all our proofs we will stay in the phasespace, the reader will not see here any application of Banach contraction principle applied to some functional equation describing the object we are after. The way we proceed is more geometric and direct, but we pay a price for our approach - to obtain smoothness, which is the 'standard' approach comes almost automatically, we need additional reasoning.

In part one of the these notes we discuss the stable and unstable manifolds of the fixed point.

The second part is concerned with the normally hyperbolic invariant manifolds.

0.1.1 Notation

By $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ we denote the set of natural, integer, rational, real and complex numbers, respectively. $\mathbb{Z}_-$ and $\mathbb{Z}_+$ are negative and nonnegative integers, respectively. By S^1 we will denote a unit circle on the complex plane.

For $\mathbb{R}^n$ we will denote the norm of x by $\|x\|$ and when in some context the formula for the norm is not specified, then it means that any norm can be used. Let $x_0 \in \mathbb{R}^s$, then $B_s(x_0, r) = \{z \in \mathbb{R}^s \mid \|x_0 - z\| < r\}$ and $B_s = B_s(0, 1)$.

For $z \in \mathbb{R}^u \times \mathbb{R}^s$ we will call usually the first coordinate, x, and the second one y. Hence $z = (x, y)$, where $x \in \mathbb{R}^u$ and $y \in \mathbb{R}^s$. We will use the projection maps $\pi_x(z) = x(z) = x$ and $\pi_y(z) = y(z) = y$. For functions f we will use also $f_x = \pi_x f$.

Let $z \in \mathbb{R}^n$ and $U \subset \mathbb{R}^n$ be a compact set and $f : U \to \mathbb{R}^n$ be continuous map, such that $z \notin f(\partial U)$. Then the local Brouwer degree [S] of f on U at z is defined and will be denoted by $\deg(f, U, z)$, see Appendix for properties of $\deg(f, U, z)$

If V, W are two vector spaces, then by $\mathrm{Lin}(V, W)$ we will denote the set of all linear maps from V to W. When $V = \mathbb{R}^k$ and $W = \mathbb{R}^m$, then we will identify $\mathrm{Lin}(\mathbb{R}^k, \mathbb{R}^m)$ with the set of matrices with k rows and m columns, which will be denoted by $\mathbb{R}^{k\times m}$.

Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. By $\mathrm{Sp}(A)$ we denote the spectrum of A, which is the set of $\lambda \in \mathbb{C}$, such that there exists $x \in \mathbb{C}^n \setminus \{0\}$, such that $Ax = \lambda x$.

Part I

Invariant manifolds for fixed points of maps

Introduction

The goal of this chapter is to present a set of tools which allow establish an explicit for the (un)stable fixed point for a map. As a byproduct we give a new, 'geometric', proof of the stable manifold theorem for hyperbolic fixed point of a map.

The chapter starts with the introduction of topological tools: h-sets, covering relations for maps on h-sets and topological disks. These are the 'global' tools, which are required to establish the existence of dynamical objects under consideration.

Later we introduce the cones on h-sets and cone conditions for maps and disks, and we state and prove our main theorems for maps. We insist on not using the inverse. The method of proof of the unstable manifold is essentially the Hadamard graph transform method. For the stable manifold, since we want to avoid the use of the inverse map, we use a different method.

The part ends an example of non-hyperbolic fixed point to which our approach applies and explanation how to apply our approach to fixed points of ODEs , without proof.

The presented material in is taken from [ZGi, KWZ, WZ, ZCC].

Comments about the method of covering relations

We believe that the topological tools introduced in this part are also of interest independent from their use in the verification of the (un)stable manifolds. The topological notion of the covering relation (Easton's 'correctly aligned windows' [E1, E2] without any differentiability assumptions) originating from the theory of the Conley index, see [ZGi] and the references given there, has been successfully applied to establish the existence of symbolic dynamics for such systems as: the Henon map[Ga1, ZN, GaZ2, CGB], the Chua circuit[Ga2], the Lorenz equations[GaZ1], the Rössler equations[ZN], the Henon-Heiles hamiltonian[AZ], PR3BP [A, WZP] or the Michelson system [W1, W2]. In all the examples listed above we are talking about the computer assisted proofs. There exist also some nontrivial applications of covering relations, not related to any computer assisted proofs, like the stability of Sharkovski order and estimates for the topological entropy for multidimensional perturbations of one-dimensional maps [MZ, ZS], the delay differential equations with small delays [WoZ] or to the Arnold diffusion [GiL, GiR].

Comments about our approach to the (un)stable manifold

The standard way to establish the hyperbolic behavior is usually through the cone fields in the tangent space which are mapped into itself by the tangent map and/or its inverse, see [T] and the references given there. Here we introduce a different approach, which is based on the two point Lapunov function for a map f, by which we understand the function of two variables $L(z_1, z_2)$, satisfying locally the following condition $L(f(z_1), f(z_2)) > L(z_1, z_2)$ for $z_1 \neq z_2$. For the hyperbolic fixed point the proposed approach appears to be equivalent to the standard one, but our method does not require the hyperbolicity (see the example in Section 4.6). The proposed approach has been applied to the study (a computer assisted proof) of the cocoon bifurcation in the Michelson system in [KWZ] and the proof of the existence of homoclinic tangency for forced damped pendulum [WZ1].

The stable manifold theorem goes back to Poincaré, Hadamard and Perron, see [Ha] and the references given there, but there still appear new proofs in the literature, the recent ones are [Ch, HL, McS]. The interesting feature of our approach is that the whole proof in made in the phase space, is local and gives explicit bounds on the size of the (un)stable manifolds. A proof, similar in sprit, but not in the realization, has been proposed by Hartman in [Ha, Exercises 5.3 and 5.4].

Our geometric approach to the proof of the stable manifold theorem should be contrasted with the standard approach see [Ha, Hal, I70, I80, Ro, C, HL], where the problem of the existence of stable manifold is rephrased as a question of the existence of fixed point in a suitable Banach space of graphs of functions or sequences. Moreover, our approach does not require that the fixed point is hyperbolic, the essential assumption is the existence of the two-point Lapunov function. In Section 4.6 we analyze a non-hyperbolic example of this type.

The results about the (un)stable manifolds for hyperbolic fixed points stated and proved in this paper are weaker than those obtained using the Perron-Irwin method [I70, I80, Ch] as we did not get the smoothness of the invariant manifolds, in our proof we obtain only that they are Lipschitz manifolds for C^1 maps. Higher smoothness follows from the results about the smoothness of NHIM discussed in chapter 8.

Also contrary to the results from [I70, I80, Ch], which are valid in Banach space, we restrict ourselves to the finite dimensional case, but it clear that our proof can be easily adapted to compact maps on the Banach space.

Chapter 1

h-sets, covering relations

The goal of this section is present the notions of h-sets and covering relations, and to state the theorem about the existence of point realizing the chain of covering relations.

1.1 h-sets and covering relations

Definition 1 *[ZGi, Definition 1] An h-set, N, is a quadruple $(|N|, u(N), s(N), c_N)$ such that*

- $|N|$ *is a compact subset of* $\mathbb{R}^n$
- $u(N), s(N) \in \{0, 1, 2, \dots\}$ *are such that* $u(N) + s(N) = n$
- $c_N : \mathbb{R}^n \to \mathbb{R}^n = \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)}$ *is a homeomorphism such that*

$$c_N(|N|) = \overline{B_{u(N)}} \times \overline{B_{s(N)}}.$$

We set

$$\begin{aligned}
\dim(N) &:= n, \\
N_c &:= \overline{B_{u(N)}} \times \overline{B_{s(N)}}, \\
N_c^- &:= \partial B_{u(N)} \times \overline{B_{s(N)}}, \\
N_c^+ &:= \overline{B_{u(N)}} \times \partial B_{s(N)}, \\
N^- &:= c_N^{-1}(N_c^-), \quad N^+ = c_N^{-1}(N_c^+).
\end{aligned}$$

Hence an h-set, N, is a product of two closed balls in some coordinate system. The numbers $u(N)$ and $s(N)$ are called the nominally unstable and nominally stable dimensions, respectively. The subscript c refers to the new coordinates given by homeomorphism c_N. Observe that if $u(N) = 0$, then $N^- = \emptyset$ and if $s(N) = 0$, then $N^+ = \emptyset$. In the sequel to make notation less cumbersome we will often drop the bars in the symbol $|N|$ and we will use N to denote both the h-sets and its support.

Sometimes we will call N^- *the exit set of* N and N^+ *the entry set of* N. These name are motivated by the Conley index theory and the role these sets will play in the context of covering relations.

Definition 2 *[ZGi, Definition 3] Let N be a h-set. We define a h-set N^T as follows*

- $|N^T| = |N|$
- $u(N^T) = s(N)$, $s(N^T) = u(N)$
- *We define a homeomorphism* $c_{N^T} : \mathbb{R}^n \to \mathbb{R}^n = \mathbb{R}^{u(N^T)} \times \mathbb{R}^{s(N^T)}$, *by*

$$c_{N^T}(x) = j(c_N(x)),$$

where $j : \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)} \to \mathbb{R}^{s(N)} \times \mathbb{R}^{u(N)}$ *is given by* $j(p,q) = (q,p)$.

∎

Observe that $N^{T,+} = N^-$ and $N^{T,-} = N^+$. This operation is useful in the context of inverse maps.

Definition 3 *[ZGi, Definition 6] Assume that N, M are h-sets, such that $u(N) = u(M) = u$ and $s(N) = s(M) = s$. Let $f : N \to \mathbb{R}^n$ be a continuous map. Let $f_c = c_M \circ f \circ c_N^{-1} : N_c \to \mathbb{R}^u \times \mathbb{R}^s$. Let w be a nonzero integer. We say that*

$$N \overset{f,w}{\Longrightarrow} M$$

(N f-covers M with degree w) iff the following conditions are satisfied

1. *there exists a continuous homotopy $h : [0,1] \times N_c \to \mathbb{R}^u \times \mathbb{R}^s$, such that the following conditions hold true*

$$\begin{aligned} h_0 &= f_c, && (1.1)\\ h([0,1], N_c^-) \cap M_c &= \emptyset, && (1.2)\\ h([0,1], N_c) \cap M_c^+ &= \emptyset. && (1.3) \end{aligned}$$

2. *If $u > 0$, then there exists a map $A : \mathbb{R}^u \to \mathbb{R}^u$, such that*

$$\begin{aligned} h_1(p,q) &= (A(p), 0), \text{ for } p \in \overline{B_u}(0,1) \text{ and } q \in \overline{B_s}(0,1), && (1.4)\\ A(\partial B_u(0,1)) &\subset \mathbb{R}^u \setminus \overline{B_u}(0,1). && (1.5) \end{aligned}$$

Moreover, we require that

$$\deg(A, \overline{B_u}(0,1), 0) = w,$$

We will call condition (1.2) the exit condition *and condition* (1.3) *will be called* the entry condition.

Note that in the case $u = 0$, if $N \overset{f,w}{\Longrightarrow} M$, then $f(N) \subset \mathrm{int} M$ and $w = 1$.

In fact in the above definition $s(N)$ and $s(M)$ can be different, see [W2, Def. 2.2].

Remark 1 *Observe, that since for any norm in $\mathbb{R}^n$ the closed unit ball is homeomorphic to $[-1,1]^n$, therefore for h-sets and covering relations we will use different norms in different contexts.*

Remark 2 *If the map A in condition 2 of Def. 3 is a linear map, then condition (1.5) implies, that*

$$\deg(A, \overline{B_u}(0,1), 0) = \pm 1.$$

Hence condition (3) is in this situation automatically fulfilled with $w = \pm 1$.

In fact, this is the most common situation in the applications of covering relations.

Most of the time we will not interested in the value of w in the symbol $N \overset{f,w}{\Longrightarrow} M$ and we will often drop it and write $N \overset{f}{\Longrightarrow} M$, instead. Sometimes we may even drop the symbol f and write $N \Longrightarrow M$.

Definition 4 *[ZGi, Definition 7] Assume N, M are h-sets, such that $u(N) = u(M) = u$ and $s(N) = s(M) = s$. Let $g : \mathbb{R}^n \supset \Omega \to \mathbb{R}^n$. Assume that $g^{-1} : |M| \to \mathbb{R}^n$ is well defined and continuous. We say that $N \overset{g}{\Longleftarrow} M$ (N g-backcovers M) iff $M^T \overset{g^{-1}}{\Longrightarrow} N^T$.*

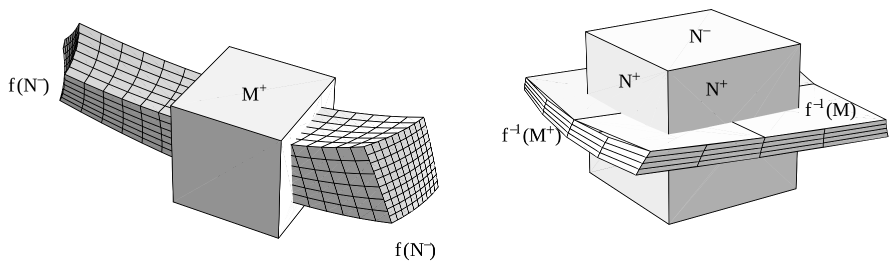

Figure 1.1: Examples of covering (left) and backcovering (right) relations. In this case $u(N) = s(N) = 1$ and $s(N) = s(M) = 2$.

The geometry of Definitions 3 and 4 is presented in Figures 1.1 and 1.2.

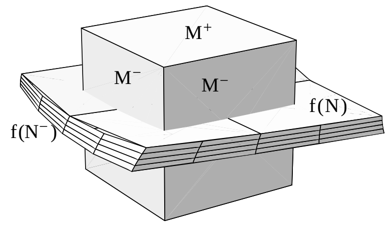

Figure 1.2: Examples of covering relations. In this case $u(N) = u(M) = 2$ and $s(N) = s(M) = 1$.

1.1.1 Main theorem about chains of covering relations

Theorem 3 (Thm. 9) *[ZGi] Assume N_i, $i = 0, \ldots, k$, $N_k = N_0$ are h-sets and for each $i = 1, \ldots, k$ we have either*

$$N_{i-1} \overset{f_i,w_i}{\Longrightarrow} N_i \tag{1.6}$$

or

$$N_i \subset \mathrm{dom}(f_i^{-1}) \quad and \quad N_{i-1} \overset{f_j,w_i}{\Longleftarrow} N_i. \tag{1.7}$$

Then there exists a point $x \in \mathrm{int} N_0$, such that

$$\begin{aligned} f_i \circ f_{i-1} \circ \cdots \circ f_1(x) &\in \mathrm{int} N_i, \quad i = 1, \ldots, k \quad (1.8)\\ f_k \circ f_{k-1} \circ \cdots \circ f_1(x) &= x \quad (1.9) \end{aligned}$$

We point the reader to [ZGi] for the proof. The basic idea of the proof of this theorem - the homotopy and the local Brouwer degree - appears in the proof Theorem 7.

The following corollary is an immediate consequence of Theorem 3.

Collorary 4 *Let N_i, $i \in \mathbb{Z}_+$ be h-sets. Assume that for each $i \in \mathbb{Z}_+$ we have either*

$$N_{i-1} \overset{f_i,w_i}{\Longrightarrow} N_i \tag{1.10}$$

or

$$N_i \subset \mathrm{dom}(f_i^{-1}) \quad and \quad N_{i-1} \overset{f_j,w_i}{\Longleftarrow} N_i. \tag{1.11}$$

Then there exists a point $x \in \mathrm{int} N_0$, such that

$$f_i \circ f_{i-1} \circ \cdots \circ f_1(x) \in \mathrm{int} N_i, \quad i \in \mathbb{Z}_+. \tag{1.12}$$

Moreover, if $N_{i+k} = N_i$ for some $k > 0$ and all i, then the point x can be chosen so that

$$f_k \circ f_{k-1} \circ \cdots \circ f_1(x) = x. \tag{1.13}$$

1.2 Examples with one unstable direction, topological horseshoe

In this section we give some examples for coverings with one unstable direction.

We consider subsets in $V = \mathbb{R} \times W$, where $W = \mathbb{R}^k$ for some $k > 0$.

Let us fix $r > 0$.

We will denote by $\mathcal{C}(r)$ the family of cylinders of the form $[a,b] \times \overline{B_W(r)}$. To each $N \in \mathcal{C}$ we assign h-set structure by $u(N) = 1$, $s(N) = \dim W$. $N^{=} L(N) \cup R(N)$, $N^{+} = H(N)$, where L, R, H are defined below.

Given $N = [a,b] \times \overline{B_W(r)} \in \mathcal{C}(r)$, we define

$$
\begin{aligned}
L(N) &= \{a\} \times \overline{B_W(r)}, \quad \text{left edge}, \\
R(N) &= \{b\} \times \overline{B_W(r)}, \quad \text{right edge}, \\
H(N) &= [a,b] \times \partial B_W(r), \quad \text{'horiz. boundary'} \\
S_L(N) &= (-\infty, a) \times B_W(r), \quad \text{left side of } N \\
S_R(N) &= (b, \infty) \times B_W(r), \quad \text{right side of } N
\end{aligned}
$$

The following lemma give sufficient conditions for covering relation between two cylinders, which avoids any homotopy in its definition.

Lemma 5 *Let $N_0, N_1 \in \mathcal{C}(r)$. $G : V \to V$. Assume that*

$$G(N_0) \subset (-\infty, \infty) \times B_W(r) \tag{1.14}$$

and one of the following two conditions hold

$$G(L(N_0)) \subset S_L(N_1), G(R(N_0)) \subset S_R(N_1) \tag{1.15}$$

$$G(L(N_0)) \subset S_R(N_1), G(R(N_0)) \subset S_L(N_1) \tag{1.16}$$

Then $N_0 \overset{G}{\Longrightarrow} N_1$

We leave to reader an easy proof.

In fact the conditions appearing in the above lemma can be used as the definition of covering relation for cylinders. Condition (1.14) means that the image of N_0 under G is contained in the 'horizontal' strip defined by N_1. Conditions (1.15) and (1.16) mean that the 'vertical' edges of N_0 are mapped to different sides of N_1 (see Fig. 1.4).

Example: Topological Smale horseshoe

Let $N_0, N_1 \in \mathcal{C}(r)$, $\mathrm{int} N_0 \cap \mathrm{int} N_1 = \emptyset$. We say that a continuous map $G : V \to V$ is *a topological Smale horseshoe* (with respect to N_0, N_1) (see Fig. 1.5) iff

$$N_i \overset{G}{\Longrightarrow} N_j, \quad \text{for } i,j = 0,1$$

From Theorem 3 we obtain easily the symbolic dynamics and all periods for G, namely

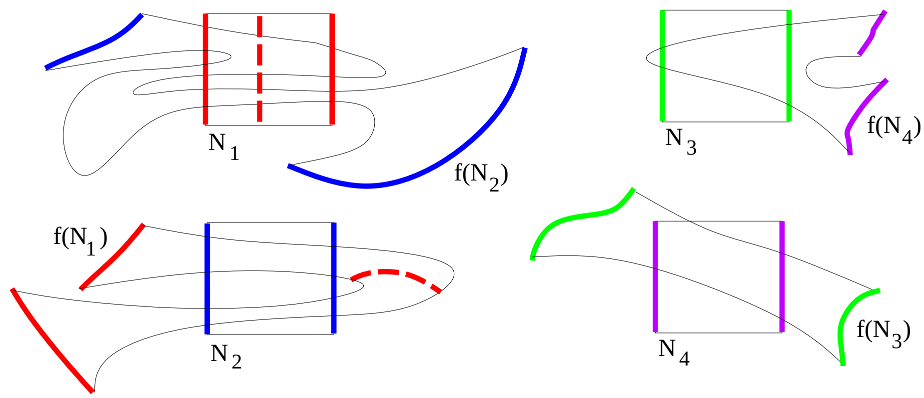

Figure 1.3: Examples of position of $f(N_i)$ with respect to N_j, vertical edges and their images are plotted using thick lines, N_2 f–covers N_1, N_1 f–covers N_2 (one can choose a subquadrangle of N_1 which f–covers N_2, for example the one defined by the left (or right) vertical edge of N_1 and the dashed line), N_3 does not f-cover N_4 (image of N_3 has nonempty intersection with horizontal edges of N_4), N_4 does not f-cover N_3 (images of horizontal edges of N_4 lie geometrically on the same side of N_3)

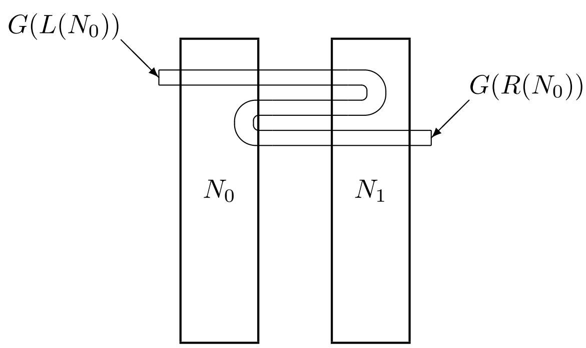

Figure 1.4: $N_0 \overset{G}{\Longrightarrow} N_1$

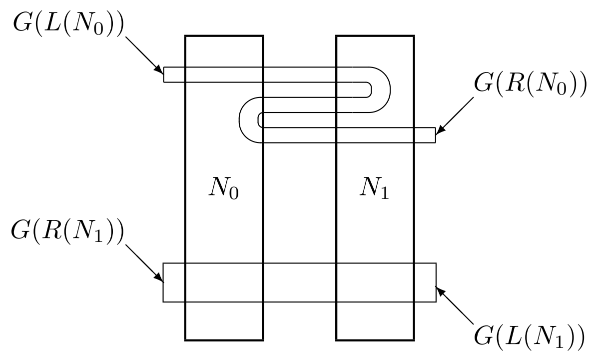

Figure 1.5: Topological Smale horseshoe.

Theorem 6 *If G is a topological Smale horseshoe with respect to N_0, N_1 then for every finite sequence $(\alpha_0, \alpha_1, \ldots, \alpha_{l-1})$, where $\alpha_i \in \{0,1\}$ there exists $x \in N_{\alpha_0}$ such that*

$$G^i(x) \in \mathrm{int} N_{\alpha_i}$$
$$G^l(x) = x$$

1.3 Natural structure of h-set

Observe that all the conditions appearing in the definition of the covering relation are expressed in 'internal' coordinates c_N and c_M. Also the homotopy is defined in terms of these coordinates. This sometimes makes the matter and the notation look a bit cumbersome. With this in mind we introduce the notion of a 'natural' structure on h-set.

Definition 5 *We will say that $N = \{(x_0, y_0)\} + \overline{B}_u(0, r_1) \times \overline{B}_s(0, r_1) \subset \mathbb{R}^u \times \mathbb{R}^s$ is an h-set with a natural structure given by :*
$u(N) = u$, $s(N) = s$, $c_N(x, y) = \left(\frac{x-x_0}{r_1}, \frac{y-y_0}{r_2}\right)$.

In context of $\mathbb{R}^2$ and $u = 1$, $s = 1$ we will sometimes write $N = z_0 + [-a, a] \times [-b, b]$. This in compatible with the above convention as a defines radius of ball $\overline{B}_u(0, a) = [-a, a]$ and b of $\overline{B}_s(0, b) = [-b, b]$.

1.4 Verification of covering relation when $u > 1$

In Section 1.2 the sufficient conditions have been given for the covering relation with $u = 1$. In the present section we discuss a way to check the covering relation

$N \overset{f}{\Longrightarrow} M$ when $u(N) > 1$ in the situation when the size of N is small enough for the linear approximation to work. This approach was used in paper [WZ] in the computer assisted proof of the existence of symbolic dynamics and heteroclinic connections in some 4D reversible map with two unstable dimensions.

Let N, M be a h-sets in $\mathbb{R}^n$ such that $u(N) = u(M) = u$ and $s(N) = s(M) = s$ and let $f : N \to \mathbb{R}^n$ be continuous. In order to prove that the covering relation $N \overset{f}{\Longrightarrow} M$ holds, it is necessary to find the homotopy $h : [0,1] \times N_c \to \mathbb{R}^u \times \mathbb{R}^s$ and a map $A : \mathbb{R}^u \to \mathbb{R}^u$ satisfying conditions (1.1-1.4).

Let $f_c = c_M \circ f \circ c_N^{-1}$. We try to find a homotopy between f_c and its derivative computed in the center of the set and projected onto unstable directions, i.e., we define

$$A : \mathbb{R}^u \ni p \to \pi_u(Df_c(0)(p, 0)) \in \mathbb{R}^u,$$

where $\pi_u : \mathbb{R}^n \to \mathbb{R}^u$ is the projection onto first u variables. We require A to be an isomorphism, then we have

$$\deg(A, \overline{B_u}(0,1), 0) = \mathrm{sgn}(\det A) = \pm 1.$$

Now we define the homotopy between f_c and $(p, q) \to (A(p), 0)$ by

$$h(t, p, q) = (1-t)f_c + t(A(p), 0), \quad \text{for } (p, q) \in \overline{B_u}(0,1) \times \overline{B_s}(0,1). \tag{1.17}$$

Obviously the homotopy (1.17) satisfies conditions (1.1) and (1.4). We need to check whether the homotopy (1.17) satisfies conditions (1.2-1.3).

Chapter 2

Topological disks

2.1 Horizontal and vertical disks in an h-set

Definition 6 *[WZ, Definition 10] Let N be an h-set. Let $b : \overline{B_{u(N)}} \to |N|$ be continuous and let $b_c = c_N \circ b$. We say that b is* a horizontal disk in N *if there exists a homotopy $h : [0,1] \times \overline{B_{u(N)}} \to N_c$, such that*

$$\begin{aligned} h_0 &= b_c & (2.1)\\ h_1(x) &= (x,0), \quad \text{for all } x \in \overline{B_{u(N)}} & (2.2)\\ h(t,x) &\in N_c^-, \quad \text{for all } t \in [0,1] \text{ and } x \in \partial B_{u(N)} & (2.3) \end{aligned}$$

Definition 7 *[WZ, Definition 11] Let N be an h-set. Let $b : \overline{B_{s(N)}} \to |N|$ be continuous and let $b_c = c_N \circ b$. We say that b is* a vertical disk in N *if there exists a homotopy $h : [0,1] \times \overline{B_{s(N)}} \to N_c$, such that*

$$\begin{aligned} h_0 &= b_c \\ h_1(x) &= (0,x), \quad \text{for all } x \in \overline{B_{s(N)}} \\ h(t,x) &\in N_c^+, \quad \text{for all } t \in [0,1] \text{ and } x \in \partial B_{s(N)}. & (2.4) \end{aligned}$$

Definition 8 *Let N be an h-set in $\mathbb{R}^n$ and b be a horizontal (vertical) disk in N.*
We will say that $x \in \mathbb{R}^n$ belongs to b, *when $b(z) = x$ for some $z \in \mathrm{dom}(b)$.*

By $|b|$ we will denote the image of b. Hence $z \in |b|$ iff z belongs to b.

2.2 Topological transversality theorem

Now we are ready to state and prove the topological transversality theorem. A simplified version of this theorem was given in [W] for the case of one unstable direction and covering relations chain without backcoverings. The argument in [W], which was quite simple and was based on the connectivity only, cannot be

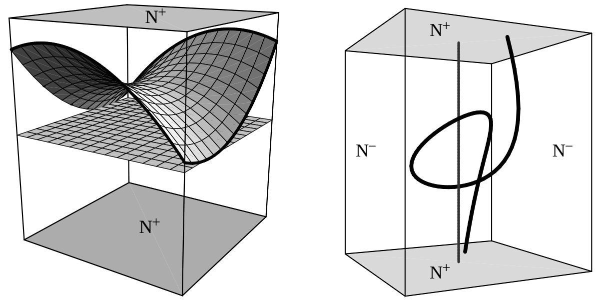

Figure 2.1: A horizontal disc in an h-set N with $u(N)=2$ and $s(N)=1$ (left). A vertical disc in an h-set N with $u(N)=2$ and $s(N)=1$ (right).

carried over to a larger number of unstable directions or to the situation when both covering and backcovering relations are present.

Theorem 7 *[WZ, Thm. 4] Let $k \geq 1$. Assume N_i, $i=0,\dots,k$, are h-sets and for each $i=1,\dots,k$ we have either*

$$N_{i-1} \overset{f_i,w_i}{\Longrightarrow} N_i \tag{2.5}$$

or

$$N_i \subset \mathrm{dom}(f_i^{-1}) \quad \text{and} \quad N_{i-1} \overset{f_j,w_i}{\Longleftarrow} N_i. \tag{2.6}$$

Assume that b_0 is a horizontal disk in N_0 and b_e is a vertical disk in N_k. Then there exists a point $x \in \mathrm{int} N_0$, such that

$$\begin{aligned} x &= b_0(t), \quad \textit{for some } t \in B_{u(N_0)}(0,1) && (2.7)\\ f_i \circ f_{i-1} \circ \cdots \circ f_1(x) &\in \mathrm{int} N_i, \quad i=1,\dots,k && (2.8)\\ f_k \circ f_{k-1} \circ \cdots \circ f_1(x) &= b_e(z), \quad \textit{for some } z \in B_{s(N_k)}(0,1) && (2.9) \end{aligned}$$

Proof: Without loss generality we can assume that

$$c_{N_i} = \mathrm{Id}, \qquad \text{for } i=0,\dots,k.$$

Then

$$f_i = f_{i,c}, \qquad \text{for } i=1,\dots,k,$$
$$N_i = N_{c,i}, \quad N_i^{\pm} = N_{i,c}^{\pm} \quad \text{for } i=0,\dots,k$$

We define $g_i = f_i^{-1}$, for those i for which we have the back-covering relation $N_{i-1} \overset{f_j,w_i}{\Longleftarrow} N_i$.

Notice that from the definition of covering relation, it follows immediately that there are $u \geq 0$, $s \geq 0$, such that $u(N_i)=u$ and $s(N_i)=s$, for all $i=0,\dots,k$.

The idea of the proof is to rewrite our problem as a zero finding problem for a suitable map, then to compute its local Brouwer degree to infer the existence of a solution.

As a tool for keeping track of the occurrences of coverings and backcoverings, we define the map dir : $\{1,\ldots,k\} \to \{0,1\}$ by $\mathrm{dir}(i) = 1$ if $N_{i-1} \overset{f_i,w_i}{\Longrightarrow} N_i$ and $\mathrm{dir}(i) = 0$ if $N_{i-1} \overset{f_j,w_i}{\Longleftarrow} N_i$. For $i = 1,\ldots,k$ let h_i be a homotopy map from the definition of covering relation for $N_{i-1} \overset{f_i,w_i}{\Longrightarrow} N_i$ or $N_{i-1} \overset{f_j,w_i}{\Longleftarrow} N_i$. In the case of a direct covering (i.e. $\mathrm{dir}(i) = 1$), the homotopy h_i satisfies

$$h_i(0,x) = f_i(x), \quad \text{where } x \in \mathbb{R}^{u+s}, \tag{2.10}$$
$$h_i(1,(p,q)) = (A_i(p),0), \quad \text{where } p \in \mathbb{R}^u \text{ and } q \in \mathbb{R}^s, \tag{2.11}$$
$$h_i([0,1],N_{i-1}^-) \cap N_i = \emptyset, \tag{2.12}$$
$$h_i([0,1],N_{i-1}) \cap N_i^+ = \emptyset. \tag{2.13}$$

In the case of a backcovering (i.e. $\mathrm{dir}(i) = 0$), the homotopy h_i satisfies

$$h_i(0,x) = g_i(x), \quad \text{where } x \in \mathbb{R}^{u+s}, \tag{2.14}$$
$$h_i(1,(p,q)) = (0,A_i(q)), \quad \text{where } p \in \mathbb{R}^u \text{ and } q \in \mathbb{R}^s, \tag{2.15}$$
$$h_i([0,1],N_i^+) \cap N_{i-1} = \emptyset, \tag{2.16}$$
$$h_i([0,1],N_i) \cap N_{i-1}^- = \emptyset. \tag{2.17}$$

Let h_t and h_z be the homotopies appearing in the definition of a horizontal and vertical disk for b_0 and b_e, respectively.

It is enough to prove that there exists $t \in B_u(0,1)$, $z \in B_s(0,1)$ and $x_i \in \mathrm{int}N_i$ for $i = 1,\ldots,k$ such that

$$\begin{aligned} b_0(t) &= x_0, \\ f_i(x_{i-1}) &= x_i, \quad \text{if } \mathrm{dir}(i) = 1, \\ g_i(x_i) &= x_{i-1}, \quad \text{if } \mathrm{dir}(i) = 0, \\ b_e(z) &= x_k. \end{aligned} \tag{2.18}$$

We will treat (2.18) as a multidimensional system of equations to be solved. To this end, let us define

$$\Pi = \overline{B_u}(0,1) \times N_1 \times \cdots \times N_{k-1} \times \overline{B_s}(0,1)$$

A point $x \in \Pi$ will be represented by $x = (t,x_1,\ldots,x_{k-1},z)$.

We define a map $F = (F_1,\ldots,F_k) : \Pi \to \mathbb{R}^{(u+s)k}$ as follows: for $i = 2,\ldots,k-1$ we set

$$F_i(t,x_1,\ldots,x_{k-1},z) = \begin{cases} x_i - f_i(x_{i-1}) & \text{if } \mathrm{dir}(i) = 1, \\ x_{i-1} - g_i(x_i) & \text{if } \mathrm{dir}(i) = 0. \end{cases}$$

For $i = 1$ we set

$$F_1(t,x_1,\ldots,x_{k-1},z) = \begin{cases} x_1 - f_1(b_0(t)) & \text{if } \mathrm{dir}(1) = 1, \\ b_0(t) - g_1(x_1) & \text{if } \mathrm{dir}(1) = 0. \end{cases}$$

For $i = k$ we define

$$F_k(t, x_1, \ldots, x_{k-1}, z) = \begin{cases} b_e(z) - f_k(x_{k-1}) & \text{if } \mathrm{dir}(k) = 1, \\ x_{k-1} - g_k(b_e(z)) & \text{if } \mathrm{dir}(k) = 0. \end{cases}$$

With this notation, solving the system (2.18) is equivalent to solving the equation $F(x) = 0$ in $\mathrm{int}\Pi$.

We define a homotopy $H = (H_1, \ldots, H_k) : [0,1] \times \Pi \to \mathbb{R}^{(u+s)k}$ as follows. For $i = 2, \ldots, k-1$ we set

$$H_i(\lambda, t, x_1, \ldots, x_{k-1}, z) = \begin{cases} x_i - h_i(\lambda, x_{i-1}) & \text{if } \mathrm{dir}(i) = 1, \\ x_{i-1} - h_i(\lambda, x_i) & \text{if } \mathrm{dir}(i) = 0. \end{cases}$$

For $i = 1$ we set

$$H_1(\lambda, t, x_1, \ldots, x_{k-1}, z) = \begin{cases} x_1 - h_1(\lambda, h_t(\lambda, t)) & \text{if } \mathrm{dir}(1) = 1, \\ h_t(\lambda, t) - h_1(\lambda, x_1) & \text{if } \mathrm{dir}(1) = 0. \end{cases}$$

For $i = k$ we define

$$H_k(\lambda, t, x_1, \ldots, x_{k-1}, z) = \begin{cases} h_z(\lambda, z) - h_k(\lambda, x_{k-1}) & \text{if } \mathrm{dir}(k) = 1, \\ x_{k-1} - h_k(\lambda, h_z(\lambda, z)) & \text{if } \mathrm{dir}(k) = 0. \end{cases}$$

Notice that $H(0, x) = F(x)$. The assertion of the theorem is a consequence of the following two lemmas, which will be proved after we complete the current proof.

Lemma 8 *For all $\lambda \in [0,1]$ the local Brouwer degree $\deg(H(\lambda, \cdot), \mathrm{int}\Pi, 0)$ is well defined and does not depend on λ. Namely, for all $\lambda \in [0,1]$ we have*

$$\deg(H(\lambda, \cdot), \mathrm{int}\Pi, 0) = \deg(H(1, \cdot), \mathrm{int}\Pi, 0).$$

Lemma 9

$$|\deg(H(1, \cdot), \mathrm{int}\Pi, 0)| = |w_1 \cdot w_2 \cdot \cdots \cdot w_k|$$

We continue the proof of Theorem 7. Since $F = H(0, \cdot)$, from the above lemmas it follows immediately that

$$\deg(F, \mathrm{int}\Pi, 0) = \deg(H(0, \cdot), \mathrm{int}\Pi, 0) = \deg(H(1, \cdot), \mathrm{int}\Pi, 0) \neq 0.$$

Hence there exists $x \in \Pi$ such that $F(x) = 0$. ∎

Proof of Lemma 8: From the homotopy property of the local Brouwer degree (see Appendix in [ZGi]) it is enough to prove that

$$H(\lambda, x) \neq 0, \quad \text{for all } x \in \partial(\Pi) \text{ and } \lambda \in [0,1]. \tag{2.19}$$

In order to prove (2.19), let us fix $x = (t, x_1, \ldots, x_{k-1}, k) \in \partial\Pi$. It is easy to see that one of the following conditions must be satisfied

$$\begin{aligned} t &\in \partial B_u(0,1), & (2.20)\\ z &\in \partial B_s(0,1), & (2.21)\\ x_i &\in N_i^+, \quad \text{for some } i = 1, \ldots, k-1, & (2.22)\\ x_i &\in N_i^- \quad \text{for some } i = 1, \ldots, k-1. & (2.23) \end{aligned}$$

We will deal with all above cases separately.

Consider first (2.20). Let us fix $\lambda \in [0,1]$. Let $x_0 = h_t(\lambda, t)$. From Def. 6 it follows that $x_0 \in N_0^-$. There are now two possibilities: either $\mathrm{dir}(1) = 1$ (direct covering) or $\mathrm{dir}(1) = 0$ (backcovering). Assume that $\mathrm{dir}(1) = 1$. From condition (2.12) it follows that $h_1(\lambda, x_0) \notin N_1$, hence $H_1(\lambda, x) \neq 0$. Assume now that $\mathrm{dir}(1) = 0$. We have $h_t(\lambda, t) \in N_0^-$ and since by (2.17) $h_1(\lambda, N_1) \cap N_0^- = \emptyset$, hence $H_1(\lambda, x) \neq 0$.

Consider now (2.21). Let us fix $\lambda \in [0,1]$ and let $x_k = h_z(\lambda, z)$. From Def. 7 it follows that $x_k \in N_k^+$. Now if $\mathrm{dir}(k) = 1$, then from condition (2.13) $h_k(\lambda, N_{k-1}) \cap N_k^+ = \emptyset$, hence $H_k(\lambda, x) \neq 0$. If $\mathrm{dir}(k) = 0$, then from (2.16) if follows, that $h_k(\lambda, x_k) \notin N_{k-1}$, hence $H_k(\lambda, x) \neq 0$.

For each of cases (2.22) and (2.23) we have to consider the following four possibilities

$$\begin{aligned} N_{i-1} &\overset{f_i}{\Longrightarrow} N_i \overset{f_{i+1}}{\Longrightarrow} N_{i+1}, & (2.24)\\ N_{i-1} &\overset{f_i}{\Longrightarrow} N_i \overset{f_{i+1}}{\Longleftarrow} N_{i+1}, & (2.25)\\ N_{i-1} &\overset{f_i}{\Longleftarrow} N_i \overset{f_{i+1}}{\Longrightarrow} N_{i+1}, & (2.26)\\ N_{i-1} &\overset{f_i}{\Longleftarrow} N_i \overset{f_{i+1}}{\Longleftarrow} N_{i+1}. & (2.27) \end{aligned}$$

Assume first that $x_i \in N_i^+$. If (2.24) or (2.25) holds true, then from (2.13) we obtain

$$h_i(\lambda, x_{i-1}) \neq x_i,$$

for every $\lambda \in [0,1]$ and every $x_{i-1} \in N_{i-1}$. If (2.26) or (2.27) is satisfied, then from (2.16) it results that

$$h_i(\lambda, x_i) \neq x_{i-1},$$

for every $\lambda \in [0,1]$ and every $x_{i-1} \in N_{i-1}$. This proves that, if $x_i \in N_i^+$, then $H(\lambda, x) \neq 0$ for any $\lambda \in [0,1]$.

Assume now that $x_i \in N_i^-$. If (2.24) or (2.26) holds true, then from (2.12) it follows that for every $\lambda \in [0,1]$ and every $x_{i+1} \in N_{i+1}$ we have

$$h_{i+1}(\lambda, x_i) \neq x_{i+1}.$$

If (2.25) or (2.27) is satisfied, then from (2.17) we obtain

$$h_{i+1}(\lambda, x_{i+1}) \neq x_i,$$

for every $\lambda \in [0,1]$ and every $x_{i+1} \in N_{i+1}$. This proves that if $x_i \in N_i^-$, then $H(\lambda, x) \neq 0$ for any $\lambda \in [0,1]$. ∎

Proof of Lemma 9: Let us represent x_i for $i = 1, \ldots, k-1$ as a pair $x_i = (p_i, q_i)$, where $p_i \in \mathbb{R}^u$ and $q_i \in \mathbb{R}^s$. In this representation the map $H(1, t, p_1, q_1, \ldots, p_{k-1}, q_{k-1}, z) = (\tilde{p}_1, \tilde{q}_1, \ldots, \tilde{p}_k, \tilde{q}_k)$ has the following form (for $\alpha = 0$)

- if $i = 2, \ldots, k-1$ then

$$\text{if } \mathrm{dir}(i) = 1, \text{ then} \quad \tilde{p}_i = (1-\alpha)p_i - A_i(p_{i-1}), \quad \tilde{q}_i = q_i, \tag{2.28}$$

$$\text{if } \mathrm{dir}(i) = 0, \text{ then} \quad \tilde{p}_i = p_{i-1}, \quad \tilde{q}_i = (1-\alpha)q_{i-1} - A_i(q_i) \tag{2.29}$$

- if $i = 1$, then

$$\text{if } \mathrm{dir}(1) = 1, \text{ then} \quad \tilde{p}_1 = (1-\alpha)p_1 - A_1(t), \quad \tilde{q}_1 = q_1 \tag{2.30}$$

$$\text{if } \mathrm{dir}(1) = 0, \text{ then} \quad \tilde{p}_1 = t, \quad \tilde{q}_1 = -A_1(q_1) \tag{2.31}$$

- if $i = k$, then

$$\text{if } \mathrm{dir}(k) = 1, \text{ then} \quad \tilde{p}_k = -A_k(p_{k-1}), \quad \tilde{q}_k = z \tag{2.32}$$

$$\text{if } \mathrm{dir}(k) = 0, \text{ then} \quad \tilde{p}_k = p_{k-1}, \quad \tilde{q}_k = (1-\alpha)q_{k-1} - A_k(z) \tag{2.33}$$

The above equations define a homotopy $C : [0,1] \times \Pi \to \mathbb{R}^{(u+s)k}$. We will show that $\deg(C(\alpha, \cdot), \mathrm{int}\Pi, 0)$ is independent of α and then we compute the degree of $C(1, \cdot)$.

