Shadowing of non-transversal heteroclinic chains

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Abstract

We present a result about the shadowing nontransversal chain of heteroclinic connections based on the idea of dropping dimensions. As an application we discuss this mechanism in a simplification of a toy model system derived by Colliander and all in the context of cubic defocusing nonlinear Schrödinger equation.

1 Introduction

In the present paper we discuss the question of shadowing a nontransversal chain of heteroclinic connections between invariant sets (fixed points, periodic orbits, etc). The motivation for us is the work [CKS+] (see also [GK]) on the transfer of energy to high frequencies in the nonlinear Schrodinger equation. From the dynamical systems perspective there is one remarkable feature of the construction in [CKS+], namely the authors were able to shadow a non-transversal highly degenerated chain of heteroclinic connections between some periodic orbits of arbitrary, but finite, length. Neither in [CKS+] or [GK] we were able to find a clear geometric picture showing how this is achieved, so it could be easily applicable to other systems. In this work we present a mechanism, which we believe that gives a geometric explanation of what is happening and we strive to establish an abstract framework, which will make it easier to apply this technique to other systems, both PDEs and ODEs, in questions related to the existence of

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diffusing orbits. The term *a diffusing orbit* relates to the Arnold's diffusion [Ar] for the perturbation of integrable Hamiltonian systems. Throughout the paper we will often call diffusing orbit an orbit shadowing a chain of heteroclinic connections, and occasionally the existence of such an orbit will be referred to as the diffusion.

In our picture we think of evolving a disk of dimension k along a heteroclinic transitions chain and when a given transition is not transversal, then we 'drop' one or more dimensions of our disk, i.e., we select a subdisk of lower dimension "parallel to expanding directions in the future". After at most k transitions, our disk is a single point and we cannot continue further. We will refer to this phenomenon as the *dropping dimensions* mechanism. While thinking about disks has some geometric appeal, we consider instead in our construction a thickened disk called h-set in the terminology of [ZGi] and our approach is purely topological (just as the one presented in [CKS+]).

The main technical tool used in our work is the notion of *covering relations* as introduced in [ZGi], which differs from the notion used under the same name in [CKS+]. The ideas about the dropping exit dimensions implicitly appear also in works [BM+, WBS], which also used the covering relations from [ZGi].

In the present work we present an abstract topological theorem about shadowing chains of covering relations with dropping dimensions, and we show how such chains of coverings can be obtained in the presence of chains of heteroclinic connections in simple examples, like a linear model, a triangular system and a more simplified Toy Model that the one in [CKS+]. We intend to treat more complicated examples, in particular NLS from [CKS+, GK] in subsequent papers.

The content of this paper can be described as follows. In Section 2 we describe the model problem with a non-transversal heteroclinic chain and state our conjecture about the possibility of shadowing arbitrarily close such chain. We also introduce an example formed by a triangular system, where the existence of the diffusion is quite obvious. In Section 3 we explain the basic geometric idea of our dropping dimensions mechanism. In Section 4 we recall from [ZGi] the notions of h-sets and the covering relation. In Section 5 we prove the main topological result on shadowing of chains of covering relations with dropping dimensions. Using the new mechanism, in the next two sections we rigorously analyze two simple models, a linear model in Section 6 and a simplified Toy Model in Section 7.

1.1 Notation

By $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ we denote the set of natural, integer, rational, real and complex numbers, respectively. We assume that $0 \in \mathbb{N}$. $\mathbb{Z}_-$ and $\mathbb{Z}_+$ are nonpositive and nonnegative integers, respectively. By S^1 we will denote the unit circle on the complex plane.

In $\mathbb{R}^n$ by e_i for $i = 1, \ldots, n$ we will denote the i-th vector from the canonical basis in $\mathbb{R}^n$, i.e. the j-th coordinate of e_i is equal to 1, when $j = i$ and 0 otherwise.

For $\mathbb{R}^n$ we will denote the norm of x by $\|x\|$ and when in some context the formula for the norm is not specified, then it means that any norm can be used. Let $x_0 \in \mathbb{R}^s$, then $B_s(x_0, r) = \{z \in \mathbb{R}^s \mid \|x_0 - z\| < r\}$ and $B_s = B_s(0, 1)$.

Sometimes, if V is a vector space with a norm, then $B_V(a, r)$ will denote an open ball in V centered at a with radius r.

For $z \in \mathbb{R}^u \times \mathbb{R}^s$ we will call usually the first coordinate, x, and the second one y. Hence $z = (x, y)$, where $x \in \mathbb{R}^u$ and $y \in \mathbb{R}^s$. We will use the projection maps $\pi_x(z) = x(z) = x$ and $\pi_y(z) = y(z) = y$. For functions $f : \mathbb{R}^u \times \mathbb{R}^s \to \mathbb{R}^u \times \mathbb{R}^s$ we will use the shortcuts $f_x = \pi_x f$ and $f_y = \pi_y f$.

Let $z \in \mathbb{R}^n$ and $U \subset \mathbb{R}^n$ be a compact set and $f : U \to \mathbb{R}^n$ be continuous map, such that $z \notin f(\partial U)$. Then the local Brouwer degree [S] of f on U at z is defined and will be denoted by $\deg(f, U, z)$, see for example Appendix [ZGi] and references given there for the properties $\deg(f, U, z)$.

If V, W are two vector spaces, then by $\mathrm{Lin}(V, W)$ we will denote the set of all linear maps from V to W. When $V = \mathbb{R}^k$ and $W = \mathbb{R}^m$, we will identify $\mathrm{Lin}(\mathbb{R}^k, \mathbb{R}^m)$ with the set of matrices with m columns and k rows, denoted by $\mathbb{R}^{k\times m}$.

If $x \in \mathbb{R}$, then $\mathrm{int}(x)$ is the integer part of x, i.e., the largest integer not greater than x.

2 Non-transverse diffusion, the statement of the problem, some examples

In this section we would like to state the geometric assumptions under which we expect to construct the orbits shadowing a non-transversal heteroclinic chain. Our approach is motivated by the work [CKS+] on the nonlinear Schrödinger equation.

The main problem in [CKS+] consists on finding an orbit which visits the neighborhoods of N invariant 1-dimensional objects in a N-dimensional complex system. Each object is connected with the previous and the following one with heteroclinic connections, so the authors look for a solution that concatenates these connections. This kind of scheme seems similar to Arnold diffusion[Ar], but we plan to explain that it is another kind of phenomenon since we do not have a transverse intersection between the invariant manifolds. In addition, the proposed mechanism could be applied to integrable systems in contrast to Arnold diffusion, which is a phenomenon that only takes place in non integrable systems.

2.1 Transverse versus Non-Transverse

In this subsection we will explain the difference between the transverse and the non-transverse situation. The hint about the idea of dropping dimensions will be given.

To do so, we are going to consider a two dimensional map with four fixed points, located at the points:

$$p_0 = (0,0) \quad p_1 = (1,0) \quad p_2 = (1,1) \quad p_3 = (2,1).$$

We are going to assume also that each point, p_i, has a one dimensional stable manifold, $\mathcal{T}^s(p_i)$, and a one dimensional unstable manifold, $\mathcal{T}^u(p_i)$ both tangent to some linear subspaces. That is

- $\mathcal{T}^s(p_0)$ is tangent to the subspace generated by $\vec{e}_2$ at p_0 and $\mathcal{T}^u(p_0)$ is tangent to the subspace generated by $\vec{e}_1$ at p_0.
- $\mathcal{T}^s(p_1)$ is tangent to the subspace generated by $\vec{e}_1$ at p_1 and $\mathcal{T}^u(p_1)$ is tangent to the subspace generated by $\vec{e}_2$ at p_1.
- $\mathcal{T}^s(p_2)$ is tangent to the subspace generated by $\vec{e}_2$ at p_2 and $\mathcal{T}^u(p_2)$ is tangent to the subspace generated by $\vec{e}_1$ at p_2.
- $\mathcal{T}^s(p_3)$ is tangent to the subspace generated by $\vec{e}_1$ at p_3 and $\mathcal{T}^u(p_3)$ is tangent to the subspace generated by $\vec{e}_2$ at p_3.

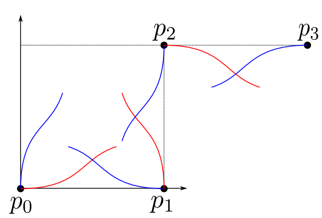
Now we are going to consider two different scenarios. The first one consists on assuming that the unstable manifold of a point p_i intersects transversally with the stable manifold of the following point p_{i+1}, that is:

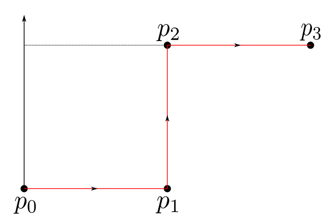
$$\mathcal{T}^u(p_i) \cap \mathcal{T}^s(p_{i+1}) \neq \emptyset$$
$$\forall q \in \mathcal{T}^u(p_i) \cap \mathcal{T}^s(p_{i+1}) \Rightarrow T_q\mathcal{T}^u(p_i) + T_q\mathcal{T}^s(p_{i+1}) = \mathbb{R}^2.$$

The second will be given by a non-transverse intersection of the manifolds and, since we are dealing with a low dimensional system, that will mean that those manifolds coincide in a branch:

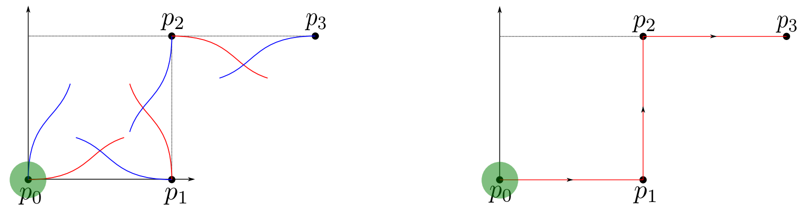
$$\mathcal{T}^u(p_i) \cap \mathcal{T}^s(p_{i+1}) \neq \emptyset$$
$$\forall q \in \mathcal{T}^u(p_i) \cap \mathcal{T}^s(p_{i+1}) \Rightarrow T_q\mathcal{T}^u(p_i) + T_q\mathcal{T}^s(p_{i+1}) = \mathbb{R}.$$

The schematic situation is the following (unstable manifolds are in red and stable manifolds are in blue):

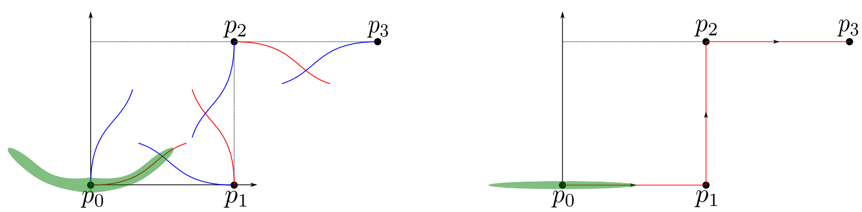




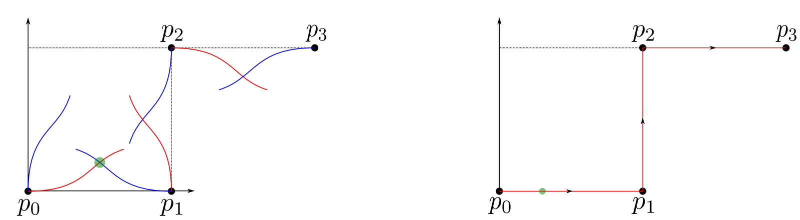
We wonder if it is possible to connect p_0 with p_3 through the map, in both situations. To do so, we consider a ball containing the first fixed point p_0:



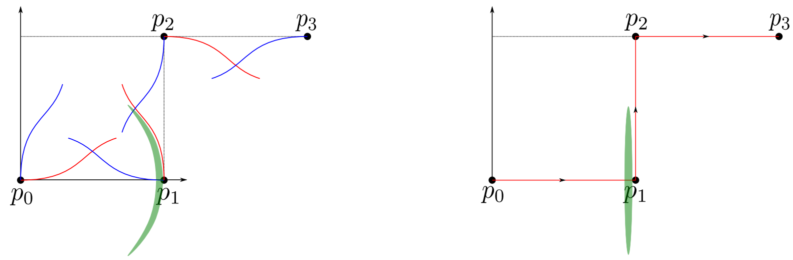
If we compute iterates of the ball through the maps we can expect that it is expanded in the unstable direction and contracted in the stable direction:



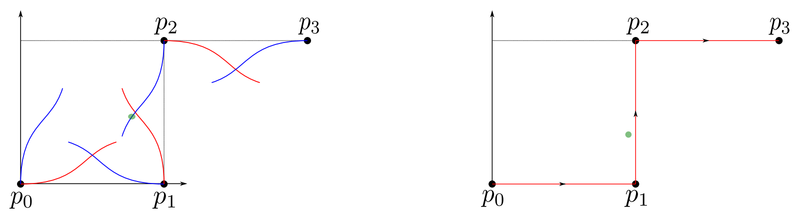
Notice that in both cases the domain intersects the stable manifold of the following fixed point, p_1. So, in the transverse case the domain contains a heteroclinic point. In the non-transverse situation this is obvious because the manifolds are coincident. We can now restrict our domain precisely around that intersection point for the transverse case and at some place in the right-hand side of the fixed point p_0 for the non-transverse case.