Lemma 10 *For any* $\alpha \in [0,1]$

$$\deg(C(\alpha, \cdot), \mathrm{int}\Pi, 0) = \deg(C(1, \cdot), \mathrm{int}\Pi, 0).$$

Proof: From the homotopy property of the local degree (see Appendix in [ZGi]), it follows that it is enough to prove that

$$C(\alpha, x) \neq 0, \quad \text{for all } x \in \partial\Pi \text{ and } \alpha \in [0,1]. \tag{2.34}$$

Let us take $x = (t, p_1, q_1, \ldots, p_{k-1}, q_{k-1}, z) \in \partial\Pi$. One of the following conditions holds true

$$\begin{aligned} t &\in S^u, \\ z &\in S^s, \\ p_i &\in S^u, \quad \text{for some } i = 2, \ldots, k-1 \\ q_i &\in S^s, \quad \text{for some } i = 2, \ldots, k-1. \end{aligned}$$

Assume that $t \in S^u$. If $\mathrm{dir}(1) = 1$, then $\|A_1(t)\| > 1$, hence $\|\tilde{p}_1\| \geq \|A_1(t)\| - \|p_1\| > 0$. If $\mathrm{dir}(1) = 0$, then $\tilde{p}_1 = t \neq 0$.

Assume that $z \in S^s$. If $\mathrm{dir}(k) = 1$, then $\tilde{q}_k = z \neq 0$. If $\mathrm{dir}(k) = 0$, then $\|A_k(z)\| > 1$ and we obtain $\|\tilde{q}_k\| \geq \|A_k(z)\| - \|q_{k-1}\| > 0$.

Assume that $p_i \in S^u$. If $\mathrm{dir}(i+1) = 1$, then $\tilde{p}_{i+1} \neq 0$, because from condition (1.5) it follows that

$$\|A_{i+1}(p_i)\| > 1 \geq \|(1-\alpha)p_{i+1}\|, \tag{2.35}$$

for any $p_{i+1} \in \overline{B_u}(0,1)$.

If $\mathrm{dir}(i+1) = 0$, then obviously $\tilde{p}_{i+1} = p_i \neq 0$.

The argument for the case $q_i \in S^s$ is similar. ∎

Now we turn to the computation of the degree of $C(1,\cdot)$. Observe that $C(1,\cdot)$ has the following form: for $i = 2,\ldots,k-1$

$$\tilde{p}_i = -A_i(p_{i-1}),\ \tilde{q}_i = q_i \quad \text{if } \mathrm{dir}(i) = 1 \tag{2.36}$$

$$\tilde{p}_i = p_{i-1},\ \tilde{q}_i = -A_i(q_i) \quad \text{if } \mathrm{dir}(i) = 0, \tag{2.37}$$

for $i = 1$

$$\tilde{p}_1 = -A_1(t),\ \tilde{q}_1 = q_1 \quad \text{if } \mathrm{dir}(1) = 1 \tag{2.38}$$

$$\tilde{p}_1 = t,\ \tilde{q}_1 = -A_1(q_1) \quad \text{if } \mathrm{dir}(1) = 0, \tag{2.39}$$

and for $i = k$

$$\tilde{p}_k = -A_k(p_{k-1}),\ \tilde{q}_k = z \quad \text{if } \mathrm{dir}(1) = 1 \tag{2.40}$$

$$\tilde{p}_k = t,\ \tilde{q}_k = -A_k(z) \quad \text{if } \mathrm{dir}(1) = 0. \tag{2.41}$$

From the product property of the degree (see Appendix in [ZGi]) it follows that

$$|\deg(C(1,\cdot),\Pi,0)| = \left|\Pi_{i\in\mathrm{dir}^{-1}(1)} \deg(-A_i,\overline{B_u}(0,1),0)\cdot \Pi_{i\in\mathrm{dir}^{-1}(0)} \deg(-A_i,\overline{B_s}(0,1),0)\right|.$$

In the formula above if $\mathrm{dir}^{-1}(j) = \emptyset$ (for $j = 0,1$), then the corresponding product is set to be equal to 1. Similarly if $u = 0$ or $s = 0$, then the corresponding product is also set equal to 1.

From Collorary 18 in [ZGi] it follows that

$$\deg(-A,U,0) = (-1)^u \deg(A,U,0). \tag{2.42}$$

This finishes the proof. ∎

Observe that a little modification of the above proof allow to draw the following conclusion

Theorem 11 *Let N_0 be an h-set. Assume that b_0 is a horizontal disk in N_0 and b_e is a vertical disk in N_0.*

Then $|b_0| \cap |b_e| \neq \emptyset$.

Chapter 3

Cone conditions

The goal of this chapter is to introduce a method, which will allow to handle relatively easily the hyperbolic structure on h-sets. This material appeared first in [KWZ, ZCC].

3.1

Definition 9 *Let $N \subset \mathbb{R}^n$ be an h-set and $Q : \mathbb{R}^n \to \mathbb{R}$ be a quadratic form*

$$Q((x,y)) = \alpha(x) - \beta(y), \qquad (x,y) \in \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)}, \tag{3.1}$$

where $\alpha : \mathbb{R}^{u(N)} \to \mathbb{R}$, and $\beta : \mathbb{R}^{s(N)} \to \mathbb{R}$ are positive definite quadratic forms.

The pair (N,Q) we be called an h-set with cones*.*

We will refer to the quadratic forms α and β as positive and negative parts of Q, respectively.

If (N,Q) is an h-set with cones, then we define a function $L_N : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$L_N(z_1,z_2) = Q(c_N(z_1) - c_N(z_2)) \tag{3.2}$$

Quite often we will drop Q in the symbol (N,Q) and we will say that N is an h-set with cones.

3.1.1 Cone conditions for horizontal and vertical disks

Definition 10 *Let (N,Q) be a h-set with cones.*

Let $b : \overline{B_u} \to |N|$ be a horizontal disk.

We will say that b satisfies the cone condition (with respect to Q) *iff for any $x_1, x_2 \in \overline{B_u}$, $x_1 \neq x_2$ holds*

$$Q(b_c(x_1) - b_c(x_2)) > 0. \tag{3.3}$$

Definition 11 *Let* (N,Q) *be a h-set with cones.*

Let $b : \overline{B_s} \to |N|$ *be a vertical disk.*

We will say that b satisfies the cone condition (with respect to Q) *iff for any* $y_1, y_2 \in \overline{B_s}$, $y_1 \neq y_2$ *holds*

$$Q(b_c(y_1) - b_c(y_2)) < 0. \tag{3.4}$$

Lemma 12 *Let* (N,Q) *be a h-set with cones and let* $b : \overline{B_u} \to |N|$ *be a horizontal disk satisfying the cone condition.*

Then there exists a Lipschitz function $y : \overline{B_u} \to \overline{B_s}$ *such that*

$$b_c(x) = (x, y(x)). \tag{3.5}$$

Analogously, if $b : \overline{B_s} \to |N|$ *is a vertical disk satisfying the cone condition, then there exists a Lipschitz function* $x : \overline{B_s} \to \overline{B_u}$

$$b_c(y) = (x(y), y)). \tag{3.6}$$

Proof: We will prove only the first assertion, the proof of the other one is analogous.

In the first part of this proof we will show that for any $x \in \text{int} B_{u(N)}$ there exists $z \in \text{int} B_{u(N)}$ and $y_x \in \overline{B}_{s(N)}$, such that

$$b_c(z) = (x, y_x). \tag{3.7}$$

For this we will use the local Brouwer degree.

In the second part using the cone condition we will show that y_x is uniquely defined and its dependence on x is Lipschitz. Then we extend the definition of $y(x)$ to $x \in \partial B_u$.

Let h be the homotopy from the definition of the horizontal disk b.

To prove (3.7) consider the homotopy $\pi_1 \circ h : [0,1] \times \overline{B}_{u(N)} \to \overline{B}_{u(N)}$, where $\pi_1 : \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)} \to \mathbb{R}^{u(N)}$ is a projection on the first component. Let us fix $x \in \text{int} B_{u(N)}$. It is easy to see that, since $x \notin \pi_1 \circ h(t, \partial B_{u(N)})$ the local Brouwer degrees in the formula below are defined and the stated equalities are satisfied by the homotopy property of the local Brouwer degree

$$\deg(\pi_1 \circ b_c, \overline{B}_{u(N)}, x) = \deg(\pi_1 \circ h_1, \overline{B}_{u(N)}, x) = \deg(\text{Id}, \overline{B}_{u(N)}, x) = 1. \tag{3.8}$$

This proves (3.7).

To prove the uniqueness of y_x, assume that there exist $z_1, z_2 \in \text{int} B_{u(N)}$ and $y_1, y_2 \in \overline{B}_{s(N)}$, $y_1 \neq y_2$ such that

$$b_c(z_1) = (x, y_1), \qquad b_c(z_2) = (x, y_2). \tag{3.9}$$

From the cone condition for b it follows that

$$0 < Q(b_c(z_1) - b_c(z_2)) = \alpha(0) - \beta(y_1 - y_2) < 0 \tag{3.10}$$

which is a contradiction. Hence we have a well defined function

$$y(x) = y_x, \qquad \text{for } x \in \text{int}B_{u(N)}. \tag{3.11}$$

Observe that from the cone condition it follows that for any $x_1, x_2 \in \text{int}B_{u(N)}$, $x_1 \neq x_2$ holds

$$A\|x_1 - x_2\|^2 \geq \alpha(x_1 - x_2) > \beta(y(x_1) - y(x_2)) \geq B\|y(x_1) - y(x_2)\|^2, \tag{3.12}$$

where A, B are some positive constants related to quadratic forms α and β, respectively.

This proves the Lipschitz condition, which allows to continuously extend the function $y(x)$ to the boundary of $B_{u(N)}$. Observe that from the closeness of $|b|$ it follows that $(x, y(x)) \in |b|$ for $x \in \partial B_{u(N)}$. ∎

3.1.2 Cone conditions for maps

Definition 12 *Assume that $(N, Q_N), (M, Q_M)$ are h-sets with cones, such that $u(N) = u(M) = u$ and let $f : N \to \mathbb{R}^{\dim(M)}$ be continuous. Assume that $N \overset{f}{\Longrightarrow} M$. We say that f satisfies the cone condition (with respect to the pair (N, M)) iff for any $x_1, x_2 \in N_c$, $x_1 \neq x_2$ holds*

$$Q_M(f_c(x_1) - f_c(x_2)) > Q_N(x_1 - x_2). \tag{3.13}$$

Definition 13 *Assume that $(N, Q_N), (M, Q_M)$ are h-sets with cones, such that $u(N) = u(M) = u$ and $s = s(N) = s(M)$ and let $f : N \to \mathbb{R}^{u+s}$ be continuous. Assume that $N \overset{f}{\Longleftarrow} M$. We say that f satisfies the cone condition (with respect to the pair $((N, Q_N), (M, Q_M))$) iff for any $y_1, y_2 \in M_c$, $y_1 \neq y_2$ holds*

$$Q_M(y_1 - y_2) > Q_N(f_c^{-1}(y_1) - f^{-1}(y_2)). \tag{3.14}$$

Observe that Definition 13 is equivalent to Definition 12 applied to map f^{-1} with respect to pair $(M^T, -Q_M), (N^T, -Q_N))$.

The cone condition in Definition 12 is expressed in coordinates associated to h-sets, in the phase space it implies that

$$L_M(f(z_1), f(z_2)) > L_N(z_1, z_2), \qquad \text{for } z_1 \neq z_2,\ z_1, z_2 \in N. \tag{3.15}$$

Below we state and prove two basic theorems relating covering relations and the cone conditions

Theorem 13 *Assume that for $i = 0, \ldots, k-1$ either*

$$N_i \overset{f_i}{\Longrightarrow} N_{i+1} \tag{3.16}$$

or

$$N_{i+1} \subset \text{dom}(f_i^{-1}) \quad \textit{and} \quad N_i \overset{f_i}{\Longleftarrow} N_{i+1}, \tag{3.17}$$

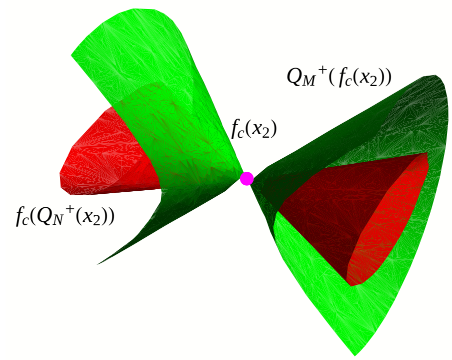

Figure 3.1: Example of cone condition for maps. In this case $u(M) = u(M) = 1$ and $s(N) = s(M) = 2$.

where all h-sets are h-sets with cones and f_i for $i = 0, \ldots, k-1$ satisfies the cone condition.

Assume that $b : \overline{B}_{s(N_k)} \to N_k$ is a vertical disk in N_k satisfying the cone condition.

Then the set of points $z \in N_0$ satisfying the following two conditions

$$f_{i-1} \circ f_{i-2} \circ \cdots \circ f_0(z) \in N_i, \qquad \text{for } i = 1, \ldots, k \tag{3.18}$$
$$f_{k-1} \circ \cdots \circ f_0(z) \in |b| \tag{3.19}$$

is a vertical disk satisfying the cone condition.

Proof: For the proof it is enough to consider the case of $k = 1$, only. For $k > 1$ the result follows by induction.

Without any loss of the generality we can assume that $N_0 = N_{0,c} = \overline{B}_{u(N_0)} \times \overline{B}_{s(N_0)}$, $N_1 = N_{1,c} = \overline{B}_{u(N_1)} \times \overline{B}_{s(N_1)}$, $f_0 = f_{0,c}$.

Consider a family of horizontal disks in N_0 $d_y : \overline{B}_{u(N_0)} \to N_0$ for $y \in \overline{B}_{s(N_0)}$

$$d_y(x) = (x, y). \tag{3.20}$$

From Theorem 7, applied to chain $N_0 \overset{f_0}{\Longrightarrow} N_1$ and disks d_y in N_0 and b in N_1 it follows that each $y \in \overline{B_{s(N_0)}}$ there exists $x \in \overline{B_{u(N_0)}}$, such that

$$f_0(x, y) \in |b|. \tag{3.21}$$

Let us fix $y \in \overline{B}_{s(N_0)}$. We will show that there exists only one x satisfying (3.21). For the proof assume the contrary, hence we have $x_1 \neq x_2$ and x_1, x_2 both satisfy (3.21).

Observe that $Q_{N_0}((x_1, y) - (x_2, y)) > 0$, hence from the fact that f_0 satisfies the cone condition it follows that

$$Q_{N_1}(f_0(x_1, y) - f_0(x_2, y))) > Q_{N_0}((x_1, y) - (x_2, y)) > 0. \tag{3.22}$$

But the above inequality is in a contradiction with the cone condition for b. Hence (3.21) defines a function $x(y)$ in a unique way.

It is easy to see that function $x(y)$ is continuous. Namely, from the compactness argument it follows that it is enough to prove that if we have a sequence of pairs (x_n, y_n), where $y_n \in \overline{B_s}$, $y_n \to \bar{y}$ for $n \to \infty$ and $x_n = x(y_n)$, $x_n \to \bar{x}$, then $f_0(\bar{x}, \bar{y}) \in |b|$, but this is an obvious consequence of the continuity of f_0 and the compactness of $|b|$.

Obviously, $b_0 : \overline{B_s} \to \overline{B_u} \times \overline{B_s}$ defined by $b_0(y) = (x(y), y)$ is a vertical disk in N_0. It remains to show that it satisfies the cone condition.

We will prove this by a contradiction. Assume that we have y_1 and y_2 such that

$$Q_{N_0}((x(y_1), y_1) - (x(y_2), y_2)) \geq 0, \tag{3.23}$$

then

$$Q_{N_1}(f_0(x(y_1), y_1) - f_0(x(y_2), y_2)) > 0, \tag{3.24}$$

hence the points $f_0(x(y_1), y_1)$ and $f_0(x(y_2), y_2)$ cannot both belong to b, because the cone condition is violated. ∎

Theorem 14 *Assume that for $i = 0, \dots, k-1$ either*

$$N_i \overset{f_i}{\Longrightarrow} N_{i+1} \tag{3.25}$$

or

$$N_{i+1} \subset \mathrm{dom}(f_i^{-1}) \quad \textit{and} \quad N_i \overset{f_i}{\Longleftarrow} N_{i+1}, \tag{3.26}$$

where all h-sets are h-sets with cones and f_i for $i = 0, \dots, k-1$ satisfies the cone condition.

Assume that $b : \overline{B}_{n(N_0)} \to N_0$ is a horizontal disk in N_0 satisfying the cone condition.

Then exists a set $Z \subset |b|$, such that for all $z \in Z$ holds

$$f_{i-1} \circ f_{i-2} \circ \cdots \circ f_0(z) \quad \in \quad N_i, \qquad \textit{for } i = 1, \dots, k \tag{3.27}$$

and $f_{k-1} \circ f_{i-2} \circ \cdots \circ f_0(Z)$ a horizontal disk in N_k satisfying the cone condition.

Proof: It is enough consider $k = 1$. Consider first the case of $N_0 \overset{f_0}{\Longleftarrow} N_1$. By the definition we have $N_1^T \overset{f_0^{-1}}{\Longrightarrow} N_0^T$ and the statement follows directly from Theorem 13.

Consider now the case of direct covering $N_0 \overset{f_0}{\Longrightarrow} N_1$. Without any loss of the generality we can assume that $N_1 = N_{1,c} = \overline{B}_{u(N_1)} \times = \overline{B}_{s(N_1)}$. Then from the cone condition for this covering relation it follows that for all $z_1, z_2 \in f(N_0 \cap |b|)$, $z_1 \neq z_2$ holds

$$Q_{N_1}(z_1 - z_2) > 0. \tag{3.28}$$

This implies that for any $x \in \overline{B_{u(N_1)}}$ there exists at most one $y \in \mathbb{B}^{s(N_1)}$, such that $(x, y) \in f(N_0 \cap |b|) \cap N_1$. From Theorem 7 it follows that such $y = y(x)$ indeed exists. We define the horizontal disk by $x \mapsto (x, y(x))$. By (3.28) it satisfies the cone condition. ∎

3.1.3 Verification of cone conditions

Assume that (N, Q_N) and (M, Q_M) are h-sets with cones and a map $f : N \to \mathbb{R}^{\dim(M)}$ is C^1.

Observe that for $x_2 \to x_1$

$$Q_M(f_c(x_2) - f_c(x_1)) - Q_N(x_2 - x_1) \to 0. \tag{3.29}$$

Hence there is no chance that the cone condition can be verified rigorously on computer [N, KWZ], by direct evaluation in interval arithmetics of $Q_M(f(x_2) - f(x_1)) - Q_N(x_2 - x_1)$.

Our intention is to give a condition, which will imply the cone condition and will be verifiable on computer.

Definition 14 *Let $U \subset \mathbb{R}^n$ and let $g : U \to \mathbb{R}^n$ be a C^1 map. We define* the interval enclosure of $Dg(U)$ *by*

$$[Dg(U)] = \left\{ A \in \mathbb{R}^{n \times n} \mid \forall_{ij} A_{ij} \in \left[\inf_{x \in U} \frac{\partial g_i}{\partial x_j}(x), \sup_{x \in U} \frac{\partial g_i}{\partial x_j}(x) \right] \right\} \tag{3.30}$$

Let $[df_c(N_c)]$ be the interval enclosure of df_c on N_c. Observe that when $\dim(M) \neq \dim(N)$ this is not a square matrix.

Lemma 15 *Assume that for any $B \in [df_c(N_c)]$, the quadratic form*

$$V(x) = Q_M(Bx) - Q_N(x) \tag{3.31}$$

is positive definite, then for any $x_1, x_2 \in N_c$ such that $x_1 \neq x_2$ holds

$$Q_M(f_c(x_1) - f_c(x_2)) > Q_N(x_1 - x_2). \tag{3.32}$$

Proof: Let us fix x_1, x_2 in N_c. We have

$$f_c(x_2) - f_c(x_1) = \int_0^1 df_c(x_1 + t(x_2 - x_1))dt \cdot (x_2 - x_1). \tag{3.33}$$

Let $B = \int_0^1 df_c(x_1 + t(x_2 - x_1))dt$. Obviously $B \in [df_c]$. Hence

$$f_c(x_2) - f_c(x_1) = B(x_2 - x_1). \tag{3.34}$$

We have

$$Q_M(f_c(x_2) - f_c(x_1)) - Q_N(x_2 - x_1) =$$
$$Q_M(B(x_2 - x_1)) - Q_N(x_2 - x_1) = V(x_2 - x_1) > 0.$$

∎

In the light of the above lemma the verification of the cone conditions can be reduced to checking that the interval matrix corresponding to the quadratic form V for various choices of $B \in [df_c(N_c)]$ given by

$$V = [df_c(N_c)]^T Q_M [df_c(N_c)] - Q_N \tag{3.35}$$

is positive definite.

Observe that, since the set of positive definite matrices in an open subset of the set symmetric matrix, then if V given by (3.35) is positive definite, then there exist $0 < a < 1 < b$, such that

$$V = [df_c(N_c)]^T Q_M [df_c(N_c)] - (1 + \epsilon) Q_N \tag{3.36}$$

is positive definite for $1+\epsilon \in (a, b)$. In fact this implies the uniform hyperbolicity see [W3, Hru1, Hru2, LL]

Chapter 4

Unstable and stable manifolds for fixed points

The goal of this chapter is to prove the existence of stable and unstable manifolds for hyperbolic fixed point for maps. The proofs are topological and do not assume that the map under the consideration is invertible.

We proceed as follows. In Section 4.1 we prove general theorems about stable and unstable manifolds under assumption of the existence of self-covering (i.e. $N \overset{f}{\Longrightarrow} N$) satisfying the cone condition. This part does not require that the fixed (periodic) point is hyperbolic.

In Section 4.3 we will show that in the neighborhood of the hyperbolic fixed point of the map we can build an h-sets, which covers itself and the cone condition holds for this relation. Then from results of Section 4.1 we obtain the stable and unstable manifold theorems for the fixed point under consideration.

The material in this chapter comes from [KWZ, ZCC].

4.1 Stable and unstable manifolds trough covering relations

Definition 15 *Consider the map $f : X \supset \mathrm{dom}(f) \to X$.*

Let $x \in X$. Any sequence $\{x_k\}_{k\in I}$, where $I \subset \mathbb{Z}$ is a set containing 0 and for any $l_1 < l_2 < l_3$ in $\mathbb{Z}$ if $l_1, l_3 \in I$, then $l_2 \in I$, such that

$$x_0 = x, \qquad f(x_i) = x_{i+1}, \qquad \text{for } i, i+1 \in I \tag{4.1}$$

will be called an orbit through x. If $I = \mathbb{Z}_-$, then we will say that $\{x_k\}_{k\in I}$ is a full backward orbit through x.

Definition 16 *Let X be a topological space and let the map $f : X \supset \mathrm{dom}(f) \to X$ be continuous.*

Let $Z \subset \mathbb{R}^n$, $x_0 \in Z$, $Z \subset \mathrm{dom}(f)$. We define

$$\begin{aligned}
W^s_Z(z_0,f) &= \{z \mid \forall_{n\geq 0} f^n(z) \in Z, \quad \lim_{n\to\infty} f^n(z) = z_0\} \\
W^u_Z(z_0,f) &= \{z \mid \exists\ \{x_n\} \subset Z \text{ a full backward orbit through } z, \text{ such that} \\
&\qquad \lim_{n\to-\infty} x_n = z_0\} \\
W^s(z_0,f) &= \{z \mid \lim_{n\to\infty} f^n(z) = z_0\} \\
W^u(z_0,f) &= \{z \mid \exists\ \{x_n\} \text{ a full backward orbit through } z, \text{ such that} \\
&\qquad \lim_{n\to-\infty} x_n = z_0\} \\
Inv^+(Z,f) &= \{z \mid \forall_{n\geq 0} f^n(z) \in Z\} \\
Inv^-(Z,f) &= \{z \mid \exists\ \{x_n\} \subset Z \text{ a full backward orbit through } z\ \}
\end{aligned}$$

If f is known from the context, then we will usually drop it and use $W^s(z_0)$, $W^s_Z(z_0)$ etc instead.

Definition 17 *Let $f : \mathbb{R}^n \supset \mathrm{dom}(f) \to \mathbb{R}^n$ be a continuous map.*

Loop of covering relations (for f) *is collection of h-sets N_i, $i = 0,\ldots,k$, $N_k = N_0$ and covering relations, such that for each $i = 1,\ldots,k$ we have either*

$$N_{i-1} \overset{f}{\Longrightarrow} N_i \tag{4.2}$$

or

$$N_i \subset \mathrm{dom}(f^{-1}) \quad \text{and} \quad N_{i-1} \overset{f}{\Longleftarrow} N_i. \tag{4.3}$$

k will be called the length of the loop.

Let L be a loop of covering relations, if additionally N_i are h-sets with cones Q_i, such that $Q_k = Q_0$ and each covering relation in the loop L we assume the cone condition. In this situation we will say that L satisfies cone conditions*.*

The following notation will be used for loops of covering relations $L = (N_0, N_1, \ldots, N_{k-1})$.

Definition 18 *Let $L = (N_0, N_1, \ldots, N_{k-1})$ is a loop of covering relations for f. We define*

$$S_L = N_0 \cap f^{-1}(N_1) \cap \cdots \cap f^{-(k-2)}(N_{k-2}) \cap f^{-(k-1)}(N_{k-1}). \tag{4.4}$$

It is easy to see that $S_{(N_0,N_1,\ldots,N_{k-1})}$ consists of points in N_0, such that

$$f^i(x) \in N_i, \qquad \text{for } i = 1,\ldots,k-1. \tag{4.5}$$

Lemma 16 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map.*

Assume that $L = (N_0,\ldots,N_{k-1})$ is a loop of covering relations for f satisfying the cone conditions.

Then there exists a unique $z_0 \in S_L$, such that

$$\begin{aligned}
f^k(z_0) &= z_0, &(4.6)\\
Inv^+(S_L,f^k) &= W^s_{S_L}(z_0,f^k), &(4.7)\\
Inv^-(S_L,f^k) &= W^u_{S_L}(z_0,f^k). &(4.8)
\end{aligned}$$

Proof: The existence of z_0 satisfying (4.6) follows directly from Theorem 3. Let us fix one such z_0.

To prove (4.7) it is enough to show that, if $f^{lk}(z) \in S_L$ for all $l \geq \mathbb{N}$, then $\lim_{l\to\infty} f^{lk}(z) = z_0$.

From the cone conditions for the loop L it follows that the function $V(z) = Q_{N_0}(c_{N_0}(z) - c_{N_0}(z_0))$ is a Lapunov function on S_L for f^k, i.e. is increasing on nonconstant orbits of f^k in S_L. By the standard Lapunov function argument it is easy to show that $Inv(S_L, f^k) = \{z_0\}$ and $\lim_{l\to\infty} f^{lk}(z) = z_0$. This finishes the proof of (4.7).

To prove (4.8) it is enough to show, that any backward orbit for f^k in S_L, $\{x_k\}_{k\in\mathbb{Z}_-}$ converges to z_0. But this is true by the same Lapunov function argument as in the previous paragraph. ∎

Theorem 17 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map.*

Assume that $L = (N_0, \ldots, N_{k-1})$ is a loop of covering relations for f satisfying the cone conditions.

Then there exists a unique $z_0 \in S_L$, such that $W^s_{S_L}(z_0, f^k)$ is a vertical disk in N_0 satisfying the cone condition.

Therefore, if c_{N_0} is an affine map, then $W^s_{S_L}(z_0, f^k)$ can be represented as a graph of a Lipschitz function over the nominally stable space in N_0.

Proof: First we show that for all $y \in \overline{B_s}$ there exists $x \in \overline{B_u}$, such that

$$z = c^{-1}_{N_0}(x, y) \in W^s_{S_L}(z_0). \tag{4.9}$$

By Lemma 16 it is equivalent to showing that

$$f^{kl}(z) \in N_0, \quad \text{for } l \in \mathbb{N}. \tag{4.10}$$

Consider a family of horizontal disks in N_0 $d_y : \overline{B_{u(N_0)}} \to N$ for $y \in \overline{B_{s(N_0)}}$

$$d_y(x) = (x, y). \tag{4.11}$$

The proof is the same for both direct- and backcovering, therefore we will just consider direct covering.

Consider an infinite chain of covering relations consisting of replicas of loop L

$$N_0 \overset{f}{\Longrightarrow} N_1 \overset{f}{\Longrightarrow} \cdots \overset{f}{\Longrightarrow} N_{k-1} \overset{f}{\Longrightarrow} N_0 \overset{f}{\Longrightarrow} \cdots N_0 \overset{f}{\Longrightarrow} \cdots \tag{4.12}$$

From Theorem 7 applied to d_y, an arbitrary vertical disk b_v in N_0 and finite chains $N_0 \overset{f}{\Longrightarrow} N_1 \overset{f}{\Longrightarrow} N \overset{f}{\Longrightarrow} \cdots \overset{f}{\Longrightarrow} N_0$ of increasing length using the compactness argument one can show (see [W2, Col. 3.10]) that for every $y \in \overline{B_s}$ there exists $x \in \overline{B_u}$, such that (4.10) holds for $z = c^{-1}_{N_0}(x, y)$.

The next step is to prove that such x is unique. Let us assume the contrary, then there exists $y \in \overline{B_s}$ and $x_1, x_2 \in \overline{B_u}$, $x_1 \neq x_2$, such that $z_i = c^{-1}_{N_0}(x_i, y)$ for $i = 1, 2$ satisfies condition (4.10). Observe that

$$Q_{N_0}(c_{N_0}(z_1) - c_{N_1}(z_2)) = \alpha(x_1 - x_2) > 0, \tag{4.13}$$

hence from the cone condition and (4.10) it follows that

$$Q_{N_0}(c_{N_0}(f^{lk}(z_1)) - c_{N_0}(f^{lk}(z_2)) > \alpha(x_1 - x_2), \qquad \text{for } l \in \mathbb{N}. \tag{4.14}$$

Passing to the limit $l \to \infty$ we obtain

$$0 = Q_{N_0}(c_{N_0}(z_0) - c_{N_0}(z_0)) = \lim_{l\to\infty} Q_{N_0}(c_{N_0}(f^{lk}(z_1)) - c_{N_0}(f^{kl}(z_2))) > \alpha(x_1 - x_2) > 0.$$

This is a contradiction. Hence we have a well defined function $x(y)$ on $\overline{B_s}$.

Obviously $W^s_{S_L}(x_0, f^k) = \{c_N^{-1}(x(y), y) \mid y \in \overline{B}_s\}$. Now we prove the cone condition for $W^s_{S_L}(x_0, f^k)$. This will imply that the map $b : \overline{B}_s \to N$, given by $b(y) = c_N^{-1}(x(y), y)$ defines a vertical disk in N.

We have to check whether

$$Q_N(c_{N_0}(z_1) - c_{N_0}(z_2)) < 0, \qquad \text{for all } z_1, z_2 \in W^s_{S_L}(x_0, f^k),\ z_1 \neq z_2 \tag{4.15}$$

Assume that (4.15) is not satisfied for some $z_1, z_2 \in W^s_{S_L}(x_0, f^k)$, $z_1 \neq z_2$. We have

$$Q_{N_0}(c_{N_0}(z_1) - c_{N_0}(z_2)) \geq 0. \tag{4.16}$$

From the cone condition it follows that for $l > 1$ holds

$$Q_{N_0}(c_{N_0}(f^{lk}(z_1)) - c_{N_0}(f^{lk}(z_2))) > Q_{N_0}(c_{N_0}(f(z_1)) - c_{N_0}(f(z_2))) > 0.$$

Passing to the limit $l \to \infty$ we obtain

$$0 = Q_{N_0}(c_{N_0}(z_0) - c_{N_0}(z_0)) = \lim_{l\to\infty} Q_{N_0}(c_{N_0}(f^{kl}(z_1)) - c_{N_0}(f^{kl}(z_2))) > Q_{N_0}(c_{N_0}(f(z_1)) - c_{N_0}(f(z_2))) > 0.$$

Which is a contradiction. This proves (4.15). ∎

The following remark will be used, when we will tackle the question of the smoothness of the stable manifold for smooth maps

Remark 18 *The proof of above theorem suggests that function $x : \overline{B}_s \to \overline{B}_u$ used to parameterize $W^s_{s_L}(z_0, f^k)$ is a limit of functions $x_l : \overline{B}_s \to \overline{B}_u$ defined for $l = 1, 2, \ldots$ by implicit equation*

$$\pi_x \circ c_{N_0} \circ f^{lk} \circ c_{N_0}^{-1}(x_l(y), y) = 0. \tag{4.17}$$

and under the constraint

$$f^{ik} \circ c_{N_0}^{-1}(x_l(y), y) \in S_L, \qquad i = 0, \ldots, l-1. \tag{4.18}$$

∎

In [ZCC] we proceeded as the above remark suggests for analytic maps to the analyticity of W^s for f analytic and x_0 hyperbolic. In the second part of these notes devoted to normally hyperbolic invariant manifold the smoothness of W^s will established for C^2 maps.

Now we would like to prove the theorem about unstable manifolds. Observe that in the case of f being non-invertible we cannot apply previous theorem to f^{-1} to obtain statement about the unstable manifold, therefore we need a different proof. It will be based on the graph transform method by Hadamard.

Theorem 19 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map.*

Assume that $L = (N_0, \ldots, N_{k-1})$ is a loop of covering relations for f satisfying the cone conditions.

Then there exists a unique $z_0 \in S_L$, such that $W^u_{S_L}(z_0, f^k)$ is a horizontal disk in N_0 satisfying the cone condition.

Therefore, if c_{N_0} is an affine map, then $W^u_{S_L}(z_0, f^k)$ can be represented as a graph of a Lipschitz function over the nominally unstable space in N_0.

Proof: We will prove the theorem for the trivial loop $L = (N)$. The modifications necessary to consider loops of arbitrary length are rather obvious, see the proof of Theorem 17.

Without any loss of the generality we can assume that $N = \overline{B}_u \times \overline{B}_s$ and $c_N = id$.

We will prove that for any $x \in B_u$ there exists $y \in B_s$, such that $(x, y) \in W^u_N(x_0)$. For any $x \in \overline{B}_u$ let v_x be a vertical disk given by

$$v_x(y) = (x, y).$$

Let $h : \overline{B}_u \to \overline{B}_u \times \overline{B}_s$ be a horizontal disk given by $h(x) = (x, 0)$.

The proof is the same for both direct- and backcovering, therefore we will just consider direct covering.

Consider a chain of covering relations consisting of k replicas of $N \overset{f}{\Longrightarrow} N$. It follows from Theorem 7 it follows that there exists a finite orbit $\{w^k_{-k}, w^k_{-k+1}, \ldots, w^k_{-1}, w^k_0\}$, such that

$$\begin{aligned} w^k_{-k}, w^k_{-k+1}, \ldots, w^k_{-1}, w^k_0 &\in N \\ f(w^k_l) = w^k_{l+1}, &\qquad l = -k, \ldots, -1 \\ w^k_{-k} \in |h|, &\qquad w^k_0 \in |v_x|. \end{aligned}$$

By applying the diagonal argument we can find an infinite backward orbit $\{w_l\}_{l \in \mathbb{Z}_- \cup \{0\}}$, such that

$$w_l \in N, \qquad l = 0, -1, -2, \ldots \tag{4.19}$$

$$f(w_l) = w_{l-1}, \qquad l < 0 \tag{4.20}$$

$$w_0 \in |v_x|. \tag{4.21}$$

Since $V(z) = Q_N(z - z_0)$ is increasing on orbits for $z \neq z_0$ (is a Lapunov function), therefore

$$\lim_{l \to -\infty} w_l = z_0. \tag{4.22}$$

We have proved that

$$w_0 \in W^u_N(z_0) \cap |v_x|. \tag{4.23}$$

We will prove that w_0 in (4.23) is uniquely defined. Let p_0 also satisfies the above condition, hence there exists a backward orbit in N through p_0 $\{p_l\}_{l \in \mathbb{Z}_- \cup \{0\}}$. We have

$$Q_N(p_0 - w_0) = -\beta(y(p_0) - y(w_0)) < 0. \tag{4.24}$$

From the cone condition for map f it follows that the function $Q_N(p_l - w_l)$ is increasing for $l < 0$, hence

$$0 > Q_N(p_0 - w_0) > Q_N(p_l - w_l) > \lim_{l \to -\infty} Q_N(p_l - w_l) = Q_N(z_0 - z_0) = 0. \tag{4.25}$$

Which is a contradiction, therefore w_0 in (4.23) is uniquely defined.

We define a horizontal disk $d : \overline{B}_u \to \overline{B}_u \times \overline{B}_s$, by $d(x) = (x, w_0)$. From the above considerations it follows that

$$W^u_N(z_0) = |d|. \tag{4.26}$$

We will show that d is satisfy the cone condition (which also implies the continuity of d)

$$Q_N(w - p) > 0, \qquad \text{for all } w, p \in |d|,\ w \neq p. \tag{4.27}$$

Assume that (4.27) does not hold. Then there exists two full backward orbits $\{w_l\}, \{p_l\}$ in N through w and p and

$$Q_N(w - p) \leq 0. \tag{4.28}$$

We have for any $l \in \mathbb{Z}_-$

$$0 \geq Q_N(w_0 - p_0) > Q_N(w_l - p_l) > \lim_{l \to -\infty} Q_N(w_l - p_l) = Q_N(z_0 - z_0) = 0.$$

But this is a contradiction, hence (4.27) is satisfied. ∎

The following remark will be used, when we will tackle the question of the smoothness of the unstable manifold for smooth maps.

Remark 20 *The proof of above theorem suggests that function $y : \overline{B}_u \to \overline{B}_s$ used to parameterize $W^u_{S_L}(z_0, f^k)$ is as a limit of functions $y_l : \overline{B}_u \to \overline{B}_s$ defined for $l = 1, 2, \ldots$ by*

$$y_l(x) = \pi_y \circ c_{N_0} \circ f^{lk} \circ c_{N_0}^{-1}(x_l(x), 0), \tag{4.29}$$

where $x_l : \overline{B}_u \to \overline{B}_u$ *for* $l = 1, 2, \ldots$ *is defined by implicit equation*

$$\pi_x \circ c_{N_0} \circ f^{lk} \circ c_{N_0}^{-1}(x_l(x), 0) = x. \tag{4.30}$$

with the constraint

$$f^{ik} \circ c_{N_0}^{-1}(x_l(x), 0) \in S_L, \qquad i = 0, \ldots, l-1. \tag{4.31}$$

In [ZCC] we proceeded as the above remark suggests for analytic maps showed the analyticity of W^s for f analytic and x_0 hyperbolic. In the second part of these notes devoted to normally hyperbolic invariant manifold the smoothness of W^s will established for C^2 maps.

4.2 Example - the multidimensional horseshoe

Assume that (N_i, Q_i) for $i = 0, 1$ are h-sets with cones. Assume that the following covering relations hold together with cone conditions

$$N_i \overset{f}{\Longrightarrow} N_j, \qquad i, j = 0, 1. \tag{4.32}$$

From Theorems 17 and 19 it follows that for any $\sigma = (\sigma_0, \ldots, \sigma_{k-1}) \in \{0,1\}^k$ that there exists a unique periodic point $z_\sigma \in N_{\sigma_0}$, such that

$$f^i(z_\sigma) \in N_{\sigma_i}, i = 0, 1, \ldots, k-1 \qquad f^k(z_0) = z_0 \tag{4.33}$$

and local stable and unstable sets of z_σ for f^k are respectively vertical and horizontal disks in N_{σ_0}. We would like to stress here, that we have a uniform bounds for both stable and unstable manifolds for periodic orbits independent of the period, just as in the case of two-dimensional horseshoe.

As example let us consider for any $u, s \in \mathbb{N}_+$ the h-sets with cones $N_i \subset \mathbb{R}^{u+s}$, $i = 0, 1$, defined as follows. Let $u_i = u$, $s_i = s$, $p_i = ((-1)^i \cdot 2, 0, \ldots, 0)$ and $c_{N_i}^{-1}(x, y) = p_i + (x, y)$ for $(x, y) \in \mathbb{R}^u \times \mathbb{R}^s$. On N_i we define $Q_{N_i}(x, y) = x^2 - y^2$.

We define the map $f : N_0 \cup N_1 \to \mathbb{R}^{u+s}$ as follows

$$f(x, y) = (A_i(5 \cdot (x - \pi_x p_i)), 0) + p_i, \qquad \text{for } (x, y) \in N_i, \text{ for } i = 0, 1$$

where $A_i : \mathbb{R}^u \to \mathbb{R}^u$ are for $i = 0, 1$ arbitrary isometries (with respect to Euclidean metric).

Observe that $f_{|N_i}$ is a uniform expansion by the factor of 5 in the $\mathbb{R}^u \times \{0\}^s$ and retraction onto 0 in the stable direction. Observe that for any of the covering relations $N_i \overset{f}{\Longrightarrow} N_j$ the derivative is a constant linear map given by

$$df_c(x, y) = (5A_i x, 0). \tag{4.34}$$

From Lemma 15 it follows that the cone conditions will be satisfied if the matrix

$$\begin{bmatrix} 5A_i^T & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \cdot \begin{bmatrix} 5A_i & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} = \begin{bmatrix} 24I & 0 \\ 0 & I \end{bmatrix} \tag{4.35}$$

is positively defined, which is clearly the case.

Since both covering relations and being positive definite are stable with respect to small perturbations, then for sufficiently small in C^1-norm maps $h : N_0 \cup N_1 \to \mathbb{R}^{u+s}$ we obtain

$$N_i \overset{f+h}{\Longrightarrow} N_j, \qquad i,j = 0,1 \tag{4.36}$$

and for all these covering relations the cone conditions are satisfied. Therefore we obtain uniform bounds for (un)stable manifolds for infinite number of periodic orbits of unbounded periods.