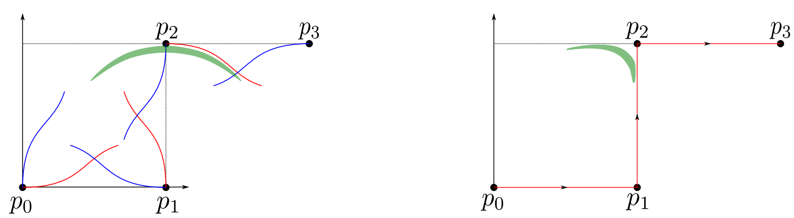
If we compute forward iterates of the restricted domain we are going to approach the fixed point p_2 since, in both cases, our domains contain heteroclinic points. The domains will not only approach p_1 but also, after some iterates, will spread to the unstable manifold of p_1:



Here we find the first big difference between the transverse and the non-transverse case. In the transverse case, since everything tends to the unstable manifold of p_1, it is clear that our domain will intersect the stable manifold of p_2. In the non-transverse situation, our domain will never cross the stable manifold of p_2 since it corresponds to an invariant curve. Then, we restrict our domain in the intersection for the transverse case and in the upper part of p_1 since we want to reach p_3:



Using, in the transverse case, the same argument as before, since our domain contains a heteroclinic point in the transverse situation, forward iterates will spread our domain on the unstable manifold of p_2. For the non transverse case we will reach the proximity of p_2 after some iterates, but our domain will be trapped and could not visit the following fixed point, p_3:



In the transverse case, we could continue and see that the domain will visit p_3.

After this schematic approach, we can see that, on the one hand, in the transverse situation there are no geometric obstructions in shadowing the heteroclinic chain. On the other hand, in the non-transverse case, we can see that, in general, we cannot visit as many invariant objects as we want. So, now, we wonder why the authors of [CKS+] can connect N periodic orbits in the Toy Model System. The main reason is the large dimension of the system and the fact that each connection takes place in a direction that has not been used in the past.

Notice that, if in the non-transverse example the last fixed point p_3 is located in a new dimension (that means that the system is three dimensional) we could continue with the argument and visit p_3.

In the next subsection we are going to generate examples for which it is clear that one can shadow a non-transverse heteroclinic chain.

2.2 Examples with diffusion in a non-transverse situation - the triangular system

In this subsection we present a simple system, the triangular system, which is an ODE defined by a polynomial of degree two.

The triangular system is

$$\dot{x} = F(x)$$

with

$$\left\{ \begin{array}{lcll} F_1(x) & = & \lambda_1 x_1 - \lambda_1 x_1^2 & \\ F_i(x) & = & (\lambda_i - \mu_i)x_i - \lambda_i x_i^2 + \mu_i x_i x_{i-1} & \quad \text{for } 1 < i \leq n \end{array} \right. \tag{1}$$

with $\lambda_i > 0$ for $1 \leq i \leq n$ and $\mu_i \in \mathbb{R}$ for $1 < i < n$.

Note that this is a triangular system and we can integrate each equation, since

$$\dot{x}_i = f_i(t)x_i + \beta x_i^2 \Rightarrow x_i(t) = \frac{e^{\int_0^t f_i(s)ds}}{\frac{1}{x_i(0)} - \beta_i \int_0^t e^{\int_0^s f_i(r)dr} ds},$$

with $f_i(t) = \lambda_i - \mu_i + \mu_i x_{i-1}(t)$ and $\beta_i = -\lambda_i$.

We can check now the linear behavior around the equilibrium points computing the derivative of the vector field:

$$DF(p_0) = \begin{pmatrix} \lambda_1 & & & & & \\ & \lambda_2 - \mu_2 & & & & \\ & & \ddots & & & \\ & & & \lambda_i - \mu_i & & \\ & & & & \ddots & \\ & & & & & \lambda_n - \mu_n \end{pmatrix}$$

$$DF(p_i) = \begin{pmatrix} -\lambda_1 & & & & & & & \\ \mu_2 & -\lambda_i & & & & & & \\ & \ddots & \ddots & & & & & \\ & & \mu_i & -\lambda_i & & & & \\ & & & & \lambda_{i+1} & & & \\ & & & & & \lambda_{i+2}-\mu_{i+2} & & \\ & & & & & & \ddots & \\ & & & & & & & \lambda_n-\mu_n \end{pmatrix}$$

so we have different possibilities of choosing the parameters.

- If $\mu_i > \lambda_i$ for all $i = 1 \dots N$, each point, p_j, has only one unstable direction defined by $\vec{e}_{j+1}$, while the rest of the directions are stable.
- If $\mu_i = \lambda_i$ for all $i = 1 \dots N$, each point, p_j, has only one unstable direction, defined by $\vec{e}_{j+1}$. All the "past" directions, defined by $\{\vec{e}_1, \dots, \vec{e}_j\}$ are stable while all the "future" directions, defined by $\{\vec{e}_{j+2}, \dots, \vec{e}_n\}$, are linearly neutral.
- If $\mu_i < \lambda_i$ for all $i = 1 \dots N$, at each point, p_j, all the "past" directions, defined by $\{\vec{e}_1, \dots, \vec{e}_j\}$, are stable while all the "future" directions, defined by $\{\vec{e}_{j+2}, \dots, \vec{e}_n)\}$, are linearly unstable.

Let us show the numerical integration of the system for these three possibilities. For the sake of concreteness we are going to assume $\lambda = \lambda_i$ and $\mu = \mu_i$ for all $i = 1, \dots, n$ in a four dimensional system. In the three cases, we are going to take the same initial condition:

$$x_1(0) = x_2(0) = x_3(0) = x_4(0) = \frac{1}{10}.$$

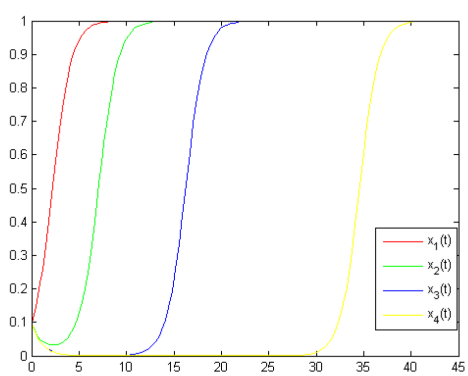


Figure 1: Solution of system (1) for $\lambda = 1$ and $\mu = 2$.

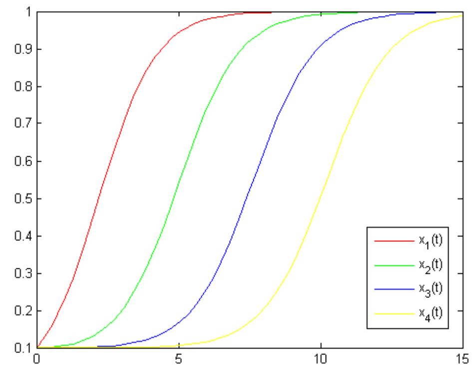


Figure 2: Solution of system (1) for $\lambda = 1$ and $\mu = 1$.

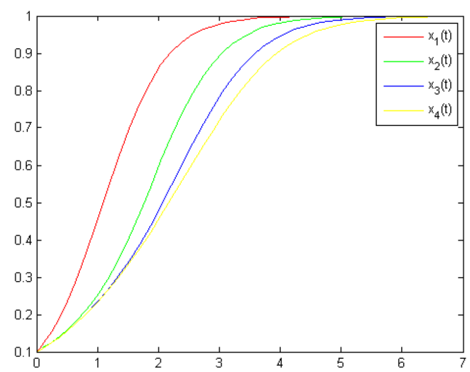


Figure 3: Solution of system (1) for $\lambda = 2$ and $\mu = 1$.

We can see that in the three cases we achieved our goal: to visit $p_4 = (1,1,1,1)$ starting close to $p_0 = (0,0,0,0)$. However it is clear that we only visit all the intermediate points $p_1 = (1,0,0,0)$, $p_2 = (1,1,0,0)$ and $p_3 = (1,1,1,0)$ in the first situation, when $\mu > \lambda$. In this case, all the directions are stable in each point, and are only activated at its turn. In the other cases, all the future directions are unstable at each point so they are activated at the very beginning although the characteristic exponent, $\lambda - \mu$, is lower than the one in the heteroclinic, λ, in the considered cases. So, if we want to visit all the intermediate points in the two last situations, we have to decrease the initial condition for the future directions. Indeed, if we take:

$$x_1(0) = \frac{1}{10}, \quad x_2(0) = x_3(0) = x_4(0) = \frac{1}{100},$$

in the case when $\mu = \lambda = 1$, we obtain:

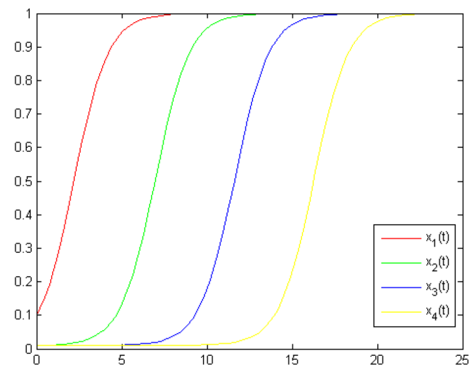


Figure 4: Solution of system (1) for $\lambda = 1$ and $\mu = 1$.

Notice that it is enough to distinguish only the first component. The weak coupling activates the component in order, since the first equation that notices the growth of x_1 is the one for x_2.

For the third case, when $\lambda > \mu$, we recall that all the future coordinates are unstable in p_0 and the linear part almost dominates in front of the coupling that would have made increase the coordinates in order. So, considering the following decreasing sequence of initial condition:

$$x_1(0) = 10^{-1}, \quad x_2(0) = 10^{-2} \quad x_3(0) = 10^{-3} \quad x_4(0) = 10^{-4},$$

we obtain:

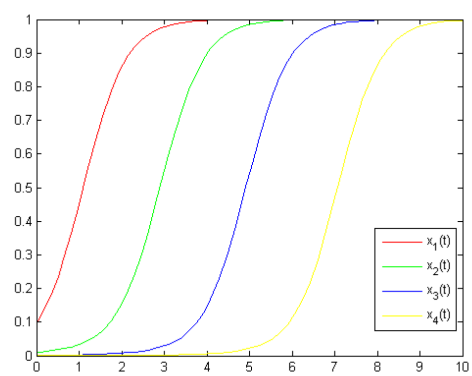


Figure 5: Solution of system (1) for $\lambda = 2$ and $\mu = 1$.

Although there are numerical evidences that the system behaves in the desired way, we could expect that just looking at the equations. It is clear that

$$\lim_{t \to \infty} x_1(t) = 1.$$

By induction, assuming

$$l_{i-1} = \lim_{t \to \infty} x_{i-1}(t)$$

and looking for equilibria of $x_i(t)$ for $t \to \infty$, it has to satisfy

$$\lim_{t \to \infty} \dot{x}_i(t) = 0,$$

i.e.

$$0 = \lambda_i l_i (1 - l_i) - \mu_i l_i (1 - l_{i-1}).$$

So, we get $l_i = 0$ or $l_i = 1$ but since $\dot{x}_i(t) > 0$ for t large enough we can conclude $l_i = 1$. This reasoning with some effort can be turned into the rigorous proof.

The main conclusion of this part is that we have obtained an easy example for which we can ensure the transition chain even if the intersection between the invariant manifolds is not transverse, regarding the high dimension and the disposition of the equilibrium points and the heteroclinic connections: each one in a new direction not used before. In addition the system is integrable by quadratures which goes against the notion of Arnold's diffusion. However, we have detected the reason why the connection could be possible. The geometric mechanism relies on the fact that we are dealing with a high dimensional system and that each new connection is defined by a direction that *has not been used before.*

2.3 The Model example

We are now in condition to present a conjecture, which probably can be proved by our method under some additional assumptions.

Let $n_i > 0$ for $i = 1, \dots, L$ and let $n_1 + n_2 + \dots + n_L = n$, $d_i = n_1 + \dots + n_i$.

For $i = 1, \dots, L$ the subspaces V_i which are spanned by $\{e_{k_{i-1}+j}\}_{j=1,\dots,n_i}$. In this notation $\mathbb{R}^n = \bigoplus_{i=1}^{L} V_i$.

We will use the following notation: for $l \in \mathbb{N}$, $0^l, 1^l$ will denote the sequences of length l consisting of l 0's or l 1's, respectively.