4.3 Stable and unstable manifolds for hyperbolic fixed points

In this section we apply theorems proved in Section 4.1 to obtain the existence of the unstable and unstable manifold for hyperbolic fixed point. The result, concerning the smoothness, is rather weak, when compared to classical results in the literature, see [HPS, I70, I80, C] and references given there, as we have only the Lipschitz condition and a suitable tangency at the fixed point. The smoothness can be obtained from our results about smoothness of NHIM.

Definition 19 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be C^1. Let $z_0 \in \mathbb{R}^n$. We say that z_0 is a* hyperbolic fixed point for f *iff $f(z_0) = z_0$ and $Sp(Df(z_0)) \cap S^1 = \emptyset$, where $Df(z_0)$ is the derivative of f at z_0.*

Theorem 21 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map. Assume that z_0 is a hyperbolic fixed point of f.*

Let $Z \subset \mathbb{R}^n$ be an open set, such that $z_0 \in Z$.

Then there exists an h-set N with cones, such that $z_0 \in \mathrm{int} N$, $N \subset Z$ and

- *$N \overset{f}{\Longrightarrow} N$ and if f is a local diffeomorphism in the neighborhood of z_0 then $N \overset{f}{\Longleftarrow} N$,*
- *$W^u_N(z_0)$ is a horizontal disk in N satisfying the cone condition*
- *$W^s_N(z_0)$ is a vertical disk in N satisfying the cone condition.*

Moreover, $W^u_N(z_0)$ can be represented as a graph of a Lipschitz function over the unstable space for the linearization of f at z_0 and tangent to it at z_0. Analogous statement is also valid for $W^s_N(z_0)$.

Proof: Let L be a linearization of f at z_0, hence $L(z) = z_0 + df(z_0)(z - z_0)$. Let u be the dimension of the unstable manifold and s of the stable manifold of L at z_0.

Then there exists a coordinate system on $\mathbb{R}^n$ and a scalar product $(\cdot,\cdot)$ such that following holds

$$df(z_0) = \begin{bmatrix} A & 0 \\ 0 & U \end{bmatrix}, \tag{4.37}$$

where $A:\mathbb{R}^u \to \mathbb{R}^u$ and $U:\mathbb{R}^s \to \mathbb{R}^s$ are linear isomorphisms, such that

$$W^u(z_0,L)=\{z_0\}+\mathbb{R}^u\times\{0\}^s,\quad W^s(z_0,L)=\{z_0\}+\{0\}^u\times\mathbb{R}^s \tag{4.38}$$
$$\|Ax\|>\|x\|,\quad \text{for } x\in\mathbb{R}^u\setminus\{0\} \tag{4.39}$$
$$\|Uy\|<\|y\|,\quad \text{for } y\in\mathbb{R}^s\setminus\{0\}, \tag{4.40}$$

where the norms are $\|x\|=\sqrt{x^2}$ and $\|y\|=\sqrt{y^2}$. We will use these coordinates in our proof.

Observe that (4.39) and (4.40) imply that matrices A^TA-Id and $Id-U^TU$ are positive definite.

For any $r>0$ we define

$$N(r)=\{z_0\}+\overline{B}_u(0,r)\times\overline{B}_s(0,r). \tag{4.41}$$

Let $Q((x,y))=\alpha x^2-\beta y^2$, where $x\in\mathbb{R}^u$ and $y\in\mathbb{R}^s$ and $\alpha>0$, $\beta>0$ are arbitrary positive reals.

It was proved in [ZCC] that if r is small enough, then

$$N(r)\overset{f}{\Longrightarrow}N(r),$$
$$Q(f(z_1)-f(z_2))>Q(z_1-z_2),\qquad \text{for for all } z_1,z_2\in N(r_0),\ z_1\neq z_2$$

and if f is a local diffeomorphism at z_0, then $N\overset{f}{\Longleftarrow}N$.

From this and from Theorems 17 and 19 follow all assertions of our theorem, with the exception of the one the concerning the tangency to $W^{u,s}(z_0,L)$ at z_0,

To prove the tangency of $W^u(z_0,f)$ to $z_0+\mathbb{R}^u\times\{0\}^s$ at $z_0=(x_0,y_0)$ it is enough to prove that for any $\epsilon>0$, there exists $r>0$, such that for any $z=(x,y(x))\in W^u_{N(r)}(z_0,f)$ holds

$$\|y(x)-y_0\|\leq\epsilon\|x-x_0\|. \tag{4.42}$$

For given α,β the set $W^u_{N(r)}(z_0,f)$ for r sufficiently small is a horizontal disk satisfying the cone condition with respect to the quadratic form $Q(x,y)=\alpha x^2-\beta y^2$. Therefore we have

$$\begin{aligned} Q((x,y(x))-(x_0,y_0)) &> 0 \\ \beta\|y(x)-y_0\|^2 &< \alpha\,\|x-x_0\|^2 \\ \|y(x)-y_0\| &< \sqrt{\alpha/\beta}\,\|x-x_0\|, \end{aligned}$$

which proves (4.42).

The proof of the tangency for $W^s(z_0,f)$ to $z_0+\{0\}^u\times\mathbb{R}^s$ at z_0 is analogous.
∎

4.4 Propagation of stable and unstable manifolds of hyperbolic fixed points for a map

Assume that z_i, $i=0,1$ are a fixed (or periodic) points of the map $f:\mathbb{R}^n\to\mathbb{R}^n$ and that we have (N_i,Q_i) h-set with cones, such that $z_i\in N_i$ and $N_i\overset{f}{\Longrightarrow}N_j$.

Assume that we would like to show that $W^u(z_0, f)$ and $W^s(z_1, f)$ intersect transversally.

Theorems 17 and 19 give us information of pieces of $W^u(z_0, f)$ and $W^s(z_1, f)$ in terms of N_0 and N_1, respectively. Usually the sizes of N_0 and N_1 are relatively small and we need to be able to get information of much larger pieces of $W^u(z_0, f)$ and $W^s(z_1, f)$. Using the tools developed in previous sections this can be achieved as follows.

First we need some approximate heteroclinic orbit, i.e. a sequence of points $v_0, v_1, \dots, v_K$, such that $f(v_i) \approx v_{i+1}$, for $i = 0, \dots, K-1$ and v_0 close to z_0 and v_K is close to z_1. Next step is find h-sets with cones (M_i, Q_{M_i}) such that, $v_i \in \mathrm{int} M_i$, the following covering relations are satisfied together with cone conditions

$$N_0 \overset{f}{\Longrightarrow} N_0 \overset{f}{\Longrightarrow} M_0 \overset{f}{\Longrightarrow} M_1 \overset{f}{\Longrightarrow} \cdots \overset{f}{\Longrightarrow} M_K \overset{f}{\Longrightarrow} N_1 \overset{f}{\Longrightarrow} N_1 \tag{4.43}$$

From Theorems 17 and 13 it follows that $W^s(z_1, f) \cap N_0$ contains a vertical disk satisfying the cone condition. Since by Theorem 19 $W^u_{N_0}(z_0, f)$ is a horizontal disk in N_0 satisfying the cone conditions, therefore we obtain a transversal intersection of $W^s(z_1, f)$ and $W^u_{N_0}(z_0, f)$. In fact to talk about transversality we need at least structure of C^1-manifold on $W^s(z_1, f)$ and $W^u_{N_0}(z_0, f)$, which is not proved in this paper, but it is known for $f \in C^1$ from [I70, I80] (see also the part of these notes devoted to the smoothness of NHIM) and the cone conditions imply that then this intersection is indeed transversal.

The obvious question arises: how to find M_i's satisfying (4.43). Without the cone conditions this was discussed and successfully used in [AZ] on the example of Henon-Heiles hamiltonian, but we believe that the same discussion applies also to the cone condition.

4.5 Relation to the standard notion of hyperboolicity

The following theorem has been proved in [ZCC]

Theorem 22 *Assume that $A : \mathbb{R}^n \to \mathbb{R}^n$ is linear map and $Q : \mathbb{R}^n \to \mathbb{R}$ is a quadratic form of signature (n_+, n_-).*

Assume, that the quadratic form V given

$$V(x) = Q(Ax) - Q(x) \tag{4.44}$$

is positive definite.

Then Q is nondegenerate, A is hyperbolic and the following conditions are satisfied

$$n_+ = \dim W^u(A), \quad n_- = \dim W^s(A) \tag{4.45}$$

$$W^u(A) \subset C^+(Q), \qquad W^s(A) \subset C^-(Q) \tag{4.46}$$

4.6 Non-hyperbolic example

The goal of this section is to provide a simple example illustrating that our theorems from Section 4.1 to obtain stable and unstable manifolds for the fixed point, which has a nonhyperbolic linear part.

In this contexts one should mention here papers [BF, F] (and an earlier paper [Mc]), where under suitable assumptions the stable set of the fixed point has been proved, using the mixture of topological and analytic arguments in the phase space, to have a manifold structure, but the analytic part there (replacing our cone conditions expressed in terms of Lapunov function) is much more elaborate and subtle and leads to results in situations, where our approach may fail.

Consider the following map $f : \mathbb{R}^2 \to \mathbb{R}^2$

$$f(x,y) = (x + x^3, y - y^3) + P(x,y), \tag{4.47}$$

where $P(x,y)$ is a polynomial, such that the degree of all nonzero terms in P is at least 4.

Observe that $z_0 = (0,0)$ is a non-hyperbolic fixed point, but a look at the dominant terms $(x + x^3, y - y^3)$, suggests that nevertheless z_0 will have a one dimensional stable and unstable manifolds tangent at z_0 to the coordinate axes.

The following theorem has been proved in [ZCC]

Theorem 23 *Consider the map f given by (4.47).*

There exists an h-set N with cones, such that $z_0 \in \operatorname{int} N$, $N \subset Z$ and

- $N \overset{f}{\Longrightarrow} N$,
- $W^u_N(z_0)$ *is a horizontal disk in N satisfying the cone condition*
- $W^s_N(z_0)$ *is a vertical disk in N satisfying the cone condition.*

Moreover, $W^u_N(z_0)$ is at z_0 tangent to the line $y = 0$ and $W^s_N(z_0)$ is at z_0 tangent to the line $x = 0$.

4.7 The Lipschitz dependence of stable and unstable manifolds on parameters

Here we would like establish the Lipschitz dependence of the stable manifolds with respect to the parameters, with effective bounds of the Lipschitz constants, which we formulate as cone conditions. These bounds has been effectively used in the computer assisted proof of the existence of homoclinic tangency for the forced pendulum in [WZ1].

The reader may also treat it as an exercise in finding cones and a warm-up for the normally hyperbolic invariant manifold discussed in the second part of these notes.

We will be using the norms for quadratic forms (identified in the sequel with symmetric matrices) which are defined by

$$|B(u,v)| \leq \|B\|\|u\|\|v\|.$$

For Euclidian norm we have

$$\|B\| = \max\{|s| \mid s \text{ in an eigenvalue of } B\}.$$

Theorem 24 *Assume that (N,Q) is an h-set in $\mathbb{R}^{u+s}$ with cones and $f_\lambda \colon N \to \mathbb{R}^{u+s}$ with $\lambda \in C$, where C is a compact interval in the parameter space, is C^1 as the function on $C \times N$ and Q has the form $Q(x,y) = \alpha(x) - \beta(y) = \sum_{i=1}^{u} a_i x_i^2 - \sum_{i=1}^{s} a_{i+u} y_i^2$.*

1. *Assume that for the covering relation $N \overset{f_\lambda}{\Longrightarrow} N$ the cone condition is satisfied for all $\lambda \in C$. Let p_λ be the unique fixed point for f_λ in N.*

2. *Let $\epsilon > 0$ and $A > 0$ be such that for all $\lambda \in C$ and $z_1, z_2 \in N$ holds*

$$Q(f_\lambda(z_1) - f_\lambda(z_2)) - (1+\epsilon)Q(z_1 - z_2) \geq A(z_1 - z_2)^2. \tag{4.48}$$

3. *Let*

$$M = \max_{\lambda \in C, z \in N} \left(\sum_i |a_i| \left\| \frac{\partial \pi_{z_i} f_\lambda}{\partial z}(z) \right\| \cdot \left\| \frac{\partial \pi_{z_i} f_\lambda}{\partial \lambda}(z) \right\| \right), \tag{4.49}$$

$$L = \|\beta\| \cdot \max_{\lambda \in C, z \in N} \left\| \frac{\partial \pi_y f_\lambda}{\partial \lambda}(z) \right\|^2. \tag{4.50}$$

4. *Let $\Gamma > 0$ be such that*

$$A - 2M\Gamma - L\Gamma^2 > 0. \tag{4.51}$$

5. *We define*

$$\delta = \frac{\Gamma^2}{\|\alpha\|}. \tag{4.52}$$

Then the set $W^s_N(p_\lambda, f_\lambda)$ for $\lambda \in C$ can be parameterized as a vertical disk in $C \times N$ for the quadratic form $\tilde{Q}(\lambda, z) = \delta Q(z) - \lambda^2$, i.e. for any pair $z_i \in W^s_N(p_{\lambda_i}, f_{\lambda_i})$, $i = 1,2$, $(\lambda_1, z_1) \neq (\lambda_2, z_2)$ holds

$$\delta Q(z_1 - z_2) - (\lambda_1 - \lambda_2)^2 < 0. \tag{4.53}$$

Before the proof let us make two observations concerning constants A, ϵ, Γ.

Remark 25 *The existence of A and ϵ in (4.48) is a consequence of the cone condition for covering relation $N \overset{f_\lambda}{\Longrightarrow} N$. We would like to have as big A as possible. This forces $\epsilon \to 0$, but ϵ is not used in the sequel.*

Remark 26 *Since $A > 0$, therefore Γ in (4.51) always exists, but it is desirable to look for largest Γ possible.*

Proof of Theorem 24: We would like to obtain that for $|\lambda_1 - \lambda_2| \leq \Gamma \|z_1 - z_2\|$ holds

$$Q(f_{\lambda_1}(z_1) - f_{\lambda_2}(z_2)) > (1+\epsilon) Q(z_1 - z_2).$$

Let B be a unique symmetric form, such that $B(u,u) = Q(u)$. Observe that

$$\begin{aligned} Q(f_{\lambda_1}(z_1) - f_{\lambda_2}(z_2)) - (1+\epsilon)Q(z_1 - z_2) = \\ Q(f_{\lambda_1}(z_1) - f_{\lambda_1}(z_2)) - (1+\epsilon)Q(z_1 - z_2) + \\ 2B(f_{\lambda_1}(z_1) - f_{\lambda_1}(z_2), f_{\lambda_1}(z_2) - f_{\lambda_2}(z_2)) + Q(f_{\lambda_1}(z_2) - f_{\lambda_2}(z_2)). \end{aligned}$$

The first term in the above expression will be estimated using (4.48).

For the third term we obtain

$$\begin{aligned} Q(f_{\lambda_1}(z_2) - f_{\lambda_2}(z_2)) \geq -\beta(\pi_y(f_{\lambda_1}(z_2) - f_{\lambda_2}(z_2))) \geq \\ -\|\beta\| \cdot \max_{\lambda \in C} \left\| \frac{\partial \pi_y f_\lambda}{\partial \lambda}(z_2) \right\|^2 \cdot (\lambda_1 - \lambda_2)^2 \geq \\ -\|\beta\| \max_{\lambda \in C, z \in N} \left\| \frac{\partial \pi_y f_\lambda}{\partial \lambda}(z) \right\|^2 \cdot \Gamma^2 \|z_1 - z_2\|^2 = -L\Gamma^2 \|z_1 - z_2\|^2. \end{aligned}$$

Finally, for the second term we have

$$\begin{aligned} |B(f_{\lambda_1}(z_1) - f_{\lambda_1}(z_2), f_{\lambda_1}(z_2) - f_{\lambda_2}(z_2))| \leq \\ \max_{\lambda \in C, z \in N} \left(\sum_i |a_i| \left\| \frac{\partial \pi_{z_i} f_\lambda}{\partial z}(z) \right\| \cdot \left\| \frac{\partial \pi_{z_i} f_\lambda}{\partial \lambda}(z) \right\| \right) \cdot \Gamma \|z_1 - z_2\|^2. \end{aligned}$$

From the above computations and (4.49–4.50) we obtain the following

$$Q(f_{\lambda_1}(z_1) - f_{\lambda_2}(z_2)) - (1+\epsilon)Q(z_1 - z_2) \geq \left(A - 2M\Gamma - L\Gamma^2\right) \|z_1 - z_2\|^2.$$

Let us fix Γ and δ as in (4.51) and (4.52).

We have proved that, if $\|\lambda_1 - \lambda_2\| \leq \Gamma \|z_1 - z_2\|$, then

$$Q(f_{\lambda_1}(z_1) - f_{\lambda_2}(z_2)) > (1+\epsilon) Q(z_1 - z_2). \tag{4.54}$$

We would like to infer from (4.54) that

$$Q(f^n_{\lambda_1}(z_1) - f^n_{\lambda_2}(z_2)) > (1+\epsilon)^n Q(z_1 - z_2), \tag{4.55}$$

but the condition (4.54) does not imply that $\|\lambda_1 - \lambda_2\| \leq \Gamma \|f_{\lambda_1}(z_1) - f_{\lambda_2}(z_2)\|$, therefore we cannot iterate (4.54).

To fix this we will use a different condition. For $\delta > 0$ we define a set $G(\delta)$ by

$$G(\delta) = \{((\lambda_1, z_1), (\lambda_2, z_2)) \in (C \times N)^2 \mid \|\lambda_1 - \lambda_2\|^2 \leq \delta Q(z_1 - z_2)\}.$$

From the cone condition it follows immediately that if $((\lambda_1, z_1), (\lambda_2, z_2)) \in G(\delta)$, then if $f_{\lambda_i}(z_i) \in N$ for $i = 1, 2$, then $((\lambda_1, f_{\lambda_1}(z_1)), (\lambda_2, f_{\lambda_2}(z_2))) \in G(\delta)$

Observe that if $((\lambda_1, z_1), (\lambda_2, z_2)) \in G(\delta)$, then

$$\|\lambda_1 - \lambda_2\|^2 \leq \delta Q(z_1 - z_2) \leq \delta\alpha(x_1 - x_2) \leq \delta\|\alpha\| \cdot \|z_1 - z_2\|^2 = \Gamma^2\|z_1 - z_2\|^2.$$

This means that any pair in $G(\delta)$ satisfies $\|\lambda_1 - \lambda_2\| \leq \Gamma\|z_1 - z_2\|$.

Therefore, if $((\lambda_1, z_1), (\lambda_2, z_2)) \in G(\delta)$, we may iterate (4.55) as long as $f^k_{\lambda_i}(z_i) \in N$, for $i = 1, 2$, and we obtain

$$Q(f^k_{\lambda_1}(z_1) - f^k_{\lambda_2}(z_2)) > (1 + \epsilon)^k Q(z_1 - z_2)., \tag{4.56}$$

We are now ready to establish the cone condition from the assertion of our theorem. If $\lambda_1 = \lambda_2$, then it is satisfied by our Thm. 17.

Now we assume that $\lambda_1 \neq \lambda_2$. We will reason by the contradiction. Assume that $z_i \in W^s_N(p_{\lambda_i}, f_{\lambda_i})$, $i = 1, 2$ are such that

$$\|\lambda_1 - \lambda_2\|^2 \leq \delta Q(z_1 - z_2), \tag{4.57}$$

therefore $((\lambda_1, z_1), (\lambda_2, z_2)) \in G(\delta)$ and $x(\lambda_1, y) \neq x(\lambda_2, y)$. Observe that by the definition of z_i, $f^j_{\lambda_i}(z_i) \in N$ for all j positive. From (4.56) it follows that for all $j > 0$

$$Q(f^j_{\lambda_1}(z_1) - f^j_{\lambda_2}(z_2)) > (1+\epsilon)^j Q(z_1 - z_2) \geq (1+\epsilon)^j \alpha(x(\lambda_1, y) - x(\lambda_2, y)) \tag{4.58}$$

Let us consider the limit $j \to \infty$. We have

$$Q(f^j_{\lambda_1}(z_1) - f^j_{\lambda_2}(z_2)) \to Q(p(\lambda_1) - p(\lambda_2))$$
$$(1 + \epsilon)^j \alpha(x(\lambda_1, y) - x(\lambda_2, y)) \to \infty.$$

We obtain a contradiction. Therefore condition (4.53) is satisfied. ∎

Chapter 5

ODEs

5.1 The stable and unstable manifolds of hyperbolic fixed points for ODEs.

Consider an ordinary differential equation

$$z' = f(z), \qquad z \in \mathbb{R}^n, \quad f \in C^2(\mathbb{R}^n, \mathbb{R}^n). \tag{5.1}$$

Let us denote by $\varphi(t,p)$ the solution of (5.1) with the initial condition $z(0) = p$. For any $t \in \mathbb{R}$ by we define a map $\varphi(t,\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ by $\varphi(t,\cdot)(x) = \varphi(t,x)$. We ignore here the question whether $\varphi(t,x)$ is defined for every (t,x), but this can be achieved by modification of f outside a large ball.

Definition 20 *Let $z_0 \in \mathbb{R}^n$. We say that z_0 is a* hyperbolic fixed point for equation (5.1) *iff $f(z_0) = 0$ and $Re\lambda \neq 0$ for all $\lambda \in Sp(df(z_0))$, where $Df(z_0)$ is the derivative of f at z_0 and $Re\lambda$ is the real part of λ.*

Let $Z \subset \mathbb{R}^n$, $z_0 \in Z$. We define

$$\begin{aligned}
W^s_Z(z_0,\varphi) &= \{z \mid \forall_{t \geq 0} \varphi(t,z) \in Z, \quad \lim_{t\to\infty} \varphi(t,z) = z_0\} && (5.2)\\
W^u_Z(z_0,\varphi) &= \{z \mid \forall_{t \leq 0} \varphi(t,z) \in Z, \quad \lim_{t\to-\infty} \varphi(t,z) = z_0\} && (5.3)\\
W^s(z_0,\varphi) &= \{z \mid \lim_{t\to\infty} \varphi(t,z) = z_0\} && (5.4)\\
W^u(z_0,\varphi) &= \{z \mid \lim_{t\to-\infty} \varphi(t,x) = z_0\} && (5.5)\\
Inv^+(Z,\varphi) &= \{z \mid \forall_{t \geq 0} \varphi(t,z) \in Z\} && (5.6)\\
Inv^-(Z,\varphi) &= \{z \mid \forall_{t \leq 0} \varphi(t,z) \in Z\} && (5.7)
\end{aligned}$$

Sometimes, when φ is known from the context it will be dropped and we will write $W^s_Z(z_0), Inv^{\pm}(Z)$ etc.

Geometric interpretation may be found in the picture 5.1.

Below we recall the notion of isolating block from the Conley index theory.

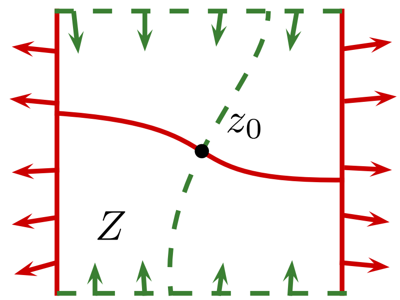

Figure 5.1: An isolating block Z for the planar ODE. The stable (vertical green dashed line) and unstable (horizontal red solid line) manifolds for φ inside Z are plotted, arrows indicate the vector field f, dashed green and solid red border lines indicate the δ-sections Σ^+ and Σ^- respectively.

Definition 21 *For $\delta > 0$ the set $\Sigma \subset \mathbb{R}^n$ is called a $\delta-$*section *for the flow φ iff $\varphi((-\delta,\delta),\Sigma)$ is an open set and the map $\sigma : \Sigma \times (-\delta,\delta) \to \varphi((-\delta,\delta),\Sigma)$ defined by $\sigma(x,t) = \varphi(t,x)$ is a homeomorphism.*

Let $B \subset \mathbb{R}^n$ be a compact set. B is called an isolating block *iff $\partial B = B^- \cup B^+$, where B^- and B^+ are closed sets, and there exists a $\delta > 0$ and two $\delta-$*sections*, Σ^+ and Σ^- such that*

$$\begin{aligned} B^+ \subset \Sigma^+, \quad & B^- \subset \Sigma^-, \\ \forall x \in B^+ \quad \forall t \in (-\delta, 0) \qquad & \varphi(t,x) \notin B \\ \forall x \in B^- \quad \forall t \in (0, \delta) \qquad & \varphi(t,x) \notin B. \end{aligned}$$

In the present paper we will use *h-sets* which are isolating blocks , which simply means that N^+ and N^- are sections of the vector field.

Definition 22 *Let N be an h-*set *in $\mathbb{R}^n$. We say that N is an isolating block for ODE* (5.1)*, iff N^- and N^+ are δ-sections for f as in Definition 21.*

Definition 23 *Let N be an h-*set*, such that c_N is an diffeomorphism. For a vector field f on $|N|$ we define a vector field on N_c by*

$$f_c(z) = Dc_N(c_N^{-1}(z))f(c_N^{-1}(z)). \tag{5.8}$$

Observe that f_c is in fact the vector field f expressed in new variables.

Definition 24 *[ZCC, Def. 13]*

Let $U \subset \mathbb{R}^n$ be such that $U = \overline{U}$ and int*$U \neq \emptyset$. Let $g : U \to \mathbb{R}^m$ be a C^1 function. We define the interval enclosure of $Dg(U)$ by*

$$[Dg(U)] := \left\{ A \in \mathbb{R}^{n\times m} : \forall_{i,j} A_{ij} \in \left[\inf_{x\in U} \frac{\partial g_i}{\partial x_j}, \sup_{x \in U} \frac{\partial g_i}{\partial x_j} \right] \right\} \tag{5.9}$$

We say that $[Dg(U)]$ is positive definite if for all $A \in [Dg(U)]$ the matrix A is positive definite.

The following two theorems about the existence and local properties of the (un)stable manifold for hyperbolic fixed points follow immediately from the proof of Theorem 26 in [ZCC].

Theorem 27 *Let $n = u + s$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 function with z_0 a hyperbolic fixed point for f, such that*

$$Df(z_0) = \begin{bmatrix} A & 0 \\ 0 & U \end{bmatrix} \tag{5.10}$$

where $A \in \mathbb{R}^{u\times u}$, $U \in \mathbb{R}^{s\times s}$, such that $A + A^T$ is positive definite and $U + U^T$ is negative definite.

Then for any $\epsilon > 0$ and for any quadratic form $Q(x,y) = ax^2 - by^2$, $x \in \mathbb{R}^u, y \in \mathbb{R}^s, a > 0, b > 0$ there exists an h-set $N = z_0 + \overline{B}_u(0,r) \times \overline{B}_s(0,r) \subset B(z_0,\epsilon)$, such that N is an isolating block for $x' = f(x)$, $W^s_N(z_0,f)$ is a vertical disk in N satisfying cone condition and $W^u_N(z_0,f)$ is a horizontal disk in N satisfying cone condition.

Theorem 28 *Assume that (N,Q) is an h-set with cones, which is an isolating block for (5.1), c_N is a diffeomorphism and that the following cone condition is satisfied:*

$$\text{the matrix} \quad [Df_c(N_c)]^T Q + Q[Df_c(N_c)] \quad \text{is positive definite.} \tag{5.11}$$

Then there exists $z_0 \in N$, $f(z_0) = 0$ and such that $W^u_N(z_0)$ is a horizontal disk in N satisfying the cone condition and $W^s_N(z_0)$ is a vertical disk in N satisfying the cone condition.

The origin of (5.11) is explained by the following computations (we assume $c_N = id$). We have

$$\frac{d}{dt}Q(z_1(t) - z_2(t)))_{|t=0} =$$
$$(f(z_1) - f(z_2))^T Q(z_1 - z_2) + (z_1 - z_2)^T Q(f(z_1) - f(z_2)) =$$
$$(z_1 - z_2)^T J^T Q(z_1 - z_2) + (z_1 - z_2)^T QJ(z_1 - z_2)$$

where

$$J = J(z_1,z_2) = \int_0^1 df(z_1 + t(z_2 - z_1))dt \in [df(N)]^T Q + Q\,[df(N)]$$

Hence for $z_1(0) \neq z_2(0)$, $z_1(0), z_2(0) \in N$ holds

$$\frac{d}{dt}Q(z_1(t) - z_2(t)))_{|t=0} \in (z_1(0) - z_(0))^T \left([df(N)]^T Q + Q\,[df(N)]\right)(z_1(0) - z_(0)) > 0. \tag{5.12}$$

So we see that the time shift by $h > 0$ for h sufficiently small satisfies the cone condition.

Part II

Normally hyperbolic invariant manifolds

In this part we present tools which allow to establish the existence of the normally hyperbolic invariant manifold (NHIM) for a map. The main feature of our approach is that it is not perturbative, the existence of NHIM for some nearby system is not assumed.

The make the presentation more readable, we restrict ourselves to the case of the torus. The main reason for this is the presence of the global chart, given by the projection $p : \mathbb{R}^m \to \mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$. However, as the reader will see, the general manifold can be also treated in this way, as we make the conscious effort, to do not use this global chart and stick to the local charts from some good atlas.

Another aspect of our presentation is that we do not assume that the map under consideration is invertible. This makes some proofs more difficult, just as in the case of the hyperbolic fixed point, the center-unstable manifold and its fibration are obtained using the graph transforms, but the center-stable manifold and its fibration are obtained by some implicit equations. We believe that this might be useful in the context of dissipative ODE (or PDEs) where the backward iterates might be not defined or hard to estimate.

In Chapter 6 we prove some technical lemmas and give definitions, which allow us to state our main theorem on NHIMs contained in Chapter 7 in Section 7.1. At the first reading the reader might skip Chapter 6 and consult it, only when needed. Chapter 7 contains the main theorems and their proofs (without establishing the smoothness). In Chapter 8 we define some geometric tools which allow us to obtain the smoothness. Finally, in Chapter 9 we apply our method to some example.

The material presented is partially new and draws heavily from [CZ1, CZ2].

Reader may also notice that we did not state in this work any definition of normally hyperbolic invariant manifold. For us NHIM is the invariant manifold we obtain from Theorem 45 in Chapter 7 plus the smoothness if the assumptions from Theorem 46 are satisfied.

Let us compare shortly our approach to NHIM with the standard theory [HPS, BB, Wi]. When applied to the perturbations of NHIM, this is the subject of the standard approach, our results are weaker, we did'nt get the full smoothness and our assumptions might require a construction of some special coordinate system to be satisfied. In fact our approach depends on charts and finite differences, the assumptions are not expressed using the geometric objects like tangent spaces etc. The main strength of our approach is that we give explicit conditions to be verified to claim the existence of NHIM. This is in principle also possible with other approaches, but it appears to be very difficult to realize.

The proposed approach (in fact its variants) has been applied recently in papers by Capinski and his coworkers [C2, CS, CR, CZ1] to some skew products and the planar restricted three body problem.

Chapter 6

Preparatory chapter: technical lemmas and basic definitions

6.1 Some technical issues

6.1.1 Is a map a diffeomorphism onto its range?

In the sequel when considering the graph transforms of disks we will face the following question in various forms.

Let $U \subset \mathbb{R}^n$ be a bounded open set and $f : U \to \mathbb{R}^n$ be a C^1 function. What conditions to impose on f so that $f : U \to f(U)$ is a diffeomorphism?

Obviously the necessary condition is that for all $x \in U$ the Jacobian matrix $Df(x)$ is a linear isomorphism, but this solves the local question only.

To formulate a criterion which we will use several times in these notes we need to introduce the notion of interval enclosure of derivatives of f over a set.

Definition 25 *Let $f : \mathbb{R}^n \supset \mathrm{dom}(f) \to \mathbb{R}^k$ be a C^1 function.*

Let $Z \subset \mathbb{R}^n$, such that $Z \subset \mathrm{dom}(f)$. We define the interval enclosure of Df on Z, denoted by $[Df(Z)] \subset \mathbb{R}^{k\times n}$, as follows.

$M \in [Df(Z)]$ iff for $i = 1, \ldots, k$ and $j = 1, \ldots, n$ $M_{ij} \in \left[\inf_{z\in Z} \frac{\partial f_j}{\partial x_j}(z), \sup_{z\in Z} \frac{\partial f_j}{\partial x_j}(z)\right]$.

Lemma 29 *Let $f : \mathbb{R}^n \supset \mathrm{dom}(f) \to \mathbb{R}^k$ be a C^1 function.*

Let $U \subset \mathbb{R}^n$ be a convex set and $U \subset \mathrm{dom}(f)$. Then

$$f(x_2) - f(x_1) = J_f(x_2, x_1)(x_2 - x_1), where \tag{6.1}$$

$$J_f(x_2, x_1) = \int_0^1 Df(x_1 + t(x_2 - x_1))dt \in [Df(U)]\,. \tag{6.2}$$

The obvious proof is left to the reader as an exercise.

Now we are ready to state and prove a lemma which partially answers our question.

Lemma 30 *Let $f : \mathbb{R}^n \supset \mathrm{dom}(f) \to \mathbb{R}^n$ be a C^1 function.*

Let $U \subset \mathbb{R}^n$ be a convex set and $U \subset \mathrm{dom}(f)$. Assume that $[Df(U)]$ is contained in the set of linear isomorphisms, then $f_{|U}$ is an injection.

Proof: From the previous lemma we know that $f(x_2)-f(x_1) = J_f(x_2,x_1)(x_2-x_1)$, but since $J_f(x_2,x_1) \in [Df(U)]$ for any $x_1, x_2 \in U$ we see that $f_{|U}$ is an injection. ∎

6.1.2 What is in the image of a map?

Continuing our considerations from the previous section we would like to find a ball around $f(q)$ such that $B(f(q),r) \subset f(U)$.

In one-dimension situation it is easy to see that if $U = x + [-\delta, \delta]$, then $\overline{B}(f(x), \delta \cdot \inf_{z\in U} |f'(z)|) \subset f(U)$.

In higher dimension the answer is similar, we just need to formulate the notion that will replace the infimum of the derivative.

Definition 26 *Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. Let $\|x\|$ be any norm on $\mathbb{R}^n$, then we define $m(A)$ as the supremum (maximum) of L, such that for all $v \in \mathbb{R}^n$ holds $\|Av\| \geq L\|v\|$.*

If $Z \subset Lin(\mathbb{R}^n, \mathbb{R}^n)$, then we set $m(Z) = \inf_{A\in Z} m(A)$.

It is easy to see that $m(A) = 0$ iff A is singular. If A is not singular, then $m(A) = \frac{1}{\|A^{-1}\|}$.

The following lemma gives an answer to our question.

Lemma 31 *Let $g : \mathbb{R}^n \supset \mathrm{dom}(g) \to \mathbb{R}^n$ be a C^1 function.*

Let $x \in \mathrm{dom}(g)$ and $r > 0$ be such that $\overline{B}(x,r) \subset \mathrm{dom}(g)$. Assume that $m([Dg(\overline{B}(x,r))]) \geq \gamma > 0$, then

$$B(g(x), \gamma r) \subset g(\overline{B}(x,r)). \tag{6.3}$$

Observe that when $n = 1$, then Dg is just a number and $m(Dg(\overline{B}(x,r))) = \inf_{z\in[-r+x,x+r]} |g'(z)|$.

Proof: The proof of (6.3) uses the local Brouwer degree argument. Let us fix x_0. For any $p \in B(g(x_0), \gamma r)$ we consider equation

$$g(x) - p = 0, \quad x \in \overline{B}(x_0, r). \tag{6.4}$$

We imbed this equation into a one parameter family of equations

$$G_t(x) - p = 0, \quad t \in [0,1] \tag{6.5}$$

where

$$G_t(x) = (1-t)g(x) + t(g(x_0) + Dg(x_0)(x - x_0)). \tag{6.6}$$

Observe that for $t = 0$ we obtain equation (6.4).

To apply the local Brouwer degree to (6.5) we need to show that for $t \in [0, 1]$ and $x \in \partial B(x_0, r)$, $G_t(x) \neq p$.

We will show that

$$\|G_t(x) - g(x_0)\| \geq \gamma r, \quad \text{for } x \in \partial B(x_0, r). \tag{6.7}$$

For any $\theta \in \overline{B}(\theta_1, r)$ from Lemma 29 we have

$$G_t(x) = (1-t)(g(x_0) + J_g(x, x_0)(x - x_0)) + t(g(x_0) + Dg(x_0)(x - x_0)) = \\ g(x_0) + ((1-t)J_g(x, x_0) + tDg(x_0))(x - x_0)$$

From the above we obtain

$$\|G_t(x) - G_t(x_0)\| \geq m\left((1-t)J_g(x, x_0) + tDg(x_0)\right)\|x - x_0\|$$

Since $(1-t)J_g(x, x_0) + tDg(x_0)$ is contained in $[Dg(B(x_0, r))]$ we obtain

$$\|G_t(x) - G_t(x_0)\| \geq \gamma\|x - x_0\|.$$

This proves (6.7).

Now we compute the degree for $t = 1$. We have linear equation

$$g(x_0) + Dg(x_0)(x - x_0) = p. \tag{6.8}$$

By our assumptions matrix $Dg(x_0)$ is nonsingular and $m(Dg(x_0)) \geq \gamma$. If x solves (6.8) then

$$\gamma\|x - x_0\| \leq \|p - g(x_0)\|. \tag{6.9}$$

Hence for $p \in B(g(x_0), \gamma r)$ the solution $x \in B(x_0, r)$ and $\deg(G_1, B(x_0, r), 0) = \pm 1$. Therefore $\deg(G_0, B(x_0, r), 0) \neq 0$ and equation (6.4) has a solution in $B(x_0, r)$.

This finishes the proof of (6.3).

∎

6.1.3 Transversality for topological manifolds

In the differential geometry the classical notion of transversal intersection between two submaninfolds U and V of the manifold M requires that each point $z \in U \cap V$ holds $T_zU + T_zV = T_zM$, where T_zM is the tangent space at Z.

In these notes we will often work with topological manifolds embedded in some vector space and we want a notion of the transversality, which when applied to differentiable manifolds gives the standard notion.

Definition 27 *A cone in a real vector space is a set K such that $\lambda K \subset K$ for any $\lambda \in \mathbb{R}$.*

Definition 28 *Let M be a topological manifold embedded in some vector space V. For $p \in M$ and $U \subset V$ an open set such that $p \in U$. We define a cone at p in M in the restriction to U, denoted by $C^U_M(p)$, as the smallest closed cone containing all vectors of the form $q - p$ for $q \in M \cap U$.*

Let M_1 and M_2 be topological submanifolds of the vector space V. Let $p \in M_1 \cap M_2$. We say that manifolds M_1 and M_2 are (cone) transversal at p iff there exists an open set $U \subset V$, $p \in U$, such that any pair of linear subspaces $W_1 \subset C^U_{M_1}(p)$ and $W_2 \subset C^U_{M_2}(p)$ of maximum dimension holds

$$W_1 + W_2 = V. \tag{6.10}$$

It is easy to see that if M_1 and M_2 are differentiable manifolds, then they intersect transversally (in classical sense) at z iff their intersection is cone transversal. There is more similarities between these notions, the cone transversality persists under small perturbation, just as in the classical case.

In this work we will encounter the following two situations of (cone) transversality described in the following lemma.

Lemma 32 **T1** *Let $V = \mathbb{R}^u \times \mathbb{R}^s$. Let $Z_1 \subset \mathbb{R}^u$ and $Z_2 \subset \mathbb{R}^s$ be some open sets. Assume that two submanifolds of V are of the following form $M_1 = \{(x, f(x)) \mid x \in Z_1\}$, $M_2 = \{(g(y), y) \mid y \in Z_2\}$ satisfy the following cone conditions*

$$\|f(x_1) - f(x_2)\| \leq \alpha_1 \|x_1 - x_2\|, \tag{6.11}$$

$$\|g(y_1) - g(y_2))\| \leq \alpha_2 \|y_1 - y_2\|. \tag{6.12}$$

Let $z_0 \in M_1 \cap M_2$. If $\alpha_1 \alpha_2 < 1$, then the intersection of M_1 and M_2 at z_0 is cone transversal.

T2 *Let $V = \mathbb{R}^c \times \mathbb{R}^u \times \mathbb{R}^s$ be equipped with the euclidian norm. Let $Z_1 \subset \mathbb{R}^{c+u}$ and $Z_2 \subset \mathbb{R}^{c+s}$ be some open sets. Assume that two submanifolds of V are of the following form $M_1 = \{(\lambda, x, f(\lambda, x)) \mid (\lambda, x) \in Z_1\}$, $M_2 = \{(\lambda, g(\lambda, y), y) \mid (\lambda, y) \in Z_2\}$ and satisfy the cone conditions*

$$\|f(\lambda_1, x_1) - f(\lambda_2, x_2)\| \leq \alpha_1 \|(\lambda_1, x_1) - (\lambda_2, x_2)\|, \tag{6.13}$$

$$\|g(\lambda_1, y_1) - g(\lambda_2, y_2))\| \leq \alpha_2 \|(\lambda_1, y_1) - (\lambda_2, y_2)\|. \tag{6.14}$$

Assume that $z_0 \in M_1 \cap M_2$. If $\alpha_1 \alpha_2 < 1$, then the intersection of M_1 and M_2 at z_0 is cone transversal.

Proof: First we deal with *T1*. Observe that for any U we have

$$C^U_{M_1}(z_0) \subset C_1 = \{(x, y) \mid \|y\| \leq \alpha_1 \|x\|\} \tag{6.15}$$

$$C^U_{M_2}(z_0) \subset C_2 = \{(x, y) \mid \|x\| \leq \alpha_2 \|y\|\}. \tag{6.16}$$

First we show that for any linear subspaces $W_1 \subset C_1$ and $W_2 \subset C_2$ the intersection $W_1 \cap W_2$ is equal to $\{0\}$. Indeed, if $(x, y) \in W_1 \cap W_2$, then

$$\|y\| \leq \alpha_1 \|x\|, \quad \|x\| \leq \alpha_2 \|y\|. \tag{6.17}$$

We multiply the above inequalites by sides to obtain

$$\|y\| \cdot \|x\| \leq \alpha_1\alpha_2\|y\| \cdot \|x\|.$$

Since $\alpha_1\alpha_2 < 1$, we obtain $x = 0$ and $y = 0$.