Assume that we have a diffeomorphism on $f : \mathbb{R}^n \times \mathbb{R}^{w_u} \times \mathbb{R}^{w_s} \to \mathbb{R}^n \times \mathbb{R}^{w_u} \times \mathbb{R}^{w_s}$, $w = w_u + w_s$ with the following properties:

- there exists a sequence of fixed points

$$\begin{aligned}
p_1 &= (0^n, 0^w), \\
p_2 &= (1, 0^{n_1-1}, 0, \ldots, 0^w), \\
p_3 &= (1, 0^{n_1-1} 1, 0^{n_2-1}, 0 \ldots, 0^w), \\
\ldots & \\
p_k &= (1, 0^{n_1-1}, 1, 0^{n_2-1}, \ldots, 1, 0^{n_k-1}, 0^{n_k+\cdots+n_L}, 0^w), \\
\ldots & \\
p_{L+1} &= (1, 0^{n_1-1}, 1, 0^{n_2-1}, \ldots, 1, 0^{n_L-1}, 0^w)
\end{aligned}$$

- for any $i = 1, \ldots, L$, the interval connecting p_i and p_{i+1} denoted by C_i

  $$C_i = \{z = (1, 0^{n_1-1}, 1, 0^{n_2-1}, \ldots, 1, 0^{n_{i-1}-1}, t, 0^{-1+n_i+\cdots+n_L}, 0^w), \quad t \in [0,1]\}$$

  is invariant under f and for any $z \in C_i$

  $$\lim_{k\to\infty} f^k(z) = p_i, \quad \lim_{k\to-\infty} f^k(z) = p_{i-1} \tag{2}$$

- at p_i the coordinate directions in V_i are exit directions and are 'dominating' for the scattering when passing by p_i. This statement is vague, because the scenario we are going to present most likely can be realized under various sets of assumptions.

Only the first n-directions really count, the others will be treated as the entry directions (in the sense of covering relations, see Section 4).

Our conjecture is

Conjecture 1 *Under the above assumptions for any $\epsilon > 0$ there exists a point z_1 and a sequence of integers $k_1 < k_2 < \cdots < k_L$, such that*

$$\begin{aligned}
\|z_1 - p_1\| &< \epsilon, \\
\|f^{k_i}(z_1) - p_{i+1}\| &< \epsilon, \quad i = 1, \ldots, L
\end{aligned}$$

Our idea of the proof of this conjecture requires a construction of covering relations (see (23), and then Conjecture 1 follows directly from Theorem 8.

We will show how a construction of suitable h-sets and coverings can be done for a linear model in Section 6 and for the simplified version of the toy model from [CKS+] in Section 7.

3 The geometric idea of dropping dimensions

In this section we will explain our idea of dropping dimensions along a non-transversal heteroclinic chain.

Let us start with a simplified version of the example from Section 2.3. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism with the following properties:

1. The points $p_i = (1, \overset{i}{\ldots}, 1, 0, \overset{n-i}{\ldots}, 0)$ are fixed under f for $i = 0 \ldots n$.
2. The segments C_i that connect the points p_{i-1} and p_i,

$$C_i = \{(1, \overset{i-1}{\ldots}, 1, t, 0, \overset{n-i}{\ldots}, 0), 0 \le t \le 1\}$$

for $1 \le i \le n$ are invariant under f and, for all $x \in C_i$:

$$\lim_{k \to \infty} f^k(x) = p_i \qquad \lim_{k \to -\infty} f^k(x) = p_{i-1}.$$

3. At each point p_i the i-th direction is stable and the $(i+1)$-th is unstable. This means:

$$Df(p_i)e_i = \mu_i e_i, \quad |\mu_i| < 1$$
$$Df(p_i)e_{i+1} = \lambda_i e_{i+1}, \quad |\lambda_i| > 1$$

4. The past directions, defined by $\vec{e_1}, \ldots, \vec{e}_{i-1}$, are contracting directions around the fixed point p_i but with a lower rate than μ_i. The future directions, defined by $\vec{e}_{i+2}, \ldots, \vec{e}_n$, are expanding directions around the fixed point p_i but with a lower rate than λ_i.

Conjecture 2 *Under the previous assumptions, for all $\epsilon > 0$ there exists a point x_0 and a sequence of integers $0 = k_0 < k_1 < \cdots < k_n$ such that:*

$$||f^{k_i}(x_0) - p_i|| < \epsilon \quad i = 0, \ldots, n.$$

Remark 3 *Notice that we connect $n+1$ points in a n dimensional space. We cannot guarantee that the result is valid for more points.*

Remark 4 *Observe that we are assuming that there are only two dominant coordinates around each fixed point. That means that this could not be applied to the Toy Model System in NLS [CKS+, GK], where we have four dominant directions. In Section 7 we treat such system.*

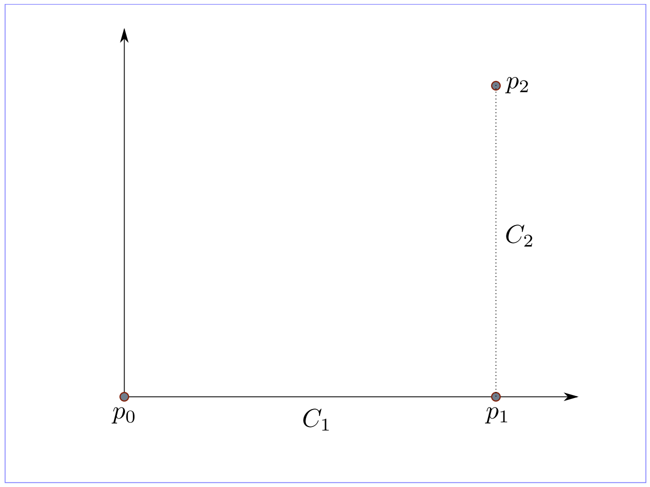
3.0.1 Sketch of the proof: dropping dimensions

Here we sketch a proof of Conjecture 2 with some pictures. We are going to consider only a two dimensional map. So, consider $f : \mathbb{R}^2 \to \mathbb{R}^2$ with three fixed points:

$$p_0 = (0,0) \quad p_1 = (1,0) \quad p_2 = (1,1),$$

with invariant segments C_1 and C_2 defined as

$$C_1 = \{(x_1, x_2) \,:\, 0 \le x_1 \le 1,\, x_2 = 0\} \quad C_2 = \{(x_1, x_2) \,:\, x_1 = 0,\, 0 \le x_2 \le 1\}$$

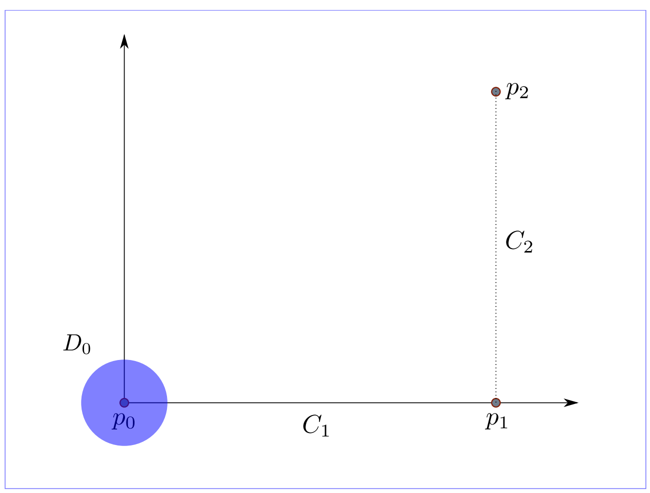


Assume also that the derivatives of the map around the fixed points have the following structure:

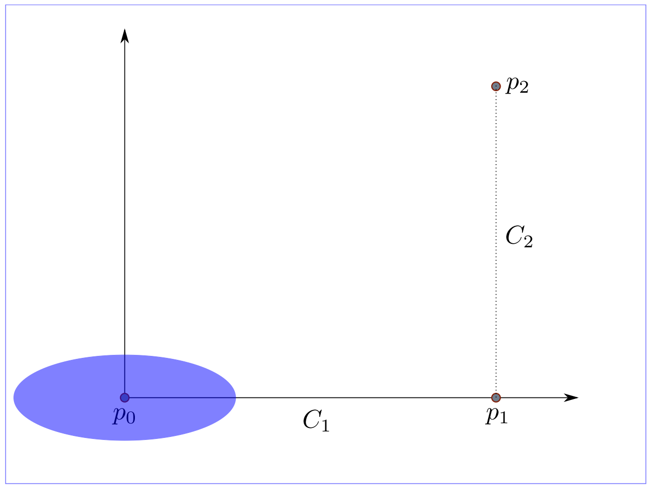
$$Df(p_0) = \begin{pmatrix} \lambda_{0,1} & 0 \\ 0 & \lambda_{0,2} \end{pmatrix} \quad Df(p_1) = \begin{pmatrix} \mu_{1,1} & 0 \\ 0 & \lambda_{1,2} \end{pmatrix} \quad Df(p_2) = \begin{pmatrix} \mu_{2,1} & 0 \\ 0 & \mu_{2,2} \end{pmatrix} \tag{3}$$

where $\lambda_{0,1}, \lambda_{0,2}, \lambda_{12} > 1$ and $0 < \mu_{1,1}, \mu_{2,1}, \mu_{2,2} < 1$.

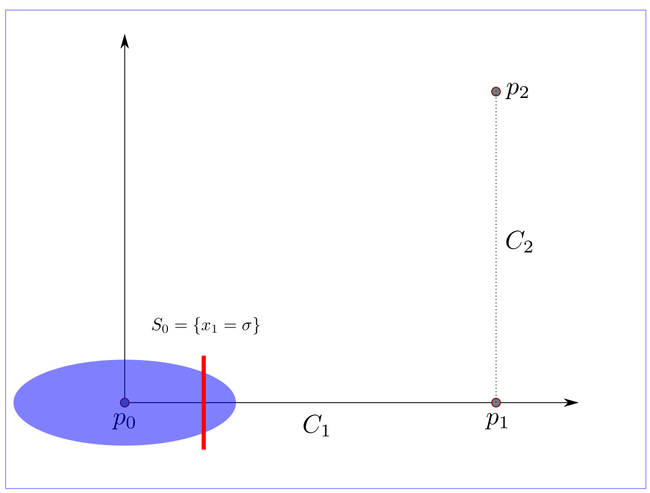
We start by considering a domain (ball) D_0 of full dimension centered around p_0.



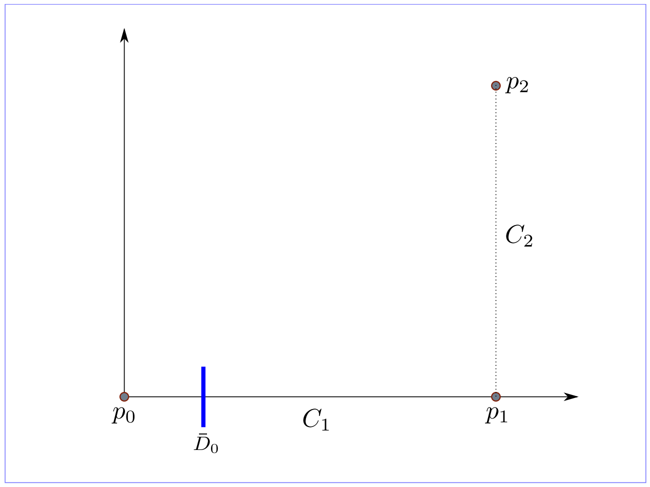
Given the linear stability from (3), we can assume that after one iteration of the map, our initial ball D_0 will be expanded in both directions:



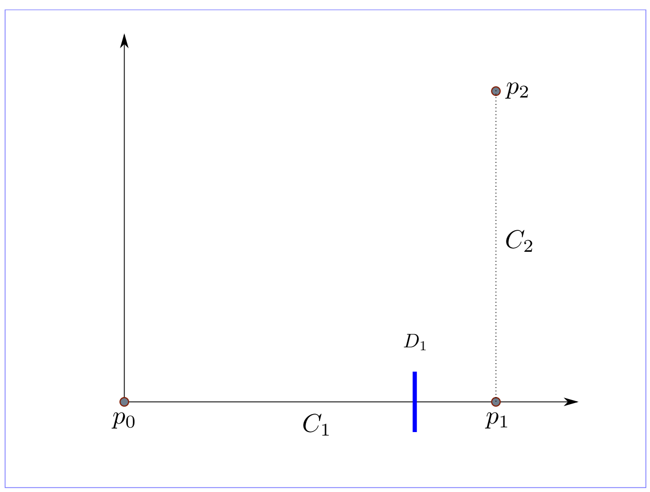
It is now time to make a decision: from all the possible directions, we are only interested in the one defined by the outgoing heteroclinic connection, that is, the segment C_1. Then we consider a section $S_0 = \{x_1 = \sigma\}$ where σ is some small parameter:



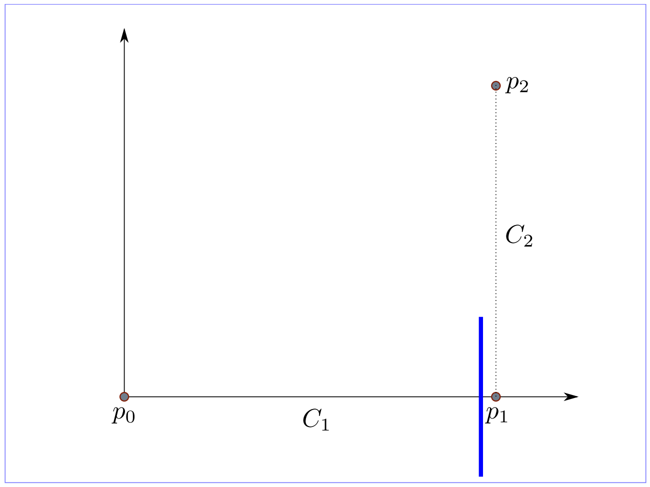
Since we are only interested in the points of our ball close to the heteroclinic connection, we intersect the domain with the section S_0. We say that we have dropped the x_1 direction. We will not use this direction in future steps. Our domain, $\bar{D}_0$, has one dimension less than D_0, that is, it has dimension one.



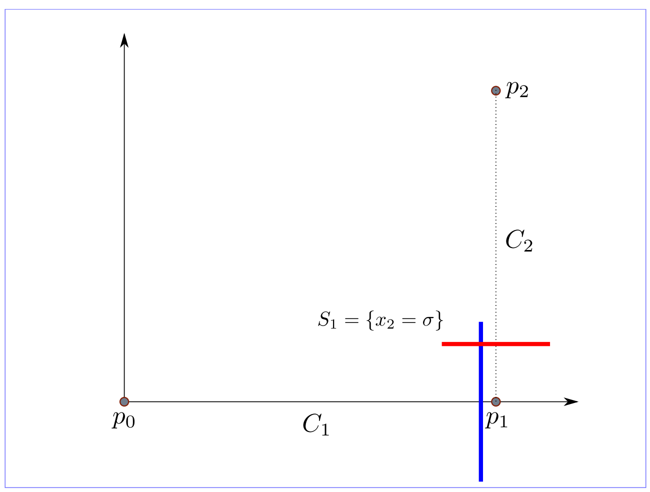
Then we continue with this domain. After several iterations of the map since the domain is close to the heteroclinic connection, we can ensure that $\bar{D}_0$ will approach p_1 and we obtain a domain D_1:



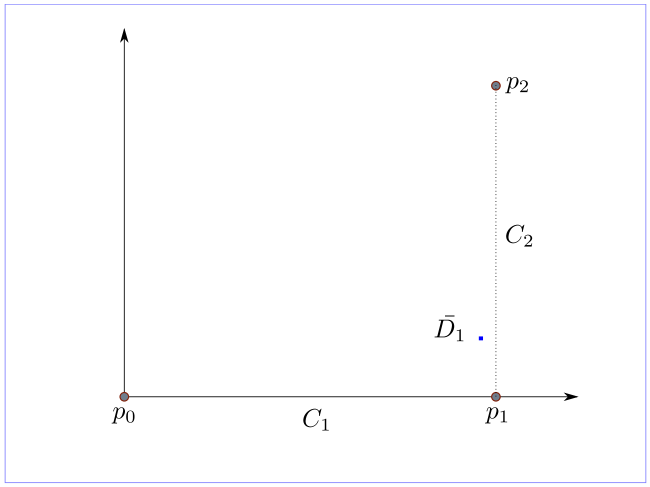
We can use, again, the linear prediction of the map, (3), to be sure that our domain is expanded in the x_2 direction.



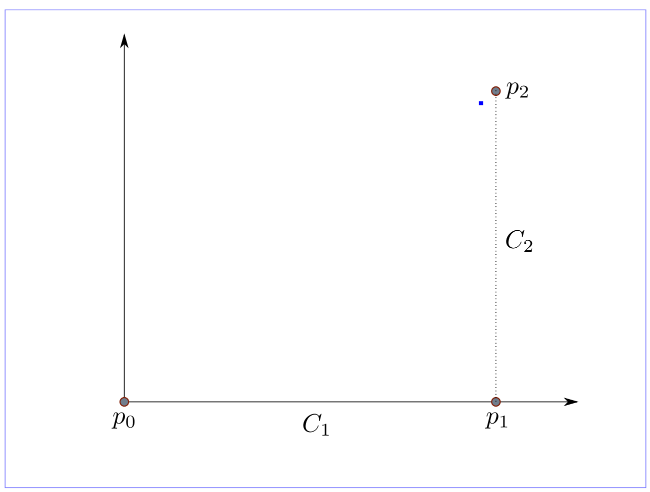
We use now the same argument. From all the possible directions that $f(D_1)$ covers, we want to escape through the one defined by the heteroclinic connection to p_2. So we put a section defined in the same spirit as before: $S_1 = \{x_2 = \sigma\}$.



We restrict now our domain in its intersection with the section S_1. The resulting domain $\bar{D}_1$ will have, then, one dimension less than D_1, which means that it will have dimension zero.



We have no more dimensions to drop, our initial domain become a single point. This point is close to the heteroclinic defined in C_2, so then, we are sure that after some iterates, it will approach the final fixed point p_2:



4 h-sets, covering relations

The goal of this section is to recall from [ZGi] the notions of h-sets and covering relations, and to state the theorem about the existence of point realizing the chain of covering relations. This will be the main technical tool in proving the existence of the orbits shadowing the heteroclinic chain in the next sections.

4.1 h-sets and covering relations

Definition 1 *[ZGi, Definition 1] An h-set N is a quadruple $(|N|, u(N), s(N), c_N)$ such that*

- $|N|$ *is a compact subset of* $\mathbb{R}^n$
- $u(N), s(N) \in \{0, 1, 2, \ldots, n\}$ *are such that* $u(N) + s(N) = n$
- $c_N : \mathbb{R}^n \to \mathbb{R}^n = \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)}$ *is a homeomorphism such that*

$$c_N(|N|) = \overline{B_{u(N)}} \times \overline{B_{s(N)}}.$$

We set

$$\begin{aligned} \dim(N) &:= n, \\ N_c &:= \overline{B_{u(N)}} \times \overline{B_{s(N)}}, \\ N_c^- &:= \partial B_{u(N)} \times \overline{B_{s(N)}}, \\ N_c^+ &:= \overline{B_{u(N)}} \times \partial B_{s(N)}, \\ N^- &:= c_N^{-1}(N_c^-), \quad N^+ = c_N^{-1}(N_c^+). \end{aligned}$$

Hence a h-set N is a product of two closed balls in some coordinate system. The numbers $u(N)$ and $s(N)$ are called the exit and entry dimensions, respectively. The subscript c refers to the new coordinates given by the homeomorphism c_N. Observe that if $u(N) = 0$, then $N^- = \emptyset$ and if $s(N) = 0$, then $N^+ = \emptyset$. In the sequel to make notation less cumbersome we will often drop the bars in the symbol $|N|$ and we will use N to denote both the h-sets and its support.

We will call N^- *the exit set of* N and N^+ *the entry set of* N. These names are motivated by the Conley index theory [C, MM] and the role that these sets will play in the context of covering relations.

Definition 2 *[ZGi, Definition 6] Assume that N, M are h-sets, such that $u(N) = u(M) = u$ and $s(N) = s(M) = s$. Let $f : N \to \mathbb{R}^n$ be a continuous map. Let $f_c = c_M \circ f \circ c_N^{-1} : N_c \to \mathbb{R}^u \times \mathbb{R}^s$. Let w be a nonzero integer. We say that*

$$N \stackrel{f,w}{\Longrightarrow} M$$

(N f-covers M with degree w) iff the following conditions are satisfied

1. *There exists a continuous homotopy $h : [0, 1] \times N_c \to \mathbb{R}^u \times \mathbb{R}^s$, such that the following conditions hold true*

$$h_0 = f_c, \tag{4}$$
$$h([0, 1], N_c^-) \cap M_c = \emptyset, \tag{5}$$
$$h([0, 1], N_c) \cap M_c^+ = \emptyset. \tag{6}$$

2. *If $u > 0$, then there exists a map $A : \mathbb{R}^u \to \mathbb{R}^u$, such that*

$$\begin{aligned} h_1(p,q) &= (A(p),0), \text{ for } p \in \overline{B_u}(0,1) \text{ and } q \in \overline{B_s}(0,1), \quad (7) \\ A(\partial B_u(0,1)) &\subset \mathbb{R}^u \setminus \overline{B_u}(0,1). \quad (8) \end{aligned}$$

Moreover, we require that

$$\deg(A, \overline{B_u}(0,1), 0) = w, \quad (9)$$

We will call condition (5) the exit condition *and condition* (6) *will be called* the entry condition.

Note that in the case $u = 0$, if $N \overset{f,w}{\Longrightarrow} M$, then $f(N) \subset \operatorname{int} M$ and $w = 1$.

In fact in the above definition $s(N)$ and $s(M)$ can be different, see [W2, Def. 2.2].

Remark 5 *If the map A in condition 2 of Def. 2 is a linear map, then condition (8) implies that*

$$\deg(A, \overline{B_u}(0,1), 0) = \pm 1.$$

Hence condition (9) is fulfilled with $w = \pm 1$.

In fact, this is the most common situation in the applications of covering relations.

Most of the time we will not be interested in the value of w in the symbol $N \overset{f,w}{\Longrightarrow} M$ and we will often drop it and write $N \overset{f}{\Longrightarrow} M$, instead. Sometimes we may even drop the symbol f, if known from the context, and write $N \Longrightarrow M$.

Definition 3 *[ZGi, Definition 7] Assume N, M are h-sets, such that $u(N) = u(M) = u$ and $s(N) = s(M) = s$. Let $g : \mathbb{R}^n \supset \Omega \to \mathbb{R}^n$. Assume that $g^{-1} : |M| \to \mathbb{R}^n$ is well defined and continuous. We say that $N \overset{g}{\Longleftarrow} M$ (N g-backcovers M) iff $M^T \overset{g^{-1}}{\Longrightarrow} N^T$.*

4.2 Main theorem about chains of covering relations

Theorem 6 (Thm. 9) *[ZGi] Assume N_i, $i = 0, \dots, k$, $N_k = N_0$ are h-sets and for each $i = 1, \dots, k$ we have either*

$$N_{i-1} \overset{f_i, w_i}{\Longrightarrow} N_i \quad (10)$$

or

$$N_i \subset \operatorname{dom}(f_i^{-1}) \quad \text{and} \quad N_{i-1} \overset{f_j, w_i}{\Longleftarrow} N_i. \quad (11)$$

Then there exists a point $x \in \operatorname{int} N_0$, such that

$$\begin{aligned} f_i \circ f_{i-1} \circ \dots \circ f_1(x) &\in \operatorname{int} N_i, \quad i = 1, \dots, k \quad (12) \\ f_k \circ f_{k-1} \circ \dots \circ f_1(x) &= x \quad (13) \end{aligned}$$

We point the reader to [ZGi] for the proof. The basic idea of the proof of this theorem is the homotopy and the local Brouwer degree.

The following corollary is an immediate consequence of Theorem 6.

Collorary 7 *Let N_i, $i \in \mathbb{Z}_+$ be h-sets. Assume that for each $i \in \mathbb{Z}_+$ we have either*

$$N_{i-1} \stackrel{f_i,w_i}{\Longrightarrow} N_i \tag{14}$$

or

$$N_i \subset \operatorname{dom}(f_i^{-1}) \quad \text{and} \quad N_{i-1} \stackrel{f_i,w_i}{\Longleftarrow} N_i. \tag{15}$$

Then there exists a point $x \in \operatorname{int} N_0$, such that

$$f_i \circ f_{i-1} \circ \cdots \circ f_1(x) \quad \in \quad \operatorname{int} N_i, \quad i \in \mathbb{Z}_+. \tag{16}$$

Moreover, if $N_{i+k} = N_i$ for some $k > 0$ and all i, then the point x can be chosen so that

$$f_k \circ f_{k-1} \circ \cdots \circ f_1(x) = x. \tag{17}$$

4.3 Natural structure of a h-set

Observe that all the conditions appearing in the definition of the covering relation are expressed in 'internal' coordinates c_N and c_M. Also the homotopy is defined in terms of these coordinates. Sometimes this makes the matter and the notation to look a bit cumbersome. With this in mind we introduce the notion of a natural structure on a h-set.

Definition 4 *We will say that $N = \{(x_0, y_0)\} + \overline{B}_u(0, r_1) \times \overline{B}_s(0, r_1) \subset \mathbb{R}^u \times \mathbb{R}^s$ is an h-*set *with a natural structure if:*
$u(N) = u$, $s(N) = s$, $c_N(x, y) = \left(\frac{x-x_0}{r_1}, \frac{y-y_0}{r_2}\right)$.