Observe that if W_1 and W_2 are of maximal dimension, then they are of the following form

$$\begin{aligned} W_1 &= \{(x, Ax) \mid x \in \mathbb{R}^u, \}, \quad A \in \mathbb{R}^{s\times u}, \quad \|A\| \leq \alpha_1\}, \\ W_2 &= \{(By, y) \mid y \in \mathbb{R}^s, \}, \quad B \in \mathbb{R}^{u\times s}, \quad \|B\| \leq \alpha_2\}. \end{aligned}$$

We see that $\dim W_1 = u$ and $\dim W_2 = s$. Since $\dim W_1 \cap W_2 = 0$ we see that $\dim(W_1 + W_2) = n$, therefore

$$W_1 + W_2 = \mathbb{R}^n.$$

This finishes the proof of *T1*.

Now we consider case *T2*. For any U we have

$$C^U_{V_1}(z_0) \subset C_1 = \{(\lambda, x, y) \mid \|y\| \leq \alpha_1\|(\lambda, x)\|\} \tag{6.18}$$

$$C^U_{V_2}(z_0) \subset C_2 = \{(\lambda, x, y) \mid \|x\| \leq \alpha_2\|(\lambda, y)\|\}. \tag{6.19}$$

Observe that the linear subspaces of maximum dimension in C_1 are of the form $W_1 = \{(\lambda, x, A \cdot (\lambda, x)^t \mid (\lambda, x) \in \mathbb{R}^{c+u}\}$, where matrix $A \in \mathbb{R}^{s\times(c+u)}$ satisfies

$$\|A\| \leq \alpha_1.$$

Analogously, the linear subspaces of maximum dimension in C_2 are of the form $W_2 = \{(\lambda, B \cdot (\lambda, y), y)^t \mid (\lambda, y) \in \mathbb{R}^{c+s}\}$, where matrix $B_1 \in \mathbb{R}^{u\times(c+s)}$ satisfies

$$\|B\| \leq \alpha_2.$$

Let us fix $W_1 \subset C_1$ and $W_2 \subset C_2$ be linear subspaces of maximum dimension. We need to show that

$$W_1 + W_2 = \mathbb{R}^c \times \mathbb{R}^u \times \mathbb{R}^s. \tag{6.20}$$

If we set $\lambda = 0$ then from the considerations in the case *T1* it follows that

$$(W_1 \cap \{\lambda = 0\}) + (W_2 \cap \{\lambda = 0\}) = \{0\} \times \mathbb{R}^u \times \mathbb{R}^s. \tag{6.21}$$

Therefore we have

$$\{(\lambda, 0, A \cdot (\lambda, 0)^t) \mid \lambda \in \mathbb{R}^c\} + (W_1 \cap \{\lambda = 0\}) + (W_2 \cap \{\lambda = 0\}) = \mathbb{R}^c \times \mathbb{R}^u \times \mathbb{R}^s. \tag{6.22}$$

Since

$$\{(\lambda, 0, A \cdot (\lambda, 0)^t) \mid \lambda \in \mathbb{R}^c\} + (W_1 \cap \{\lambda = 0\}) + (W_2 \cap \{\lambda = 0\}) \subset W_1 + W_2,$$

from (6.22) we obtain (6.20). ∎

6.2 Coverings, cone conditions

The goal of this section is to set up the structure for our NHIM, which will be diffeomorphic with manifold Λ. To make this simple we will focus on Λ being a torus.

Let $\Lambda = \mathbb{T} = \mathbb{T}^m$ be an m-dimensional torus, i.e. $\mathbb{T}^m = (\mathbb{R}/\mathbb{Z})^m$. We consider a C^k map $f : \mathbb{T} \times \mathbb{R}^u \times \mathbb{R}^s \to \mathbb{T} \times \mathbb{R}^u \times \mathbb{R}^s$, where $k \geq 1$.

Let us denote by D the set

$$D = \Lambda \times \overline{B}_u(0, R) \times \overline{B}_s(0, R). \tag{6.23}$$

.

In the sequel we will also use $D^{\pm}$ defined by

$$D^- = \Lambda \times (\partial \overline{B}_u(0, R)) \times \overline{B}_s(0, R), \tag{6.24}$$
$$D^+ = \Lambda \times \overline{B}_u(0, R) \times (\partial \overline{B}_s(0, R)). \tag{6.25}$$

If $Z \subset \Lambda$ we define

$$D(Z) = Z \times \overline{B}_u(0, R) \times \overline{B}_s(0, R), \tag{6.26}$$
$$D(Z)^- = Z \times (\partial \overline{B}_u(0, R)) \times \overline{B}_s(0, R), \tag{6.27}$$
$$D(Z)^+ = Z \times \overline{B}_u(0, R) \times (\partial \overline{B}_s(0, R)). \tag{6.28}$$

In fact we will study $f : \mathbb{R}^m \times \mathbb{R}^u \times \mathbb{R}^s \to \mathbb{R}^m \times \mathbb{R}^u \times \mathbb{R}^s$ of class C^k and $\mathbb{Z}^m$-periodic in the first variable, i.e.

$$f(\lambda + l, x, y) = f(\lambda, x, y), \qquad l \in \mathbb{Z}^m. \tag{6.29}$$

In the sequel we will write f as (f_λ, f_x, f_y) , where $f_\lambda = \pi_\lambda f : \mathbb{R}^m \times \mathbb{R}^u \times \mathbb{R}^s \to \mathbb{R}^m$, $f_x = \pi_x f : \mathbb{R}^m \times \mathbb{R}^u \times \mathbb{R}^u \to \mathbb{R}^u$ and $f_y = \pi_y f : \mathbb{R}^m \times \mathbb{R}^u \times \mathbb{R}^u \to \mathbb{R}^s$

We assumed here that our manifold Λ is connected. Including the case of a manifold several connected components is trivial and occasionally we will mention what needs to be modified in our reasoning.

6.2.1 Assumptions about local charts

The goal of this section to formulate the structure Λ and D, which will allow to formulate our results on NHIM.

On $\mathbb{R}^m \times \mathbb{R}^u \times \mathbb{R}^s$ we will use the euclidian norm.

Definition 29 *Let $\alpha_v > 0$. We define a quadratic form $Q_v : \mathbb{R}^m \times \mathbb{R}^u \times \mathbb{R}^s \to \mathbb{R}$*

$$Q_v(\lambda, x, y) = -\lambda^2 - x^2 + \alpha_v^2 y^2 \tag{6.30}$$

Definition 30 *Let $\alpha_h > 0$. We define a quadratic form $Q_v : \mathbb{R}^m \times \mathbb{R}^u \times \mathbb{R}^s \to \mathbb{R}$*

$$Q_h(\lambda, x, y) = -\lambda^2 - y^2 + \alpha_h^2 x^2 \tag{6.31}$$

Definition 31 *For $z \in \mathbb{R}^{m+u+s}$ and $i \in \{v,h\}$ we define positive and negative cones by*

$$Q_i^+(z) = \{q \mid Q_i(z-q) > 0\} \tag{6.32}$$
$$Q_i^-(z) = \{q \mid Q_i(z-q) \leq 0\} \tag{6.33}$$

We require that **the compability condition between Q_h and Q_v**

$$\alpha_v \cdot \alpha_h > 1. \tag{6.34}$$

Lemma 33 $Q_v^-(z) \cap Q_h^-(z) \cap \{\lambda = \pi_\lambda z\} = \{z\}$

Proof: We can assume that $z = 0$. Then for $(0,x,y) \in Q_v^-(0) \cap Q_h^-(0)$ holds

$$-\|x\|^2 + \alpha_v^2 \|y\|^2 \leq 0, \quad \text{and} \quad -\|y\|^2 + \alpha_h^2 \|x\|^2 \leq 0,$$
$$\|x\|^2 \geq \alpha_v^2 \|y\|^2 \geq \alpha_v^2 \alpha_h^2 \|x\|^2$$

This in view of (6.34) implies that $x = y = 0$. ∎

In paper [CZ1] we introduced the notion of good atlas on Λ and generally everything there was quite cumbersome. But the basic principle was that for any $q \in D$ the positive cones $Q_v^+(q) \cap D$, $Q_h^+(q) \cap D$ are contained in the domain is some chart map from the good atlas. Observe that since we assumed that our phasespace is of the form $\Lambda \times \mathbb{R}^u \times \mathbb{R}^s$, we have the coordinates (x,y) given. In applications, this might might pose some problems the coordinate charts should be constructed for all variables.

In the simple situation Λ being the torus, we are in a very nice situation, namely we have the covering $p : \mathbb{R}^m \to \mathbb{T}^m = (\mathbb{R}/\mathbb{Z})^m$, which gives us the set of good charts being the restriction of p to some balls, such that $p : B_m(z,\delta) \to \mathbb{T}$ is a homeomorphism on its image.

Let R_Λ be such that for $q \in D$ $\pi_\lambda(Q_v^+(q) \cap D) \subset B_m(\pi_\lambda q, R_\lambda)$ and $\pi_\lambda(Q_h^+(q) \cap D) \subset B_m(\pi_\lambda q, R_\lambda)$, and $p_{|B(\lambda,R_\Lambda)}$ is a homeomorphism onto its image. Observe that this imposes the following conditions on $\alpha_{v,h}$, R and the 'size' of the torus D_Λ($D_\Lambda = 1$ in our setting)

$$R_\Lambda < D_\Lambda/2, \quad R_\Lambda \geq 2\alpha_v R, \quad R_\Lambda \geq 2\alpha_h R. \tag{6.35}$$

For the map f we want the following if U is a good chart then $f(U)$ is contained in some chart which is locally good. In the case of torus this assumption is implied by the following condition

$$\|\pi_\lambda(f(\lambda_1,x_1,y_1) - f(\lambda_2,x_2,y_2))\| < D_\Lambda/2, \quad \|\lambda_1 - \lambda_2\| \leq R_\Lambda. \tag{6.36}$$

Quite often we will use the sentence $z_1, z_2 \in D$ *belong to (are in) the same good chart* by this in our case we mean that $\pi_\lambda(z_1 - z_2) \leq 2R_\Lambda$, because then indeed $\pi_\lambda z_i \in \overline{B}_m(\pi_\lambda(z_i + z_2)/2, R_\Lambda)$.

We introduce the following notation for the set of points which are same good chart with a given point q,

$$P(q) = \{z \in D \mid \|\pi_\lambda z - \pi_\lambda q\| \leq 2R_\Lambda\}. \tag{6.37}$$

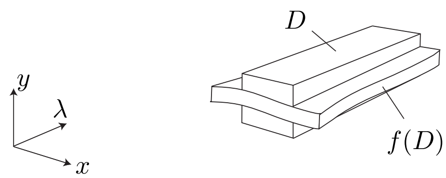

Figure 6.1: Forward covering relation.

We finish this section, with the definition formalizing the fact that we have the set of good charts in our situation.

Definition 32 *We will say that Q_v, Q_h, f satisfy the* compatibility conditions *if conditions* (6.34), (6.36) *are satisfied.*

It should be stressed that the conditions in the above definition are tailored the set $\mathbb{T}\times\mathbb{R}^u\times\mathbb{R}^s$. Possible way to approach this problem for general manifold Λ is discussed in [CZ1].

6.2.2 Covering

Now we introduce topological assumptions on f which we refer to as a "covering relation".

Definition 33 *We say that $f : D \to \Lambda\times\mathbb{R}^u\times\mathbb{R}^s$* satisfies the (forward) covering relation *if for any $\lambda^* \in \Lambda$ there exists a $\lambda^{**} \in \Lambda$, a homotopy $H : [0,1] \times D(\overline{B_m}(\lambda^*, R_\Lambda)) \to \Lambda \times \mathbb{R}^u \times \mathbb{R}^s$ and a linear map $A : \mathbb{R}^u \to \mathbb{R}^u$ such that*

$$
\begin{aligned}
H_0 &= f_{|D(\overline{B}_m(\lambda^*,r_\Lambda))}, &(6.38)\\
H([0,1], D(\overline{B}_m(\lambda^*, R_\Lambda))^-) \cap D &= \emptyset, &(6.39)\\
H([0,1], D(\overline{B}_m(\lambda^*, R_\Lambda)) \cap D^+ &= \emptyset, &(6.40)\\
H_1(\lambda, x, y) &= (\lambda^{**}, Ax, 0), &(6.41)\\
A\,(\partial B_u(0,R)) &\subset \mathbb{R}^u \setminus \overline{B}_u(0,R). &(6.42)
\end{aligned}
$$

In the sequel we will call condition (6.39) *the* exit condition *and* (6.40) the entry condition.

Intuitively, when f satisfies the covering relation then we have a topological expansion on the x coordinate and a topological contraction on the y coordinate (see Figures 6.1, 6.2).

Lemma 34 *Assume that f satisfies the covering relation on D.*
Then

$$f(D^-) \cap D = \emptyset, \tag{6.43}$$

$$\pi_y(f(D) \cap D) \subset B_s(0,R). \tag{6.44}$$

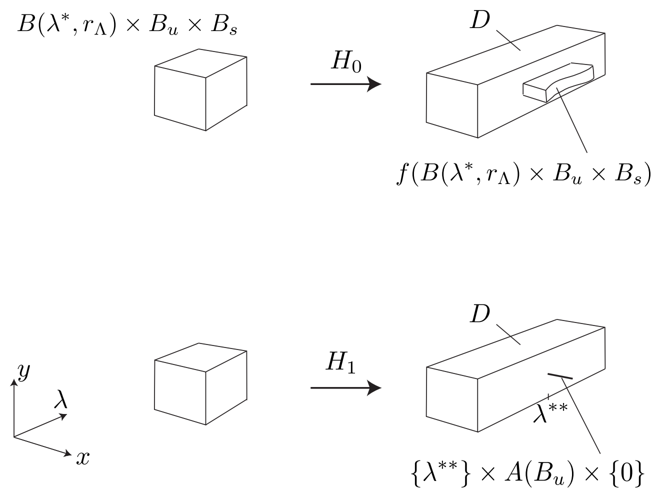

Figure 6.2: A homotopy for covering.

Proof: (6.43) is the direct consequence of the exit condition (6.39).

To prove (6.44) observe that from (6.38,6.40) it follows that

$$\pi_y \left(f(D(\overline{B}_m(\lambda^*, r_\Lambda))) \cap D \right) \subset B_s(0, R)$$

∎

6.2.3 Backward cone condition for map f

Definition 34 *We say that f satisfies the local backward cone condition, if for any $z_1 \neq z_2$, $\|\pi_\lambda z_1 - \pi_\lambda z_2\| < 2R_\Lambda$ (i.e. z_1 and z_2 are in the same good chart) holds:*

$$Q_v(f(z_1) - f(z_2)) < 0, \quad \textit{if} \quad Q_v(z_1 - z_2) \leq 0. \tag{6.45}$$

Definition 35 *We say that f satisfies* the backward cone condition *if the following condition holds:*
if $z_1 \neq z_2 \in D$ are such that $Q_v(f(z_1) - f(z_2)) \geq 0$, then $Q_v(z_1 - z_2) > 0$.

The following lemma is obvious.

Lemma 35 *If f satisfies the backward cone condition, then it satisfies the local backward cone condition.*

Let us define

$$\beta_v = \sup_{z \in D} \left(\alpha_v \left\| \frac{\partial f_y}{\partial(\lambda, x)}(z) \right\| + \left\| \frac{\partial f_y}{\partial y}(z) \right\| \right) \tag{6.46}$$

$$\eta_v = \inf_{z \in D} \left(m \left(\left[\frac{\partial f_{(\lambda,x)}}{\partial(\lambda, x)}(P(z)) \right] \right) - \frac{1}{\alpha_v} \sup_{z \in D} \left\| \frac{\partial f_{(\lambda,x)}}{\partial y}(z) \right\| \right) \tag{6.47}$$

In the sequel the following two conditions will play a crucial role

$$\beta_v < \eta_v \tag{6.48}$$
$$\beta_v < 1 \tag{6.49}$$

Lemma 36 *Let α_v, Q_v and f be such that (6.48) holds.*
Then f satisfies the local backward cone condition
If $\|\pi_\lambda(z_1) - \pi_\lambda(z_2)\| < 2R_\Lambda$ and $Q_v(z_1 - z_2) \leq 0$, then and

$$\|\pi_{(\lambda,x)}(f(z_1) - f(z_2))\| \geq \eta_v \|\pi_{(\lambda,x)} z_1 - \pi_{(\lambda,x)} z_2\|. \tag{6.50}$$

Proof: Let us denote by θ the pair (λ, x). We have

$$f(\theta_2, y_2) - f(\theta_1, y_1) = \int_0^1 Df(t(\theta_2 - \theta_1, y_2 - y_1) + (\theta_1, y_1))dt \cdot (\theta_2 - \theta_1, y_2 - y_1)^T$$

Hence for $Q_v((\theta_1, y_1) - (\theta_1, y_1)) \leq 0$ we have

$$\|\pi_\theta(f(\theta_2, y_2) - f(\theta_1, y_1))\| \geq$$
$$m\left(\left[\frac{\partial f_\theta}{\partial \theta}(P(\theta_1, y_1))\right]\right)\|\theta_2 - \theta_1\| - \sup_{z\in D}\left\|\frac{\partial f_\theta}{\partial y}(z)\right\| \cdot \|y_2 - y_1\| \geq$$
$$\left(m\left(\left[\frac{\partial f_\theta}{\partial \theta}\right]\right) - \frac{1}{\alpha_v}\left\|\frac{\partial f_\theta}{\partial y}\right\|\right) \cdot \|\theta_2 - \theta_1\| \geq \eta_v \|\theta_2 - \theta_1\|$$

and

$$\|\pi_y(f(\theta_2, y_2) - f(\theta_1, y_1))\| \leq \left\|\frac{\partial f_y}{\partial \theta}\right\| \cdot \|\theta_2 - \theta_1\| + \left\|\frac{\partial f_y}{\partial y}\right\| \cdot \|y_2 - y_1\| \leq$$
$$\left(\left\|\frac{\partial f_y}{\partial \theta}\right\| + \frac{1}{\alpha_v}\left\|\frac{\partial f_y}{\partial y}\right\|\right) \cdot \|\theta_2 - \theta_1\|$$

To verify the local backward cone condition, we want the following inequality to hold

$$\alpha_v \|\pi_y(f(\theta_1, y_1) - f(\theta_1, y_1))\| < \|\pi_\theta(f(\theta_1, y_1) - f(\theta_1, y_1))\|, \tag{6.51}$$

which leads to

$$\alpha_v\left(\left\|\frac{\partial f_y}{\partial \theta}\right\| + \frac{1}{\alpha_v}\left\|\frac{\partial f_y}{\partial y}\right\|\right) = \beta_v < m\left(\left[\frac{\partial f_\theta}{\partial \theta}\right]\right) - \frac{1}{\alpha_v}\left\|\frac{\partial f_\theta}{\partial y}\right\| = \eta_v. \tag{6.52}$$

∎

From Lemma 36 follows the following statement which we will also alleviate to the lemma status since it will be used frequently in the sequel.

Lemma 37 *If $z_1, z_2 \in D$ are such that $\|\pi_\lambda f^i(z_1) - \pi_\lambda f^i(z_2)\| < 2R_\Lambda$ for $i = 0, \ldots, k$ and $Q_v(z_1 - z_2) \leq 0$, then*

$$\|\pi_\theta(f^k(z_1) - f^k(z_2))\| \geq \eta_v^k \|\pi_\theta z_1 - \pi_\theta z_2\|. \tag{6.53}$$

6.2.4 Forward cone condition

Definition 36 *We say that f satisfies* the forward cone condition *if the following conditions holds:*
there exists $m_h > 1$, such that for $z_1 \neq z_2$, $z_1, z_2 \in D$, $Q_h(z_1 - z_2) > 0$ holds

$$Q_h(f(z_1) - f(z_2)) \quad > \quad Q_h(z_1 - z_2), \tag{6.54}$$
$$\|\pi_x f(z_1) - \pi_x f(z_2)\| \quad > \quad m_h \|\pi_x z_1 - \pi_x z_2\|. \tag{6.55}$$

Lemma 38 *Assume that f satisfies the forward cone condition, then $z_1, z_2 \in D$ are such that $f(z_1) \in D$, $f(z_2) \in D$, $\|\pi_\lambda f(z_1) - \pi_\lambda f(z_2)\| < 2R_\Lambda$ and $Q_h(f(z_1) - f(z_2)) \leq 0$, then either $\|\pi_\lambda z_1 - \pi_\lambda z_2\| \geq 2R_\Lambda$ or $Q_h(z_1 - z_2) < 0$.*

Let us define

$$\beta_h \quad = \quad \sup_{z \in D} \left(\left\| \frac{\partial f_{(\lambda,y)}}{\partial(\lambda, y)}(z) \right\| + \frac{1}{\alpha_h} \cdot \left\| \frac{\partial f_{(\lambda,y)}}{\partial x}(z) \right\| \right) \tag{6.56}$$
$$\eta_h \quad = \quad \inf_{z \in D} \left(m \left(\left[\frac{\partial f_x}{\partial x}(P(z)) \right] \right) - \alpha_h \sup_{z \in D} \left\| \frac{\partial f_x}{\partial(\lambda, y)}(z) \right\| \right) \tag{6.57}$$

In the sequel the following two conditions will play a crucial role

$$\eta_h > 1, \quad \eta_h > \beta_h \tag{6.58}$$

Lemma 39 *Assume* (6.58). *Then f satisfies the forward cone condition.*

Proof: We will denote be θ the pair of coordinates (λ, y).
Let $z_i = (\theta_i, x_i) \in D$, $i = 1, 2$. If $Q_h(z_1 - z_2) \geq 0$, then

$$\|\theta_1 - \theta_2\| \leq \alpha_h \|x_1 - x_2\|. \tag{6.59}$$

We have

$$f(\theta_2, x_2) - f(\theta_1, x_1) = \int_0^1 Df(t(\theta_2 - \theta_1, x_2 - x_1) + (\theta_1, x_1))dt \cdot (\theta_2 - \theta_1, x_2 - x_1)^T$$

Hence from (6.59) it follows that

$$\|\pi_x(f(\theta_2, x_2) - f(\theta_1, x_1))\| \geq \eta_h \|x_2 - x_1\|,$$

which proves (6.55) - the expansion in the x-direction, and

$$\|\pi_\theta(f(\theta_2, x_2) - f(\theta_1, x_1))\| \leq$$
$$\sup_{z \in D} \left\| \frac{\partial f_\theta}{\partial \theta}(z) \right\| \|\theta_2 - \theta_1\| + \sup_{z \in D} \left\| \frac{\partial f_\theta}{\partial x}(z) \right\| \cdot \|x_2 - x_1\| \leq$$
$$\left(\alpha_h \sup_{z \in D} \left\| \frac{\partial f_\theta}{\partial \theta}(z) \right\| + \sup_{z \in D} \left\| \frac{\partial f_\theta}{\partial x}(z) \right\| \right) \cdot \|x_2 - x_1\| = \alpha_h \beta_h \|x_2 - x_1\|$$

Now we check whether $Q_h(f(z_1) - f(z_2)) > 0$. We have

$$\alpha_h^2(\pi_x(f(z_1) - f(z_2)))^2 - (\pi_\theta(f(z_1) - f(z_2)))^2 \geq \\ \alpha_h^2\eta_h^2\|x_2 - x_1\|^2 - \alpha_h^2\beta_h^2\|x_2 - x_1\|^2 = \\ \left(\alpha_h^2(\eta_h^2 - \beta_h^2)\right)\|x_1 - x_2\| > 0$$

∎

Lemma 40 *Assume* (6.58).

If $z_1, z_2 \in D$ are in the same good chart and $Q_h(f(z_1) - f(z_2)) \leq 0$, then

$$\|\pi_{\lambda,y}(f(z_1) - f(z_2))\| \leq \beta_h\|\pi_{\lambda,y}(z_1 - z_2)\|. \tag{6.60}$$

Proof: We will denote by θ the pair of coordinates (λ, y).

From Lemma 39 we know that f satisfies the forward cone condition. This implies that if $Q_h(f(z_1) - f(z_2)) \leq 0$ then $Q_h(z_1 - z_2) < 0$. Hence

$$\alpha_h\|\pi_x(z_1 - z_2)\| < \|\pi_\theta(z_1 - z_2)\|. \tag{6.61}$$

Therefore we have

$$\|f_\theta(z_1) - f_\theta(z_2)\| \leq \sup_{z\in D}\left\|\frac{\partial f_\theta}{\partial\theta}(z)\right\| \cdot \|\pi_\theta(z_1 - z_2)\| + \\ \sup_{z\in D}\left\|\frac{\partial f_\theta}{\partial x}(z)\right\| \cdot \frac{1}{\alpha_h}\|\pi_\theta(z_1 - z_2)\| = \beta_h\|\pi_\theta(z_1 - z_2)\|$$

∎

Lemma 41 *Assume* (6.58).

If $z_1, z_2 \in D$ are in the same good chart, $f^j(z_1), f^j(z_2)$ are in good chart for $j = 1, 2, \ldots, k-1$ and $Q_h(f^k(z_1) - f^k(z_2)) \leq 0$, then

$$\|\pi_{\lambda,y}(f^k(z_1) - f^k(z_2))\| \leq \beta_h^k\|\pi_{\lambda,y}(z_1 - z_2)\|. \tag{6.62}$$

Proof: From Lemma 39 we know that f satisfies the forward cone condition. This and our assumptions imply that $Q_h(f^j(z_1) - f^j(z_2)) < 0$ for $j = 0, \ldots, k-1$. Hence from Lemma 40 we obtain

$$\|\pi_{\lambda,y}(f^k(z_1) - f^k(z_2))\| \leq \beta_h\|\pi_{\lambda,y}(f^{k-1}(z_1) - f^{k-1}(z_2))\| \leq \beta_h^k\|\pi_{\lambda,y}(z_1 - z_2)\|.$$

∎

6.3 Disks and their images

We assume that D, Q_h, Q_v is as in the previous section and we have a good atlas on Λ.

The goal of this section is to define several types of disks of in D and the investigation of how they behave under the application of the map satisfying the cone condition.

6.3.1 Definitions of various types of disks

Definition 37 *A continuous function* $b : \overline{B}_u(0,R) \to D$ *is called a* horizontal disk *in* D*, if* $\pi_x b(x) = x$*. We say that it satisfies the cone condition, if for any* $x_1 \neq x_2$*, such that* $x_i \in \overline{B}_u(0,R)$ *holds*

$$Q_h(b(x_1) - b(x_2)) > 0. \tag{6.63}$$

Observe that from our assumptions it follows immediately, that every horizontal disk b is contained in some good chart.

Definition 38 *A continuous function* $b : \overline{B}_s(0,R) \to D$ *is called a* vertical disk *in* D*, if* $\pi_y b(y) = y$*. We say that it satisfies the cone condition, if for any* $y_1 \neq y_2$*, such that* $y_i \in \overline{B}_s(0,R)$ *holds*

$$Q_v(b(x_1) - b(x_2)) > 0. \tag{6.64}$$

Observe that from our assumptions it follows immediately, that every vertical disk b is contained in some good chart.

Definition 39 *A continuous function* $b : \Lambda \times \overline{B}_u(0,R) \to D$ *is called a* center-horizontal disk *in* D*, if* $\pi_{(\lambda,x)} b(\lambda, x) = (\lambda, x)$*. We say that it satisfies the cone condition, if for any* $\theta_1 = (\lambda_1, x_1)$ *and* $\theta_2 = (\lambda_2, x_2)$*, such that* $\|\lambda_1 - \lambda_2\| < 2R_\Lambda$ *holds*

$$Q_v(b(\theta_1) - b(\theta_2)) \leq 0. \tag{6.65}$$

Definition 40 *A continuous function* $b : \Lambda \times \overline{B}_s(0,R) \to D$ *is called a* center-vertical disk *in* D*, if* $\pi_{(\lambda,y)} b(\lambda, y) = (\lambda, y)$*. We say that it satisfies the cone condition, if for any* $\theta_1 = (\lambda_1, y_1)$ *and* $\theta_2 = (\lambda_2, y_2)$*, such that* $\|\lambda_1 - \lambda_2\| < 2R_\Lambda$ *holds*

$$Q_h(b(\theta_1) - b(\theta_2)) \leq 0. \tag{6.66}$$

Sometimes, we will identify the disk with its support and say that $q \in b$, which means that there exists $p\mathrm{dom}(b)$ such that $b(p) = q$.

6.3.2 Lemmas on images of disks, the graph transforms

Lemma 42 *Let* $k \geq 1$ *and* f *be of class* C^k*. We assume that* f *satisfies the covering relation on* D *and* f *satisfies* (6.58) *(the forward cone condition).*

Let b *be a horizontal disk in* D *satisfying the cone condition. Then* $f \circ b(\overline{B}_u(0,R)) \cap D$ *is a horizontal disk in* D *satisfying the cone condition, such that*

$$\pi_y(f \circ b(\overline{B}_u(0,R)) \cap D) \subset \mathrm{int}\overline{B}_s(0,R). \tag{6.67}$$

If b *is* C^k*, then* $\left(f \circ b(\overline{B}_u(0,R))\right) \cap D$ *is of class* C^k*.*

Proof: From Lemma 39 it follows that f satisfies the forward cone condition.

First we show that

$$f \circ b(\overline{B}_u(0,R)) \cap D \neq \emptyset. \tag{6.68}$$

We prove that for every $x_e \in \overline{B}_u(0,R)$ there exists $x \in \overline{B}_u(0,R)$, such that

$$\pi_x f \circ b(x) = x_e. \tag{6.69}$$

Let $\lambda^* = \pi_\lambda b(0)$. Observe that from our assumptions follows that

$$\pi_\lambda b(x) \in B_m(\lambda^*, R_\Lambda). \tag{6.70}$$

We will use the homotopy from the covering relation. Let us denote this homotopy by H_t.

We imbed equation (6.69) into a one parameter family of equations

$$\pi_x H_t(b(x)) = x_e. \tag{6.71}$$

For $t = 0$ (6.71) becomes (6.69).

Observe from the definitions of the covering relation (Def. 33) and of the horizontal disk in D it follows that for $x \in \partial\overline{B}_u(0,R)$ $\pi_x H_t(b(x)) \neq x_e$. Therefore $\deg(\pi_x H_t \circ b, \mathrm{int}\overline{B}_u(0,R), x_e)$ is defined and by the homotopy property of the degree

$$\deg(\pi_x H_t \circ b, \mathrm{int}\overline{B}_u(0,R), x_e) = \deg(A, \mathrm{int}\overline{B}_u(0,R), x_e) = \mathrm{sgn}\det A = \pm 1 \tag{6.72}$$

where A the linear map from the definition of the homotopy.

Therefore we have a solution of (6.69). Now we will establish its uniqueness. This is an immediate consequence of the cone condition, namely $Q_h(f(b(x_1)) - f(b(x_2))) > 0$ implies that $f_x(b(x_1)) \neq f_x(b(x_2))$.

(6.67) is a direct consequence of Lemma 34.

It remains to prove that $\pi_x f \circ b$ is a diffeomorphism if b is C^1.

$$\|Df_x \circ b(x)\| \geq m\left(\frac{\partial f_x}{\partial x}(b(x))\right) - \left\|\frac{\partial f_x}{\partial(\lambda,y)}(b(x))\right\| \cdot \|Db(x)\| \geq$$
$$m\left(\frac{\partial f_x}{\partial x}(b(x))\right) - \alpha_h \left\|\frac{\partial f_x}{\partial(\lambda,y)}(b(x))\right\| \geq \eta_h > 0.$$

Hence for all $x \in \overline{B}_u(0,R)$ the matrix $Df_x \circ b(x)$ is nonsingular, hence $f_x \circ b$ is diffeomorphism onto its image.

∎

Definition 41 *Let a C^1 map b be a horizontal disk in D satisfying the cone condition. We will define* the graph transform *of b denoted by $\mathcal{G}_h(b)$ as the horizontal disk in D satisfying the cone condition obtained in Lemma 42 from the disk b*

Lemma 43 *Let $k \geq 1$ and f be of class C^k. We assume that f satisfies the covering relation on D, f satisfies* (6.58)*(the forward cone condition) and the backward cone condition on D.*

Let a C^k map b be a center-horizontal disk in D satisfying the cone condition. Then $f \circ b(\Lambda \times \overline{B}_u(0,R)) \cap D$ is a center horizontal disk in D satisfying the cone condition of class C^k and such that

$$\pi_y(f \circ b(\overline{B}_u(0,R)) \cap D) \subset \mathrm{int}\overline{B}_s(0,R). \tag{6.73}$$

Proof: We will perform the proof in the case when D is connected, the proof in a general case is a trivial modification of the present one.

First, we will prove that

$$\left(f \circ b(\Lambda \times \overline{B}_u(0,R))\right) \cap D \neq \emptyset. \tag{6.74}$$

Indeed, for any $\lambda \in \Lambda$ let us consider the horizontal disk $b^\lambda : \overline{B}_u(0,R) \to D$ given by $b^\lambda(x) = b(\lambda, x)$. We will argue it satisfies the cone condition. Namely, we know that for any $x_1 \neq x_2$ holds

$$0 \geq Q_v(b(\lambda,x_1) - b(\lambda,x_2)) = \alpha_v^2(\pi_y b^\lambda(x_1) - \pi_y b^\lambda(x_2))^2 - (x_1 - x_2)^2,$$

hence

$$(x_1 - x_2)^2 \geq \alpha_v^2(\pi_y b^\lambda(x_1) - \pi_y b^\lambda(x_2))^2.$$

From the above and Lemma 33 it follows that

$$Q_h(b^\lambda(x_1) - b^\lambda(x_2)) > 0, \tag{6.75}$$

which establishes the cone condition for b^λ as a horizontal disk in D.

From Lemma 42 it follows that $f \circ b^\lambda(\overline{B}_u(0,R)) \cap D$ is a horizonal disk in D, in particular this implies (6.74).

In the remainder of the proof we will use notation $\theta = (\lambda, x)$.

We will now show that $\pi_\theta f \circ b$ is an open map, in fact it is a local diffeomorphism. Namely, we have

$$D\pi_\theta f \circ b(\theta) = \frac{\partial f_\theta}{\partial \theta}(b(\theta)) + \frac{\partial f_\theta}{\partial y}(b(\theta))\frac{\partial b_y}{\partial \theta}(\theta). \tag{6.76}$$

Due to the fact that b satisfies the cone condition, we have

$$\left\| \frac{\partial b_y}{\partial \theta} \right\| \leq 1/\alpha_v. \tag{6.77}$$

Therefore we obtain

$$m(D\pi_\theta f \circ b(\theta)) \geq m\left(\frac{\partial f_\theta}{\partial \theta}(b(\theta))\right) - \frac{1}{\alpha_v}\left\|\frac{\partial f_\theta}{\partial y}(b(\theta))\right\| \geq \eta_v > 0. \tag{6.78}$$

Hence $\pi_\theta f \circ b$ is a local diffeomorphism, hence an open map. Therefore $\pi_\theta f \circ b(\Lambda \times \mathrm{int}\overline{B}_u(0,R))$ is an open set. From the covering relation and Lemma 34 we know that the points $b(\theta)$ for $\theta \in \Lambda \times \partial\overline{B}_u(0,R)$ are mapped by f out of the set D

$$\pi_\theta f \circ b(\Lambda \times \mathrm{int}\overline{B}_u(0,R)) \cap (\Lambda \times \overline{B}_u(0,R)) = \pi_\theta f \circ b(\Lambda \times \overline{B}_u(0,R)) \cap (\Lambda \times \overline{B}_u(0,R)). \tag{6.79}$$

Therefore the set $\pi_\theta f \circ b(\Lambda \times \mathrm{int}\overline{B}_u(0,R)) \cap (\Lambda \times \overline{B}_u(0,R))$ is both open and closed in $\Lambda \times \overline{B}_u(0,R)$ and since it is also nonempty, therefore we infer that

$$\pi_\theta f \circ b(\Lambda \times B_u(0,R)) \cap (\Lambda \times \overline{B}_u(0,R)) = \Lambda \times \overline{B}_u(0,R). \tag{6.80}$$

We need to show that the map $\pi_\theta f \circ b$ is an injection on $(\pi_\theta f \circ b)^{-1}(\Lambda \times \overline{B}_u(0,R))$. This is a direct consequence of the backward cone condition. To show this, assume that there exists $\theta_1 \neq \theta_2$ in $\Lambda \times \overline{B}_u(0,R)$ such that $\pi_\theta f(b(\theta_1)) = \pi_\theta f(b(\theta_2))$. Then $Q_v(f(b(\theta_1)) - f(b(\theta_2))) = \alpha_v(\pi_y f(b(\theta_1)) - \pi_y f(b(\theta_2)))^2 \geq 0$. Therefore the backward cone condition implies that $Q_v(b(\theta_1)-b(\theta_2)) > 0$, which contradicts the cone condition for center horizontal disks.

(6.73) is a direct consequence of Lemma 34.

∎

Definition 42 *Let b be a C^1 center horizontal disk satisfying the cone condition. The* graph transform *of b denoted by $\mathcal{G}_{ch}(b)$ we will be the center horizontal disk obtained in Lemma 43.*

Chapter 7

NHIM as Lipschitz manifolds

7.1 The main results on NIHM

In this section we gather together all assumptions, we define various invariant sets and state the main theorem.

Our assumptions will be divided into levels (layers). The first level will give us the existence plus the Lipschitz condition, the second level when added to the first one will give us the smoothness.

Definition 43 *The following assumptions will be referred to as* the standard assumptions

A1 *f is C^1, $\Lambda = \mathbb{T}^m = (\mathbb{R}/D_\Lambda\mathbb{Z})^m$*

A2 *there exist $\alpha_v > 0$ and $\alpha_h > 0$ such the compatibility conditions in the sense of Def. 32 are satisfied,*

BCC *f satisfies the backward cone condition*

CC *Let us define*

$$\beta_v = \sup_{z\in D}\left(\alpha_v\left\|\frac{\partial f_y}{\partial(\lambda,x)}(z)\right\| + \left\|\frac{\partial f_y}{\partial y}(z)\right\|\right) \tag{7.1}$$

$$\eta_v = \inf_{z\in D}\left(m\left(\left[\frac{\partial f_{(\lambda,x)}}{\partial(\lambda,x)}(P(z))\right]\right) - \frac{1}{\alpha_v}\sup_{z\in D}\left\|\frac{\partial f_{(\lambda,x)}}{\partial y}(z)\right\|\right) \tag{7.2}$$

$$\beta_h = \sup_{z\in D}\left(\left\|\frac{\partial f_{(\lambda,y)}}{\partial(\lambda,y)}(z)\right\| + \frac{1}{\alpha_h}\cdot\left\|\frac{\partial f_{(\lambda,y)}}{\partial x}(z)\right\|\right) \tag{7.3}$$

$$\eta_h = \inf_{z\in D}\left(m\left(\left[\frac{\partial f_x}{\partial x}(P(z))\right]\right) - \alpha_h\sup_{z\in D}\left\|\frac{\partial f_x}{\partial(\lambda,y)}(z)\right\|\right) \tag{7.4}$$

We assume that

$$\beta_v < \eta_v \quad (7.5)$$
$$\beta_v < 1 \quad (7.6)$$
$$\eta_h > 1, \quad \eta_h > \beta_h \quad (7.7)$$

CR *f satisfies the covering relation on D.*

Let us comment on the standard assumptions

Remark 44 *The conditions contained in **CC** imply the local backward cone condition and the forward cone condition.*

*Assumption **BCC** about the backward cone condition is of global character. In particular, it rules out that points $q_1, q_2 \in D$ such that $\pi_\lambda q_1$ and $\pi_\lambda q_2$ are not close can be mapped to the same fiber over Λ, i.e. $\pi_\lambda f(q_1) = \pi_\lambda f(q_2)$. This assumption is necessary, as the counter example take the Smale-Williams solenoid, it will satisfy all the assumptions on the expansion and contraction rates we are going to impose in the sequel, but the resulting invariant set is not a manifold.*

*Assumption **CR** in the case of Λ non connected, we change as follows. Let $\Lambda_0 \cup \Lambda_1 \cup \dots \Lambda_{l-1}$ be a decomposition into connected components. We require that the exists a loop of covering relations with $D_i = \Lambda_i \times \overline{B}_u(0,R) \times \overline{B}_s(0,R)$ covering $D_{(i+1) \mod l}$*

We give here definitions of our main heroes - the invariant sets and their (un)stable which will appear in sequel.

Definition 44 *Let us define the following sets*

- *$q \in W^{cu}$ iff $q \in D$ and there is full backward trajectory of q in D.*
- *$q \in W^{cs}$ iff $f^k(q) \in D$ for $k \in \mathbb{Z}_+$*
- *$\tilde{\Lambda} = W^{cu} \cap W^{cs}$. (This is our invariant set in D)*
- *let $q \in W^{cu}$. $z \in W^u_q$ iff $z \in W^{cu}$ and $Q_h(f^{-k}(z) - f^{-k}(q)) > 0$ for $k \in \mathbb{Z}_+$, where $f^{-i}(q)$ and $f^{-i}(z)$ are unique full backward trajectories in D through q and z, respectively.*
- *let $q \in W^{cs}$. $z \in W^s_q$ iff $z \in W^{cs}$ and $(Q_v(f^k(z) - f^k(q)) > 0$ or $f^k(z) = f^k(q))$ for $k \in \mathbb{Z}_+$*

The following theorem summarizes our first level results related to the invariant set in D and manifolds related to it.

Theorem 45 *Assume the standard assumptions. Then*

- *W^{cu} is a center horizontal disk in D satisfying the cone condition. Moreover, $f_{|W^{cu}}$ is an injection, which has continuous inverse.*

- *W^{cs} is a center vertical disk in D satisfying the cone condition.*
- *$\tilde{\Lambda}$ is a graph of the Lipschitz function over χ. By this we mean the following: there exists $\chi : \Lambda \to D$ and L such that $\pi_\lambda \chi(\lambda) = \lambda$ and if $\|\lambda_1 - \lambda_2\| < 2R_\Lambda$, then*

$$\|\pi_{x,y}(\chi(\lambda_1) - \chi(\lambda_2))\| \leq L\|\lambda_1 - \lambda_2\|. \tag{7.8}$$

- *for every $q \in W^{cu}$ the unstable fiber W^u_q has the following properties*
  - *W^u_q is a horizontal disk in D satisfying the cone condition.*
  - *W^u_q depends continuously on q*
  - *W^u_q intersects W^{cs} and $\tilde{\Lambda}$ in single point (the same point)*

$$W^u_q = \{z \in W^{cu} \mid \exists C > 0, \gamma \geq \eta_h \ \left\| f^{-k}_{|W^{cu}}(z) - f^{-k}_{|W^{cu}}(q) \right\| \leq C\gamma^{-k}, \ k \in \mathbb{Z}_+ \right\}$$

- *for every $q \in W^{cs}$ the stable fiber W^s_q has the following properties*
  - *W^s_q is a vertical disk in D satisfying the cone condition.*
  - *W^s_q depends continuously on q*
  - *W^s_q intersects W^{cu} and $\tilde{\Lambda}$ in a single point (the same point)*

$$W^s_q = \{z \in W^{cs} \mid \exists C > 0, 0 < \gamma < \min(\eta_v, 1) \ \left\| f^k(z) - f^k(q) \right\| \leq C\gamma^k, \ k \in \mathbb{Z}_+ \}$$

The next layer of our results is concerned with the smoothness of $W^{cu,cs}$, $\tilde{\Lambda}$ and $W^{u,s}_q$. It requires stronger assumptions.