4.4 The operation of dropping exit dimensions

Definition 5 *Assume that we have a decomposition $\mathbb{R}^n = \mathbb{R}^{u_1} \oplus \mathbb{R}^t \oplus \mathbb{R}^{s_1}$ and the norm for $(x_1, x_2, x_3) \in \mathbb{R}^{u_1} \oplus \mathbb{R}^t \oplus \mathbb{R}^{s_1}$ is $\|(x_1, x_2, x_3)\| = \max(\|x_1\|, \|x_2\|, \|x_3\|)$.*

Assume that N is an h-set, with $u(N) = u_1 + t$ and $s(N) = s_1$. In view of the norm on $\mathbb{R}^n$ we have

$$c_{|N|} = \left(\overline{B}_{u_1} \oplus \overline{B}_t\right) \oplus \overline{B}_{s_1} \tag{18}$$

where the parentheses enclose the exit directions.

Let us denote by V the subspace $\{0\} \times \mathbb{R}^t \times \{0\}$. We define a new h-set $R_V(N)$ by setting

- $|R_V(N)| = |N|$
- $u(R_V(N)) = u_1$, $s(R_V(N)) = s_1 + t$
- $c_{R_V(N)} = c_N$

Roughly speaking $R_V(N)$ is obtained from N by relabeling some exit coordinates in N as entry directions.

5 The mechanism of dropping dimensions - the main topological theorem

5.1 h-sets M_i and $\widetilde{M}_i$

Our setting is motivated by Conjecture 1. For the sake of completeness we recall here from Section 2 some assumptions.

Let $n_i > 0$ for $i = 1, \dots, L$ and let $n_1 + n_2 + \cdots + n_L = n$ and $w = w_u + w_s$, where $w_u, w_s \in \mathbb{N}$.

For $i = 1, \dots, L$ the subspaces V_i, $\dim V_i = n_i$. Let W_u and W_s be two subspaces of dimensions w_u and w_s, respectively.

In $\mathbb{R}^n \times \mathbb{R}^{w_u} \times \mathbb{R}^{w_s}$ we will represent points as $(z_1, \dots, z_L, z_{L+1}, z_{L+2})$, where $z_i \in V_i$ for $i = 1, \dots, L$, $z_{L+1} \in W_u$ and $z_{L+2} \in W_s$. In each of the spaces V_i, W_j we have some fixed basis and an isomorphism with some $\mathbb{R}^d$ equipped with the metric, so we can define balls in this subspace.

First, we put sections (hyperplanes of codimension n_i) in the vicinity of each point p_i (not to be confused with the Poincaré sections for ODEs):
the exit section S_i is given by conditions

$$z_i = \delta_i = (\Delta_i, 0^{n_i - 1}) \in V_i, \quad i = 1, \dots, L, \tag{19}$$

where $\Delta_i > 0$.

We also define

$$\delta_{L+1} = 0. \tag{20}$$

For $1 \leq i \leq L + 1$ we define set M_i (centered on the section S_i for $i \leq L$ and on p_{L+1} for M_{L+1})

$$M_i = p_i + \delta_i + \left(\Pi_{j=1}^{L} \overline{B}_{n_j}(0, t_{i,j})\right) \times \overline{B}_{w_u}(0, t_{i,L+1}) \times \overline{B}_{w_s}(0, t_{i,L+2}), \tag{21}$$

where $t_{i,j}$ are positive real numbers.

We equip the set M_i with two different h-set structures. To define the first one, denoted by M_i for $i = 1, \dots, L, L + 1$, we declare as the exit directions $V_i \oplus V_{i+1} \oplus \cdots \oplus V_L \oplus W_u$ and $V_1 \oplus V_2 \oplus \cdots V_{i-1} \oplus W_s$ as the entry directions. For the second one, we set

$$\widetilde{M}_i = R_{V_i}(M_i), \quad i = 1, \dots, L. \tag{22}$$

This means that (see Def. 5) we declare as the exit directions $V_{i+1} \oplus \cdots \oplus V_L \oplus W_u$ (i.e. when compared to M_i we drop the subspace V_i of exit directions). We will not need $\widetilde{M}_{L+1}$.

Observe that $\widetilde{M}_L$ and M_{L+1} have W_u as the exit directions and M_L has $V_L \oplus W_u$ as the exit directions.

Now we assume that

$$\widetilde{M}_i \overset{f^{l_i}}{\Longrightarrow} M_{i+1} \quad i = 1, \dots, L. \tag{23}$$

The above covering relations are expected by combining the transition where we drop the connecting direction (plus others we decide to treat from that point on as the entry ones) with the local hyperbolic behavior near p_{i+1}, while in other directions for both covering relations where the dynamics might not help us we just adjust the sizes to obtain the correct inequalities. For this purpose we need to increase the sizes during the transition if these are treated as the entry directions or to decrease the sizes if they are treated as the exit ones.

5.2 The main topological shadowing theorem

About the same time as this work was under development the theorem about shadowing a chain covering relations with decreasing number of exit directions appeared in works [BM+, WBS], where a slightly different technique of proof has been used, but it still is based on the same covering relations we are using.

Theorem 8 *Assume that the following covering relations are satisfied*

$$\widetilde{M_i} = R_{V_i}(M_i) \overset{f^{l_i}}{\Longrightarrow} M_{i+1} \quad i=1,\dots,L. \tag{24}$$

Let $k_i = \sum_{j=1}^{i} l_j$.

Then there exists q*, such that*

$$\begin{aligned} q &\in M_1, \\ f^{k_i}(q) &\in M_{i+1} \quad i=1,\dots,L. \end{aligned}$$

Proof: Equation (21) allows us to introduce the coordinates on M_i through the map

$$\mathcal{C}_i : M_i \to \Pi_{i=1}^{L} \overline{B_{V_i}}(0,1) \times \overline{B_{W_u}}(0,1) \times \overline{B_{W_s}}(0,1), \tag{25}$$

$$\mathcal{C}_i(z_1,\dots,z_L,z_{L+1},z_{L+2}) = \left(\frac{z_1 - p_{i,1} - \delta_{i,1}}{t_{i,1}}, \dots, \frac{z_{L+2} - p_{i,L+2} - \delta_{i,L+2}}{t_{i,L+2}} \right).$$

Observe that the above coordinates $\mathcal{C}_i$, up to a permutation required to put the exit direction first, are the ones from the natural structure of h-set.

From now on we will use these coordinates. Without any loss of generality we will assume that $M_i = M_{i,c} = \Pi_{i=1}^{L} \overline{B_{V_i}}(0,1) \times \overline{B_{W_u}}(0,1) \times \overline{B_{W_s}}(0,1)$.

We will prove the following statement, which implies the assertion of our theorem.

For any $\bar{y} \in B_{W_s}(0,1)$, $\bar{x} \in B_{W_u}(0,1)$, $\eta_i \in B_{V_i}(0,1)$, $i=1,\dots,L$, there exists q, such that

$$q \in M_1, \ \pi_{W_s}(q) = \bar{y}, \ \pi_{V_1}(q) = \eta_1, \tag{26}$$

$$f^{k_i}(q) \in M_{i+1}, \ \pi_{V_{i+1}} f^{k_i}(q) = \eta_{i+1} \quad i=1,\dots,L, \tag{27}$$

$$\pi_{W_u}(f^{k_{L+1}}(q)) = \bar{x}. \tag{28}$$

In the sequel we will denote f^{l_i} by f_i.

To obtain $q_1 \in M_1$ satisfying (26–28) it is enough to find a sequence $\{q_i\}_{i=1}^{L+1}$ satisfying the following conditions

$$
\begin{aligned}
y(q_1) - \bar{y} &= 0, && && (29)\\
z_i(q_i) - \eta_i &= 0, && i = 1, \dots, L && (30)\\
f_i(q_i) - q_{i+1} &= 0, && i = 1, \dots, L, && (31)\\
z_{L+1}(q_{L+1}) - \bar{x} &= 0. && && (32)
\end{aligned}
$$

which we will consider in the set

$$D = \Pi_{i=1}^{L+1} M_i.$$

Let us remind the reader that the supports of M_i and $\widetilde{M}_i$ coincide, but $M_i^{\pm}$ and $\widetilde{M}_i^{\pm}$ differ.

Observe that the number of equations in system (29–32) coincides with the number of variables in D. Indeed the equation count goes as follows:

- (29) gives w_s equations
- (30) consists of $n_1 + n_2 + \cdots + n_L = n$ equations
- (31) consists of $L \cdot (n + w_u + w_s)$ equations
- (32) gives w_u equations,

which gives $(L+1)(n + w_u + w_s)$ equations in the system, which coincides with the dimension of set D.

If $w_u = 0$, then $\bar{x} = 0$ and equation (32) is dropped from further considerations when defining maps F, H_t. Analogously, when $w_s = 0$ then $\bar{y} = 0$ and we drop equation (29).

Let us denote by F the map given by the left hand side of system (29–32). We have for $q = (q_1, \dots, q_{L+1}) \in D$

$$F(q) = \begin{pmatrix} y(q_1) - \bar{y} & \\ z_i(q_i) - \eta_i & i = 1, \dots, L \\ f_i(q_i) - q_{i+1} & i = 1, \dots, L, \\ z_{L+1}(q_{L+1}) - \bar{x} & \end{pmatrix} \tag{33}$$

We will prove that system (29–32) has a solution in D, by using the homotopy argument to show that the local Brouwer degree $\deg(F, \operatorname{int} D, 0)$ is nonzero.

Let h_i for $i = 1, 2, \dots, L$ be homotopies from the covering relations (24).

We imbed F into a one-parameter family of maps (a homotopy) H_t, as follows

$$H_t(q) = \begin{pmatrix} y(q_1) - (1-t)\bar{y} & \\ z_i(q_i) - (1-t)\eta_i & i = 1, \dots, L \\ h_{t,i}(q_i) - q_{i+1} & i = 1, \dots, L, \\ z_{L+1}(q_{L+1}) - (1-t)\bar{x} & \end{pmatrix} \tag{34}$$

It is easy to see that $H_0(q) = F(q)$.

We show that if $q \in \partial D$ then for all $t \in [0,1]$ $H_t(q) \neq 0$ holds. This will imply that $\deg(H_t, D, 0)$ is defined for all $t \in [0,1]$ and does not depend on t.

Let $q \in \partial D$. Then for some $i = 1, \dots, L+1$ $q_i \in \partial M_i$. We will use the following decomposition of ∂M_i for $i = 1, \dots, L$: $\partial M_i = M_i^+ \cup (\widetilde{M}_i^+ \cap M_i^-) \cup \widetilde{M}_i^-$, while for $i = L+1$, since $\widetilde{M}_{L+1}$ is not defined, $\partial M_i = M_i^+ \cup M_i^-$. It may happen that $M_{L+1}^- = \emptyset$.

- the case $q_i \in M_i^+$.

  If $i > 1$, then we consider $\widetilde{M}_{i-1} \overset{f_{i-1}}{\Longrightarrow} M_i$ and we see from (6) that $q_i \notin h_{t,i}(q_{i-1})(M_{i-1})$. Therefore in this case $h_{t,i}(q_{i-1}) - q_i \neq 0$.

  If $i = 1$, then $y(q_1) \neq (1-t)\bar{y}$, because in this case $y(q_1) \in \partial B_{w_s}(0,1)$, hence $\|y(q_1)\| = 1 > \|\bar{y}\|$.

- the case $i \leq L$ and $q_i \in \widetilde{M}_i^+ \cap M_i^-$. Then $q_i \in \partial B_{V_i}(0,1)$, hence $\|z_i(q_i)\| = 1 > \|\eta_i\|$.

- the case $i \leq L$ and $q_i \in \widetilde{M}_i^-$.

  From the exit condition (5) in covering relation $\widetilde{M}_i \overset{f_i}{\Longrightarrow} M_{i+1}$ it follows $h_{t,i}(q_i) \notin M_{i+1}$, therefore $h_{t,i}(q_i) - q_{i+1} \neq 0$.

- the case $i = L+1$ and $q_{L+1} \in M_{L+1}^-$.

  We have $\|z_{L+1}(q_{L+1})\| = 1 > \|\bar{x}\|$.

We have proved that $\deg(H_t, \operatorname{int} D, 0)$ is defined. By the homotopy invariance we have

$$\deg(F, \operatorname{int} D, 0) = \deg(H_1, \operatorname{int} D, 0). \tag{35}$$

In the sequel the points in M_i (and $\tilde{M}_i$) will be denoted by $(z_{i,1}, \dots, z_{i,L}, z_{i,L+1}, y_i)$, where $z_{i,k} \in V_k$ for $k = 1, \dots, L$, $z_{i,L+1} \in W_u$ and $y_i \in W_s$.

Observe that $H_1(q) = 0$ is the following system of *linear equations*

$$\begin{aligned} y_1 &= 0, \\ z_{1,1} &= 0, \\ (0, A_1(z_{1,2}, \dots, z_{1,L+1}), 0) - (z_{2,1}, z_{2,2}, \dots, z_{2,L+1}, y_2) &= 0, \\ z_{2,2} &= 0, \\ (0, 0, A_2(z_{2,3}, \dots, z_{2,L+1}), 0) - (z_{3,1}, z_{3,2}, \dots, z_{3,L+1}, y_3) &= 0, \\ &\dots \\ (0, \dots, A_L(z_{L,L+1}), 0) - (z_{L+1,1}, z_{L+1,2}, \dots, z_{L+1,L+1}, y_{L+1}) &= 0, \\ z_{L+1,L+1} &= 0, \end{aligned}$$

where A_i is a linear map which appears at the end of the homotopy h_i.

It is not hard to see that $q = 0$ is the only solution of this system. For the proof observe that $y_i = 0$ for $i = 1, 2, \dots$ because the first term in each equation involving A_i has zero on the last (y) coordinate. To prove that $z_{i,j} = 0$ for

$i, j = 1, \ldots, L, L+1$, we should start from the two bottom equations to infer that $z_{L+1,i} = 0$ for $i = 1, \ldots, L+1$, and since A_L is an isomorphism then also $z_{L,L+1} = 0$. Now we consider $z_{L,i}$ from the next two equations from the bottom and so on.

Therefore $\deg(H_1, \operatorname{int} D, 0) = \pm 1$.

This and (35) implies that

$$\deg(F, \operatorname{int} D, 0) = \pm 1 \tag{36}$$

hence there exists a solution of equation $F(q) = 0$ in D. This finishes the proof. ∎

5.3 Generalization

In the theorem below we allow chains of coverings relations combined with dropping some directions.

Theorem 9 *Assume that we have h-sets N_i and M_j (and $\tilde{M}_j$ when some exit dimensions have been dropped) and the following covering relations are satisfied*

$$\begin{array}{rl} N_{0,0} \overset{f_{0,0}}{\Longrightarrow} N_{0,1} \overset{f_{0,1}}{\Longrightarrow} & \cdots \overset{f_{0,i_0}}{\Longrightarrow} N_{0,i_0+1} = M_0, \\ \tilde{M}_0 = N_{1,0} \overset{f_{1,0}}{\Longrightarrow} N_{1,1} \overset{f_{1,1}}{\Longrightarrow} & \cdots \overset{f_{1,i_1}}{\Longrightarrow} N_{1,i_1+1} = M_1, \\ \tilde{M}_1 = N_{2,0} \overset{f_{2,0}}{\Longrightarrow} N_{2,1} \overset{f_{2,1}}{\Longrightarrow} & \cdots \overset{f_{2,i_1}}{\Longrightarrow} N_{2,i_2+1} = M_2, \\ & \cdots \\ \tilde{M}_L = N_{L,0} \overset{f_{L,0}}{\Longrightarrow} N_{L,1} \overset{f_{L,1}}{\Longrightarrow} & \cdots \overset{f_{L,i_1}}{\Longrightarrow} N_{L,i_L+1} = M_L. \end{array}$$

Then there exists $q_0, \ldots, q_l$, such that

$$\begin{array}{rcl} q_k & \in & N_{k,0}, \quad f_{k,j} \circ \cdots \circ f_{k,1} \circ f_{k,0}(q_k) \in N_{k,i_j+1}, \quad j = 0, \ldots, i_k, \quad k = 0, \ldots, L \\ q_{k+1} & = & f_{k,i_k} \circ \cdots \circ f_{k,1} \circ f_{k,0}(q_k), \quad k = 0, \ldots, L-1. \end{array}$$

Proof: Conceptually the same as the proof of Theorem 8. ∎

6 Diffusion in the linear model

We would like to prove now Conjecture 2 for f being a linear model. To formulate the precise assumptions about our linear model we need first to introduce some notations.

Let $z = (x_1, \ldots, x_n)$. Define for $i = 0, \ldots, n$, $z_i = (z_{i,p}, z_{i,\mathrm{inc}}, z_{i,\mathrm{out}}, z_{i,f})$ where

- $z_{i,p} = (x_1, \ldots, x_{i-1})$ are the past coordinates
- $z_{i,\mathrm{inc}} = x_i$ is the incoming coordinate

- $z_{i,\text{out}} = x_{i+1}$ is the outgoing coordinate
- $z_{i,f} = (x_{i+2}, \dots, x_n)$ are the future coordinates.

These are the local coordinates around each fixe point p_i.

We assume that we have a sequence of linear maps: f_i for $i = 0, \dots, n$ and affine maps $f_{i-1,i}$ for $i = 1, \dots, n$.

The maps f_i will correspond to the map close to the fixed point p_i and we will call them *local maps*. The maps $f_{i-1,i}$ will correspond to the maps that connect two consecutive fixed points, that we will call them *transition maps*.

We assume that the map f is equal to the linear map f_i around the fixed point p_i and defined as $f_{i-1,i}$ close to the heteroclinic connections.

6.1 Local maps

Let $f_i(z_i) = (f_{i,p}(z_i), f_{i,\text{inc}}(z_i), f_{i,\text{out}}(z_i), f_{f,i}(z_i))$ the decomposition of the map f_i in terms of the previous splitting of the coordinates z_i. We assume that:

$$
\begin{aligned}
f_{i,p}(z_i) &= A_{i,p} z_{i,p} \\
f_{i,\text{inc}}(z_i) &= \mu_i z_{i,\text{inc}} \\
f_{i,\text{out}}(z_i) &= \lambda_i z_{i,\text{out}} \\
f_{f,i}(z_i) &= A_{i,f} z_{i,f},
\end{aligned}
$$

where $A_{i,p}$ and $A_{i,f}$ are matrices that satisfy

$$
|A_{i,p} z_{i,p}| \leq \mu_{i,p} |z_{i,p}| \qquad |A_{i,f} z_{i,f}| \geq \lambda_{i,f} |z_{i,f}|,
$$

with $\mu_i \leq \mu_{i,p} < 1$ and $1 < \lambda_{i,f} \leq \lambda_i$. The last relations are not needed in our argument, but we include them to point out that the dominant directions are the ones defined by $z_{i,\text{inc}}$ and $z_{i,\text{out}}$. The norm that we are using here and for the rest of the proof is the maximum norm, $|.| = ||.||_\infty$.

For each $i = 0, \dots, n$, we want to define h-sets that will be centered in the following points $q_{i,\text{inc}}$ and $q_{i,\text{out}}$:

- $q_{i,\text{inc}} = (0, \sigma, 0, 0)$
- $q_{i,\text{out}} = (0, 0, \sigma, 0)$,

where $\sigma > 0$ is some small parameter that does not depend on the fixed point we are dealing with. Notice that $q_{i,\text{inc}}$ is located close to the fixed point p_i in the direction of the incoming heteroclinic, defined by the segment C_i. The point $q_{i,\text{out}}$ is also located close to the fixed point p_i but in the direction of the outgoing heteroclinic, defined by the segment C_{i+1}. Let $\delta > 0$ satisfying $\delta \leq \sigma$. Define the sets:

$$
N_i^{\text{inc}} = \{z_i \in \mathbb{R}^n, : \, |z_i - q_{i,\text{inc}}| \leq \delta\} \tag{37}
$$

$$
N_i^{\text{out}} = \{z_i \in \mathbb{R}^n, : \, |z_i - q_{i,\text{out}}| \leq \delta\}. \tag{38}
$$

Notice that they are boxes of size δ. We equip these sets with an h-set structure. We declare the directions $(z_{i,p}, z_{i,\mathrm{inc}})$ as entry directions and $(z_{i,\mathrm{out}}, z_{i,f})$ as exit directions in both cases.

It is time to relate the the h-sets (37) and (38) after some iterates of the map f_i.

Lemma 10 *There exists an integer k_i such that the following covering relation hold:*

$$N_i^{inc} \overset{f_i^{k_i}}{\Longrightarrow} N_i^{out},$$

for all $i = 0, \ldots, n-1$.

Proof: Since the map is linear we only have to prove that the entry (stable) components of N_i^{inc} are mapped inside N_i^{out} and that the exit (unstable) directions of N_i^{inc} cover the exit components of N_i^{out}. This is, the boundary of the exit directions of N_i^{inc} is mapped outside N_i^{out}.

Let us start with the past components. We have to show that $\left|f_{i,p}^{k_i}(z_i)\right| \leq \delta$ for $z_i \in N_i^{\mathrm{inc}}$. But

$$\left|f_{i,p}^{k_i}(z_i)\right| = \left|A_{i,p}^{k_i} z_{i,p}\right| \leq \mu_{i,p}^{k_i} \left|z_{i,p}\right| \leq \mu_{i,p}^{k_i}\delta,$$

and the requested inequality holds since $0 < \mu_{i,p} \leq 1$.

Consider the incoming component, $z_{i,\mathrm{inc}}$. We want k_i such that $f_{i,\mathrm{inc}}^{k_i}(z_i) \leq \delta$ for $z_i \in N_i^{\mathrm{inc}}$. But

$$\left|f_{i,\mathrm{inc}}^{k_i}(z_i)\right| = \mu_i^{k_i} \left|z_{i,\mathrm{inc}}\right| \leq \mu_i^{k_i} \left(\sigma + \delta\right).$$

If we take

$$k_i \geq \frac{\ln \frac{\sigma+\delta}{\delta}}{\ln \mu_i^{-1}},$$

we obtain the desired inequality.

Now we want to study the exit components. Take $z_i \in N_i^{\mathrm{inc}}$ such that its outgoing component $z_{i,\mathrm{out}}$ satisfies $\left|z_{i,\mathrm{out}}\right| = \delta$. We want to see that

$$\left|f_{i,\mathrm{out}}^{k_i}(z_i)\right| \geq \sigma + \delta.$$

Notice that we have:

$$\left|f_{i,\mathrm{out}}^{k_i}(z_i)\right| = \lambda_i^{k_i} \left|z_{i,\mathrm{out}}\right| = \lambda_i^{k_i}\delta.$$

If we take k_i such that

$$k_i \geq \frac{\ln \frac{\sigma+\delta}{\delta}}{\ln \lambda_i},$$

we obtain the desired inequality.

Finally, for the future components we proceed in the same way. Take $z_i \in N_i^{\text{inc}}$ such that its future component $z_{i,f}$ satisfies $|z_{i,f}| = \delta$. We want to see that

$$\left| f_{i,f}^{k_i}(z_i) \right| \geq \delta.$$

But

$$\left| f_{i,f}^{k_i}(z_i) \right| = \left| A_{i,f}^{k_i} z_{i,f} \right| \geq \lambda_{i,f}^{k_i} |z_{i,f}| = \lambda_{i,f}^{k_i} \delta,$$

and the requested inequality holds since $\lambda_{i,f} > 1$.

To finish the proof we take

$$k_i = \max \left\{ \frac{\ln \frac{\sigma+\delta}{\delta}}{\ln \mu_i^{-1}}, \frac{\ln \frac{\sigma+\delta}{\delta}}{\ln \lambda_i} \right\}.$$

∎

6.2 Dropping of one direction

Now we are going to equip N_i^{out} with another h-set structure, $\widetilde{N_i^{\text{out}}}$. We are going to put the outgoing coordinate $z_{i,\text{out}}$ in the set of entry directions. Notice that $\widetilde{N_i^{\text{out}}}$ is the same as N_i^{out} as sets. We are only changing the declaration of entry and exit coordinates, that is, the h-set structure.

Notice that it is precisely at this moment where we have lost the outgoing direction. This argument is equivalent to the one in Section 3.0.1 where we intersect some domain with a section of co-dimension one located in the desired outgoing direction. Notice that $\widetilde{N_i^{\text{out}}}$ have the same number of entry (and exit) components than N_{i+1}^{inc}.

6.3 Transition along the heteroclinic connection

We define the map close to the heteroclinic segment just as a translation, $f_{i,i+1}$. For points in $\widetilde{N_i^{\text{out}}}$, that is for points of the form $q_{i,\text{out}} + z_i$ with $|z_i| \leq \delta$ the map $f_{i,i+1}$ is defined as:

$$f_{i,i+1}(q_{i,\text{out}} + z_i) = q_{i+1,\text{inc}} + z_i.$$

Notice that, with the transition written in this way we do not have to perform a change of variables that would locate the fixed point p_{i+1} at the origin. The change is included in the transition.

Our goal is to prove that $\widetilde{N_i^{\text{out}}}$ covers N_{i+1}^{inc}. If we write the transition map in terms of z_i and z_{i+1} we have:

$$\begin{aligned} (z_{i,p}, z_{i,\text{inc}}) &= z_{i+1,p} \\ z_{i,\text{out}} &= z_{i+1,\text{inc}} \\ z_{i,f} &= (z_{i+1,\text{out}}, z_{i+1,f}). \end{aligned}$$

With this relation and the fact that we are using the maximum norm we can conclude that:

Lemma 11 *The following covering relation hold:*

$$\widetilde{N_i^{out}} \overset{f_{i,i+1}}{\Longrightarrow} N_{i+1}^{inc},$$

for all $i = 0, \ldots, n-1$.