Theorem 46 *Standard assumptions and assume $f \in C^2$. Then*

- *Let*

$$\beta_{cu} = \sup_{q \in D} \left(\left\| \frac{\partial f_y}{\partial y}(q) \right\| + \frac{1}{\alpha_v} \left\| \frac{\partial f_{(\lambda,x)}}{\partial y}(q) \right\| \right) \tag{7.9}$$

$$\tilde{\eta}_v = \inf_{z \in D} \left(m\left(\frac{\partial f_{(\lambda,x)}}{\partial(\lambda,x)}(z) \right) - \frac{1}{\alpha_v} \left\| \frac{\partial f_{(\lambda,x)}}{\partial y}(z) \right\| \right) \tag{7.10}$$

  If $\frac{\beta_{cu}}{\tilde{\eta}_v^2} < 1$, then W^{cu} is C^1.

- *Let*

$$\eta_{cs} = \inf_{q \in D} \left(m\left(\frac{\partial f_x}{\partial x}(q) \right) - \frac{1}{\alpha_h} \left\| \frac{\partial f_{(\lambda,y)}}{\partial x}(q) \right\| \right). \tag{7.11}$$

  If $\eta_{cs>0}$ and $\frac{\beta_h^2}{\eta_{cs}} < 1$, then W^{cs} is C^1.

- *If $\frac{\beta_{cu}}{\tilde{\eta}_v^2} < 1$ and $\frac{\beta_h^2}{\eta_{cs}} < 1$, then $\tilde{\Lambda}$ is C^1.*

- *Let*

$$\begin{aligned}\mu_u &= \sup_{q\in D}\left(\left\|\frac{\partial f_{(\lambda,y)}}{\partial(\lambda,y)}(q)\right\| + \alpha_h \left\|\frac{\partial f_x}{\partial(\lambda,y)}(q)\right\|\right), &(7.12)\\ \tilde{\eta}_h &= \inf_{z\in D}\left(m\left(\frac{\partial f_x}{\partial x}(z)\right) - \alpha_h\left\|\frac{\partial f_x}{\partial(\lambda,y)}(z)\right\|\right) &(7.13)\end{aligned}$$

  If

$$\frac{\mu_u}{\tilde{\eta}_h^2} < 1, \quad (7.14)$$

  then for every $q \in W^{cu}$ the unstable fiber W^u_q is C^1.

- *Let*

$$\mu_s = \inf_{q\in D}\left(m\left(\frac{\partial f_{(\lambda,x)}}{\partial(\lambda,x)}(q)\right) - \alpha_v\left\|\frac{\partial f_y}{\partial(\lambda,x)}(q)\right\|\right). \quad (7.15)$$

  If

$$\mu_s > 0, \quad \frac{\beta_v^2}{\mu_s} < 1, \quad (7.16)$$

  then for any $q \in W^{cs}$ the stable fiber W^s_q is C^1.

In the above theorem we have introduced $\tilde{\eta}_h$ and $\tilde{\eta}_v$. They are related to η_h and η_v - they differ as the follows: with $\tilde{\eta}_{h,v}$ we take $m(\cdot)$ in the point, in the version without the tilde sign the infimum is taken over $P(z)$. Obviously, we have $\eta_{h,v} \leq \tilde{\eta}_{h,v}$.

7.2 The center-unstable manifold

In this section we assume the standard assumptions from Def. 43 in Section 7.1.

The goal of this section to prove that the set W^{cu} is a center horizontal disk.

Theorem 47 *W^{cu} is a center-horizontal disk in D satisfying the cone condition.*

Moreover, it has the following properties.

- $\pi_y(W^{cu}) \subset \mathrm{int}\overline{B}_s(0,R)$.
- *For each $q \in W^{cu}$ there exists a unique $p \in W^{cu}$ such that $f(p) = q$.*
- *The map $f^{-1}_{|W^{cu}} : W^{cu} \to W^{cu}$ satisfies the Lipschitz condition: there exist $\epsilon > 0$ and L such that if $\|\pi_{(\lambda,x)}(q_1 - q_2)\| \leq \epsilon$, then*

$$\|\pi_{(\lambda,x)}(f^{-1}_{|W^{cu}}(q_1) - f^{-1}_{|W^{cu}}(q_2)\| \leq \frac{1}{\eta_v}\|\pi_{(\lambda,x)}(q_1 - q_2)\|. \quad (7.17)$$

Proof: We use notation $\theta = (\lambda, x)$. Using Lemma 43 we can define a sequence of center horizontal disks by $b_0(\lambda, x) = (\lambda, 0, 0)$, $b_{i+1} = \mathcal{G}_{ch}(b_i)$ $i > 0$.

This implies that

$$\forall \theta \in \Lambda \times \overline{B}_u(0,R)\, \forall k \in \mathbb{Z}_+\, \exists y \in \mathrm{int}\overline{B}_s(0,R), \theta_k \in \Lambda \times \overline{B}_u(0,R) \quad f^k(\theta_k, 0) = (\theta, y) \tag{7.18}$$

From (7.18) it follows that for any $\theta_0 \in \Lambda \times \overline{B}_u(0,R)$ there exist backward orbits of f in D through z_0, with $\pi z_0 = \theta_0$ of arbitrary length

$$\pi_\theta z_0 = \theta_0, \qquad f(z_i) = f(z_{i+1}),\ i = -k, -k+1, \ldots, -1, 0. \tag{7.19}$$

Using the compactness of D and applying the diagonal argument gives us a point $y_{\theta_0} \in B_s(0,R)$, such that there exists a full backward orbit of (θ_0, y_{θ_0}) for f.

This shows that $\pi_\theta W^{cu} = \Lambda \times \overline{B}_u(0,R)$.

Now we show the cone condition on W^{cu}, i.e. if $z_1 \neq z_2$ $z_1, z_2 \in W^{cu}$ and $\|\pi_\lambda z_1 - \pi_\lambda z_2\| < 2R_\Lambda$ (which means that both points are in a good chart), then

$$Q_v(z_1 - z_2) \leq 0. \tag{7.20}$$

The argument will be by the contradiction, we assume that $Q_v(z_1 - z_2) \geq 0$.

Let $\{z_i^k\}_{k \in \mathbb{Z}_-}$ be a full backward orbit of z_i for f in D, $i = 1, 2$. From the backward cone condition it follows immediately that

$$Q_v(z_1^k - z_2^k) \geq 0, \quad k \in \mathbb{Z}_- \tag{7.21}$$

(7.21) implies that

$$\alpha_v \|\pi_y z_1^k - \pi_y z_2^k\| \geq \|\pi_\theta z_1^k - \pi_\theta z_2^k\|, \quad k \in \mathbb{Z}_- \tag{7.22}$$

hence for any $k \in \mathbb{Z}$ holds

$$\begin{aligned}
\|\pi_y(z_1^{-k+1} - z_2^{-k+1})\| = \|\pi_y(f(z_1^{-k}) - f(z_2^{-k}))\| \leq \\
\sup_{z \in D} \left\| \frac{\partial f_y}{\partial y}(z) \right\| \cdot \|\pi_y(z_1^{-k} - z_2^{-k})\| + \sup_{z \in D} \left\| \frac{\partial f_y}{\partial \theta}(z) \right\| \cdot \|\pi_\theta(z_1^{-k} - z_2^{-k})\| \leq \\
\left(\sup_{z \in D} \left\| \frac{\partial f_y}{\partial y} \right\| + \alpha_v \sup_{z \in D} \left\| \frac{\partial f_y}{\partial \theta} \right\| \right) \|\pi_y(z_1^{-k} - z_2^{-k})\| \leq \beta_v \|\pi_y(z_1^{-k} - z_2^{-k})\|.
\end{aligned}$$

For iterates of f we obtain

$$\|\pi_y(z_1 - z_2)\| = \|\pi_y(f^k(z_1^{-k}) - f^k(z_2^{-k}))\| \leq \beta_v^k \|\pi_y(z_1^{-k} - z_2^{-k})\| \leq 2\beta_v^k R$$

Passing to the limit $k \to \infty$ we obtain $\|\pi_y(z_1 - z_2)\| = 0$, and since $Q_v(z_1 - z_2) \geq 0$ we get $z_1 = z_2$.

This proves the cone condition for W^{cu} and implies also that π_θ is a homeomorphism between W^{cu} and $\Lambda \times \overline{B}_u(0,R)$.

The condition $\pi_y(W^{cu}) \subset \mathrm{int}\overline{B}_s(0,R)$ follows immediately from Lemma 34 and the following obvious fact: $f(W^{cu}) \cap D = W^{cu}$.

Now we will show that for any $q \in W^{cu}$ there exists a unique $p \in W^{cu}$ such that $f(p) = q$. Since $f(W^{cu}) \cap D = W^{cu}$ we see that such $p \in W^{cu}$ exists. To prove the uniqueness assume that exist $p_1, p_2 \in W^{cu}$, such that $p_1 \neq p_2$ and $f(p_1) = f(p_2)$. From the backward cone condition it follows that $Q_v(p_1 - p_2) > 0$, but this contradicts the cone condition for W^{cu}.

Now we turn to the estimation of the Lipschitz constant for $\pi_\theta f^{-1}_{W^{cu}}$.

Let $b : \Lambda \times \overline{B}_u(0, R) \to D$ be the center horizontal disk such that

$$W^{cu} = \{b(\theta) \mid \theta \in \Lambda \times \overline{B}_u(0, R)\}$$

Let $g : W^{cu} \to \Lambda \times \mathbb{R}^u$ given by $g(\theta) = \pi_\theta f(b(\theta))$.

From Lemma 36 we know that if $\|\theta_1 - \theta_2\| \leq 2R_\Lambda$, then

$$\|g(\theta_1) - g(\theta_2)\| \geq \eta_v \|\theta_1 - \theta_2\|, \tag{7.23}$$

from this we would like to argue that

$$\|\theta_1 - \theta_2\| \geq \eta_v \|g^{-1}(\theta_1) - g^{-1}(\theta_2)\|. \tag{7.24}$$

The only issue here is: whether θ_1 and θ_2 are close enough to be contained in the same good chart.

From the definition of g it follows that for each $\theta_1 \in \Lambda \times \text{int}\overline{B}_u(0, R)$ and $r > 0$, such that $r < 2R_\Lambda$ and $\overline{B}(\theta_1, r) \subset \Lambda \times \overline{B}_u(0, R)$ (to make sure that we are in the domain of f and in the good chart) holds

$$m(\left[Dg(\overline{B})\right]) \geq \eta_v. \tag{7.25}$$

From this and Lemma 31 we obtain for each $\theta_1 \in \Lambda \times \text{int}\overline{B}_u(0, R)$ and $r > 0$, such that $r < 2R_\Lambda$ and $B(\theta_1, r) \subset \Lambda \times \overline{B}_u(0, R)$ (to make sure that we are in the domain of f) holds

$$B(g(\theta_1), r_1) \subset g(B(\theta_1, r_1/\eta_v)) \tag{7.26}$$

Observe that (7.26) gives us immediately the Lipschitz constant for g^{-1} as follows. First of all $B(g(\theta_1), \eta_v r)$ is expressed in some coordinates, which may not be a good chart on Λ in its full domain. Therefore we require that $r_1 = \eta_v r < 2R_\Lambda$. Then by applying g^{-1} to (7.26) we obtain

$$g^{-1}(B(g(\theta_1), r_1)) \subset B(\theta_1, r_1/\eta_v). \tag{7.27}$$

∎

7.2.1 Convergence of graph transforms of center-horizontal disks

To prove that W^{cu} is C^1 we will need to represent W^{cu} as the limit in C^1 topology of graphs of smooth functions. Here we make the first step in this direction.

From the reasoning in the proof of Theorem 47 it follows immediately the following estimate for two center horizontal disks in D satisfying the cone condition b_1, b_2

$$\|\mathcal{G}_{ch}^{i+j}(b_1) - \mathcal{G}_{ch}^{i}(b_2)\| \leq 2\beta_v^i R. \tag{7.28}$$

Theorem 48 *Let b any center horizontal disk in D satisfying the cone condition. Then $\mathcal{G}^i(b)$ converges uniformly to W^{cu}.*

Proof: The uniform convergence follows immediately from (7.28). The limit must be the fixed point of $\mathcal{G}_{ch}$, hence it must coincide with W^{cu}. ∎

7.3 The center stable manifold

In this section we assume the standard assumptions defined in Section 7.1.

The goal of this section is to establish the existence of the center stable manifold.

Lemma 49 *For every $(\lambda, y) \in \Lambda \times \overline{B}_s(0, R)$ there exists a unique $x_{cs}(\lambda, y) \in \mathrm{int}\overline{B}_u(0, R)$, such that $f^k(\lambda, x_{cs}(\lambda, y), y) \in D$ for $k \in \mathbb{Z}_+$.*

Proof: Let us fix (λ, y) and consider $b : \overline{B}_u(0, R) \to D$ a horizontal disk in D satisfying the cone condition given by $b(x) = (\lambda, x, y)$. From Lemma 42 we obtain a sequence of horizontal disks $b_{i+1} = \mathcal{G}_h(b)$, which implies that for any $k \in \mathbb{Z}_+$, there exists $x_k \in \mathrm{int}\overline{B}_u(0, R)$ such that

$$f^i(\lambda, x_k, y) \in D, \quad i = 0, \ldots, k. \tag{7.29}$$

Using the compactness of $\overline{B}_u(0, R)$ we can find x_{cs}, such that

$$f^i(\lambda, x_{cs}, y) \in D, \quad i \in \mathbb{Z}_+. \tag{7.30}$$

The forward cone condition implies the uniqueness of x_{cs}. For the proof assume that there exists x_1, such that

$$f^i(\lambda, x_1, y) \in D, \quad i \in \mathbb{Z}_+. \tag{7.31}$$

Then from the forward cone condition it follows that for $i \in \mathbb{Z}_+$

$$Q_h(f^i(\lambda, x_1, y) - f^i(\lambda, x_{cs}, y)) > 0, \tag{7.32}$$

$$\|\pi_x(f^i(\lambda, x_1, y) - f^i(\lambda, x_{cs}, y))\| \geq \eta_h^i \|x_1 - x_{cs}\| \to \infty \tag{7.33}$$

but $\|\pi_x(f^i(\lambda, x_1, y) - f^i(\lambda, x_{cs}, y))\|$ is bounded by $2R$. Hence x_{cs} cannot be different from x_1. ∎

Theorem 50 *Let x_{cs} be the function from Lemma 49. Then*

$$W^{cs} = \{(\lambda, x_{cs}(\lambda, y), y) \mid (\lambda, y) \in \Lambda \times \overline{B}_s(0, R)\}. \tag{7.34}$$

Moreover, W^{cs} is a center vertical disk in D satisfying the cone condition and such that

$$\pi_x W^{cs} \subset \mathrm{int}\overline{B}_u(0, R). \tag{7.35}$$

Proof: We will show that W^{cs} satisfies the cone condition: if $z_1 \neq z_2$, $z_1, z_2 \in W^{cs}$ and $\|\pi_\lambda z_1 - \pi_\lambda z_2\| < 2R_\Lambda$, then

$$Q_h(z_1 - z_2) \leq 0. \tag{7.36}$$

Assume the contrary, i.e.

$$Q_h(z_1 - z_2) > 0. \tag{7.37}$$

Then from the forward cone condition it follows that for $i \in \mathbb{Z}_+$

$$Q_h(f^i(z_1) - f^i(z_2) > 0, \tag{7.38}$$

$$\|\pi_x(f^i(z_1) - f^i(z_2))\| \geq \eta_h^i \|\pi_x z_1 - \pi_x z_2\| \to \infty \tag{7.39}$$

but $\|\pi_x(f^i(z_1) - f^i(z_2)\|$ is bounded by $2R$. Hence we obtained a contradiction. This establishes the cone condition for W^{cs}.

Observe that the continuity of x_{cs} is implied by the cone condition.

Condition (7.35) is an immediate consequence of Lemma 34. ∎

7.3.1 W^{cs} as the limit of graphs of functions

To prove that W^{cs} is C^1 we will need to represent W^{cs} as the limit in C^1 topology of graphs of smooth functions. Here we make the first step in this direction.

For any $k \in \mathbb{Z}_+$ and $(\lambda, y) \in \Lambda \times \overline{B}_s(0, R)$ we consider the following problem

$$\pi_x f^k(\lambda, x, y) = 0 \tag{7.40}$$

under the constraint

$$f^i(\lambda, x, y) \in D. \quad i = 0, 1, \ldots, k \tag{7.41}$$

From Lemma 42 it follows immediately that this problem has a unique solution $x_k(\lambda, y)$ which is as smooth as f.

Lemma 51 *Let $d_k : \Lambda \times \overline{B}_s(0, R) \to D$ be a center vertical disk in D given by $d_k(\lambda, y) = (\lambda, x_k(\lambda, y), y)$.*

Then d_k satisfies the cone condition (as a center vertical disk) and sequence d_k converges uniformly to W^{cs}.

Proof: We have to prove that if $\|\lambda_1 - \lambda_2\| < 2R_\Lambda$, then $Q_h(d_k(\lambda_1, y_1) - d_k(\lambda_2, y_2)) \leq 0$. We will argue by the contradiction. Assume that $Q_h(d_k(\lambda_1, y_1) - d_k(\lambda_2, y_2)) > 0$. Then from (7.41) and the forward cone condition it follows that for $i = 0, 1, \ldots, k$ holds

$$Q_h(f^i(d_k(\lambda_1, y_1)) - f^i(d_k(\lambda_2, y_2)) > 0, \tag{7.42}$$

$$\|\pi_x(f^k(d_k(\lambda_1, y_1)) - f^i(d_k(\lambda_2, y_2)))\| \geq \eta_h^k \|d_k(\lambda_1, y_1) - d_k(\lambda_2, y_2)\| \to \infty \tag{7.43}$$

but $\|d_k(\lambda_1, y_1) - d_k(\lambda_2, y_2)\|$ is bounded by $2R$. Hence we obtained a contradiction. This establishes the cone condition for d_k.

To prove the uniform convergence of d_k we show the Cauchy condition for this sequence. Let $k, j \in \mathbb{Z}_+$. We have $Q_h(d_{k+j}(\lambda, y) - d_{k+j}(\lambda, y)) \geq 0$. Then from (7.41) and the forward cone condition it follows that for $i = 0, 1, \ldots, k$ holds

$$Q_h(f^i(d_k(\lambda, y)) - f^i(d_{k+j}(\lambda, y)) \geq 0, \tag{7.44}$$

$$\|\pi_x(f^k(d_k(\lambda, y)) - f^i(d_{k+j}(\lambda, y)))\| \geq \eta_h^k \|d_k(\lambda, y) - d_{k+j}(\lambda, y)\|. \tag{7.45}$$

Since $\|\pi_x(f^k(d_k(\lambda, y)) - f^i(d_{k+j}(\lambda, y)))\| \leq 2R$ we obtain

$$\|d_k(\lambda, y) - d_{k+j}(\lambda, y)\| \leq \frac{2R}{\eta_h^k}. \tag{7.46}$$

This proves the uniform convergence of d_k to some disk $\bar{d}$. Observe that $|\bar{d}| = W^{cs}$, because for each $(\lambda, y) \in \Lambda \times \overline{B}_s(0, R)$ $d(\lambda, y) = \lim_{k\to\infty} d_k(\lambda, y)$ and

$$f^i(d_k(\lambda, y)) \in D, \quad i = 0, \ldots, k. \tag{7.47}$$

Let us fix i in (7.47) and pass to the limit with k. We obtain that for all $i \in \mathbb{Z}_+$ $f^i(d(\lambda, y)) \in D$.

∎

7.4 Invariant manifold

In this section we assume the standard assumptions defined in Section 7.1.

The goal of this section is to prove that the invariant manifold exists, later we will show its normally hyperbolic behavior.

Theorem 52 *There exists $\chi : \Lambda \to \mathrm{int} D$, such that $\pi_\lambda(\chi(\lambda)) = \lambda$ and $\tilde{\Lambda} = \{\chi(\lambda) \mid \lambda \in \Lambda\}$.*

$\tilde{\Lambda}$ satisfies the following set of cone conditions: if $\|\lambda_1 - \lambda_2\| < 2R_\Lambda$, then

$$\|\pi_x \chi(\lambda_1) - \pi_x \chi(\lambda_2)\| \leq \left(\frac{1+\alpha_v^2}{\alpha_v^2\alpha_h^2 - 1}\right)^{1/2} \|\lambda_1 - \lambda_2\|,$$

$$\|\pi_y \chi(\lambda_1) - \pi_y \chi(\lambda_2)\| \leq \left(\frac{1+\alpha_h^2}{\alpha_v^2\alpha_h^2 - 1}\right)^{1/2} \|\lambda_1 - \lambda_2\|.$$

The intersection of W^{cs} and W^{cu} along $\tilde{\Lambda}$ is cone transversal.

Proof: Let us fix $\lambda_0 \in \Lambda$. We will show that there exists $p \in D$, such that

$$W^{cu} \cap W^{cs} \cap \{z \in D,\ \pi_\lambda = \lambda_0\} = \{p\} \tag{7.48}$$

Let $b(\lambda, x) = (\lambda, x, y_{cu}(\lambda, x))$ be the center horizontal disk in D satisfying the cone condition representing W^{cu} and let $g(\lambda, x) = (\lambda, x, y_{cu}(\lambda, x))$ be the center vertical disk in D satisfying the cone condition representing W^{cs}. Since $\lambda = \lambda_0$

is fixed we will drop it as an argument in b and g. We see that $\pi_{(x,y)}b$ and $\pi_{(x,y)}g$ are horizontal and vertical disks in an h-set $\overline{B}_u(0,R) \times \overline{B}_s(0,R)$ endowed with a natural structure, respectively. Bt Thm. 11 these disks intersect, i.e. there exists (x_0, y_0) such that $\pi_{(x,y)}b(\lambda_0, x_0) = \pi_{(x,y)}g(\lambda_0, y_0)$.

To prove the uniqueness let us assume that $\{p_1, p_2\} \in W^{cu} \cap W^{cs} \cap \{z \in D,\ \pi_\lambda = \lambda_0\}$. Then from the cone conditions for W^{cu} and W^{cs} it follows that

$$Q_h(p_1 - p_2) \leq 0, \quad Q_v(p_1 - p_2) \leq 0. \tag{7.49}$$

Since $\pi_\lambda p_1 = \pi_\lambda p_2 = \lambda_0$, then from Lemma 33 it follows that $p_1 = p_2$.

We define $\chi(\lambda_0) = p$, where p is the unique point satisfying (7.48). Observe that from Theorems 47 and 50 it follows that $\pi_{(x,y)}(\chi(\lambda)) \in \mathrm{int}(\overline{B}_u(0,R) \times \overline{B}_s(0,R))$.

The cone transversality follows from the case *T2* in Lemma 32.

Now we estimate the Lipschitz constant for $\tilde{\chi} = \pi_{(x,y)}\chi$. Assume that $\lambda_1 - \lambda_2| < 2R_\Lambda$. Let us denote by $(x_i, y_i) = \pi_{(x,y)}(\chi(\lambda_i))$ for $i = 1, 2$. From the cone conditions for W^{cu} and W^{cs} we have

$$-\|y_1 - y_2\|^2 + \alpha_h^2\|x_1 - x_2\|^2 \leq \|\lambda_1 - \lambda_2\|^2, \tag{7.50}$$
$$-\|x_1 - x_2\|^2 + \alpha_v^2\|y_1 - y_2\|^2 \leq \|\lambda_1 - \lambda_2\|^2. \tag{7.51}$$

Let us denote by $\tilde{x} = \left(\frac{\|x_1 - x_2\|}{\|\lambda_1 - \lambda_2\|}\right)^2$, $\tilde{y} = \left(\frac{\|y_1 - y_2\|}{\|\lambda_1 - \lambda_2\|}\right)^2$. Then (7.50, 7.51) is equivalent to

$$-\tilde{y} + \alpha_h^2\tilde{x} \leq 1,$$
$$-\tilde{x} + \alpha_v^2\tilde{y} \leq 1,$$

which gives the following inequalities

$$\alpha_h^2\tilde{x} - 1 \leq \tilde{y} \leq \frac{1 + \tilde{x}}{\alpha_v^2}. \tag{7.52}$$

System of inequalities (7.52) defines a set S on the plane $(\tilde{x}, \tilde{y})$. We would like to find $(L_x, L_y) \in \mathbb{R}_+^2$, such that for all $(\tilde{x}, \tilde{y}) \in S$, holds $\tilde{x} \leq L_x$ and $\tilde{y} \leq L_y$. Such (L_x, L_y) exist and are given by

$$L_x = \frac{1 + \alpha_v^2}{\alpha_v^2\alpha_h^2 - 1}, \quad L_y = \frac{1 + \alpha_h^2}{\alpha_v^2\alpha_h^2 - 1}. \tag{7.53}$$

Observe that the denominator $\alpha_v^2\alpha_h^2 - 1$ due to condition (6.34).

Hence we proved that

$$\|x_1 - x_2\| \leq \left(\frac{1 + \alpha_v^2}{\alpha_v^2\alpha_h^2 - 1}\right)^{1/2} \|\lambda_1 - \lambda_2\|, \quad \|y_1 - y_2\| \leq \left(\frac{1 + \alpha_h^2}{\alpha_v^2\alpha_h^2 - 1}\right)^{1/2} \|\lambda_1 - \lambda_2\|$$

∎

7.5 Unstable fibers

In this section we assume the standard assumptions defined in Section 7.1.

The goal of this section is to establish the existence of the foliation of W^{cu} into unstable fibers W^u_q for $q \in W^{cu}$.

In this section $\theta = (\lambda, y)$.

From Theorem 47 it follows that $f^{-1}_{|W^{cu}} : W^{cu} \to W^{cu}$ is a Lipschitz function. In this section we will drop the subscript and just write $f^{-1}(q)$ instead of $f^{-1}_{|W^{cu}}(q)$, because the argument will be always from W^{cu}.

For $q \in D$ we define h_q is a horizontal disk in D satisfying the cone condition by $h_q(x) = (\pi_\lambda q, x, \pi_y q)$.

Let $q \in W^{cu}$. Consider a sequence of horizonal disks in D satisfying cone condition

$$d_{k,q} = \mathcal{G}^k_h(h_{f^{-k}(q)}). \tag{7.54}$$

Lemma 53 *If $f^j(z_i) \in Q^+_h(f^j(q))$, $i = 1, 2$ and $j = 0, 1, \dots, k$ and*

$$Q_h(f^k(z_1) - f^k(z_2)) < 0, \tag{7.55}$$

then

$$\|\pi_\theta(f^k(z_1) - f^k(z_2))\| \leq 4R\alpha_h \left(\frac{\beta_h}{\eta_h}\right)^k, \tag{7.56}$$

$$\|f^k(z_1) - f^k(z_2)\| \leq 4R(\alpha_h + 1) \left(\frac{\beta_h}{\eta_h}\right)^k. \tag{7.57}$$

Proof: Our assumption $f^j(z_i) \in Q^+_h(f(q))$, $i = 1, 2$ and $j = 0, 1, \dots, k$ implies that $f^j(z_1), f^j(z_2)$ are both contained in good charts for $j = 0, 1, \dots, k-1$.

Therefore from Lemma 41 it follows that

$$\|\pi_\theta(f^k(z_1) - f^k(z_2))\| \leq \beta^k_h \|\pi_\theta(z_1 - z_2)\|. \tag{7.58}$$

We estimate $\|\pi_\theta(z_1 - z_2)\|$ using the expansion in the x-direction. We have for $i = 1, 2$

$$2R \geq \|\pi_x(f^k(z_i) - f^k(q))\| \geq \eta_h \|\pi_x(f^{k-1}(z_i) - f^{k-1}(q))\| \geq \eta^k_h \|\pi_x(z_i - q)\|,$$

hence we obtain

$$\|\pi_x(z_i - q)\| \leq \frac{2R}{\eta^k_h}. \tag{7.59}$$

Since $z_i \in Q^+_h(q)$ we get

$$\|\pi_\theta(z_i - q)\| \leq \frac{2R\alpha_h}{\eta^k_h}.$$

From the triangle inequality we obtain

$$\|\pi_\theta(z_1 - z_2)\| \leq \|\pi_\theta(z_1 - q)\| + \|\pi_\theta(z_2 - q)\| \leq \frac{4R\alpha_h}{\eta^k_h}. \tag{7.60}$$

By combining the above inequality with (7.58) we obtain

$$\|\pi_\theta(f^k(z_1) - f^k(z_2))\| \leq 4R\alpha_h \left(\frac{\beta_h}{\eta_h}\right)^k.$$

Since $Q_h(f^k(z_1) - f^k(z_2)) < 0$, from the above inequality we obtain

$$\|f^k(z_1) - f^k(z_2)\| \leq 4(1 + 1/\alpha_h)R\alpha_h \left(\frac{\beta_h}{\eta_h}\right)^k.$$

∎

Lemma 54 *$d_{k,q}$ converge uniformly to a horizontal disk d_q. d_q satisfies the cone condition and $\{d_q(x) \mid x \in \overline{B}_u(0,R)\} = W^u_q$.*

Proof:
Observe first that from the definition of graph transform it follows that for each $k \in \mathbb{Z}_+$ and for each $x \in \overline{B}_u(0,R)$ the point $d_{k,q}(x)$ has a backward orbit of length $k+1$, z_i for $i = 0, -1, \ldots, -k$ such that

$$\begin{aligned} z_0 = d_{k,q}, \quad f(z_i) = z_{i+1} \quad i = -k, -k+1, \ldots, -1 \\ Q_h(z_i - f^{-i}(q)) > 0, \quad i = -k, -k+1, \ldots, -1, 0. \end{aligned}$$

Let us fix $x \in \overline{B}_u(0,R)$. Let k, j be positive integers. From the above observation we can find (define) z_1 and z_2 as follows. Let z_1 be such that $f^i(z_1) \in Q_h^+(f^{-k+i}(q))$ for $i = 0, 1, \ldots, k$ and $f^k(z_1) = d_{q,k}(x)$, analogously z_2 be such that $f^i(z_2) \in Q_h^+(f^{-k+i}(q))$ for $i = 0, 1, \ldots, k$ and $f^k(z_2) = d_{q,k+j}(x)$.

Observe that $Q_h(f^k(z_2) - f^k(z_1)) = -\|\pi_\theta(f^k(z_2) - f^k(z_1))\|^2 = -\|f^k(z_2) - f^k(z_1)\|^2$. Assume that $f^k(z_2) \neq f^k(z_1)$, then from Lemma 53 applied to z_1, z_2 and $f^{-k}(q)$ it follows that

$$\|d_{k,q}(x) - d_{k+j,q}(x)\| = \|\pi_\theta(d_{k,q}(x) - d_{k+j,q}(x))\| \leq 4R\alpha_h \left(\frac{\beta_h}{\eta_h}\right)^k. \tag{7.61}$$

Therefore we have a Cauchy sequence. Let us denote the limit by d_q.

We show that for all $x \in \overline{B}_u(0,R)$ $d_q(x) \in W^{cu}$. We need to construct a full backward orbit through $d_q(x)$. Let us consider backward orbits through $d_{q,k}(x)$ of length $k+1$. From Lemma 53 it follows that they converge pointwise to fullbackward orbit through $d_q(x)$. Therefore $d_q(x) \in W^{cu}$ for $x \in \overline{B}_u(0,R)$.

From this it follows that $f^{-k}(q)$ is well defined for all $k \in \mathbb{Z}_+$.

From this reasoning it follows also that

$$\mathcal{G}_h(d_{f^{-1}(q)}) = d_q. \tag{7.62}$$

By passing to the limit in the cone condition for disks $d_{k,q}$ we obtain

$$Q_h(d_q(x_1) - d_q(x_2)) \geq 0, \tag{7.63}$$

$$Q_h(f^{-k}(d_q(x)) - f^{-k}(q)) \geq 0. \tag{7.64}$$

In both cases we need strong inequalites.

Let us first deal with (7.63). From (7.62) it follows that for any $x_1, x_2 \in \overline{B}_u(0,R)$, $x_1 \neq x_2$ there exists $p_1, p_2 \in \overline{B}_u(0,R)$ such that $f(d_{f^{-1}(q)}(p_i)) = d_q(x_i)$, $i = 1, 2$. Since by (7.63) we have $Q_h(d_{f^{-1}(q)}(p_1) - d_{f^{-1}(q)}(p_2)) \geq 0$, then from the forward cone condition we obtain that $Q_h(d_q(x_1) - d_q(x_2)) > 0$.

The case of (7.64) is analogous. Assume $d_q(x) \neq q$. Then since by (7.64) holds $Q_h(f^{-k-1}(d_q(x)) - f^{-k-1}(q)) \geq 0$, then the forward cone condition implies that $Q_h(f^{-k}(d_q(x)) - f^{-k}(q)) > 0$.

This shows that $\{d_q(x) \mid x \in \overline{B}_u(0,R)\} \subset W^u_q$.

For the other direction let us consider $z \in W^{cu}$, such that $f^{-k}(z) \in Q_h + (f^{-k}(q))$ for all $k \in \mathbb{Z}_+$. Let $x = \pi_x z$. We will show that $z = d_q(x)$.

Assume the contrary, then from Lemma 53 it follows for any k holds

$$\|z - d_q(x)\| = \|\pi_\theta(z - d_q(x))\| \leq 4R\alpha_h \left(\frac{\beta_h}{\eta_h}\right)^k \to 0, \quad k \to \infty.$$

Therefore $z = d_q(x)$. ∎

For $(q,x) \in W^{cu} \times \overline{B}_u(0,R)$ we define a function $d^u(q,x) = d_q(x) \in W^{cu}$, where d_q is the function defined in Lemma 54.

Lemma 55 $d^u : W^{cu} \times \overline{B}_u(0,R) \to W^{cu}$ *is continuous.*

Proof: Assume that $\lim_{n\to\infty}(q_n, x_n) = (\bar{q}, \bar{x})$. Due to the compactness of D we can also assume that $d^u(q_n, x_n) \to p$.

We have to show that $d^u(\bar{q}, \bar{x}) = p$.

From the definition of W^u_q it follows that for $n, k \in \mathbb{Z}_+$ holds

$$Q_h(f^{-k}(d^u(q_n, y_n)) - f^{-k}(q_n)) > 0. \tag{7.65}$$

Passing to the limit for $n \to \infty$ we obtain for $k \in \mathbb{Z}_+$

$$Q_h(f^{-k}(p) - f^{-k}(\bar{q})) \geq 0. \tag{7.66}$$

The forward cone condition implies that if $Q_h(f^{-k}(p) - f^{-k}(\bar{q})) = 0$, then $Q_h(f^{-(k+1)}(p) - f^{-(k+1)}(\bar{q})) < 0$. This forces all inequalities in (7.66) to be strong.

Therefore $p = d^u(\bar{q}, \bar{x})$.

∎

Lemma 56 *Let $q \in W^{cu}$. Then the intersection $W^u_q \cap W^{cs}$ consists of a single point and is cone transversal.*

The intersection $W^u_q \cap \tilde{\Lambda}$ consists of a single point.

Proof:

Let d^u be as in Lemma 55 and x_{cs} be as in Theorem 50.

Let us fix $q \in W^{cu}$. Let an open set $U \subset \Lambda$ be such that $\pi_\theta(Q_h^+(q)) \subset U$, U is contained in good chart. We want to show that we can find $(\lambda, y) \in \mathrm{int}U \times \overline{B}_s(0,R)$ and $x \in \mathrm{int}\overline{B}_u(0,R)$ such that

$$(\lambda, x_{cs}(\lambda, y), y) = d^u(q,x) \tag{7.67}$$

Since we are good chart we can rewrite this equation as

$$F(\lambda, y, x) = (\lambda, x_{cs}(\lambda, y), y) - d^u(q,x) = 0. \tag{7.68}$$

We consider this equation for $(\lambda, y, x) \in \overline{U} \times \overline{B}_s(0,R) \times \overline{B}_u(0,R)$. We want to show that the local Brouwer degree $\deg(F, \mathrm{int}\overline{U} \times \overline{B}_s(0,R) \times \overline{B}_u(0,R), 0) \neq 0$, this implies the existence of solution.

First we show that for $(\lambda, y, x) \in \partial\overline{U} \times \overline{B}_s(0,R) \times \overline{B}_u(0,R)$ we have $F(\lambda, y, x) \neq 0$, hence the local Brouwer degree is defined.

For the proof we consider three cases: $\lambda \in \partial U$, $y \in \partial\partial\overline{B}_s(0,R)$ and $x \in$ *xdom*. Let $\lambda \in \partial U$. In this situation we know for any $x \in \overline{B}_u(0,R)$ holds $\pi_\lambda(d^u(q,x)) \in \pi_\lambda(Q_h^+(q)) \subset U$, hence $\pi_\lambda F(\lambda, y, x) \neq 0$.

Let $y \in \partial\overline{B}_s(0,R)$, we have $\pi_y d^u(q,x) \subset \mathrm{int}\overline{B}_s(0,R)$, hence $\pi_y F(\lambda, y, x) \neq 0$.

If $x \in \partial\overline{B}_u(0,R)$, we have $|x_{cs}(\lambda, y)| < R$ and $\pi_x d^u(q,x) = x$, hence $\pi_x F(\lambda, y, x) \neq 0$. This shows that $\deg(F, \mathrm{int}\overline{U} \times \overline{B}_s(0,R) \times \overline{B}_u(0,R), 0) \neq 0$ is defined.

Let us define a homotopy $G_t(\lambda, y, x) = (1-t)F(\lambda, y, x) + t((\lambda, 0, y) - (\pi_\lambda q, x, \pi_y q))$ for $t \in [0,1]$. The same argument which worked for F, shows that for any point $z \in \partial U \times \overline{B}_s(0,R) \times \overline{B}_u(0,R)$ we have $F(z) \neq 0$ works also for G_t. Hence $\deg(F, \mathrm{int}U \times \overline{B}_s(0,R) \times \overline{B}_u(0,R), 0) = \deg(G_1, \mathrm{int}U \times \overline{B}_s(0,R) \times \overline{B}_u(0,R), 0)$. For $t = 1$ equation (7.68) becomes

$$(\lambda, 0, y) - (\pi_\lambda q, x, \pi_y q) = 0. \tag{7.69}$$

The solution is given by $\lambda = \pi_\lambda q$, $y = \pi_y q$, $x = 0$ and belongs to $\mathrm{int}\overline{U} \times \overline{B}_s(0,R) \times \overline{B}_u(0,R)$, therefore since the system (7.69) is linear, it has nonzero degree. Therefore $\deg(F, \mathrm{int}\overline{U} \times \overline{B}_s(0,R) \times \overline{B}_u(0,R), 0) \neq 0$, This clearly has a nonzero degree, hence (7.68) has a solution.

The solution is unique, because if $q_1, q_2 \in W_q^u \cap W^{cs}$, then q_1 and q_2 must be contained in a common good chart, hence the cone condition for W^{cs} and W_q^u apply and we have, respectively,

$$Q_h(q_1 - q_2) \leq 0, \quad Q_h(q_1 - q_2) > 0, \tag{7.70}$$

which is a contradiction.

In fact the cones for W^{cs} and W_q^u are separated, as we have a possibility to change α_h within our assumptions. The cone transversality follows from the case *T1* in Lemma 32.

Since $\tilde{\Lambda} \cap W_q^u \subset W^{cs} \cap W_q^u \subset W^{cs} \cap W^{cu} = \tilde{\Lambda}$ we see that $\tilde{\Lambda} \cap W_q^u$ contains exactly one point.

∎

Lemma 57 *If $z \in W^u_q$, then $\|f^{-k}(z) - f^{-k}(q)\| \leq 2R(1+\alpha_h)\eta_h^{-k}$ for $k \in \mathbb{Z}_+$.*

Proof: Since $f^{-k}(z) \in Q_v^+(f^{-k}(q))$ for $k \in \mathbb{Z}_+$, from the forward cone condition we obtain

$$\|\pi_x(z-q)\| \geq \eta_h \|\pi_x(f^{-1}(z) - f^{-1}(q))\| \geq \\ \eta_h^2 \|\pi_x(f^{-2}(z) - f^{-2}(q))\| \geq \cdots \geq \eta_h^k \|\pi_x(f^{-k}(z) - f^{-k}(q))\|$$

Since $\|\pi_x(z-q)\| \leq 2R$ we obtain

$$\|\pi_x(f^{-k}(z) - f^{-k}(q))\| \leq 2R\eta_h^{-k}. \tag{7.71}$$

Since in $f^{-k}(z) \in Q_h^+(f^{-k}(q))$ we get the following estimate

$$\|f^{-k}(z) - f^{-k}(q)\| \leq (1+\alpha_h)\|\pi_x(f^{-k}(z) - f^{-k}(q))\| \leq 2R(1+\alpha_h)\eta_h^{-k}.$$

∎

Lemma 58 *If $z, q \in W^{cu}$ and there exists $C > 0$ and $\gamma \geq \eta_h$, such that $\|f^{-k}(z) - f^{-k}(q)\| \leq C\gamma^{-k}$ for $k \in \mathbb{Z}_+$, then $z \in W^u_q$.*

Proof: We will reason by the contradiction.

There exists k_0 such that $f^{-k}(z)$ and $f^{-k}(q)$ are in good charts for $k \geq k_0$. Let k_0 be the smallest number with this property. Observe that it must be that $k_0 = 0$ or

$$Q_h(f^{-k_0}(z) - f^{-k_0}(q)) < 0. \tag{7.72}$$

Indeed if $Q_h(f^{-k_0}(z) - f^{-k_0}(q)) \geq 0$ and $k_0 > 0$, then from the forward cone condition it follows that $Q_h(f^{-(k_0-1)}(z) - f^{-(k_0-1)}(q)) > 0$, which implies that $f^{-(k_0-1)}(z), f^{-(k_0-1)}(q)$ belong to the same good chart.