6.4 The conclusion

By combining Lemmas 10,11 with Theorem 9 we obtain the following

Theorem 12 *Under the previous assumptions, for all* $\epsilon > 0$ *there exists a point* x_0 *and a sequence of integers* $0 = k_0 < k_1 < \cdots < k_n$ *such that:*

$$||f^{k_i}(x_0) - p_i|| < \epsilon \quad i = 0, \ldots, n.$$

7 Diffusion in a simplified Toy Model from [CKS+]

7.1 Our model

Our model is a simplification of the toy model system from [CKS+, GK]. It can be defined as the composition of the following systems parameterized by $j \in \mathbb{Z}$.

Our phase space is defined by a sequence of coordinate charts indexed by $j \in \mathbb{Z}$. Each of these maps is has at its center a fixed point $\mathbb{T}_j$ (which corresponds to the j-th torus in the toy model system). These coordinate charts are global, the dynamics is defined in each chart (this is again forced by the symplectic reduction in the toy model from [CKS+, GK]).

The j-th node chart uses the following coordinates

- $c_k \in \mathbb{C}$, $k \leq j-2$ or $k \geq j+2$
- $y_-, x_-, y_+, x_+ \in \mathbb{R}$.

7.1.1 Transition map between two adjacent nodes

Let

$$\omega = e^{i2\pi/3} = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right). \tag{39}$$

For $c \in \mathbb{C}$ we set

$$c = \omega c^- + \omega^2 c^+, \tag{40}$$

where $c^-, c^+ \in \mathbb{R}$.

The above decomposition means that we represent a complex number c in the basis $\{\omega, \omega^2\}$ over the field $\mathbb{R}$. For the future use we define two functions $g_1 : \mathbb{R}^2 \to \mathbb{C}$ and $g_2 : \mathbb{C} \to \mathbb{R}^2$ by

$$g_1(c^-, c^+) = \omega c^- + \omega^2 c^+, \qquad g_2(c) = g_1^{-1}(c). \tag{41}$$

The transition between the charts the j-th and the $(j+1)$-th acts as follows (the variables without tildes are the ones referring to the j-th node chart, while those with tildes denote the variables with respect to the $(j+1)$-th chart)

$$\begin{aligned}
\tilde{c}_{k\leq j-2} &= c_{k\leq j-2} \\
\tilde{c}_{j-1} &= g_1(x_-, y_-) \\
\tilde{x}_- &= y_+ \\
\tilde{y}_- &= x_+ \\
(\tilde{x}_+, \tilde{y}_+) &= g_2(c_{j+2}) \\
\tilde{c}_{k\geq j+3} &= c_{k\geq j+3}.
\end{aligned}$$

Observe that from the above we can also read what is the transition in the other direction.

7.1.2 Local evolution close to the fixed point

Let $\sigma > 0$ be fixed.

The evolution in the j-th node chart is given by the following ODE

$$\dot{y}_- = -y_- + O(x_-(y_+)^2), \tag{42}$$
$$\dot{x}_- = x_- + O(y_-(x_+)^2), \tag{43}$$
$$\dot{y}_+ = -y_+ + O(x_+(y_-)^2), \tag{44}$$
$$\dot{x}_+ = x_+ + O(y_+(x_-)^2), \tag{45}$$
$$\dot{c_k} = ic_k, \qquad k \leq j-2 \,\text{or}\, k \geq j+2, \tag{46}$$

where we assume that all $O()$ terms satisfy

$$|O(z)| \leq K|z|. \tag{47}$$

Let us stress each $O(\dots)$ function may depend on all variables.

7.2 Fixed points and heteroclinic connections

Observe that in our system we have fixed points $\mathbb{T}_j$ parameterized by $j \in \mathbb{Z}$ given in the j-th node centered coordinates by

$$c_{k\leq j-2} = 0, \quad c_{k\geq j+2} = 0, \quad x_- = y_- = x_+ = y_+ = 0. \tag{48}$$

For the fixed point $\mathbb{T}_j$ the directions $c_{k\leq j-2}, c_{k\geq j+2}$ are the center directions, x_-, x_+ are unstable directions and y_-, y_+ are stable directions.

Consecutive fixed points $\mathbb{T}_j$ and $\mathbb{T}_{j+1}$ are connected by a heteroclinic connection escaping the neighborhood of $\mathbb{T}_j$ along the solution $c_k(t) = 0$ for $k \leq j-2$ or $k \geq j+2$, $x_-(t) = y_-(t) = y_+(t) = 0$, $x_+(t) = \sigma e^t$ for $t \leq 0$ and continued later in the coordinates centered on $\mathbb{T}_{j+1}$ as $c_j(t) = 0$ for $k \leq j-1$ or $k \geq j+3$, $x_-(t) = x_+(t) = y_+(t) = 0$, $y_-(t) = \sigma e^t$ for $t \geq 0$.

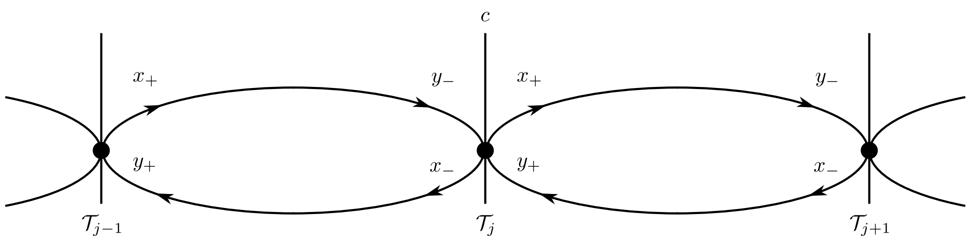


Figure 6: The heteroclinic connections in the toy model

7.3 Scattering estimates

We consider the system (42–45).

Let us set

$$\sigma' = 1.01\sigma. \tag{49}$$

We look for the solution such that

$$y_-(0) = \eta, \quad x_-(0) = e^{-2T}a_0, \quad y_+(0) = e^{-T}b_0, \quad x_+(0) = e^{-T}d_0 \tag{50}$$

where $\eta \in (0, \sigma')$ and $a_0 \in a$, $b_0 \in b$, $d_0 \in d$ and a, b, d are intervals such that

$$a, b, d \subset [-T^k, T^k], \quad k \geq 1. \tag{51}$$

We will try to use the iteration.

In the sequel we will use the following notation

$$\alpha = [-1, 1].$$

7.3.1 The 0-th order approximation

$$\begin{aligned} y_-(t) &= \eta e^{-t} \\ x_-(t) &= a_0 e^{-2T} e^{t} \\ y_+(t) &= b_0 e^{-T} e^{-t} \\ x_+(t) &= d_0 e^{-T} e^{t}. \end{aligned}$$

The estimates for the nonlinear terms (we skip the absolute value sign for a, b, d in the estimates for the nonlinear terms) are

$$K|x_-(t)(y_+(t))^2| \leq K(e^{-2T}ae^{t})(e^{-T}be^{-t})^2 \leq e^{-4T}e^{-t}\left(Kab^2\right) \leq e^{-3T}e^{-t}$$

for T large enough to satisfy

$$e^T > Kab^2. \tag{52}$$

$$\begin{aligned} K|y_-(t)(x_+(t))^2| &\leq K(\eta e^{-t})(e^{-T}de^{t})^2 \leq e^{-2T}e^{t}\left(K\sigma' d^2\right), \\ K|x_+(t)(y_-(t))^2| &\leq K\left(e^{-T}de^{t}\right)\left(\eta e^{-t}\right)^2 \leq e^{-T}e^{-t}\left(Kd\sigma'^2\right), \\ K|y_+(t)(x_-(t))^2| &\leq K\left(e^{-T}be^{-t}\right)\left(e^{-2T}ae^{t}\right)^2 = e^{-5T}e^{t}\left(Kba^2\right) \leq e^{-4T}e^{t} \end{aligned}$$

for T large enough to satisfy

$$e^T > Kba^2. \tag{53}$$

We obtain the following estimates in 1-st approximation

$$\begin{aligned}
y_-(t) &\in \eta e^{-t} + \alpha e^{-t} t e^{-3T} \\
x_-(t) &\in e^{-2T} a_0 e^t + \alpha t e^{-2T} e^t \left(K\sigma' d^2\right) \\
y_+(t) &\in e^{-T} b_0 e^{-t} + \alpha t e^{-T} e^{-t} \left(K d \sigma'^2\right) \\
x_+(t) &\in e^{-T} d_0 e^t + \alpha t e^{-4T} e^t.
\end{aligned}$$

7.3.2 Next iterate

$$\begin{aligned}
K|x_-(t)(y_+(t))^2| &\leq K(e^{-2T}e^t)(a + tK\sigma' d^2)(e^{-T}e^{-t})^2 \left(b + tKd\sigma'^2\right)^2 = \\
&\quad e^{-4T}e^{-t}\left(K(a + tK\sigma' d^2)\right)\left(b + tKd\sigma'^2\right)^2 \leq e^{-3T}e^{-t} \\
K|y_-(t)(x_+(t))^2| &\leq Ke^{-t}\left(\sigma' + te^{-3T}\right)(e^{-T}e^t)^2\left(d + te^{-3T}\right)^2 = \\
&\quad e^{-2T}e^t\left(K\left(\sigma' + te^{-3T}\right)\left(d + te^{-3T}\right)^2\right) \leq e^{-2T}e^t\left(2K\sigma'\left(d + \sigma'\right)^2\right) \\
K|x_+(t)(y_-(t))^2| &\leq K\left(e^{-T}e^t\right)\left(d + te^{-3T}\right)e^{-2t}\left(\sigma' + te^{-3T}\right)^2 = \\
&\quad e^{-t}e^{-T}\left(K\left(d + te^{-3T}\right)\left(\sigma' + te^{-3T}\right)^2\right) \leq e^{-t}e^{-T}\left(4K\sigma'^2(d + \sigma')\right) \\
K|y_+(t)(x_-(t))^2| &\leq K\left(e^{-T}e^{-t}\right)\left(b + tKd\sigma'^2\right)\left(e^{-2T}e^t\right)^2\left(a + tK\sigma'^2 d\right)^2 = \\
&\quad e^{-5T}e^t\left(K\left(b + tKd\sigma'^2\right)\left(a + tK\sigma'^2 d\right)^2\right) \leq e^{-4T}e^t
\end{aligned}$$

provided T is large enough for the following conditions to hold for $t \in [0, T]$

$$\begin{aligned}
e^T &> \left(K(a + tK\sigma' d^2)\right)\left(b + tKd\sigma'^2\right)^2, \\
te^{-3T} &< \sigma', \\
e^T &> \left(K\left(b + tKd\sigma'^2\right)\left(a + tK\sigma'^2 d\right)^2\right).
\end{aligned}$$

These bounds give us the following estimates in 2-nd approximation

$$\begin{aligned}
y_-(t) &\in \eta e^{-t} + \alpha e^{-t} t e^{-3T} \\
x_-(t) &\in e^{-2T} a_0 e^t + \alpha t e^{-2T} e^t \left(2K\sigma'\left(d + \sigma'\right)^2\right) \\
y_+(t) &\in e^{-T} b_0 e^{-t} + \alpha t e^{-T} e^{-t} \left(4K\sigma'^2(d + \sigma')\right) \\
x_+(t) &\in e^{-T} d_0 e^t + \alpha t e^{-4T} e^t.
\end{aligned}$$

Observe that we have obtained the same formula for $y_-(t)$ and $x_+(t)$ in the second approximation as in the first approximation.

7.3.3 Next iterate

Therefore the bounds for terms $y_-(x_+)^2$ and $x_+(y_-)^2$ will be the same, so we just compute bounds for $x_-(y_+)^2$ and $y_+(x_-)^2$.