In the case of $k_0 = 0$, since $z \notin W^u_q$ it follows that there exists k_1, such that

$$Q_h(f^{-k_1}(z) - f^{-k_1}(q)) < 0. \tag{7.73}$$

Now in both cases we are in the following situation, there exist $k_1 \geq 0$, such that for all $k \geq k_1$ the pair $f^{-k}(z), f^{-k}(q)$ is contained in some good chart for all $k \geq k_1$ and (7.73).

From the forward cone condition it follows that

$$Q_h(f^{-k}(z) - f^{-k}(q)) < 0, \quad k > k_1. \tag{7.74}$$

The above condition and Lemma 41 it follows that for any $j > 0$ holds

$$\left\|\pi_\theta\left(f^{-(k_1+j)}(z) - f^{-(k_1+j)}(q)\right)\right\| \geq \frac{1}{\beta_h^j}\left\|\pi_\theta\left(f^{-k_1}(z) - f^{-k_1}(q)\right)\right\|, \quad j \in \mathbb{Z}_+.$$

Therefore for $j \in \mathbb{Z}_+$ we have

$$C \geq \gamma^{(k_1+j)}\left\|f^{-(k_1+j)}(z) - f^{-(k_1+j)}(q)\right\| \geq \eta_h^{k_1+j}\left\|f^{-(k_1+j)}(z) - f^{-(k_1+j)}(q)\right\| \geq \\ \eta_h^{k_1}\left(\frac{\eta_h}{\beta_h}\right)^j \left\|\pi_\theta\left(f^{-k_1}(z) - f^{-k_1}(q)\right)\right\| \to \infty, \quad j \to \infty.$$

Therefore we obtained the contradiction, hence $z \in W^u_q$.

∎

7.6 Stable fibers

In this section we assume the standard assumptions defined in Section 7.1.

The goal of this section is to establish the existence of the foliation of W^{cs} into unstable fibers W^s_q for $q \in W^{cs}$.

In this section $\theta = (\lambda, x)$.

7.6.1 Local covering relations along the trajectory

Let $N((\theta,y),r_1,r_2) = (\theta,y) + \overline{B}_{m+u}(0,r_1) \times \overline{B}_s(0,r_2)$ an h-set with a natural structure.

Lemma 59 *Let $\gamma < 1$ be such that $\beta_v < \gamma < \eta_v$. For any $z = (\theta, y) \in int\ D$ and $r > 0$, such that $f(z) \in D$, $N(z, r, r/\alpha_v) \subset D$, $\pi_\lambda N(z, r, r/\alpha_v)$ is contained in a good chart,*

holds

$$N(z, r, r/\alpha_v) \overset{f}{\Longrightarrow} N(f(z), \gamma r, \gamma r/\alpha_v) \tag{7.75}$$

Proof:

First let us notice that $N(f(z), \gamma r, \gamma r/\alpha_v)$ is contained in a good chart.

Observe first that

$$\forall q \in N(z, r, r/\alpha_v)^+ \quad Q_v(q - z) \geq 0. \tag{7.76}$$

To establish the covering relation we need to define a homotopy satisfying the entry and exit conditions, which at the end will have a affine map depending only on the 'unstable' variable.

In this proof our homotopy will be split in three parts. The first part will be a contraction in the domain, so the value will depend only on the 'unstable' variable. The second one will make the map flat and the third one will end with an affine map.

The first homotopy is given by $H(t, \theta, y) = f(z + (\theta, (1 - t)y))$. Observe that the homotopy is really a deformation retraction in the domain of f on to the center-unstable direction, i.e. $\overline{B}_{m+u}(0, r)$, preserving the set $N(z, r, r/\alpha_v)^-$, composed with the map itself. From this it follows that it is enough to verify the exit and entry conditions for H, it is enough to do it for the map f, only. Hence we need to verify

$$f(N(z, r, r/\alpha_v)^-) \cap N(f(z), \gamma r, \gamma r/\alpha_v) = \emptyset \tag{7.77}$$

$$N(f(z), \gamma r, \gamma r/\alpha_v)^+ \cap f(N(z, r, r/\alpha_v)) = \emptyset \tag{7.78}$$

For the proof of the exit condition (7.77) let us take a point from $N(z, r, r/\alpha_v)^-$ given by $z + (\theta, y)$, where $\|\theta\| = r$ and $\|y\| \leq \frac{r}{\alpha_v}$. We have

$$\|\pi_\theta(f(z + (\theta, y)) - f(z))\| \geq \left(m\left(\left[\frac{\partial f_\theta}{\partial \theta}\right] \right) - \frac{1}{\alpha_v} \left\| \frac{\partial f_\theta}{\partial y} \right\| \right) \cdot r \geq \eta_v r > \gamma r.$$

For the proof of the entry condition (7.78) we consider two cases. Either we have $Q_v(\theta, y) \geq 0$ ($\alpha_v \|y\| \geq |\theta|$) or $Q_v(\theta, y) < 0$ ($\alpha_v \|y\| < |\theta|$).

Consider first the case $Q_v(\theta, y) < 0$. From Lemma 36 it follows that

$$Q_v(f(z + (\theta, y)) - f(z)) < 0, \tag{7.79}$$

hence it does not intersect $N(f(z), \gamma r, \gamma r/\alpha_v)^+$, due to (7.76).

Now we assume that $Q_v(\theta, y) \geq 0$. We have

$$\|\pi_y(f(z + (\theta, y)) - f(z))\| \leq \left\| \frac{\partial f_y}{\partial y} \right\| \cdot \|y\| + \left\| \frac{\partial f_y}{\partial \theta} \right\| \cdot |\theta| \leq$$
$$\left(\left\| \frac{\partial f_y}{\partial y} \right\| + \alpha_v \left\| \frac{\partial f_y}{\partial \theta} \right\| \right) \|y\| \leq \left(\left\| \frac{\partial f_y}{\partial y} \right\| + \alpha_v \left\| \frac{\partial f_y}{\partial \theta} \right\| \right) \frac{r}{\alpha_v} \leq \beta_v \frac{r}{\alpha_v} < \gamma \frac{r}{\alpha_v}.$$

This shows the entry condition (7.78).

Observe that at the end of the homotopy H we have the map

$$H_1(\theta, y) = f(z + (\theta, 0)). \tag{7.80}$$

We define now the second homotopy, which makes H_1 'flat'. We set

$$K(t, \theta, y) = (1 - t) f(z + (\theta, 0)) + t \left(f_\theta(z + (\theta, 0), f_y(z)) \right). \tag{7.81}$$

Since H_1 satisfies the exit condition (7.77), then the same holds for K_t because $\pi_\theta K(t, \theta, y) = \pi_\theta H_1(\theta, y)$.

Regarding the entry condition (7.78) we consider two cases. If $\pi_\theta K_t(\theta, y) \notin \overline{B_1}(\pi_\theta f(z), \gamma r)$, then there is nothing to prove. In the other case: $\pi_\theta K_t(\theta, y) \in \overline{B_1}(\pi_\theta f(z), \gamma r)$ from previous estimates it follows that $\pi_y K_t(\theta, y) \in B_s(f_y(z), \gamma r/\alpha_v)$.

We have

$$K_1(\theta, y) = (f_\theta(z + (\theta, 0)), f_y(z)). \tag{7.82}$$

We will homotope this map further to obtain the affine map as required at the end of the homotopy for covering relations. Let us set

$$G(t, \theta, y) = (1 - t) \left(f_\theta(z + (\theta, 0)), f_y(z) \right) + t \left(f(z) + \left(\frac{\partial f_\theta}{\partial \theta}(z) \cdot \theta, 0 \right) \right), \tag{7.83}$$

The entry condition is satisfied, because $\pi_y G_t(\theta, y) = f_y(z)$ for all $t \in [0, 1]$ and (θ, y).

Now we establish the exit condition. We have

$$\pi_\theta \left(G_t(\theta, y) - f(z) \right) = (1 - t) \int_0^1 \frac{\partial f_\theta}{\partial \theta}(z + t(\theta, 0)) dt \cdot \theta + t \frac{\partial f_\theta}{\partial \theta}(z) \cdot \theta \in$$
$$\left[\frac{\partial f_\theta}{\partial \theta}(N(z, r, r/\alpha_v)) \right] \cdot \theta.$$

Therefore, we obtain for $(\theta, y) \in N(z, r, r/\alpha_v)^-$

$$\| \pi_\theta \left(G_t(\theta, y) - f(z) \right) \| \geq \eta_v r > \gamma r. \tag{7.84}$$

This proves the exit condition for the homotopy K.

At the end of our homotopy we have the following map

$$(\theta, y) \mapsto f(z) + \left(\frac{\partial f_\theta}{\partial \theta}(z) \cdot \theta, 0 \right) \tag{7.85}$$

We know that $m\left(\frac{\partial f_\theta}{\partial \theta}(z)\right) \geq \eta_v > 0$, hence $\frac{\partial f_\theta}{\partial \theta}(z)$ is an isomorphism. This finishes the proof.

∎

7.6.2 Existence and properties of W^s_q

Lemma 60 *There exists a constant C, such that:*
if $z, q \in D$, $z \in W^s_q$, then

$$\|f^k(q) - f^k(z)\| \leq C\beta_v^k, \qquad k \in \mathbb{Z})+ \tag{7.86}$$

Proof: Since $Q_v(f^k(q) - f^k(z)) \geq 0$ for $k \in \mathbb{Z}_+$, therefore we have

$$\alpha_v \|\pi_y(f^k(q) - f^k(z))\| \geq \|\pi_\theta(f^k(q) - f^k(z))\|. \tag{7.87}$$

Using the above inequality we obtain

$$\|\pi_y(f(z) - f(q))\| \leq \left\| \frac{\partial f_y}{\partial y} \right\| \cdot \|\pi_y(z-q)\| + \left\| \frac{\partial f_y}{\partial \theta} \right\| \cdot \|\pi_\theta(z-q)\| \leq$$
$$\left(\left\| \frac{\partial f_y}{\partial y} \right\| + \alpha_v \left\| \frac{\partial f_y}{\partial \theta} \right\| \right) \|\pi_y(z-q)\| \leq \beta_v \|\pi_y(z-q)\|.$$

For iterates of f we obtain

$$\|\pi_y(f^k(z) - f^k(q))\| \leq \beta_v^k \|\pi_y(z-q)\|$$

and finally

$$\|f^k(z) - f^k(q)\| \quad \leq \quad (1+\alpha_v)\|\pi_y(f^k(z) - f^k(q))\| \leq \beta_v^k (1+\alpha_v)\|\pi_y(z-q)\|$$

∎

Lemma 61 *Let $\gamma < 1$ be such that $\beta_v < \gamma < \eta_v$. For any open set $U \subset D$ and such that $\overline{U} \cap W^{cs} \subset \mathrm{int} D$, there exists $r > 0$, such that for any $z \in U \cap W^{cs}$ the following conditions are satisfied:*

- *$N(f^k(z), \gamma^k r, \gamma^k r/\alpha_v) \subset D$ for $k \in \mathbb{Z}_+$*
- *for $k \in \mathbb{Z}_+$ the $\pi_\theta N(f^k(z), \gamma^k r, \gamma^k r/\alpha_v)$ is contained in a good chart (could be different for different k)*

Proof: For the first assertion observe that $D \cap f(D) \cap f^{-1}(D) \subset \Lambda \times B_u(0, R-\delta) \times B_s(0, R-\delta)$ for some $\delta > 0$

The second assertion is easily obtained by taking sufficiently small r. ∎

The lemma below shows that our description of W^s_q capture the classical definition.

Lemma 62 *If $z, q \in W^{cs}$ and there exist constants $0 < \rho < \min(1, \eta_v)$ and $A > 0$, such that*

$$\|f^k(z) - f^k(q)\| \leq A\rho^k, \tag{7.88}$$

then $z \in W^s_q$.

Proof: Let us take $\rho < \gamma < \min(1, \eta_v)$. Let $r = r(q, \gamma)$ be as in Lemma 61. Then there exists k_0, such that

$$f^k(z) \in N(f^k(q), \gamma^k r, \gamma^k r/\alpha_v), \quad k \geq k_0 \tag{7.89}$$

Let k_0 be minimal with this property.

If $f^{k_0}(z) = f^{k_0}(q)$, then our assertion hold as the consequence the backward cone condition (assumption BCC). Therefore we assume that $f^{k_0}(z) \neq f^{k_0}(q)$.

We show that $Q_v(f^{k_0}(z) - f^{k_0}(q)) > 0$. Namely, if this is not the case then $Q_v(f^{k_0}(z) - f^{k_0}(q)) \leq 0$, then from Lemma 37 we obtain

$$\|\pi_\theta(f^{k_0+k}(z) - f^{k_0+k}(q))\| \geq \eta_v^k \|\pi_\theta f^{k_0}(z) - \pi_\theta f^{k_0}(q)\| > 0. \tag{7.90}$$

Observe that we also have

$$\|\pi_\theta(f^{k_0+k}(z) - f^{k_0+k}(q))\| \leq \gamma^{k_0+k} r. \tag{7.91}$$

But $\gamma < \eta_v$, hence (7.90) and (7.91) are incompatible. We obtained a contradiction, therefore

$$Q_v(f^{k_0}(z) - f^{k_0}(q)) > 0. \tag{7.92}$$

From the backward cone condition (assumption BCC) it follows that

$$Q_v(f^k(z) - f^k(q)) > 0, \quad k < k_0. \tag{7.93}$$

Therefore $k_0 = 0$ and $z \in W^s_q$. ∎

Lemmas 60 and 62 immediately imply the following characterization of W^s_q.

Theorem 63 *Assume that $z, q \in W^{cs}$. Then $z \in W^s_q$ iff z, q are in the same good chart and there exists $A > 0$ such that $\|f^k(z) - f^k(q)\| \leq A\beta_v^k$ for $k \in \mathbb{Z}_+$.*

We will now proceed toward proving that W^s_q is a vertical disk in D satisfying the cone condition.

Theorem 64 *Let $\gamma < 1$ be such that $\beta_v < \gamma < \eta_v$, $q = (\theta, y) \in (int\ D) \cap W^{cs}$ and $r > 0$, such that $N(f^k(z), \gamma^k r, \gamma^k r/\alpha_v) \subset D$ and $N(f^k(z), \gamma^k r, \gamma^k r/\alpha_v)$ is* *contained in good chart (might be different for different k) for all $k \in \mathbb{Z}$ (i.e. as obtained in Lemma 61)*

Let

$$S_q(\gamma, r) = \{z \in D \mid f^k(z) \in N(f^k(q), \gamma^k r, \gamma^k r/\alpha_v), \quad k \in \mathbb{Z}_+\}. \tag{7.94}$$

Then $W^s_q \cap S_q(\gamma, r)$ is a vertical disk in $N(q, r, r/\alpha_v)$ satisfying cone condition $Q_v(z_1 - z_2) > 0$ for $z_1 \neq z_2$.

Proof: Let us fix q, r, γ. Consider an infinite chain of covering relations

$$\begin{aligned} N(q, r, r/\alpha_v) \overset{f}{\Longrightarrow} N(f(q), \gamma r, \gamma r/\alpha_v) \overset{f}{\Longrightarrow} N(f^2(q), \gamma^2 r, \gamma^2 r/\alpha_v) \overset{f}{\Longrightarrow} \\ \ldots \overset{f}{\Longrightarrow} N(f^k(q), \gamma^k r, \gamma^k r/\alpha_v) \overset{f}{\Longrightarrow} \ldots \end{aligned} \tag{7.95}$$

From Col. 4 it follows that for any $y \in \overline{B}_s(0, r/\alpha_v)$ there exists $\theta_s(y) \in B_{m+u}(0, r)$ such that

$$f^k(q + (\theta_s(y), y)) \in N(f^k(q), \gamma^k r, \gamma^k r/\alpha_v), \quad k = 0, 1, 2, \ldots. \tag{7.96}$$

Hence $q + (\theta_s(y), y) \in W^s_q$. The uniqueness of $\theta_s(y)$ is a consequence of the cone condition for W^s_q, which we will prove now.

We will reason by a contradiction. Assume that $z_1, z_2 \in W^s_q \cap S_q(\gamma, r)$ are such that

$$Q_v(z_1 - z_2) \leq 0. \tag{7.97}$$

Observe that for pairs of points realizing chain of covering relations (7.95) we can apply Lemma 37 to obtain

$$\|\pi_\theta(f^k(z_1) - f^k(z_2))\| \geq \eta_v^k \|\pi_\theta z_1 - \pi_\theta z_2\| > 0. \tag{7.98}$$

Observe that we also have

$$\|\pi_\theta(f^k(z_1) - f^k(z_2))\| \leq 2\gamma^k r. \tag{7.99}$$

But $\gamma < \eta_v$, hence (7.98) and (7.99) are incompatible. We obtained a contradiction. ∎

Theorem 65 *W^s_q is a vertical disk in D satisfying cone condition $Q_v(z_1 - z_2) > 0$ for $z_1, z_2 \in W^s_q$ $z_1 \neq z_2$.*

Proof: Let us observe that by Theorem 63 $q \in W^s_z$ iff $z \in W^s_z$ and if $q \in W^s_z$ and $z \in W^s_p$, then $q \in W^s_p$, therefore we can patch together local pieces of W^s_q obtained in Theorem 64, to obtain the horizontal disk in D $\theta_s : \overline{B}_s(0, R) \to D$, such that $\pi_y \theta_s(y) = y$ and $W^s_q = \{\theta_s(y) \mid y \in \overline{B}_s(0, R)\}$. It remains to show the cone condition. We know that it holds locally.

The cone condition is equivalent to the following inequality

$$\|\pi_\theta \theta_s(y_1) - \pi_\theta \theta_s(y_2)\| < \alpha_v \|y_1 - y_2\|. \tag{7.100}$$

Now we know that (7.100) holds locally. Now if $y_b \neq y_e$, then on the segment $[y_b, y_e]$ we can chose points $y_0 = y_b, y_1, y_2, \dots, y_k = y_e$, such that to each pair y_i, y_{i+1} (7.100) applies. Therefore we have

$$\|\pi_\theta \theta_s(y_b) - \pi_\theta \theta_s(y_e)\| \leq \sum_{i=1}^{k} \|\pi_\theta \theta_s(y_{i-1}) - \pi_\theta \theta_s(y_i)\| <$$
$$\alpha_v \sum_{i=1}^{k} \|y_{i-1} - y_i\| = \alpha_v \|y_b - y_e\|.$$

∎

Theorem 66 *Let us denote $S(q,y)$ the vertical disk equal to W^s_q. .*
Then the function $S : W^{cs} \times \overline{B}_s(0,R) \to W^{cs}$ is continuous.

Proof: Assume that $\lim_{n\to\infty}(q_n, y_n) = (\bar{q}, \bar{y})$. Due to the compactness of D we can also assume that $S(q_n, y_n) \to p$.

We have to show that $S(\bar{q}, \bar{y}) = p$.

From the definition of W^s_q it follows that for $n, k \in \mathbb{Z}_+$ holds

$$Q_v(f^k(S(q_n, y_n)) - f^k(q_n)) \geq 0. \tag{7.101}$$

Passing to the limit for $n \to \infty$ we obtain for $k \in \mathbb{Z}_+$

$$Q_v(f^k(p) - f^k(\bar{q})) \geq 0. \tag{7.102}$$

The backward cone condition (in fact the local backward condition) implies that if $Q_v(f^k(p) - f^k(\bar{q})) = 0$ and $f^k(p) \neq f^k(q)$, then $Q_v(f^{k+1}(p) - f^{k+1}(\bar{q})) < 0$. This forces all inequalities in (7.102) to be strong as long $f^k(p) \neq f^k(q)$.

Therefore $p = S(\bar{q}, \bar{y})$.

∎

Theorem 67 *Let $q \in W^{cs}$. Then the intersection $W^s_q \cap W^{cu}$ consists of a single point and is cone transversal.*

The intersection $W^s_q \cap \tilde{\Lambda}$ consists of a single point.

Proof: Let $y_{cu} : \Lambda \times \overline{B}_u(0,R) \to \mathrm{int}\overline{B}_s(0,R)$ be such that

$$W^{cu} = \{(\lambda, x, y_{cu}) \mid (\lambda, x) \in \Lambda \times \overline{B}_u(0,R)\}. \tag{7.103}$$

Let $S : W^{cs} \times \overline{B}_s(0,R) \to D$, be such that $S(q, \dot{)}$ is a vertical disk in D satisfying the condition and representing W^s_q.

Let us fix $q \in W^{cs}$. Let an open set $U \subset \Lambda$ be such that $\pi_\lambda(\underline{Q^+_v}(q)) \subset U$, U is contained in good chart. We want to find $(\lambda, x) \in U \times \overline{B}_u(0,R)$ and $y \in \overline{B}_s(0,R)$ such that

$$(\lambda, x, y_{cu}(\lambda, x)) = S(q, y). \tag{7.104}$$

Since we are all possible solutions are a good chart we can rewrite this equation as

$$F(\lambda, x, y) = (\lambda, x, y_{cu}(\lambda, x)) - S(q, y) = 0 \qquad (7.105)$$

We consider equation (7.105) for $(\lambda, x, y) \in \overline{U} \times \overline{B}_u(0, R) \times \overline{B}_s(0, R)$. We will use the local Brouwer degree $\deg(F, \mathrm{int}\overline{U} \times \overline{B}_u(0, R) \times \overline{B}_s(0, R), 0)$ to show the existence of a solution of (7.105).

First we show that for $(\lambda, x, y) \in \partial U \times \overline{B}_u(0, R) \times \overline{B}_s(0, R)$ we have $F(\lambda, x, y) \neq 0$. This implies that the local Brouwer degree is defined.

For the proof we consider three cases: $\lambda \in \partial U$, $x \in \partial xdom$ and $y \in \partial\overline{B}_s(0, R)$. Let $\lambda \in \partial U$. In this situation we know for any $x \in \overline{B}_u(0, R)$ holds $\pi_\lambda(S(q, x)) \in \pi_\lambda(Q_v^+(q)) \subset U$, hence $\pi_\lambda F(\lambda, x, y) \neq 0$.

If $x \in \partial\overline{B}_u(0, R)$ we have $|\pi_x S(q, y)| < R$ hence $\pi_x F(\lambda, x, y) \neq 0$

Let $y \in \partial\overline{B}_s(0, R)$, we have $|y_{cu}| < R$ and $\pi_y S(q, y) = y$, hence $\pi_y F(\lambda, x, y) \neq 0$. This implies that $\deg(F, \mathrm{int}\overline{U} \times \overline{B}_u(0, R) \times \overline{B}_s(0, R), 0)$ is defined.

Let us define a homotopy $G_t(\lambda, x, y) = (1 - t)F(\lambda, x, y) + t((\lambda, x, 0) - (\pi_\theta q, \pi_x q, y))$ for $t \in [0, 1]$. The same argument which worked for F, shows that for any point $z \in \partial U \times \overline{B}_u(0, R) \times \overline{B}_s(0, R)$ we have $F(z) \neq 0$ works also for G_t. Hence $\deg(F, \mathrm{int}U \times \overline{B}_u(0, R) \times \overline{B}_s(0, R), 0) = \deg(G_1, \mathrm{int}U \times \overline{B}_u(0, R) \times \overline{B}_s(0, R), 0)$. For $t = 1$ equation (7.105) becomes

$$(\lambda, x, 0) - (\pi_\theta q, \pi_x q, y) = 0. \qquad (7.106)$$

The solution of (7.106) is given by $\lambda = \pi_\lambda q$, $x = \pi_x q$, $y = 0$ and belongs to $\mathrm{int}U \times \overline{B}_u(0, R) \times \overline{B}_s(0, R)$, therefore since the system (7.69) is linear, it has nonzero degree. Therefore $\deg(F, \mathrm{int}U \times \overline{B}_u(0, R) \times \overline{B}_s(0, R), 0) \neq 0$, hence also (7.105) has the solution.

The solution is unique, because if $q_1, q_2 \in W_q^s \cap W^{cu}$, then q_1 and q_2 must be contained in a common good chart, hence the cone condition for W^{cu} and W_q^s apply and we have, respectively,

$$Q_v(q_1 - q_2) \leq 0, \quad Q_v(q_1 - q_2) > 0, \qquad (7.107)$$

which is a contradiction.

In fact the cones for W^{cu} and W_q^s are separated, as we have a possibility to change α_v within our assumptions. The cone transversality follows from the case *T1* in Lemma 32.

Since $\tilde{\Lambda} \cap W_q^s \subset W^{cu} \cap W_q^s \subset W^{cu} \cap W^{cs} = \tilde{\Lambda}$ we see that $\tilde{\Lambda} \cap W_q^s$ contains exactly one point.

∎

7.7 Hölder dependence of W_q^s with respect to q

In this section we assume the standard assumptions defined in Section 7.1

We would like to show that W_q^s for $q \in W^{cs}$ satisfies the Hölder condition with respect to q. The same technique of the proof works for W_q^u

In this section $\theta = (\lambda, x)$.

Theorem 68 *Let $\theta_s : W^{cs} \times \overline{B}_s(0,R) \to \Lambda$, be a function such that for $q \in W^{cs}$ holds $W^s_q = \{(\theta_s(q,y),y) \mid y \in \overline{B}_s(0,R)\}$.*

Then θ_s satisfies Hölder with respect to q with Hölder exponent independent of y and $q \in W^{cs}$.

Proof: Let L be the Lipschitz constant for the map f (the maximum of Lipschitz constants over all good charts), if necessary we increase it to have $L \geq 1$.

Let $q_1, q_2 \in W^{cs}$, $y \in \mathbb{Z}_+$. Let us consider points $(\theta_s(q_i,y),y)$, $i = 1,2$. Since $Q_v((\theta_s(q_1,y),y) - (\theta_s(q_2,y),y)) \leq 0$, then from Lemma 37 it follows immediately that for any $k \in \mathbb{Z}_+$ the following inequality holds

$$\|\pi_\theta(f^k(\theta_s(q_1,y),y)) - f^k(\theta_s(q_2,y),y)))\| \geq \eta_v^k \|\theta_s(q_1,y) - \theta_s(q_2,y)\| \quad (7.108)$$

as long as $\pi_\theta f^k(\theta_s(q_1,y),y)), \pi_\theta f^k(\theta_s(q_2,y),y))$ belong to the same good chart (maybe different good charts for different k).

The number of iterates for which (7.108) can be estimated as follows

$$L^k(\|q_1 - q_2\|) < \tilde{\Delta} = \Delta - 2R\alpha_v, \quad (7.109)$$

where Δ is the size of good chart over the whole Λ. Indeed, the distance between $\pi_\theta f^k(q_1)$ and $\pi_\theta f^k(q_2)$ is estimated by $L^k\|q_1 - q_2\|$ to estimate $\pi_\theta(f^k(q_i) - f^k(\theta_s(q_i,y),y))$ we use the fact that $Q_v(f^k(q_i) - f^k(\theta_s(q_i,y),y)) > 0$, which gives an estimate on the difference by $2R\alpha_v$.

Now we derive an upper bound for $\|\pi_\theta(f^k(\theta_s(q_1,y),y)) - f^k(\theta_s(q_2,y),y)))\|$.

From Theorem 65 it follows that there exists a constant $C > 0$, such that

$$\begin{aligned}
\|\pi_\theta(f^k(\theta_s(q_1,y),y)) - f^k(\theta_s(q_2,y),y)))\| &\leq \\
\|f^k(\theta_s(q_1,y),y)) - f^k(\theta_s(q_2,y),y))\| &\leq \\
\|f^k(\theta_s(q_1,y),y)) - f^k(q_1)\| + \|f^k(q_1) - f^k(q_2)\| &+ \\
\|f^k(q_2) - f^k(\theta_s(q_2,y),y))\| \leq C\beta_v^k + L^k\|q_1 - q_2\| &+ \\
C\beta_v^k = 2C\beta_v^k + L^k\|q_1 - q_2\|&.
\end{aligned}$$

Combining the above estimates we obtain

$$\|\theta_s(q_1,y) - \theta_s(q_2,y)\| \leq 2C\left(\frac{\beta_v}{\eta_v}\right)^k + \left(\frac{L}{\eta_v}\right)^k \|q_1 - q_2\|.$$

It should be stressed that (7.110) holds for given k only if (7.109) is satisfied.

Since $\frac{\beta_v}{\eta_v} < 1$ it is easy to see that $\theta_s(q_2,y)$ is uniformly continuous with respect to q, with constants independent from y.

Now we will show that $\theta_s(\cdot,y)$ satisfies the Hölder condition, by which we mean that there exist E, $\gamma > 0$, $\kappa > 0$ such that

$$\|\theta_s(q_1,y) - \theta_s(q_2,y)\| \leq E\|q_1 - q_2\|^\gamma, \quad \forall y \in \overline{B}_s(0,R), \quad \|q_1 - q_2\| \leq \kappa \quad (7.110)$$

Our strategy is as follows. We fix $\delta = \|q_1 - q_2\|$ and for each δ we chose an integer $k = k(\delta)$ for which we evaluate (7.110) to check condition (7.110), for some constants E and γ, which will be determined during the procedure. After this is accomplished we need to check that (7.109) is satisfied for δ small enough.

Let us fix y and let $\rho(q) = \theta_s(q, y)$. Our point of departure is (7.110), which we rewrite as follows with $C_1 = 1$ and $C_2 = 2C$

$$\|\rho(q_1) - \rho(q_2, y)\| \leq C_2 \left(\frac{\beta_v}{\eta_v}\right)^k + C_1 \left(\frac{L}{\eta_v}\right)^k \|q_1 - q_2\| \tag{7.111}$$

Let us denote $\delta = \|q_1 - q_2\|$. For any γ and $k \in \mathbb{Z}_+$

$$\frac{\|\rho(q_1) - \rho(q_2)\|}{\|q_1 - q_2\|^\gamma} \leq C_2 \left(\frac{\beta_v}{\eta_v}\right)^k \delta^{-\gamma} + C_1 \left(\frac{L}{\eta_v}\right)^k \delta^{1-\gamma}. \tag{7.112}$$

Observe that (7.110) holds if there exists constants E_1, E_2 such that for each $0 < \delta < \kappa$ there exists $k \in \mathbb{Z}_+$ such that the following inequalities are satisfied

$$C_2 \left(\frac{\beta_v}{\eta_v}\right)^k \delta^{-\gamma} \leq E_1 \tag{7.113}$$

$$C_1 \left(\frac{L}{\eta_v}\right)^k \delta^{1-\gamma} \leq E_2. \tag{7.114}$$

Let us fix values of E_1, E_2, γ to be specified later.

Let us fix $\delta > 0$. The strategy is as follows: first from (7.113) we compute k and then we insert it to (7.114), which will give an inequality, which should hold for any $0 < \delta < \kappa$, this will produce bound for γ.

From (7.113) we obtain

$$\begin{aligned} \left(\frac{\eta_v}{\beta_v}\right)^k &\geq \frac{C_2 \delta^{-\gamma}}{E_1} \\ k \ln\left(\frac{\eta_v}{\beta_v}\right) &\geq \ln\frac{C_2}{E_1} - \gamma \ln \delta \end{aligned} \tag{7.115}$$

We set

$$k_0 = \left(\ln\frac{C_2}{E_1} - \gamma \ln \delta\right) \left(\ln\left(\frac{\eta_v}{\beta_v}\right)\right)^{-1}. \tag{7.116}$$

Observe that k_0 can be very large for δ very small, from now on we assume that δ is sufficiently small, such that $k_0 > 0$ (the size of δ depends on γ and E_1, which will assigned the values later). We set $k = k(\delta) = \lfloor k_0 + 1 \rfloor$, where $\lfloor x \rfloor$ is the integer part of $x \in \mathbb{R}$. With this choice of k equation (7.113) is satisfied.

Now we work with (7.114). Since

$$\left(\frac{L}{\eta_v}\right)^k \leq \left(\frac{L}{\eta_v}\right)^{k_0+1}$$

then (7.114) is satisfied if the following inequality is satisfied

$$\ln C_1 + \left(1 + \left(\ln \frac{C_2}{E_1} - \gamma \ln \delta\right)\left(\ln\left(\frac{\eta_v}{\beta_v}\right)\right)^{-1}\right) \ln\left(\frac{L}{\eta_v}\right) + (1-\gamma)\ln\delta \leq \ln E_2.$$

After an rearrangement we obtain the following

$$(\ln \delta)\left(-\gamma \frac{\ln\left(\frac{L}{\eta_v}\right)}{\ln\left(\frac{\eta_v}{\beta_v}\right)} + (1-\gamma)\right) \leq -\ln C_1 +$$

$$\ln E_2 - \left(1 + \left(\ln \frac{C_2}{E_1}\right)\left(\ln\left(\frac{\eta_v}{\beta_v}\right)\right)^{-1}\right)\ln\left(\frac{L}{\eta_v}\right)$$

Since $0 < \delta < \kappa$, then we need the coefficient on lhs in the last equation by $\ln\delta$ to be nonnegative, then we can take $E_1 = C_2$ and E_2 large enough for the right hand side to be positive.

$$1 - \gamma\left(1 + \frac{\ln \frac{L}{\eta_v}}{\ln \frac{\eta_v}{\beta_v}}\right) \geq 0$$

$$1 + \frac{\ln \frac{L}{\eta_v}}{\ln \frac{\eta_v}{\beta_v}} = \frac{\ln \frac{L}{\beta_v}}{\ln \frac{\eta_v}{\beta_v}}$$

$$\gamma \leq \frac{\ln \frac{\eta_v}{\beta_v}}{\ln \frac{L}{\beta_v}}.$$

It remains to verify (7.109) for $k = k(\delta)$. We set $\gamma = \frac{\ln \frac{\eta_v}{\beta_v}}{\ln \frac{L}{\beta_v}}$, $E_1 = C_2$.

We rewrite (7.109) as

$$L^{k(\delta)}\delta \leq \tilde{\Delta}. \tag{7.117}$$

It is easy to see that it is enough to have

$$L^{k_0+1}\delta \leq \tilde{\Delta}, \tag{7.118}$$

where k_0 is given by (7.116). After substituting the values for γ and E_1 we obtain

$$k_0 = -\frac{\ln \delta}{\ln \frac{L}{\beta_v}}. \tag{7.119}$$

Condition (7.118) is equivalent to the following inequality

$$(k_0 + 1)\ln L + \ln \delta \leq \ln \tilde{\Delta}. \tag{7.120}$$

An easy computation shows that

$$(k_0 + 1)\ln L + \ln\delta = \left(1 - \frac{\ln \delta}{\ln \frac{L}{\beta_v}}\right)\ln L + \ln \delta =$$

$$\left(1 - \frac{\ln L}{\ln \frac{L}{\beta_v}}\right)\ln\delta + \ln L = \left(\frac{-\ln \beta_v}{\ln L - \ln \beta_v}\right)\ln \delta + \ln L.$$

Therefore we end up with the following inequality

$$\left(\frac{-\ln\beta_v}{\ln L-\ln\beta_v}\right)\ln\delta+\ln L\leq\ln\tilde{\Delta} \tag{7.121}$$

Since $\frac{-\ln\beta_v}{\ln L-\ln\beta_v}>0$, we see that for δ small enough (7.121) is satisfied. This completes the proof. ∎

Chapter 8

Smoothness

The goal of this chapter is to improve the results obtained so far and to show that if $f \in C^2$ then $\tilde{\Lambda}$, $W^{cu,cs}$, $W_q^{u,s}$ are C^1 under some additional conditions.

8.1 Jet evolution

The goal of this section is to provide a tool, which will allows to prove the C^1 smoothness of W^{cs}, W^{cu}, $W_q^{s,u}$.

Let $x \in \mathbb{R}^u$, $y \in \mathbb{R}^s$ and $M > 0$.

The 'unstable' and 'stable' jets are given by

$$J_u(z_0, A, M) = \{z_0 + (x, Ax + y), \quad \|y\| \leq M\|x\|^2\} \tag{8.1}$$

where $A : \mathbb{R}^u \to \mathbb{R}^s$ is a linear map (a matrix).

$$J_s(z_0, A, M) = \{z_0 + (x + Ay, y), \quad \|y\|^2 \leq M\|x\|\} \tag{8.2}$$

where $A : \mathbb{R}^s \to \mathbb{R}^u$ is a linear map (a matrix).

For $\delta > 0$ we define

$$\begin{aligned} J_u(z_0, A, M, \delta) &= J_u(z_0, A, M) \cap \overline{B}(0, \delta) \\ J_s(z_0, A, M, \delta) &= J_s(z_0, A, M) \cap \overline{B}(0, \delta) \end{aligned}$$

The above defined jets are devised to control the second derivatives of functions. The following lemmas explain this relation.

Lemma 69 *Assume that $g : \mathbb{R}^u \supset \mathrm{dom}(g) \to \mathbb{R}^s$ is a C^2 function. Let $x_0 \in \mathrm{dom}(g)$, $M > \|D^2 g(x_0\|$. Then there exists $\delta > 0$, such that*

$$\{(x, g(x)) \mid \|x - x_0\| \leq \delta\} \subset J_u((x_0, g(x_0)), Dg(x_0), M/2). \tag{8.3}$$

Proof: From the Taylor formula it follows that if $\|x - x_0\| \leq \delta$ holds

$$\|g(x) - g(x_0) - Dg(x_0)(x - x_0)\| \leq M\|x - x_0\|^2/2. \tag{8.4}$$

Therefore for $\|x - x_0\| \leq \delta$ we have

$$(x, g(x)) = (x_0, g(x_0)) + (x - x_0, Dg(x_0)(x - x_0) + y)$$

where $y = g(x) - g(x_0) - Dg(x_0)(x - x_0)$ satisfies $\|y\| \leq (M/2)\|x - x_0\|^2$.
Hence (8.3) is satisfied. ∎

Lemma 70 *Assume that $g : \mathbb{R}^s \supset \mathrm{dom}(g) \to \mathbb{R}^u$ is a C^2 function. Let $y_0 \in \mathrm{dom}(g)$, $\|D^2 g(y_0\| < M$. Then there exists $\delta > 0$, such that*

$$\{(g(y), y) \mid \|y - y_0\| \leq \delta\} \cap J_s((g(y_0), y_0), Dg(y_0), 2/M) = \emptyset \tag{8.5}$$

Proof: From the Taylor formula it follows that if $\|y - y_0\| \leq \delta$ holds

$$\|g(y) - g(y_0) - Dg(y_0)(y - y_0)\| < M\|y - y_0\|^2/2. \tag{8.6}$$

Hence if we define $x = g(y) - g(y_0) - Dg(y_0)(y - y_0)$, then

$$\frac{2}{M}\|x\| < \|y - y_0\|^2. \tag{8.7}$$

This implies that point $(g(y), x)$ does not belong to $J_s\left((g(y_0), y_0), Dg(y_0), \frac{2}{M}\right)$
∎

The crucial property of J_u and J_s is that the above lemmas can be reversed to give bounds on the second derivative.

Lemma 71 *Assume that $g : \mathbb{R}^u \supset \mathrm{dom}(g) \to \mathbb{R}^s$ is a C^2 function. Let $x_0 \in \mathrm{dom}(g)$ and assume that there exists $\delta > 0$, such that*

$$\{(x, g(x)) \mid \|x - x_0\| \leq \delta\} \subset J_u((x_0, g(x_0)), Dg(x_0), M/2). \tag{8.8}$$

Then $\|D^2 g(x_0)\| \leq M$.

Lemma 72 *Assume that $g : \mathbb{R}^s \supset \mathrm{dom}(g) \to \mathbb{R}^u$ is a C^2 function. Let $y_0 \in \mathrm{dom}(g)$ and assume that there exists $\delta > 0$, such that*

$$\{(g(y), y) \mid \|y - y_0\| \leq \delta\} \cap J_s((g(y_0), y_0), Dg(y_0), 2/M) = \emptyset \tag{8.9}$$

Then $\|D^2 g(y_0\| < M$.

The proof of above lemmas is an easy consequence of the following fact.

Lemma 73 *Let $M \in \mathbb{R}$ be a fixed positive number.*

Let $U \subset \mathbb{R}^n$ be an open set and let $b : U \to \mathbb{R}$ be a C^2 function, which satisfies for all $p, p + h \in U$

$$|b(p + h) - b(p) - Db(p)h| \leq M\|h\|^2. \tag{8.10}$$

Then

$$\|D^2 b(p)\| \leq 2M, \tag{8.11}$$

$$\left|\frac{\partial^2 b}{\partial p_i \partial p_j}(p)\right| \leq 4M, \qquad p \in U, \quad i, j = 1, \ldots, n. \tag{8.12}$$

Proof: From the Taylor formula with an integral remainder it follows that for any $p, p+h \in U$, such that $[p, p+h] \subset U$ holds

$$b(p+h) - b(p) - Db(p)h = \int_0^1 (1-t)D^2b(p+th)(h,h)dt, \tag{8.13}$$

where $D^2f(z)(h,h) = \sum_{j,k} \frac{\partial^2 f}{\partial y_j y_k}(z)h_j h_k$.

From (8.10) and (8.13) it follows that

$$\left| \int_0^1 (1-t)D^2b(p+th)(h,h)dt \right| \leq M\|h\|^2. \tag{8.14}$$

Let $e \in S^n$ and let $h = se$ for $s \in [0,1]$. From the continuity of D^2b we have

$$\begin{aligned} \int_0^1 (1-t)D^2b(p+th)(h,h)dt &= s^2 \int_0^1 (1-t)D^2b(p+tse)(e,e)dt \\ &= s^2 \int_0^1 (1-t)D^2b(p)(e,e)dt + s^2\epsilon(p,e,s) \\ &= \frac{s^2}{2}D^2b(p)(e,e) + s^2\epsilon(p,e,s), \end{aligned}$$

where $\epsilon(p,e,s) \to 0$ for $s \to 0$ (for fixed p).

Therefore we obtain from (8.14)

$$\left| \frac{s^2}{2}D^2b(p)(e,e) \right| - s^2|\epsilon(p,e,s)| \leq Ms^2.$$

After dividing by s^2 and passing to the limit $s \to 0$ we obtain for all $p \in U$ and $e \in S^n$

$$|D^2b(p)(e,e)| \leq 2M.$$

Second order partial derivatives of b can be obtained from $D^2b(p)(e,e)$ as follows. Let $\{e_i\}_{i=1,\dots,n}$ denote the canonical basis in $\mathbb{R}^n$, then

$$\begin{aligned} \frac{\partial^2 b}{\partial p_i p_j}(p) &= D^2b(p)(e_i,e_j) \\ &= \frac{1}{2}\left(D^2b(p)(e_i+e_j, e_i+e_j) - D^2b(p)(e_i,e_i) - D^2b(p)(e_j,e_j)\right). \end{aligned}$$

We end up with the following estimate

$$\left| \frac{\partial^2 b}{\partial p_i p_j}(p) \right| \leq \frac{1}{2}(2 \cdot 2M + 2M + 2M) = 4M.$$

∎

8.1.1 Motivation

We will discuss here heuristically the use of the unstable and stable jets on the example of the hyperbolic fixed point. Let (x, y) are the unstable and stable coordinates respectively.