We have

$$\begin{aligned} K|x_-(t)(y_+(t))^2| &\leq K(e^{-2T}e^t)(a+t2K\sigma'\,(d+\sigma')^2)(e^{-T}e^{-t})^2\left(b+t4K\sigma'^2(d+\sigma')\right)^2 = \\ & e^{-4T}e^{-t}\left(K(a+t2K\sigma'\,(d+\sigma')^2)\left(b+t4K\sigma'^2(d+\sigma')\right)^2\right) \leq e^{-3T}e^{-t} \\ K|y_+(t)(x_-(t))^2| &\leq K\left(e^{-T}e^{-t}\right)\left(b+t4K\sigma'^2(d+\sigma')\right)\left(e^{-2T}e^t\right)^2(a+t2K\sigma'\,(d+\sigma')^2)^2 = \\ & e^{-5T}e^t\left(K\left(b+t4K\sigma'^2(d+\sigma')\right)(a+t2K\sigma'\,(d+\sigma')^2)^2\right) \leq e^{-4T}e^t \end{aligned}$$

provided T is large enough for the following conditions to hold for $t \in [0,T]$

$$\begin{aligned} e^T &\geq (K(a+t2K\sigma'\,(d+\sigma')^2)\left(b+t4K\sigma'^2(d+\sigma')\right)^2 \\ e^T &\geq K\left(b+t4K\sigma'^2(d+\sigma')\right)(a+t2K\sigma'\,(d+\sigma')^2)^2 \end{aligned}$$

Therefore we have established the following theorem

Theorem 13 *Consider (42–46) satisfying (47) with initial conditions (50) satisfying (51).*

Assume that $T \geq T_0 > 1$ is large enough, so that the following inequalities are satisfied

$$\begin{aligned} e^T &\geq (K(T^k+T2K\sigma'\left(T^k+\sigma'\right)^2)\left(T^k+T4K\sigma'^2(T^k+\sigma')\right)^2 && (54) \\ e^T &\geq K\left(T^k+T4K\sigma'^2(T^k+\sigma')\right)(T^k+T2K\sigma'\left(T^k+\sigma'\right)^2)^2 && (55) \\ Te^{-3T} &< \sigma'. && (56) \end{aligned}$$

Then for $t \in [0,T]$

$$\begin{aligned} y_-(t) &\in \eta e^{-t} + \alpha e^{-t}te^{-3T} \\ x_-(t) &\in e^{-2T}a_0e^t + \alpha te^{-2T}e^t\left(2K\sigma'\,(|d|+\sigma')^2\right) \\ y_+(t) &\in e^{-T}b_0e^{-t} + \alpha te^{-T}e^{-t}\left(4K\sigma'^2(|d|+\sigma')\right) \\ x_+(t) &\in e^{-T}d_0e^t + \alpha te^{-4T}e^t. \end{aligned}$$

In the variables c_p and c_f the scattering by $\mathbb{T}_j$ acts as the rotation map. Hence in the context of Theorem 13 the following formulas hold

$$c_{\leq j-2}(T) = c_{\leq j-2}(0)e^{iT}, \quad c_{\geq j+2}(T) = c_{\geq j+2}(0)e^{iT}. \qquad (57)$$

7.4 Construction of the covering relations

The goal of this section is to construct a sequence of coverings for our model using the estimates obtained in Theorem 13.

We will have two types of h-sets in the j-th chart around $\mathbb{T}_j$, N^j_{in} and N^j_{out}, such that the following covering relations are satisfied

$$N^j_{in} \overset{\varphi_T}{\Longrightarrow} N^j_{out}, \tag{58}$$

$$R_{<x_+,y_+>} N^j_{out} = \widetilde{N}^j_{out} \overset{Id}{\Longrightarrow} N^{j+1}_{in}. \tag{59}$$

In relation (58) the map is the shift along the trajectory by the time T. The map in relation (59) is formally an identity map, but since N^j_{out} and N^{j+1}_{in} are defined in terms of different coordinate charts its representation will no longer be the identity.

In the derivation we will use

- γ - will be used for the sizes in the entry directions
- r - will be used for the sizes in the exit directions

Observe that when we are dropping some directions the sizes in these directions become very close to zero (to set them to zero will not change anything, but it will require slight changes in Theorem 9).

By c_p (past modes) we will denote the collection $\{c_k\}_{k\leq j-2}$ and by c_f (future modes) we will denote the collection $\{c_k\}_{k\geq j+2}$. On c_p and c_f we use the sup norm, i.e. $\|c_p\| = \sup_{k\leq j-2}|c_k|$.

The structure of h-sets N^j_{in} and N^j_{out} is defined as follows (we are using the j-th chart):

- the entry variables: c_p, x_-, y_-
- the exit variables: x_+, y_+, c_f
- parameters of N^j_{in}:

  For the entry directions:

$$\begin{aligned} |c_p| &\leq \gamma^j_{in}(c_p)e^{-T}, \\ y_- &\in \sigma + \alpha\gamma^j_{in}(y_-)e^{-T}, \qquad \gamma^j_{in}(y_-)\approx 0, \\ |x_-| &\leq \gamma^j_{in}(x_-)e^{-2T}, \qquad \gamma^j_{in}(x_-)\approx 0. \end{aligned}$$

  For the exit directions:

$$\begin{aligned} |x_+| &\leq r^j_{in}(x_+)e^{-T}, \quad r^j_{in}(x_+) = 2.1\sigma, \\ |y_+| &\leq r^j_{in}(y_+)e^{-T}, \\ |c_f| &\leq r^j_{in}(c_f)e^{-T}. \end{aligned}$$

- parameters of N^j_{out}:

  For the entry directions:

$$\begin{aligned} |y_-| &\leq \gamma^j_{out}(y_-)e^{-T}, \\ |x_-| &\leq \gamma^j_{out}(x_-)e^{-T}, \\ |c_p| &\leq \gamma^j_{out}(c_p)e^{-T}. \end{aligned}$$

For the exit directions:

$$\begin{aligned} x_+ &\in \sigma + \alpha r^j_{out}(x_+)e^{-T}, \quad r^j_{out}(x_+) \approx 0, \\ |y_+| &\leq r^j_{out}(y_+)e^{-2T}, \quad r^j_{out}(y_+) \approx 0 \\ |c_f| &\leq r^j_{out}(c_f)e^{-T}. \end{aligned}$$

7.4.1 Covering $N^j_{in} \overset{\varphi_T}{\Longrightarrow} N^j_{out}$

We use Theorem 13 for the shift along the trajectory by time T with

$$\begin{aligned} \eta &= \sigma + \alpha\gamma^j_{in}(y_-)e^{-T}, \quad a = \alpha\gamma^j_{in}(x_-), \\ b &= \alpha r^i_{in}(y_+), \quad d = \alpha r^i_{in}(x_+) = 2.1\sigma. \end{aligned} \tag{60}$$

In order to make Theorem 13 applicable we require that

$$\gamma^j_{in}(y_-)e^{-T} < \sigma' - \sigma = 0.01\sigma. \tag{61}$$

The conditions for $N^j_{in} \overset{\varphi_T}{\Longrightarrow} N^j_{out}$ are

- entry conditions:
  - for c_p variables:
  $$\gamma^j_{in}(c_p) < \gamma^j_{out}(c_p). \tag{62}$$
  - for x_- from Theorem 13 and (60) we have the following condition
  $$\gamma^j_{in}(x_-)e^{-T} + Te^{-T}(2K\sigma'(3.1\sigma')^2) < \gamma^j_{out}(x_-)e^{-T}$$
  therefore it is enough to have
  $$T(2.1K\sigma'(3.1\sigma')^2) \leq \gamma^j_{out}(x_-), \tag{63}$$
  if $\gamma^j_{in}(x_-) \approx 0$, which will turn out to be compatible with other conditions. In fact we had replaced 2 by 2.1 and σ by σ' (σ appears in d) in order to make an explicit margin for $\gamma^j_{in}(x_-)$ given by
  $$\gamma^j_{in}(x_-) < T(0.1K\sigma'(3.1\sigma')^2). \tag{64}$$
  - for y_- it is enough to have
  $$\sigma' e^{-T} + Te^{-4T} < \gamma^j_{out}(y_-)e^{-T}.$$
  We see that from (56) it is enough to have the following
  $$2\sigma' \leq \gamma^j_{out}(y_-). \tag{65}$$
- exit conditions:

– x_+

$$r^j_{in}(x_+) - Te^{-3T} > \sigma + r^j_{out}(x_+)e^{-T},$$

which in view of assumption (56),(49) and (60) is satisfied , if

$$r^j_{out}(x_+)e^{-T} < 0.09\sigma. \quad (66)$$

Obviously (66) is perfectly compatible with $r^j_{out}(x_+) \approx 0$, which is to be expected as this is the direction which will be dropped in the next covering relation.

– y_+, from Theorem 13 and (60) it follows that the following estimate is enough

$$r^j_{in}(y_+)e^{-2T} - Te^{-2T}\left(4K\sigma'^2(r^j_{in}(x_+) + \sigma')\right) > r^j_{out}(y_+)e^{-2T}. \quad (67)$$

Observe that this is the direction which is dropped, hence any $r^j_{out}(y_+) > 0$, $r^j_{out}(y_+) \approx 0$ is good for our construction. Therefore we can demand (we entered the known value of $r^j_{in}(x_+)$)

$$r^j_{in}(y_+) \geq T\left(4.1K\sigma'^2(3.1\sigma')\right), \quad (68)$$

which leaves the margin for $r^j_{out}(y_+)$ given by

$$r^j_{out}(y_+) < T\left(0.1K\sigma'^2(3.1\sigma')\right). \quad (69)$$

– c_f

$$r^j_{in}(c_f) > r^j_{out}(c_f). \quad (70)$$

7.4.2 Covering relation (59)

Let $L \geq 1$ be the Lipschitz constant which holds for both functions g_1 and g_2, where on $\mathbb{R}^2$ the max-norm is used, while in $\mathbb{C}$ we use the euclidian norm.

The conditions are as follows.

In the entry directions

$$\begin{aligned} L(\gamma^j_{out}(y_-) + \gamma^j_{out}(x_-)) &\leq \gamma^{i+1}_{in}(c_p), & (71)\\ \gamma^j_{out}(c_p) &< \gamma^{j+1}_{in}(c_p), & (72)\\ r^j_{out}(x_+) &< \gamma^{j+1}_{in}(y_-), & (73)\\ r^j_{out}(y_+) &< \gamma^{j+1}_{in}(x_-). & (74)\end{aligned}$$

In the exit directions

$$\begin{aligned} r^j_{out}(c_f) &\geq Lr^{j+1}_{in}(x_+), & (75)\\ r^j_{out}(c_f) &\geq Lr^{j+1}_{in}(y_+), & (76)\\ r^j_{out}(c_f) &> r^{j+1}_{in}(c_f). & (77)\end{aligned}$$

7.4.3 Solving the inequalities for coverings

We have to find the following set of parameters $\gamma^j_{in,out}(c_p, x_-, y_-)$ and $r^j_{in,out}(x_+, y_+, c_f)$, such that the equations derived above are satisfied.

We split these parameters in two groups: the ones related to dropped directions $\gamma^j_{in}(y_-)$, $\gamma^j_{in}(x_-)$, $r^j_{out}(x_+)$, $r^j(y_+)$ and the remaining ones.

In the first group we effectively should obtain

$$\gamma^j_{in}(y_-) = 0, \quad \gamma^j_{in}(x_-) = 0, \quad r^j_{out}(x_+) = 0, \quad r^j(y_+) = 0.$$

The conditions involving these parameters are (64), (66), (69), (73) and (74). It is clear that these can be easily satisfied with all these parameters being very close to zero.

Now we deal with the other directions. We already have set the value for $r^j_{in}(x_+)$ and also set the following following parameters (compare (63),(65), (68))

$$\begin{aligned} r^j_{in}(x_+) &= 2.1\sigma, && (78)\\ \gamma^j_{out}(x_-) &= TQ_1, \quad Q_1 = (2.1K\sigma'(3.1\sigma')^2), && (79)\\ \gamma^j_{out}(y_-) &= 2\sigma, && (80)\\ r^j_{in}(y_+) &= TQ_2, \quad Q_2 = \left(4.1K\sigma'^2(3.1\sigma')\right), && (81) \end{aligned}$$

The remaining inequalities involve only the sizes for variables c_p and c_f. These are as follows:

- for the entry directions (see (62), (71), (72))

$$\begin{aligned} \gamma^j_{in}(c_p) &< \gamma^j_{out}(c_p) \\ L(2\sigma' + TQ_1) &\leq \gamma^{j+1}_{in}(c_p), \\ \gamma^i_{out}(c_p) &< \gamma^{j+1}_{in}(c_p), \end{aligned}$$

- for the exit directions (see (70), (75), (76) and (77)

$$\begin{aligned} r^j_{in}(c_f) &> r^j_{out}(c_f), \\ r^j_{out}(c_f) &\geq 2.1L\sigma, \\ r^j_{out}(c_f) &\geq TLQ_2, \\ r^j_{out}(c_f) &> r^{j+1}_{in}(c_f). \end{aligned}$$

It is clear that there exist $\gamma^j_{in}(c_p)$, $\gamma^j_{out}(c_p)$ satisfying

$$Q_3T < \gamma^j_{in}(c_p) < \gamma^j_{out}(c_p) < \gamma^{j+1}_{in}(c_p) < 2Q_3T, \qquad (82)$$

where $Q_3 = L\left(\frac{2\sigma'}{T} + Q_1\right)$. With such sequence we have solved the inequalities for the entry directions.

In the exit direction the situation is similar. We just take any sequence satisfying

$$\max(TLQ_2, 2.1L\sigma) \le r^j_{out}(c_f) < r^j_{in}(c_f) < r^j_{out}(c_f) < 2\max(TLQ_2, 2.1L\sigma). \tag{83}$$

In (82) and (83) we introduced also an upper bound which is $O(T)$, so now all sizes are $O(T)$ times a suitable weight function (e^{-T} or e^{-2T}). Observe that this bound allows us to use Theorem 13 with $k = 2$, for $T \ge \max(2LQ_2, 2Q_3, Q_2, Q_1, 1)$

7.5 The conclusion

From the chain of coverings constructed above we obtain

Theorem 14 *For the system discussed in this section, for all $\epsilon > 0$ there exists a point x_0 close to $\mathbb{T}_0$ whose trajectory is ϵ close to the infinite chain of heteroclinic connections $\mathbb{T}_0 \to \mathbb{T}_1 \to \ldots$*

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