Let $b_0(x) = (x, 0)$ be the horizonal disk and let $b_i = \mathcal{G}^i(b_0)$ be its graph transforms. From the cone cone condition we know the functions $\tilde{b}_i = \pi_y b_i$ satisfy Lipschitz condition with the constant L.

Now assume that there exists a constant $M > 0$, such that for any horizontal disk satisfying the cone condition holds for some $\delta > 0$

$$f(J_u(b(x_0), D\tilde{b}(z_0), M, \delta)) \subset J_u(f(b(x_0)), D\mathcal{G}(\pi_x f(b(x_0))), M). \tag{8.15}$$

Then using the induction argument it follows the definition of the unstable jet that

$$\|\tilde{b}(x+h) - \tilde{b}(x) - D\tilde{b}(x)h\| \leq M\|h\|^2 \tag{8.16}$$

which due to Lemma 73 leads to bound for $\|D^2\tilde{b}\|$, which implies the equicontinuity of $D\tilde{b}_i$. Then we use the Ascoli Arzela lemma to obtain a sequence converging uniformly together with their derivatives.

The stable jets will be applied to the functions that are solutions of equations $\pi_y f^k(x_k(y), y) = 0$ for $k \in \mathbb{Z}_+$.

For simplicity we let us consider the case $k = 1$. We will show how we can use the stable jet to give a bound for the second derivative of Dx_1.

Observe that the vertical disk $y \mapsto (x_1(y), y)$ is mapped by f into the disk $y \mapsto (0, y)$.

Assume that there exist a constant $M > 0$ such that for some $\delta > 0$ holds

$$f(J_s((x_1(y_0), y_0), Dx_1(y_0), M, \delta)) \subset J_s(f(x_1(y_0), y_0)), 0, M). \tag{8.17}$$

Let us observe that $\pi_x f(x_1(y_0), y_0) = 0$, hence $\pi_x z = 0$ for $z \in J_s(f(x_1(y_0), y_0)), 0, M)$ is satisfied for $z = f(x_1(y_0), y_0)$.

Therefore $(x_1(y_0 + h), y_0 + h) \notin J_s((x_1(y_0), y_0), Dx_1(y_0), M, \delta)$ for h sufficiently small. This means that

$$M\|x_1(y_0 + h) - x(y_0) - Dx_1(y_0)h\| \leq \|h\|^2.$$

Just as before this gives us bound for the second derivative of x_1.

Our goal in the remainder of this chapter will to to formulate the conditions, which will guarantee that (8.15) and (8.17) are satisfied for some $M > 0$.

8.1.2 Unstable jets

First let us state the local lemma in good coordinates.

Lemma 74 *Let $D = \overline{B}(0, \delta_0) \subset \mathbb{R}^u \times \mathbb{R}^s$. Assume that $f : D \to \mathbb{R}^u \times \mathbb{R}^s$ is a C^2 map, such that*

$$f(0) = 0, \quad \frac{\partial f_y}{\partial x}(0) = 0. \tag{8.18}$$

Let C, B be positive constants, such that

$$\|D^2 f(D)\| \leq 2C, \quad \left\|\frac{\partial f_x}{\partial y}(0)\right\| \leq B. \tag{8.19}$$

Let

$$\mu = \left\|\frac{\partial f_y}{\partial y}(0)\right\| \tag{8.20}$$

$$\lambda = m\left(\frac{\partial f_x}{\partial x}(0)\right). \tag{8.21}$$

Assume that

$$\lambda > 0, \quad \frac{\mu}{\lambda^2} < 1. \tag{8.22}$$

Then there exists $M_1 = M_1\left(C, B, 1/\lambda, \frac{\mu}{\lambda^2}\right)$, such that for any $M_1 < M$ there exists $\delta = \delta(M) \leq \delta_0$ such that

$$f(J_u(0,0,M,\delta)) \subset J_u(0,0,M) \tag{8.23}$$

M_1 is bounded for C, B $1/\lambda$ bounded and $\frac{\mu}{\lambda^2} < \rho < 1$.

Proof: Let us introduce the following notations

$$D_{11} = \frac{\partial f_x}{\partial x}(0), \quad D_{12} = \frac{\partial f_x}{\partial y}(0), \quad D_{22} = \frac{\partial f_y}{\partial y}(0),$$

then

$$f(x,y) = (D_{11}x + D_{12}y, D_{22}y) + (\|x\|^2 + \|y\|^2)g(x,y), \tag{8.24}$$

where $\|g(x,y)\| \leq C$.

Let $(x,y) \in J_u(0,0,M,\delta_0)$. Let $(x_1,y_1) = f(x,y)$. We have

$$\|x_1\| \geq m(D_{11})\|x\| - \|D_{12}\| \cdot \|y\| - C(\|x\|^2 + \|y\|^2) \geq \\ \lambda\|x\| - BM\|x\|^2 - C\|x\|^2(1 + M^2\|x\|^2)$$

It is easy to see that for $\|x\|$ small enough our lower bound for $\|x_1\|$ is positive.

$$\|y_1\| \leq \|D_{22}\| \cdot \|y\| + C(\|x\|^2 + \|y\|^2) \leq \|x\|^2(M\mu + C(1 + M^2\|x\|^2))$$

Hence for $\|x\| \leq (1/M)^{1+\alpha}$ with $\alpha > 0$ holds

$$\frac{\|y_1\|}{\|x_1\|^2} \leq \frac{\|x\|^2(M\mu + C(1 + M^2\|x\|^2))}{(\lambda\|x\| - BM\|x\|^2 - C\|x\|^2(1 + M^2\|x\|^2))^2} = \\ M\frac{\mu + \frac{C}{M}(1 + M^2\|x\|^2)}{(\lambda - BM\|x\| - C\|x\|(1 + M^2\|x\|^2))^2} \leq \\ M\frac{\mu + \frac{C}{M}(1 + M^{-2\alpha})}{\lambda^2\left(1 - B/(\lambda M^\alpha) - C(1 + 1/M^{2\alpha})/(\lambda M^{1+\alpha})\right)^2}$$

It is easy to see that be taking $M > M_1$ large enough we obtain our assertion. ∎

Now we want to formulate a general theorem about the propagation of the unstable jets.

Theorem 75 *Let $D = \overline{B}(q, \delta_0) \subset \mathbb{R}^u \times \mathbb{R}^s$. Assume that $f : D \to \mathbb{R}^u \times \mathbb{R}^s$ is a C^2 map. Assume that*

$$\{Df(q)(x, A_0 x) \mid x \in \mathbb{R}^u\} \subset \{(x, A_1 x) \mid x \in \mathbb{R}^u\}. \tag{8.25}$$

Let C, B and L be positive constants, such that

$$\|D^2 f(D)\| \leq 2C, \quad \left\| \frac{\partial f_x}{\partial y}(q) \right\| \leq B, \quad \|A_0\| \leq L, \quad \|A_1\| \leq L. \tag{8.26}$$

Let

$$\mu = \left\| \frac{\partial f_y}{\partial y}(q) - A_1 \frac{\partial f_x}{\partial y}(q) \right\| \tag{8.27}$$

$$\lambda = m\left(\frac{\partial f_x}{\partial x}(q) + \frac{\partial f_x}{\partial y}(q) A_0 \right). \tag{8.28}$$

Assume that

$$\lambda > 0, \quad \frac{\mu}{\lambda^2} < 1. \tag{8.29}$$

Then there exists $M_1 = M_1\left(C, B, L, 1/\lambda, \frac{\mu}{\lambda^2}\right)$, such that for any $M > M_1$ there exists $\delta = \delta(M) \leq \delta_0$ such that

$$f(J_u(q, A_0, M, \delta)) \subset J_u(f(q), A_1, M) \tag{8.30}$$

M_1 is bounded for C, B, L, $1/\lambda$ bounded and $\frac{\mu}{\lambda^2} < \rho < 1$.

Proof: We would like to change the coordinates around q and $\bar{q} = f(q)$, so that the map f in these coordinates will be like in the assumptions of Lemma 74.

Let us denote $q = (x_q, y_q)$ and $\bar{q} = (x_{\bar{q}}, y_{\bar{q}})$.

Let (x_0, y_0) be the new coordinates in the neighborhood of q given by $\Phi_{q, z_0 \to z}$

$$\begin{aligned} x &= x_q + x_0, \\ y &= y_q + A_0 x_0 + y_0 \end{aligned}$$

The inverse transformation is $\Phi_{q, z \to z_0}$

$$\begin{aligned} x_0 &= x - x_q, \\ y_0 &= y - y_q - A_0(x - x_q) \end{aligned}$$

Analogously (x_1, y_1) be coordinates around $f(q)$ given by $\Phi_{f(q), z_1 \to z}$

$$\begin{aligned} x &= x_q + x_1, \\ y &= y_q + A_1 x_1 + y_1 \end{aligned}$$

The inverse transformation is $\Phi_{f(q), z \to z_1}$

$$\begin{aligned} x_1 &= x - x_q, \\ y_1 &= y - y_q - A_1(x - x_q) \end{aligned}$$

Now let $\tilde{f}(x_0, y_0) = \Phi_{f(q),z\to z_1}(f(\Phi_{q,z_0\to z}(x_0,y_0)))$, i.e. we express f in new coordinates.

Observe that in coordinates (x_0, y_0) the set $J_u(q, A_0, M)$ is just $J_u(0, 0, M)$, i.e. $\Phi_{q,z\to z_0}(J_u(q, A_0, M)) = J_u(0, 0, M)$. Analogously, $\Phi_{f(q),z\to z_1}(J_u(f(q), A_1, M)) = J_u(0, 0, M)$.

Obviously

$$\tilde{f}(0) = 0. \tag{8.31}$$

Now we compute the derivative of $\tilde{f}$. We have

$$D\tilde{f}(0) = \begin{bmatrix} I & 0 \\ -A_1 & I \end{bmatrix} \cdot \begin{bmatrix} Df_{11} & Df_{12} \\ Df_{21} & Df_{22} \end{bmatrix} \cdot \begin{bmatrix} I & 0 \\ A_0 & I \end{bmatrix} = \begin{bmatrix} D\tilde{f}_{11} & D\tilde{f}_{12} \\ D\tilde{f}_{21} & D\tilde{f}_{22} \end{bmatrix},$$

where

$$\begin{aligned} D\tilde{f}_{11} &= Df_{11} + Df_{12}A_0, \\ D\tilde{f}_{12} &= Df_{12} \\ D\tilde{f}_{21} &= -A_1 Df_{11} + Df_{21} + (-A_1 Df_{12} + Df_{22})A_0 \\ D\tilde{f}_{22} &= -A_1 Df_{12} + Df_{22}. \end{aligned}$$

Observe that from our assumption (8.25) follows that

$$D\tilde{f}_{21} = 0. \tag{8.32}$$

This is obvious if one thinks about the interpretation of coordinates (x_0, y_0) and (x_1, y_1). Assumption (8.25) states that hyperplane $y_0 = 0$ is mapped by Df into the hyperplane $y_1 = 0$. Now we will confirm this by the formal computation.

We have

$$Df(q)(x, A_0x) = (Df_{11}x + Df_{12}A_0x, Df_{21}x + Df_{22}A_0x).$$

Condition (8.25) implies that

$$A_1(Df_{11}x + Df_{12}A_0x) = Df_{21}x + Df_{22}A_0x \qquad \forall x \in \mathbb{R}^u. \tag{8.33}$$

We can drop x to obtain

$$Df_{11}A_0 + Df_{12} = A_1(Df_{21}A_0 + Df_{22}A_0). \tag{8.34}$$

Hence indeed (8.32) holds.

Now we apply Lemma 74. To do this observe that

$$\begin{aligned} \|D\Phi_{\dots}\| &\leq 1 + L, \\ \|D^2\tilde{f}\| &\leq \|D^2 f\|(1+L)^3 \\ \frac{\partial \tilde{f}_x}{\partial y}(0) &= \frac{\partial f_x}{\partial y}(q), \end{aligned}$$

■

Observe that under the assumptions of Theorem 75 matrix A_1 in (8.25) is uniquely determined. Namely, for any $x \in \mathbb{R}^u$ we have for some x_1

$$Df(q)(x, A_0 x) = (Df_{11}x + Df_{12}A_0 x, Df_{21}x + Df_{22}A_0 x) = (x_1, A_1 x_1).$$

From (8.29) it follows that the matrix $Df_{11} + Df_{12}A_0$ is invertible, hence

$$A_1 = (Df_{21}x + Df_{22}A_0)(Df_{11} + Df_{12}A_0)^{-1}. \tag{8.35}$$

8.1.3 Stable jets

First let us state the local lemma in good coordinates.

Lemma 76 *Let $D = \overline{B}(0, \delta_0) \subset \mathbb{R}^u \times \mathbb{R}^s$. Assume that $f : D \to \mathbb{R}^u \times \mathbb{R}^s$ is a C^2 map, such that*

$$f(0) = 0, \quad \frac{\partial f_x}{\partial y}(0) = 0. \tag{8.36}$$

Let C, B be positive constants, such that

$$\|D^2 f(D)\| \le 2C, \quad \left\| \frac{\partial f_y}{\partial x}(0) \right\| \le B. \tag{8.37}$$

Let

$$\mu = \left\| \frac{\partial f_y}{\partial y}(0) \right\| \tag{8.38}$$

$$\lambda = m\left(\frac{\partial f_x}{\partial x}(0) \right). \tag{8.39}$$

Assume that

$$\lambda > 0, \quad \frac{\mu^2}{\lambda} < 1. \tag{8.40}$$

Then there exists $M_2 = M_2\left(C, B, 1/\lambda, \frac{\mu^2}{\lambda}\right) > 0$, such that for any $M < M_2$ there exists $\delta = \delta(M) \le \delta_0$ such that

$$f(J_s(0, 0, M, \delta)) \subset J_s(0, 0, M). \tag{8.41}$$

Moreover, if for some $K > 0$ and $\rho < 1$ holds $C, B, 1/\lambda \in [0, K]$ and $\frac{\mu^2}{\lambda} < \rho$, then $M_2 > \epsilon(K, \rho) > 0$.

Proof: Let us introduce the following notations

$$D_{11} = \frac{\partial f_x}{\partial x}(0), \quad D_{21} = \frac{\partial f_y}{\partial x}(0), \quad D_{22} = \frac{\partial f_y}{\partial y}(0),$$

then

$$f(x, y) = (D_{11}x, D_{21}x + D_{22}y) + (\|x\|^2 + \|y\|^2)g(x, y), \tag{8.42}$$

where $\|g(x,y)\| \leq C$.

Let $(x,y) \in J_s(0,0,M,\delta_0)$. Let $(x_1,y_1)=f(x,y)$. We have

$$\begin{aligned}\|x_1\| &\geq m(D_{11})\|x\| - C(\|x\|^2+\|y\|^2) \geq \lambda\|x\| - \|x\|C(\|x\|+M) = \\ &\|x\|(\lambda - C(\|x\|+M))\end{aligned}$$

It is easy to see that for $\|x\|$ and M small enough our lower bound for $\|x_1\|$ is positive.

$$\begin{aligned}\|y_1\| &\leq \|D_{21}\|\cdot\|x\| + \|D_{22}\|\cdot\|y\| + C(\|x\|^2+\|y\|^2) \leq \\ &B\cdot\|x\| + \mu M^{1/2}\|x\|^{1/2} + \|x\|C(\|x\|+M) = \\ &M^{1/2}\|x\|^{1/2}\left(\mu + \left(\frac{\|x\|}{M}\right)^{1/2}(B + C(\|x\|+M))\right).\end{aligned}$$

Therefore we obtain for $\|x\| \leq M^2$

$$\begin{aligned}\frac{\|y_1\|^2}{\|x_1\|} &\leq M\frac{\left(\mu + \left(\frac{\|x\|}{M}\right)^{1/2}(B+C(\|x\|+M))\right)^2}{\lambda - C(\|x\|+M)} \leq \\ &M\frac{\left(\frac{\mu}{\lambda^{1/2}} + \left(\frac{M}{\lambda}\right)^{1/2}\left(B + C(M^2+M)\right)\right)^2}{1 - C(M^2+M)/\lambda}\end{aligned}$$

It is easy to see by taking $M < M_2$ small enough we obtain our assertion. ∎

Now we want to formulate a general theorem about the propagation of stable jets.

Theorem 77 *Let $D = \overline{B}(q,\delta_0) \subset \mathbb{R}^u \times \mathbb{R}^s$. Assume that $f: D \to \mathbb{R}^u \times \mathbb{R}^s$ is a C^2 map.*

Assume that

$$\{Df(q)(A_0 y, y) \mid y \in \mathbb{R}^s\} \subset \{(A_1 y, y) \mid y \in \mathbb{R}^s\}. \tag{8.43}$$

Let C, B and L be positive constants, such that

$$\|D^2 f(D)\| \leq 2C, \quad \left\|\frac{\partial f_y}{\partial x}(q)\right\| \leq B, \quad \|A_0\| \leq L, \quad \|A_1\| \leq L. \tag{8.44}$$

Let

$$\mu = \left\|\frac{\partial f_y}{\partial y}(q) + \frac{\partial f_y}{\partial x}(q)A_0\right\| \tag{8.45}$$

$$\lambda = m\left(\frac{\partial f_x}{\partial x}(q) - A_1\frac{\partial f_y}{\partial x}(q)\right). \tag{8.46}$$

Assume that

$$\lambda > 0, \quad \frac{\mu^2}{\lambda} < 1. \tag{8.47}$$

Then there exists $M_2 = M_2\left(C, B, L, 1/\lambda, \frac{\mu^2}{\lambda}\right) > 0$*, such that for any* $M < M_2$ *there exists* $\delta = \delta(M) \leq \delta_0$ *such that*

$$f(J_s(q, A_0, M, \delta)) \subset J_s(f(q), A_1, M) \tag{8.48}$$

Moreover, if for some $K > 0$ *and* $\rho < 1$ *holds* $C, B, 1/\lambda, L \in [0, K]$ *and* $\frac{\mu^2}{\lambda} < \rho$*, then* $M_2 > \epsilon(K, \rho) > 0$*.*

Proof: We would like to change the coordinates around q and $\bar{q} = f(q)$, so that the map f in these coordinates will be like in the assumptions of Lemma 76.

Let us denote $q = (x_q, y_q)$ and $\bar{q} = (x_{\bar{q}}, y_{\bar{q}})$.

Let (x_0, y_0) be the new coordinates in the neighborhood of q given by $\Phi_{q,z_0 \to z}$

$$\begin{aligned} x &= x_q + A_0 y_0 + x_0, \\ y &= y_q + y_0 \end{aligned}$$

The inverse transformation is $\Phi_{q,z \to z_0}$

$$\begin{aligned} x_0 &= x - x_q - A_0(y - y_q), \\ y_0 &= y - y_q \end{aligned}$$

Analogously (x_1, y_1) be coordinates around $\bar{q}$ given by $\Phi_{f(q),z_1 \to z}$

$$\begin{aligned} x &= x_{\bar{q}} + A_1 y_1 + x_1, \\ y &= y_{\bar{q}} + y_1 \end{aligned}$$

The inverse transformation is $\Phi_{f(q),z \to z_1}$

$$\begin{aligned} x_1 &= x - x_{\bar{q}} - A_1(y - y_{\bar{q}}), \\ y_1 &= y - y_{\bar{q}}. \end{aligned}$$

Now let $\tilde{f}(x_0, y_0) = \Phi_{f(q),z \to z_1}(f(\Phi_{q,z_0 \to z}(x_0, y_0)))$, i.e. we express f in new coordinates.

Observe that in coordinates (x_0, y_0) the set $J_s(q, A_0, M)$ is just $J_s(0, 0, M)$, i.e. $\Phi_{q,z \to z_0}(J_s(q, A_0, M)) = J_s(0, 0, M)$. Analogously, $\Phi_{f(q),z \to z_1}(J_s(f(q), A_1, M)) = J_s(0, 0, M)$.

Obviously

$$\tilde{f}(0) = 0. \tag{8.49}$$

Now we compute the derivative of $\tilde{f}$. We have

$$D\tilde{f}(0) = \left[\begin{array}{cc} I & -A_1 \\ 0 & I \end{array}\right] \cdot \left[\begin{array}{cc} Df_{11} & Df_{12} \\ Df_{21} & Df_{22} \end{array}\right] \cdot \left[\begin{array}{cc} I & A_0 \\ 0 & I \end{array}\right] = \left[\begin{array}{cc} D\tilde{f}_{11} & D\tilde{f}_{12} \\ D\tilde{f}_{21} & D\tilde{f}_{22} \end{array}\right],$$

where

$$\begin{aligned} D\tilde{f}_{11} &= Df_{11} - A_1 Df_{21}, \\ D\tilde{f}_{12} &= Df_{11}A_0 - A_1 Df_{21}A_0 + Df_{12} - A_1 Df_{22} \\ D\tilde{f}_{21} &= Df_{21} \\ D\tilde{f}_{22} &= Df_{21}A_0 + Df_{22}. \end{aligned}$$

Observe that from our assumption (8.43) follows that

$$D\tilde{f}_{12} = 0. \tag{8.50}$$

(8.50) is obvious if one thinks about the interpretation of coordinates (x_0, y_0), (x_1, y_1). Assumption (8.43) states that the hyperplane $x_0 = 0$ is mapped by Df into the hyperplane $x_1 = 0$. Now we will confirm this by the formal computation.

We have

$$Df(q)(A_0 y, y) = ((Df_{11}A_0 + Df_{12})y, (Df_{21}A_0 + Df_{22})y).$$

Condition (8.43) implies that

$$(Df_{11}A_0 + Df_{12})y = A_1(Df_{21}A_0 + Df_{22})y, \qquad \forall y \in \mathbb{R}^s. \tag{8.51}$$

We can drop y to obtain

$$Df_{11}A_0 + Df_{12} = A_1(Df_{21}A_0 + Df_{22}). \tag{8.52}$$

Hence indeed (8.50) holds.

Now we apply Lemma 76. To do this observe that

$$\begin{aligned} \|D\Phi_{\dots}\| &\leq 1 + L, \\ \|D^2\tilde{f}\| &\leq \|D^2 f\|(1+L)^3 \\ \frac{\partial \tilde{f}_y}{\partial x}(0) &= \frac{\partial f_y}{\partial x}(q), \end{aligned}$$

∎

Observe that in the assumptions of Theorem 77 matrix A_1 does not need to be unique. It is unique when $Df_{21}A_0 + Df_{22}$ is invertible. In such situation

$$A_1 = (Df_{11}A_0 + Df_{12})(Df_{21}A_0 + Df_{22})^{-1}. \tag{8.53}$$

To see an example of nonuniqueness we take $x, y \in \mathbb{R}$ $f(x, y) = (x, 0)$. Then $f(J_s(0, 0, M)) \subset \mathbb{R} \times \{0\} \subset J_s(0, a, M)$ for any $a \in \mathbb{R}$ and $M \in \mathbb{R}_+$.

8.2 Smoothness of the center-unstable manifold and unstable fibers

The goal of this section is to prove the smoothness of W^{cu} and fibers W^u_q if $f \in C^2$. The assume the standard assumptions defined in Section 7.1.

The proofs are smoothness of W^{cu} and W^u_q are almost identical, they are based on the graph transform for center-horizontal and horizontal disks and estimates for for the 'unstable' jets in the sense of Section 8.1.

8.2.1 Smoothness of the center-unstable manifold

Theorem 78 *Assume $f \in C^2$.*

Let

$$\begin{aligned}\beta_{cu} &= \sup_{q\in D}\left(\left\|\frac{\partial f_y}{\partial y}(q)\right\| + \frac{1}{\alpha_v}\left\|\frac{\partial f_{(\lambda,x)}}{\partial y}(q)\right\|\right)\\ \tilde{\eta}_v &= \inf_{z\in D}\left(m\left(\frac{\partial f_{(\lambda,x)}}{\partial(\lambda,x)}(z)\right) - \frac{1}{\alpha_v}\left\|\frac{\partial f_{(\lambda,x)}}{\partial y}(z)\right\|\right)\end{aligned}$$

We assume that

$$\frac{\beta_{cu}}{\tilde{\eta}_v^2} < 1. \tag{8.54}$$

Then the center-unstable manifold W^{cu} is of class C^1 in the following sense, there exists a C^1 function $\chi_{cu} : \Lambda \times \overline{B}_u(0,R) \to \text{int}\overline{B}_s(0,R)$, such that

$$W^{cu} = \{(\theta, \chi_{cu}(\theta)) | \theta \in \Lambda \times \overline{B}_u(0,R)\}.$$

Proof:

We use notation $\theta = (\lambda, x)$.

The existence of W^{cu} and χ_{cu} has been established in Theorem 47. It remains to show that (8.54) implies that χ_{cu} is C^1.

To achieve this we use the graph transform of center-horizontal disks in D satisfying cone condition defined in Section 7.2.1. Let $b : \Lambda\times\overline{B}_u(0,R) \to D$ given by $b(\theta) = (\theta, 0)$. From Lemma 43 and Theorem 48 it follows that $b_i = \mathcal{G}^i_{ch}(b)$ are center-horizontal disks in D satisfying the cone condition of class C^2 and b_i uniformly converge to $\theta \mapsto (\theta, \chi_{cs}(\theta))$. In the sequel we will use the following notation. $\tilde{b}_i = \pi_y b$.

Observe that the cone condition for b_i implies that $\pi_y b_i$ has the derivative bounded by $1/\alpha_v$. Indeed, since $Q_v(b_i(\theta_1) - b_i(\theta_2)) \leq 0$ if $\|\theta_1 - \theta_2\| < 2R_\Lambda$, then we obtain

$$-(\theta_1 - \theta_2)^2 + \alpha_v^2(\tilde{b}_i(\theta_1) - \tilde{b}_i(\theta_2))^2 \leq 0 \tag{8.55}$$

hence

$$\|\tilde{b}_i(\theta_1) - \tilde{b}_i(\theta_2)\| \leq \|\theta_1 - \theta_2\|/\alpha_v. \tag{8.56}$$

Hence

$$\|D\tilde{b}_i\| \leq 1/\alpha_v. \tag{8.57}$$

Since the derivatives are uniformly bounded, the idea of the proof is to show that $D^2\tilde{b}_i$ are uniformly bounded, which will imply the equicontinuity of $D\tilde{b}_i$ and will allow to apply the Ascoli-Arzela lemma.

The bound for the second derivatives will obtained from Theorem 75. To verify assumptions of this theorem we define

$$\mu = \sup_{i\in\mathbb{Z}_+,\theta\in\Lambda\times\overline{B}_u(0,R)}\left\|\frac{\partial f_y}{\partial y}(b_i(\theta)) - \frac{\partial\tilde{b}_{i+1}}{\partial\theta}(\pi_\theta f(b_i(\theta)))\frac{\partial f_\theta}{\partial y}(b_i(\theta))\right\| \tag{8.58}$$

$$\rho = \inf_{i\in\mathbb{Z}_+,\theta\in\Lambda\times\overline{B}_u(0,R)} m\left(\frac{\partial f_\theta}{\partial\theta}(b_i(\theta)) + \frac{\partial f_\theta}{\partial y}(b_i(\theta))\frac{\partial\tilde{b}_i}{\partial\theta}(\theta)\right). \tag{8.59}$$

We want check whether $\lambda > 0$ and $\frac{\mu}{\lambda^2} < 1$.

Using (8.57) we obtain

$$\lambda \geq \inf_{q \in D} \left(m \left(\frac{\partial f_\theta}{\partial \theta}(q) \right) - \left\| \frac{\partial f_\theta}{\partial y}(q) \right\| \cdot \alpha_v^{-1} \right) = \tilde{\eta}_v > 0.$$

For μ we have the following estimate

$$\mu \leq \sup_{q \in D} \left(\left\| \frac{\partial f_y}{\partial y}(q) \right\| + \left\| \frac{\partial f_\theta}{\partial y}(q) \right\| \alpha_v^{-1} \right) = \beta_{cu}.$$

From our assumptions it follows that $\frac{\mu}{\lambda^2} < 1$.

Let B and C be constants such that $\|D^2 f(D)\| \leq 2C$, $\left\| \frac{\partial f_x}{\partial y}(q) \right\| \leq B$. Let $M = M_1(C, B, 1/\alpha_v, \frac{\mu}{\lambda^2})$ be a constant obtained from Theorem 75.

Then there exists $\delta > 0$ such that for any $i \in \mathbb{Z}_+$ and $\theta \in \Lambda$ holds

$$f(J_u(b_i(\theta), D\tilde{b}_i(\theta), M, \delta)) \subset J_u(f(b_i(\theta)), D\tilde{b}_{i+1}(\pi_\theta f(b_i(\theta))), M) \tag{8.60}$$

Observe for $i = 0$ we have a 'flat' center horizontal disk $b_0(\theta) = (\theta, 0)$, hence for each θ

$$b_0(\theta + h) \subset J_u(b_0(\theta), 0, M), \qquad h \text{ sufficently small} \tag{8.61}$$

Using (8.60,8.61) by an induction argument we obtain that for each θ and i there exists δ (it could be in fact be the same) such that

$$b_i(\theta + h) \subset J_u(b_i(\theta), D\tilde{b}_i(\theta), M), \qquad \|h\| \leq \delta \tag{8.62}$$

This implies that

$$\|\tilde{b}_i(q + h) - \tilde{b}_i(q) - D\tilde{b}_i(q)h\| \leq M^2 h \tag{8.63}$$

It is easy to prove that (8.63) implies a uniform bound on $D^2\tilde{b}_i(q)$.

Now by using the Arzela-Ascoli lemma we can prove that $\tilde{b}_i$ converge in C^1-norm to a C^1 function χ.

∎

8.2.2 Smoothness of the unstable fibers

Theorem 79 *Assume $f \in C^2$.*

Let

$$\begin{aligned} \mu_u &= \sup_{q \in D} \left(\left\| \frac{\partial f_{(\lambda,y)}}{\partial(\lambda, y)}(q) \right\| + \alpha_h \left\| \frac{\partial f_x}{\partial(\lambda, y)}(q) \right\| \right), \\ \tilde{\eta}_h &= \inf_{z \in D} \left(m \left(\frac{\partial f_x}{\partial x}(z) \right) - \alpha_h \left\| \frac{\partial f_x}{\partial(\lambda, y)}(z) \right\| \right) \end{aligned}$$

If

$$\frac{\mu_u}{\tilde{\eta}_h^2} < 1, \tag{8.64}$$

then for every $q \in W^{cu}$ the unstable fiber W^u_q is C^1.

Proof: We proceed as in the proof of Theorem 78. We use the graph transform of horizontal disks in D satisfying the cone condition defined in Def. 41 and for $q \in W^{cu}$ we consider the sequence of horizontal disks in D satisfying the cone condition $d_{k,q} = \mathcal{G}_h^k(h_{f^{-k}}(q))$ defined in Section 7.5 by (7.54). Let us remind the reader that the disk h_q was defined by $h_q(x) = (\pi_\lambda q, x, \pi_y q)$.

Let us observe that

$$\mathcal{G}_h(d_{k-1,f^{-1}(q)}) = d_{k,q}. \tag{8.65}$$

In the reminder of the proof we use notation $\theta = (\lambda, y)$.

It was shown in Section 7.5 (see Lemma 54) that for each $q \in W^{cu}$ $d_{k,q}$ converge uniformly for d_q (in C^0 norm) to W^u_q. We would like to show that it is converging in C^1 norm, which will imply that W^u_q is C^1. We intend to use the Ascoli-Arzela lemma, we need for this a uniform bound for the first and second derivatives of $d_{k,q}$ with respect to x.

Let Z be a set of all horizontal disks in D satisfying the cone condition of class C^1. The bounds for derivatives of $b \in Z$ are obtained from the cone condition for horizontal disks. We have

$$Q_h(b(x_1) - b(x_2)) \geq 0, \tag{8.66}$$

which implies that

$$\|D\pi_y b\| \leq \alpha_h, \quad b \in Z. \tag{8.67}$$

This is gives us bound for the first derivative of $d_{k,q}$

The bound for the second derivatives will obtained from Theorem 75. In that theorem x will be the 'unstable' direction and θ will play the role of y variable. We define

$$\mu = \sup_{b\in Z, x\in \overline{B}_u(0,R), f(x)\in D} \left\| \frac{\partial f_\theta}{\partial \theta}(b(x)) - D\pi_\theta \mathcal{G}_h(b)(\pi_x f(b(x))) \frac{\partial f_x}{\partial y}(b(x)) \right\| \tag{8.68}$$

$$\rho = \inf_{b\in Z, x\in \overline{B}_u(0,R), f(x)\in D} m\left(\frac{\partial f_x}{\partial x}(b(x)) + \frac{\partial f_x}{\partial y}(b(x)) D\pi_\theta b(x) \right). \tag{8.69}$$

We want check whether $\rho > 0$ and $\frac{\mu}{\rho^2} < 1$.

Using (8.67) we obtain

$$\rho \geq \inf_{q\in D} m\left(\frac{\partial f_x}{\partial x}(q) - \alpha_h \left\| \frac{\partial f_x}{\partial y}(q) \right\| \right) = \tilde{\eta}_h.$$

For μ we have the following estimate

$$\mu \leq \sup_{q\in D} \left(\left\| \frac{\partial f_\theta}{\partial \theta}(q) \right\| + \alpha_h \left\| \frac{\partial f_x}{\partial y}(q) \right\| \right) = \mu_u.$$

From our assumptions it follows that $\frac{\mu_u}{\rho^2} < 1$.

Let B and C be constants such that $\|D^2 f(D)\| \leq 2C$, $\left\| \frac{\partial f_x}{\partial \theta}(q) \right\| \leq B$. Let $M = M_1(C, B, 1/\alpha_v, \frac{\mu_u}{\rho^2})$ be a constant obtained from Theorem 75.

Consider now the disks $d_{k,q}$. Each of them in a iterate of graph transform of the flat disk of the form h_p for some $p \in D$. We have

$$h_p(x+v) \subset J_u(h_p(x), 0, M), \qquad v \text{ sufficently small.} \tag{8.70}$$

Hence from Theorem 75 it follows that there exists $\delta > 0$ such that for any $k \in \mathbb{Z}_+$ and $\theta \in \Lambda$ holds

$$f(J_u(d_{k,q}(x), D\pi_\theta d_{k,q}(x), M, \delta)) \subset J_u(f(d_{k,q}(x)), \pi_\theta \mathcal{G}_h(\pi_x f(d_{k,q}(x))), M) \tag{8.71}$$

Using (8.65) (8.71,8.70) by an induction argument we obtain that for each θ and i there exists δ (it could be in fact be the same) such that

$$d_{k,q}(\theta + h) \subset J_u(d_{k,q}(x), D\pi_\theta d_{k,q}(x), M), \qquad \|h\| \leq \delta \tag{8.72}$$

This implies that

$$\|\pi_\theta d_{k,q}(x+h) - \pi_\theta d_{k,q}(q) - D\pi_\theta d_{k,q}(x)h\| \leq M^2 h \tag{8.73}$$

It is easy to prove that (8.73) implies a uniform bound on $D^2\pi_\theta d_{k,q}$.

Now by using the Arzela-Ascoli lemma we can prove that $\pi_\theta d_{k,q}$ converge in C^1-norm to a C^1 function.

∎

8.3 Smoothness of W^{cs} and the stable fibers

8.3.1 Smoothness of W^{cs}

We assume the standard assumptions defined in Section 7.1 and we assume that $f \in C^2$.

The goal of this section is to show that W^{cs} is smooth.

Theorem 80 *Let*

$$\eta_{cs} = \inf_{q \in D} \left(m\left((\frac{\partial f_x}{\partial x}(q) \right) - \frac{1}{\alpha_h} \left\| \frac{\partial f_{(\lambda,y)}}{\partial x}(q) \right\| \right). \tag{8.74}$$

If $\eta_{cs} > 0$ *and* $\frac{\beta_h^2}{\eta_{cs}} < 1$*, then* W^{cs} *is* C^1.

Proof: It was shown in Lemma 51 that W^{cs} is a uniform limit of center-vertical disks in D satisfying cone condition and defined as follows:
for any $k \in \mathbb{Z}_+$ and $(\lambda, y) \in \Lambda \times \overline{B}_s(0, R)$ we consider the following problem

$$\pi_x f^k(\lambda, x, y) = 0 \tag{8.75}$$

under the constraint

$$f^i(\lambda, x, y) \in D. \quad i = 0, 1, \ldots, k \tag{8.76}$$

has a unique solution $x_k(\lambda, y) \in C^2$. The center-horizontal disks converging to W^{cs} are given by

$$d_k(\lambda, y) = (\lambda, x_k(\lambda, y), y). \tag{8.77}$$

Observe that $d_0(\lambda, y) = (\lambda, 0, y)$.

From now in the remainder of the proof we will use notation $\theta = (\lambda, y)$.

The cone condition implies that for any center-vertical disk b holds for $\|\theta_1 - \theta_2\|$ sufficiently small

$$\alpha_h(b(\theta_1) - b(\theta_2)) \leq \theta^2 \tag{8.78}$$

hence

$$\|Dx_k\| \leq 1/\alpha_h, \quad k \in \mathbb{Z}_+. \tag{8.79}$$

We will show that the family of functions $x_k = \pi_x d_k$ has a uniformly bounded second derivative. For this we will use Theorem 77.

Observe that

$$0 = \pi_x f^k(\theta, x_k(\theta)) = \pi_x f^{k-1}(f(\theta, x_k(\theta))).$$

Therefore we obtain

$$\begin{aligned} f(d_k(\theta)) &\in \{d_{k-1}(q) \mid q \in \Lambda \times \overline{B}_s(0, R)\}, \\ f(\theta, x_k(\theta)) &= (\pi_\theta f(\theta, x_k(\theta)), x_{k-1}(\pi_\theta f(\theta, x_k(\theta)))). \end{aligned}$$

Moreover that map f maps the tangent space to d_k and $d_k(\theta)$ into a tangent space of $d_{k-1}(\pi_\theta f(\theta, x_k(\theta)))$.

We consider the stable jets $J_s(d_k(\theta), Dx_k, M)$ for $\theta \in \Lambda \times \overline{B}_s(0, R)$ and $k \in \mathbb{Z}_+$ and using Theorem 77 we want to find $M > 0$, such that for sufficiently small $\delta > 0$ holds

$$f(J_s(d_k(\theta), Dx_k(\theta), M, \delta)) \subset f(J_s(d_{k-1}(\pi_\theta f(\theta, x_k(\theta))), Dx_{k-1}(\pi_\theta f(\theta, x_k(\theta))), M)). \tag{8.80}$$

Let

$$\mu = \sup_{\theta \in \Lambda \times \overline{B}_s(0,R), k \in \mathbb{Z}_+, k>0} \left\| \frac{\partial f_y}{\partial y}(d_k(\theta)) + \frac{\partial f_y}{\partial x}(d_k(\theta)) Dx_k(\theta) \right\| \tag{8.81}$$

$$\rho = \inf_{\theta \in \Lambda \times \overline{B}_s(0,R), k \in \mathbb{Z}_+, k>0} m\left(\frac{\partial f_x}{\partial x}(d_k(\theta)) - D_{k-1}(\pi_\theta f(d_k(\theta))) \frac{\partial f_y}{\partial x}(\theta) \right) \tag{8.82}$$

We need to have

$$\rho > 0, \quad \frac{\mu^2}{\rho} < 1. \tag{8.83}$$

Using (8.79) we obtain

$$\mu \leq \sup_{q \in D} \left(\left\| \frac{\partial f_y}{\partial y}(q) \right\| + \left\| \frac{\partial f_y}{\partial x}(d_k(\theta)) \right\| \alpha_h^{-1} \right) = \beta_h$$

and

$$\rho \geq \inf_{q\in D} \left(m\left(\frac{\partial f_x}{\partial x}(q)\right) - \alpha_h^{-1}\left\|\frac{\partial f_y}{\partial x}(\theta)\right\|\right) = \eta_{cs}.$$

Therefore from our assumptions follows that (8.83) are satisfied. Hence there exists $M > 0$ (independent of k) such that for $k \in \mathbb{Z}_+$ holds (8.80). After k iteration of (8.80) we obtain for some $\delta > 0$ (remember that d_0 is flat $x_0(\theta) = 0$)

$$f^k(J_s(d_k(\theta), Dx_k(\theta), M, \delta)) \subset J_s(f^k(d_k(\theta)), 0, M). \tag{8.84}$$

Observe that (8.84) means the following: let $q \in \Lambda \times \overline{B}_s(0, R)$ if $\|(\theta_0, x_0)\| \leq \delta$, $\|\theta_0\|^2 \leq M\|x_0\|$, then

$$f^k(d_k(q) + (Dx_k(q)\theta_0 + x_0, \theta_0)) = f^k(d_k(q)) + (\theta_1, y_1),$$

where $\|\theta_1\|^2 \leq M\|x_1\|$.

Since $\pi_x f^k(d_k(q)) = \pi_x f^k(q)$, therefore for $z \in J_s(d_k(\theta), Dx_k(\theta), M, \delta)$ we have $\pi_x f^k(z) \neq \pi_x f^k(q)$ if $z \neq d_k(q)$.

Hence $d_k(\theta_1) \notin J_s\left(d_k(\theta), Dx_k(\theta), M, \delta\right)$ for $0 < \|\theta_1 - \theta\|$ sufficiently small. From this observation we obtain

$$\|x_k(\theta_1) - x_k(\theta) - Dx_k(\theta)(\theta_1 - \theta)\| \leq \frac{1}{M}\|\theta_1 - \theta\|^2. \tag{8.85}$$

It is easy to prove that (8.85) implies a uniform bound on $D^2x_k(q)$.

Now by using the Arzela-Ascoli lemma we can prove that x_k converge in C^1-norm to a C^1 function.

∎

8.3.2 Smoothness of stable fibers

We assume the standard assumptions defined in Section 7.1 and we assume that $f \in C^2$. The goal of this section is to show that the stable fibers are smooth. The proof is more involved than the previous proofs of smoothness, because in this case we do not have functions defined on the common domain that converge uniformly. Instead W^s_q is patched from local pieces.

We use notation $\theta = (\lambda, x)$.

Lemma 81 *Let $\gamma < 1$ be such that $\beta_v < \gamma < \eta_v$. Let $U \subset D$ be an open set and such that $\overline{U} \cap W^{cs} \subset \mathrm{int} D$. Let $r = r(U, \gamma)$ be as in Lemma 61.*

For any $q \in U \cap W^{cs}$ and for any $y \in \overline{B}_s(\pi_y q, r/\alpha_v)$ and any $k \in \mathbb{Z}_+$ there exists $\theta_k(q, y)$ a unique solution of the problem given by conditions (8.86,8.87)

$$\pi_\theta\left(f^k(\theta_k(q,y),y)\right) - f^k(q)) = 0 \tag{8.86}$$

$$Q_v(f^j(\theta_k(q,y),y)) - f^j(q)) \geq 0, \qquad j = 0, 1, 2, \ldots, k-1 \tag{8.87}$$

Functions $\theta_k(q, y)$ are C^2 and for a fixed q satisfy the cone condition

$$Q_v((\theta_k(q, y_1), y_1) - (\theta_k(q, y_2), y_2)) \geq 0, \tag{8.88}$$

Moveover, for fixed q $\theta_k(q,\cdot)$ converges uniformly to vertical disk in $N(q,r,r/\alpha_v)$ which is equal to $W^s_q \cap S_q(\gamma,r)$, where (compare Theorem 64)

$$S_q(\gamma,r)=\{z\in D \mid f^j(z)\in N(f^j(q),\gamma^j r,\gamma^j r/\alpha_v),\quad j\in\mathbb{Z}_+\} \tag{8.89}$$

Proof: For $k\in\mathbb{Z}_+$ we consider the following equation

$$F(\theta,q,y)=\pi_\theta f^k(\theta,y)-\pi_\theta f^k(q)=0, \tag{8.90}$$

in the set (compare Theorem 64)

$$S_q(\gamma,r,k)=\{z\in D \mid f^j(z)\in N(f^j(q),\gamma^j r,\gamma^j r/\alpha_v),\quad j=0,1,2,\ldots,k-1\} \tag{8.91}$$

We want to apply the implicit function theorem to (8.86). The equation itself is given by C^2 function, hence it is enough to show, that the solution exists, is unique and $\frac{\partial F}{\partial \theta}(z)$ is an linear isomorphism for any $z\in S_q(\gamma,r,k)$

For any y, q and k the solution of (8.90) in $S_q(\gamma,r,k)$ exists as the consequence of Theorem 7. Let us denote this solution by θ_k. We will show that this solution is unique.

Let $\theta'\neq\theta$ be such that $(\theta',y)\in S_q(\gamma,r,k)$. Then from Lemma 37 it follows that

$$\|\pi_\theta(f^k(\theta',y)-f^k(\theta_k,y))\|\geq\eta_v^k\|\theta'-\theta_k\|. \tag{8.92}$$

Hence $\pi_\theta f^k(\theta',y)\neq\pi_\theta f^k(\theta_k,y)=\pi_\theta f^k(q)$. This shows the uniqueness of the solution of (8.90).

We need to show that $\theta_k(q,y)$ satisfies (8.87). We will reason by a contradiction. If $Q_v(f^j(\theta_k(q,y))-f^j(q))<0$, for some $j<k$ then by Lemma 37 it follows that

$$0=\|\pi_\theta(f^k(\theta_k(q,y),y)-f^k(q))\|\geq\eta_v^{k-j}\|\pi_\theta(f^j(\theta_k(q,y))-f^j(q))\|>0.$$

Hence indeed (8.87) is satisfied.

Now we show that $\frac{\partial\pi_\theta f^k}{\partial\theta^k}(z)$ is a linear isomorphism for any $z=(\lambda,y)\in S_q(\gamma,r,k)$. From Lemma 37 for $z_1=(\lambda,y), z_2=(\lambda_2,y)\in S_q(\gamma,r,k)$,

$$\|\pi_\theta(f^k(\lambda,y)-f^k(\lambda_2,y))\|\geq\eta_v^k\|\lambda_1-\lambda_2\| \tag{8.93}$$

Passing to the limit $\lambda_2\to\lambda$ we obtain for any vector $v\in\mathbb{R}^s$

$$\left\|\frac{\partial\pi_\theta f^k}{\partial\theta^k}(\lambda,y)v\right\|\geq\eta_v^k\|v\|. \tag{8.94}$$

This shows that $\frac{\partial\pi_\theta f^k}{\partial\theta^k}(z)$ is a linear isomorphism.

To verify (8.88) we reason by a contradiction. Assume that $Q_v((\theta_k(q,y_1),y_1)-(\theta_k(q,y_2),y_2))<0$, then from Lemma 37 it follows that

$$0=\|\pi_\theta(f^k(\theta_k(q,y_1),y_1)-f^k(\theta_k(q,y_2),y_2))\|\geq\eta_v^k\|\theta_k(q,y_1)-\theta_k(q,y_2)\|>0.$$

Hence indeed (8.88) is satisfied.

Uniform convergence. Let $j < k$. We have from Lemma 37

$$\|\pi_\theta(f^j(\theta_j(q,y),y) - f^j(\theta_k(q,y),y))\| \geq \eta_v^j \|\theta_j(q,y) - \theta_k(q,y)\|. \tag{8.95}$$

But $\|\pi_\theta(f^j(\theta_j(q,y),y) - f^j(\theta_k(q,y),y))\| \leq 2\gamma^j r$, hence we obtain

$$\|\theta_j(q,y) - \theta_k(q,y)\| \leq 2r \left(\frac{\gamma}{\eta_v}\right)^j. \tag{8.96}$$

We see that $\theta_k(q,\cdot)$ is a Cauchy sequence. Let $\theta_s(q,y)$ be the limit function.

Observe that we have

$$Q_v(f^j(\theta_s(q,y)) - f^j(q)) \geq 0, j \in \mathbb{Z}_+. \tag{8.97}$$

Therefore the graph $\{(\theta_s(q,y),y) \mid y \in \overline{B}_s(\pi_y q, r/\alpha_v)\}$ is equal to $W_q^s \cap S_q(\gamma,r)$.

∎

Theorem 82 *Let*

$$\mu_s = \inf_{q \in D}\left(m\left(\frac{\partial f_\theta}{\partial \theta}(q)\right) - \alpha_v \left\|\frac{\partial f_y}{\partial \theta}(q)\right\|\right). \tag{8.98}$$

Assume that

$$\mu_s > 0, \quad \frac{\beta_v^2}{\mu_s} < 1. \tag{8.99}$$

Then for all $q \in W^{cs}$ the stable fiber W_q^s is C^1 in the following sense: there exists a C^1 function $\theta_s : \overline{B}_s(0,R) \to \mathbb{T}$, such that

$$W_q^s = \{(\theta_s(y),y) \mid y \in \overline{B}_s(0,R)\}.$$

Proof:

From Lemma 81 it follows that for any $q \in W^{cs} \cap \mathrm{int} D$ there exists $r(q)$, such that $y \in \overline{B}_s(\pi_y q, r(q))$ and any $k \in \mathbb{Z}_+$ there exists $\theta_k(q,y)$ a unique solution of the equation

$$\pi_\theta\left(f^k(\theta_k(q,y),y) - f^k(q)\right) = 0 \tag{8.100}$$

and satisfying the following condition

$$Q_v(f^j(\theta_n(q,y),y)) - f^j(q)) \geq 0, \qquad j = 0,1,2,\ldots,k-1 \tag{8.101}$$

Functions $\theta_k(q,y)$ are C^2 and for a fixed q satisfy the cone condition

$$Q_v((\theta_k(q,y_1),y_1) - (\theta_k(q,y_2),y_2)) \geq 0, \tag{8.102}$$

which gives

$$\begin{aligned} -(\theta_k(q,y_1) - \theta_k(q,y_2))^2 + \alpha_v^2(y_1-y_2)^2 &\geq 0 \\ |\theta_k(q,y_1) - \theta_k(q,y_2)| &\leq \alpha_v \|y_1 - y_2\| \\ \left\|\frac{\partial \theta_k}{\partial y}\right\| &\leq \alpha_v. \end{aligned} \tag{8.103}$$

We define $\theta_0(q,y) = \pi_\theta q$.

We will show that the family of functions $\{\theta_k(q,\cdot)\}$ has uniformly bounded second derivatives, i.e. there exists $M > 0$, such that

$$\left\| \frac{\partial^2 \theta_k}{\partial y^2}(q,y) \right\| \leq M, \qquad \forall q \in D, y \in \overline{B}_s(0,R), k \in \mathbb{Z}_+. \tag{8.104}$$

For this we will use Theorem 77 about the propagation of the stable jets.

Observe first that for any $y \in B_s(\pi_y q, r(q))$

$$\begin{aligned} f(\theta_k(q,y),y) \in \{(\theta_{k-1}(f(q),y),y) \mid y \in \overline{B}_s(\pi_y f(q), \gamma^k r(q))\}, \quad k = 1,2,\ldots \\ \overline{B}_s(\pi_y f(q), \gamma^k r(q)) \subset \overline{B}_s(\pi_y f(q), r(f^k(q))) \end{aligned}$$

The tangent spaces to $\{(\theta_k(q,y),y) \mid y \in \overline{B}_s(\pi_y q, r(q))\}$ are mapped into tangent spaces of $y \mapsto (\theta_{k-1}(f(q),y),y)$, but this map does not need to have a full rank. The tangent space is always of the form $\{(\theta_k(q,y),y) + \left(\frac{\partial \theta_k}{\partial y} y_0, y_0\right) \mid y \in \mathbb{R}^s\}$.

To verify the assumptions of Thm. 77 for J_s along the sequence $(\theta_k(q,y),y)$ let us define

$$\begin{aligned} \mu &= \sup_{q \in W^{cs}, y \in \overline{B}_s(\pi_y q, r(q)), k \in \mathbb{Z}_+} \left\| \frac{\partial f_y}{\partial y}(\theta_k(q,y),y) + \frac{\partial f_y}{\partial \theta}(\theta_k(q,y),y) \frac{\partial \theta_k}{\partial y}(q,y) \right\|, \\ \rho &= \inf_{q \in W^{cs}, y \in \overline{B}_s(\pi_y q, r(q)), k \in \mathbb{Z}_+} m \left(\frac{\partial f_\theta}{\partial \theta}(\theta_k(q,y),y) - \frac{\partial \theta_{k+1}}{\partial y}(f(\theta_k(q,y),y), \pi_y f(\theta_k(q,y),y)) \frac{\partial f_y}{\partial \theta}(\theta_k(q,y) \right. \end{aligned}$$

We need to check that

$$\rho > 0, \quad \frac{\mu^2}{\rho} < 1. \tag{8.105}$$

Observe that from (8.103) it follows that

$$\begin{aligned} \mu &\leq \sup_{q \in D} \left(\left\| \frac{\partial f_y}{\partial y}(q) \right\| + \left\| \frac{\partial f_y}{\partial \theta} \right\| \alpha_v \right) = \beta_v \\ \rho &\geq \inf_{q \in D} \left(m\left(\frac{\partial f_\theta}{\partial \theta}(q) \right) - \alpha_v \left\| \frac{\partial f_y}{\partial \theta}(q) \right\| \right) = \mu_s. \end{aligned}$$

Therefore conditions (8.105) are satisfied. Hence from Theorem 77 it follows there exists $M > 0$ (independent of q, y) and δ, such that

$$f^k \left(J_s \left((\theta_k(q,y),y), \frac{\partial \theta_k}{\partial y}(q,y), M, \delta \right) \right) \subset J_s(f^k(\theta_k(q,y),y), 0, M). \tag{8.106}$$

Observe that (8.106) means the following: if $\|(\theta_0, y_0)\| \leq \delta$, $\|y_0\|^2 \leq M\|\theta\|$, then

$$f^k \left((\theta_k(q,y),y) + \left(\frac{\partial \theta_k}{\partial y}(q,y) y_0 + \theta_0, y_0 \right) \right) = f^k(\theta_k(q,y),y) + (\theta_1, y_1),$$

where $\|y_1\|^2 \leq M\|\theta_1\|$.

Since $\pi_\theta f^k(\theta_k(q,y),y) = \pi_\theta f^k(q)$, therefore for $z \in J_s\left((\theta_k(q,y),y), \frac{\partial\theta_k}{\partial y}(q,y), M, \delta\right)$ we have $\pi_\theta f^k(z) \neq \pi_\theta f^k(q)$ if $z \neq (\theta_k(q,y),y)$.

Hence $(\theta_k(q,y_1),y_1) \notin J_s\left((\theta_k(q,y),y), \frac{\partial\theta_k}{\partial y}(q,y), M, \delta\right)$ for $0 < \|y_1 - y\|$ sufficiently small. From this observation we obtain

$$\left\|\theta_k(q,y_1) - \theta_k(q,y) - \frac{\partial\theta_k}{\partial y}(q,y)(y_1 - y)\right\| \leq \frac{1}{M}\|y_1 - y\|^2. \tag{8.107}$$

It is easy to prove that (8.107) implies a uniform bound on $\frac{\partial^2\theta_k}{\partial y^2}(q,y)$.

Now by using the Arzela-Ascoli lemma we can prove that $\theta_k(q,\cdot)$ converge in C^1-norm to a C^1 function.

∎

Chapter 9

Example

9.1 Example

We consider perturbation of the following map $f : S^1 \times \mathbb{R}^u \times \mathbb{R}^s$

$$f(\lambda, x, y) = (\lambda + \nu + \frac{\epsilon}{2\pi}\sin(2\pi\lambda), 2x, y/2) + \delta g(\lambda, x, y). \tag{9.1}$$

where $g(\lambda + 1, x, y) = g(\lambda, x, y)$ and $g \in C^2$. The parameter $\nu \in [0, 1]$ and $\epsilon \in [-1/4, 1/4]$.

For $\delta = 0$ the map f has an invariant manifold - the central circle $\Lambda = S^1 \times \{0\} \times \{0\}$, but the dynamics on Λ depending on (ν, ϵ) can be either rigid rotation (typical from the measure point of view for ϵ small) or can posses sinks and repelling points (typical from the topological point of view). This phenomenon is related to so called *Arnold tongues* [ArV].

Our goal is to illustrate that independent on the detailed dynamics for the unperturbed map our approach yields the invariant manifold.

Let $D = S^1 \times \overline{B}_u(0, R) \times \overline{B}_s(0, R)$.

We gather some estimates in a form of lemma.

Lemma 83 *Let $\delta = 0$. Assume that $|\epsilon| < 1$. Then*

$$m\left(\left[\frac{\partial f_{(\lambda,x)}}{\partial(\lambda, x)}(D)\right]\right) \geq 1 - |\epsilon|, \tag{9.2}$$

$$m\left(\left[\frac{\partial f_x}{\partial x}(D)\right]\right) = 2, \tag{9.3}$$

$$\sup_{z\in D}\left\|\frac{\partial f_{(\lambda,y)}}{\partial(\lambda, y)}(z)\right\| = 1 + |\epsilon|. \tag{9.4}$$

Proof: An easy computation show that

$$\left[\frac{\partial f_{(\lambda,x)}}{\partial(\lambda, x)}(D)\right] = \begin{bmatrix} [1 - |\epsilon|, 1 + |\epsilon|] & 0 \\ 0 & 2 \end{bmatrix}.$$

This proves (9.2).

$$\frac{\partial f_{(\lambda,y)}}{\partial(\lambda,y)}(z) = \begin{bmatrix} 1+\epsilon\cos(2\pi\lambda) & 0 \\ 0 & 1/2 \end{bmatrix}$$

∎

Let us take $\alpha_h < 1$, $\alpha_v < 1$ but close to 1.

First we establish the conditions related to good atlas. We have $D_\Lambda = 1$, hence $R_\Lambda < 1/2$. This gives us upper bound for $R < 1/2R_\Lambda$ (we assume that α_h and α_v are smaller than 1 but very close to 1).

Condition (6.36) is implied by the following inequality

$$\delta \max_{z\in D} \|D\pi_\lambda g(z)\| \cdot \|(R_\Lambda, R, R)\| + (1+|\epsilon|)R_\Lambda < 1/2. \tag{9.5}$$

After some elementary calculations we obtain (we substitute R with $R_\Lambda/2$)

$$R_\Lambda \left(\delta \max_{z\in D} \|D\pi_\lambda g(z)\| \sqrt{3/2} + 1 + |\epsilon|\right) < 1/2. \tag{9.6}$$

From this we obtain

$$R < \left(\delta \max_{z\in D} \|D\pi_\lambda g(z)\| \sqrt{3/2} + 1 + |\epsilon|\right)^{-1} /4. \tag{9.7}$$

We summarize the above considerations as the following lemma.

Lemma 84 *Assume $R < 1/4$. If (9.7) holds, then there exist $\alpha_v \approx 1$ and $\alpha_h \approx 1$, such the compatibility conditions in the sense of Def. 32 are satisfied,*

Lemma 85 *If (9.7) is satisfied and the following conditions hold*

$$R > \delta|\pi_x g(D)|, \tag{9.8}$$
$$R/2 > \delta|\pi_y g(D)|. \tag{9.9}$$

then f satisfies the covering relation on D.

Now we turn the cone conditions. Following definition 43 we define

$$\beta_v = 1/2 + \delta \sup_{z\in D}\left(\alpha_v \left\|\frac{\partial g_y}{\partial(\lambda,x)}(z)\right\| + \left\|\frac{\partial g_y}{\partial y}(z)\right\|\right) \tag{9.10}$$

$$\eta_v = 1 - |\epsilon| - \delta\left(\sup_{z\in D}\left\|\frac{\partial g_{(\lambda,x)}}{\partial(\lambda,x)}(z)\right\| + \frac{1}{\alpha_v}\sup_{z\in D}\left\|\frac{\partial g_{(\lambda,x)}}{\partial y}(z)\right\|\right) \tag{9.11}$$

$$\beta_h = 1 + |\epsilon| + \delta \sup_{z\in D}\left(\left\|\frac{\partial g_{(\lambda,y)}}{\partial(\lambda,y)}(z)\right\| + \frac{1}{\alpha_h}\cdot\left\|\frac{\partial g_{(\lambda,y)}}{\partial x}(z)\right\|\right) \tag{9.12}$$

$$\eta_h = 2 - \delta\left(\sup_{z\in D}\left\|\frac{\partial g_x}{\partial x}(z)\right\| + \alpha_h \sup_{z\in D}\left\|\frac{\partial g_x}{\partial(\lambda,y)}(z)\right\|\right) \tag{9.13}$$

The conditions (7.5), (7.6), (7.7) are implied by the following ones

$$\delta\left(\sup_{z\in D}\left(\alpha_v\left\|\frac{\partial g_y}{\partial(\lambda,x)}(z)\right\|+\left\|\frac{\partial g_y}{\partial y}(z)\right\|\right)+\right.$$
$$\left.\sup_{z\in D}\left\|\frac{\partial g_{(\lambda,x)}}{\partial(\lambda,x)}(z)\right\|+\frac{1}{\alpha_v}\sup_{z\in D}\left\|\frac{\partial g_{(\lambda,x)}}{\partial y}(z)\right\|\right)<1/2-|\epsilon|$$
$$\delta\left(\left(\sup_{z\in D}\left\|\frac{\partial g_x}{\partial x}(z)\right\|+\alpha_h\sup_{z\in D}\left\|\frac{\partial g_x}{\partial(\lambda,y)}(z)\right\|\right)+\right.$$
$$\left.\sup_{z\in D}\left(\left\|\frac{\partial g_{(\lambda,y)}}{\partial(\lambda,y)}(z)\right\|+\frac{1}{\alpha_h}\cdot\left\|\frac{\partial g_{(\lambda,y)}}{\partial x}(z)\right\|\right)\right)<1-|\epsilon|.$$

To make expressions easily manageable (but likely not optimal) observe that all the norms of partial derivatives of g appearing in the above expressions can be estimated by $\sup_{z\in D}\|Dg\|$. Using this we obtain the following conditions

$$\delta\left(2+\alpha_v+1/\alpha_v\right)\sup_{z\in D}\|Dg(z)\|<1/2-|\epsilon|,$$
$$\delta\left(2+\alpha_h+1/\alpha_h\right)\sup_{z\in D}\|Dg(z)\|<1-|\epsilon|.$$

Observe that we can replace α_i+1/α_i, where $i\in\{v,h\}$ by 2 in the above inequalities, because if the inequality holds after such substitution we can always find $\alpha_i<1$ close to 1 for which the original inequality holds.

Therefore we are left with one equation

$$4\delta\sup_{z\in D}\|Dg(z)\|<1/2-|\epsilon|. \tag{9.14}$$

Summarizing, we have proved the following theorem

Theorem 86 *Let $\epsilon\in[-1/4,1/4]$ and $\nu\in[0,1]$. If*

$$\begin{aligned}
\delta\left(\sup_{z\in D}\|Dg(z)\|\right) &< \frac{1}{16},\\
R &< \frac{1}{4}\left(\delta\max_{z\in D}\|D\pi_\lambda g(z)\|\sqrt{3/2}+5/4\right)^{-1},\\
R &< 1/4,\\
\delta|\pi_x g(D)| &< R,\\
\delta|\pi_y g(D)| &< R/2.
\end{aligned}$$

Then there exists $\alpha_v<1$ and $\alpha_h<1$, such that the standard assumptions from Section 7.1 are satisfied for the map (9.1) on the set $D=S^1\times\overline{B}_u(0,R)\times\overline{B}_s(0,R)$. Therefore the conclusions of Theorem 45 apply to this map.

Now we will do a similar analysis for the assumptions of Theorem 46. As above we will bound all the partial derivatives of g by $\|Dg\|$, set $\alpha_v=\alpha_h=1$ and $|\epsilon|\leq 1/4$.

$$\beta_{cu}=1/2+2\delta\|Dg\|. \tag{9.15}$$

The condition $\frac{\beta_{cu}}{\tilde{\eta}_v^2} < 1$ becomes

$$1/2 + 2\delta\|Dg\| < (3/4 - 2\delta\|Dg\|)^2. \tag{9.16}$$

The important quantity here is $\Delta = \delta\left(\sup_{z\in D}\|Dg(z)\|\right)$. Theorem 86 introduces an upper bound $\Delta < 1/16$.

From (9.16) we obtain

$$5\Delta < \frac{1}{16} + 4\Delta^2.$$

It is easy to see that ignoring the term $4\Delta^2$ does not change the range of Δ much, hence

$$\Delta < \frac{1}{80}. \tag{9.17}$$

Now we compute η_{cs}. We have

$$\eta_{cs} = 2 - 2\delta\|Dg\|.$$

The condition $\frac{\beta_h^2}{\eta_{cs}} < 1$ becomes

$$(5/4 + 2\Delta)^2 < 2 - 2\Delta.$$

It is easy to see that it is satisfied for Δ as in (9.17).

If follows from Theorem 46 and the above computation that if (9.17) holds together with assumptions of Theorem 86 then W^{cu}, W^{cs} and $\tilde{\Lambda}$ are C^1.

Now we investigate the conditions implying the smoothness of stable and unstable fibers. We have

$$\mu_u = 5/4 + 2\Delta,$$
$$\mu_s = 3/4 - 2\Delta.$$

Condition $\frac{\mu_u}{\tilde{\eta}_h^2} < 1$ implying the smoothness of W_q^u becomes

$$5/4 + 2\Delta < (2 - 2\Delta)^2$$

which is equivalent to

$$10\Delta < \frac{11}{4} + 4\Delta^2.$$

It is easy to see that this inequality is satisfied if (9.17) holds.

Condition $\frac{\beta_v^2}{\mu_s} < 1$ implying the smoothness of W_q^s becomes

$$(1/2 + 2\Delta)^2 < 3/4 - 2\Delta,$$

which is equivalent to

$$4\Delta + 4\Delta^2 < 1/2. \tag{9.18}$$

It is easy to see that this inequality is satisfied if (9.17) holds.

Summarizing we have proved the following theorem

Theorem 87 *Let $\epsilon \in [-1/4, 1/4]$ and $\nu \in [0, 1]$. If*

$$\begin{aligned} \delta \left(\sup_{z \in D} \|Dg(z)\| \right) &< \frac{1}{80}, \\ R &< \frac{1}{4} \left(\delta \max_{z \in D} \|D\pi_\lambda g(z)\| \sqrt{3/2} + 5/4 \right)^{-1}, \\ R &< 1/4, \\ \delta|\pi_x g(D)| &< R, \\ \delta|\pi_y g(D)| &< R/2. \end{aligned}$$

Then there exists $\alpha_v < 1$ and $\alpha_h < 1$, such that the standard assumptions from Section 7.1 are satisfied for the map (9.1) on the set $D = S^1 \times \overline{B}_u(0, R) \times \overline{B}_s(0, R)$. The sets W^{cu}, W^{cs}, W^u_q and W^s_q are graphs of C^1 functions.

Appendix

9.2 Appendix

9.2.1 Local Brouwer degree

In this section we list the basic properties of *the local Brouwer degree* which are relevant for us in this paper. The proofs can be found in [Sch, Ch. III].

For $n = 0$ we have $\mathbb{R}^n = \{0\}$. We have only one self map for this space, namely $f(0) = 0$. We formally define *the local Brouwer degree of f at 0 in the set* $\{0\}$ by

$$\deg(f, \{0\}, 0) = 1$$

Assume $n > 0$. Let $D \subset \mathbb{R}^n$ be an open set and $f : S \to \mathbb{R}^n$ be continuous, $D \subset S$ and $c \in \mathbb{R}^n$. Suppose that

$$\text{the set} \quad f^{-1}(c) \cap D \quad \text{is compact.} \tag{9.19}$$

Then *the local Brouwer degree of f at c in the set D* is defined. We denote it by $\deg(f, D, c)$.

If $\overline{D} \subset \mathrm{dom}(f)$ and $\overline{D}$ is compact, then (9.19) follows from the condition

$$c \notin f(\partial D). \tag{9.20}$$

Let us summarize the properties of the local Brouwer degree

Degree is an integer.

$$\deg(f, D, c) \in \mathbb{Z}. \tag{9.21}$$

Solution property.

$$\text{If} \quad \deg(f, D, c) \neq 0, \quad \text{then there exists } x \in D \text{ with } f(x) = c. \tag{9.22}$$

Homotopy property. Let $H : [0, 1] \times D \to \mathbb{R}^n$ be continuous. Suppose that

$$\bigcup_{\lambda \in [0,1]} H_\lambda^{-1}(c) \cap D \quad \text{is compact.} \tag{9.23}$$

Then

$$\forall \lambda \in [0, 1] \quad \deg(H_\lambda, D, c) = \deg(H_0, D, c). \tag{9.24}$$

If $[0, 1] \times \overline{D} \subset \mathrm{dom}(H)$ and $\overline{D}$ is compact, then (9.23) follows from the following condition

$$c \notin H([0, 1], \partial D). \tag{9.25}$$

Local degree is a locally constant function. Assume D is bounded and open.

If p and q belong to the same component of $\mathbb{R}^n \setminus f(\partial D)$, then

$$\deg(f, D, p) = \deg(f, D, q). \tag{9.26}$$

Excision property. Suppose that $E \subset D$, E is open and

$$f^{-1}(c) \cap D \subset E. \tag{9.27}$$

Then

$$\deg(f, E, c) = \deg(f, D, c). \tag{9.28}$$

Local degree for affine maps. Suppose that $f(x) = A(x - x_0) + c$, where A is a linear map and $x_0 \in \mathbb{R}^n$. If the equation $A(x) = 0$ has no nontrivial solutions (i.e. if $Ax = 0$, then $x = 0$) and $x_0 \in D$, then

$$\deg(f, D, c) = \operatorname{sgn}(\det A). \tag{9.29}$$

Product property Let $U_i \subset \mathbb{R}^{n_i}$, $c_i \in \mathbb{R}^{n_i}$, $f_i : U_i \to \mathbb{R}^{n_i}$, for $i = 1, 2$. The map $(f_1, f_2) : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ is given by $(f_1, f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2)$. We have

$$\deg((f_1, f_2), U_1 \times U_2, (c_1, c_2)) = \deg(f_1, U_1, c_1) \cdot \deg(f_2, U_2, c_2), \tag{9.30}$$

whenever the right hand side is defined.

Multiplication property Let $D \subset \mathbb{R}^n$ be bounded and open. Let $f : \bar{D} \subset \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^n$ are two continuous mappings and Δ_i the bounded components of $\mathbb{R}^n \setminus f(\partial D)$. Then

$$\deg(g \circ f, D, p) = \sum_{\Delta_i} \deg(g, \Delta_i, p) \deg(f, D, \Delta_i), \tag{9.31}$$

where $\deg(f, D, \Delta_i) = \deg(f, D, q_i)$ for some $q_i \in \Delta_i$. From equation (9.26) it follows that this definition of $\deg(f, D, \Delta_i)$ does not depend on the choice of q_i.

Addition property. If $D = \bigcup_{i \in I} D_i$, where each D_i is open, the family $\{D_i\}$ is disjoint and $\partial D_i \subset \partial D$, then for every $c \notin f(\partial D)$:

$$\deg(f, D, c) = \sum_{i \in I} \deg(f, D_i, c). \tag{9.32}$$

From Multiplication property and formula (9.29) we obtain immediately

Collorary 88 *Let $D \subset \mathbb{R}^n$ be open and bounded. Let $A : D \to \mathbb{R}^n$, be continuous and $0 \notin A(\partial D)$,*

$$\deg(-A, U, 0) = (-1)^n \deg(A, U, 0). \tag{9.33}$$

As the consequence of Addition and Excision property we obtain the following

Collorary 89 *Suppose that D is a finite union of open sets $D = \bigcup_{i=1}^{n} D_i$ such that the sets $f_{|D_i}^{-1}(c)$ are mutually disjoint and $c \notin f(\partial D_i)$. Then*

$$\deg(f, D, c) = \sum_{i=1}^{n} \deg(f_{|D_i}, D_i, c). \tag{9.34}$$

Here is another important consequence of above properties

Collorary 90 *Assume $V \subset \mathbb{R}^n$ is bounded and open. Let $f : \overline{V} \to \mathbb{R}^n$ be a C^1-map. Assume that $c \in \mathbb{R}^n \setminus f(\partial V)$ is a regular value for f, i.e. for each $x \in f^{-1}(c)$ the Jacobian matrix of f at x denoted by $Df(x)$ is nonsingular, then*

$$\deg(f, V, c) = \sum_{x \in f^{-1}(c)} \operatorname{sgn}(\det Df(x)).$$

9.2.2 The degree of maps $S^n \to S^n$

In this section we recall some relevant facts on the degree of maps $S^n \to S^n$ see for example [DG, Ch. 7.5].

Definition 45 *Let $n \geq 1$. The degree of a continuous map $f : S^n \to S^n$ is a unique integer $d(f)$ such that $f_*(u) = d(f) \cdot u$, for any generator $u \in H_n(S^n)$, where $H_n(S^n)$ is n-th homology group of S^n and $f_* : H_n(S^n) \to H_n(S^n)$ is the induced homomorphism.*

For $n = 0$ we define the degree, $d(f)$, as follows, $S^0 = \{-1, 1\}$. We set

$$d(f) = \begin{cases} 1, & \text{if } f(1) = 1 \text{ and } f(-1) = -1, \\ -1, & \text{if } f(1) = -1 \text{ and } f(-1) = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{9.35}$$

Theorem 91 (H. Hopf) *Let $n \geq 1$. Then $f, g : S^n \to S^n$ are homotopic if and only if $d(f) = d(g)$.*

Lemma 92 *Let $u > 0$, Assume that $A : \overline{B_u}(0,1) \to \mathbb{R}^u$ is a continuous map, such that*

$$0 \notin A(\partial B(0,1)).$$

Let the map $s_A : S^{u-1} \to S^{u-1}$ be given by

$$s_A(x) = \frac{A(x)}{\|A(x)\|}.$$

Then

$$\deg(A, \overline{B_u}(0,1), 0) = d(s_A). \tag{9.36}$$

References

Bibliography

[A] G. Arioli, Periodic orbits, symbolic dynamics and topological entropy for the restricted 3-body problem. *Comm. Math. Phys.* 231 (2002), no. 1, 1–24.

[ArV] V. I. Arnold, Geometrical Methods In The Theory Of Ordinary Differential Equations, Springer-Verlag (1988), ISBN 0-387-96649-8.

[AZ] G. Arioli, P. Zgliczyński, Symbolic dynamics for the Hénon-Heiles Hamiltonian on the critical level. *J. Differential Equations* 171 (2001), no. 1, 173–202.

[BF] I. Baldomá and E. Fontich, Stable manifolds associated to fixed points with linear part equal to identity, *JDEq*, 197 (2004) 45–72.

[BB] P. Berger, A. Bounemoura, *A geometrical proof of the persistence of normally hyperbolic submanifolds* http://arxiv.org/abs/1109.3280

[C] M.J. Capiński, Covering Relations and the Existence of Topologically Normally Hyperbolic Invariant Sets, *Discrete and Continuous Dynamical Systems A.*, 23 (2009), 705–725

[C2] M.J. Capiński, *Lyapunov Orbits at L2 and transversal Intersections of Invariant Manifolds in the Jupiter-Sun Planar Restricted Circular Three Body Problem.* To appear in SIAM Journal on Applied Dynamical Systems.

[CS] M.J. Capiński, C. Simó, Computer assisted proof for normally hyperbolic invariant manifolds, *Nonlinearity* 25 (2012), 1997–2026

[CR] M. J. Capiński, P. Roldán, *Existence of a Center Manifold in a Practical Domain around L_1 in the Restricted Three Body Problem,* SIAM J. Appl. Dyn. Syst. 11 (2012), pp. 285-318.

[CZ1] M.J. Capiński, P. Zgliczyński, *Cone Conditions and Covering Relations for Topologically Normally Hyperbolic Invariant Manifolds*, *Discrete Cont. Dyn. Sys. A*, 30 (2011) 641–670

[CZ2] M.J. Capiński, P. Zgliczyński, *Geometric Proof of the Normally Hyperbolic Invariant Manifold Theorem*, in preparation

[Ch] M. Chaperon, Stable manifolds and the Perron-Irwin method. *Ergodic Theory Dynam. Systems* 24 (2004), no. 5, 1359–1394.

[CGB] T. Csendes, B. Garay, B. Bánhelyi, A verified optimization technique to locate chaotic regions of Hnon systems. *J. Global Optim.* 35 (2006), no. 1, 145–160

[DG] J. Dugundji and A. Granas, *Fixed Point Theory,* Monografie Matematyczne 61, PWN, Warszawa, 1982.

[E1] R. Easton, Isolating blocks and symbolic dynamics, *J. Differential Equations* 17 (1975) 96–118.

[E2] R. Easton, Homoclinic phenomena in Hamiltonian systems with several degrees of freedom, *J. Differential Equations* 29 (1978) 241–252.

[Fen] N. Fenichel, *Asymptotic Stability with Rate Conditions*, Indiana Univ. Math. J. 23 No. 12 (1975), 1109–1137

[F] E. Fontich, Stable curves asymptotic to a degenerate fixed point, *Nonlinear Analysis*, 35(1999) 711-733

[Ga1] Z. Galias, Rigorous numerical studies of the existence of periodic orbits for the Hénon map. *SCAN-97 (Lyon). J.UCS* 4 (1998), no. 2, 114–124

[Ga2] Z. Galias, Positive topological entropy of Chua's circuit: a computer assisted proof. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* 7 (1997), no. 2, 331–349

[GaZ1] Z. Galias, P. Zgliczyński, Computer assisted proof of chaos in the Lorenz equations. *Phys. D* 115 (1998), no. 3-4, 165–188

[GaZ2] Z. Galias, P. Zgliczyński, Abundance of homoclinic and heteroclinic orbits and rigorous bounds for the topological entropy for the Hénon map. *Nonlinearity* 14 (2001), no. 5, 909–932

[GiL] M. Gidea, R. de la Llave, Topological methods in the instability problem of Hamiltonian systems. *Discrete Contin. Dyn. Syst.* 14 (2006), no. 2, 295–328.

[GiR] M. Gidea, C. Robinson, Shadowing orbits for transition chains of invariant tori alternating with Birkhoff zones of instability. *Nonlinearity* 20 (2007), no. 5, 1115–1143.

[Hal] J. Hale, *Ordinary Differential Equations*, 1980 , Wiley-Interscience, New York, London, Sydney

[Ha] P. Hartman, *Ordinary Differential Equations*, 1964 John Wiley & Sons, Inc., New York, London, Sydney

[Hi] M. Hirsh, *Differential Topology*, Graduate Texts in Mathematics, No. 33. Springer-Verlag, New York-Heidelberg, 1976

[HPS] M.W. Hirsch, C.C. Pugh and M. Shub, *Invariant manifolds*, Lecture Notes in Mathematics vol. 583, 1997

[HL] M. Holland, S. Luzzatto, A new proof of the stable manifold theorem for hyperbolic fixed points on surfaces. *J. Difference Equ. Appl.* 11 (2005), no. 6, 535–551

[Hru1] S. L. Hruska, A numerical method for constructing the hyperbolic structure of complex Henon mappings, *Found. Comput. Math.*, 6 (2006), pp. 427455.

[Hru2] S. L. Hruska, Rigorous numerical models for the dynamics of complex Henon mappings on their chain recurrent sets, *Discrete Contin. Dynam. Systems*, 15 (2006), pp. 529558.

[I70] M. C. Irwin, On the stable manifold theorem, *Bull. London Math. Soc.* 2 (1970), 196–198.

[I80] M. C. Irwin, A new proof of the pseudostable manifold theorem, *J. London Math. Soc.* 21 (1980), 557–566.

[Jo] C. K. R. T. Jones, *Geometric singular perturbation theory.* Dynamical systems (Montecatini Terme, 1994), 44118, Lecture Notes in Math., 1609, Springer, Berlin, 1995.

[KWZ] H. Kokubu, D. Wilczak and P. Zgliczyński, Rigorous verification of the existence of cocoon bifurcation for the Michelson system , *Nonlinearity* 20 (2007) 2147-2174

[LL] Li, Ming-Chia; Lyu, Ming-Jiea, Topological dynamics for multidimensional perturbations of maps with covering relations and Liapunov condition, *JDEq* 250 (2011) 799–812

[Mc] R. McGehee, A Stable Manifold Theorem for Degenerate Fixed Points with Applications to Celestial Mechanics, *JDEq*, 14, 70-88(1973).

[McS] R. McGehee, E. Sander, A new proof of the stable manifold theorem. *Z. Angew. Math. Phys.* 47 (1996), no. 4, 497–513.

[MZ] M. Misiurewicz, P. Zgliczyński, Topological entropy for multidimensional perturbations of one-dimensional maps. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* 11 (2001), no. 5, 1443–1446

[Mo] R.E. Moore, *Interval Analysis.* Prentice Hall, Englewood Cliffs, N.J., 1966

[N] A. Neumaier, *Interval methods for systems of equations*, Cambridge University Press, 1990.

[NR] A. Neumaier and T. Rage, Rigorous chaos verification in discrete dynamical system, *Physica D* 67 (1993) 327–346

[O] J. Ombach, Computation of the local stable and unstable manifolds. *Univ. Iagel. Acta Math.* No. 32 (1995), 129–136

[RNS] T. Rage, A. Neumaier, C. Schlier, Rigorous verification of chaos in a molecular model, *Phys. Rev. E* 50 (1994), 2682–2688

[Ro] Robbin, J. W. Stable manifolds of semi-hyperbolic fixed points, *Illinois J. Math.* 15 (1971), 595–609.

[Sch] J.T. Schwartz, *Nonlinear Functional Analysis*, 1969, Gordon and Breach Science Publishers Inc., New York.

[S] J. Smoller, Shock Waves and Reaction -Diffusion Equations. Springer, 1983.

[T] W. Tucker, *A Rigorous ODE solver and Smale's 14th Problem*, Foundations of Computational Mathematics, (2002), Vol. 2, Num. 1, 53-117

[Wi] St. Wiggins, *Normally hyperbolic invariant manifolds in dynamical systems.* Applied Mathematical Sciences, 105. Springer-Verlag, New York, 1994. x+193 pp. ISBN: 0-387-94205-X

[W] D. Wilczak, *Chaos in the Kuramoto-Sivashinsky equations - a computer assisted proof,* J. Differential Equations, 194(2003), 433–459

[W1] D. Wilczak, Symmetric heteroclinic connections in the Michelson system – a computer assisted proof, *SIAM J. App. Dyn. Sys.* Vol.4, No.3, 489-514 (2005).

[W2] D. Wilczak, The existence of Shilnikov homoclinic orbits in the Michelson system: a computer assisted proof, *Foundations of Computational Mathematics*, 6, 495-535, (2006).

[W3] D. Wilczak, Uniformly hyperbolic attractor of the Smale-Williams type for a Poincar map in the Kuznetsov system, *SIAM Journal on Applied Dynamical Systems*, Vol. 9, No. 4, 1263-1283 (2010).

[WZP] D. Wilczak and P. Zgliczyński, Heteroclinic connections between periodic orbits in planar restricted circular three-body problem—a computer assisted proof. *Comm. Math. Phys.* 234 (2003), no. 1, 37–75.

[WZ] D. Wilczak and P. Zgliczyński, Topological method for symmetric periodic orbits for maps with a reversing symmetry, *Discrete Cont. Dyn. Sys. A*, 17, 629–652 (2007)

[WZ1] D. Wilczak and P. Zgliczyński, *Computer assisted proof of the existence of homoclinic tangency for the Hénon map and for the forced-damped pendulum*, SIAM Journal on Applied Dynamical Systems 8, Issue 4, pp. 1632–1663 (2009)

[WoZ] K. Wójcik and P. Zgliczyński, Topological horseshoes and delay differential equations. *Discrete Contin. Dyn. Syst.* 12 (2005), no. 5, 827–852

[ZN] P. Zgliczyński, Computer assisted proof of chaos in the Rössler equations and in the Hénon map. *Nonlinearity* 10 (1997), no. 1, 243–252

[ZS] P. Zgliczyński, *Sharkovskii's theorem for multidimensional perturbations of one-dimensional maps.* Ergodic Theory Dynam. Systems 19 (1999), no. 6, 1655–1684.

[ZGi] P. Zgliczyński and M. Gidea, *Covering relations for multidimensional dynamical systems*, Journal of Differential Equations 202/1, 33-58 (2004)

[ZCC] P. Zgliczyński, *Covering relations, cone conditions and the stable manifold theorem*, Journal of Differential Equations. Volume 246 issue 5, 1774-1819 (2009)