Covering relations, cone conditions and the stable manifold theorem

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Abstract

We show how to effectively link covering relations with cone conditions. We give a new, 'geometric', proof of the stable manifold theorem for hyperbolic fixed point of a map.

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1 Introduction

The topological notion of the covering relation (Easton's 'correctly aligned windows' [E1, E2] without any differentiability assumptions) originating from the theory of the Conley index, see [GiZ] and the references given there, has been successfully applied to establish the existence of symbolic dynamics for such systems as: the Henon map[Ga1, ZN, GaZ2, CGB], the Chua circuit[Ga2], the Lorenz equations[GaZ1], the Rössler equations[ZN], the Henon-Heiles hamiltonian[AZ], PR3BP [A, WZP] or the Michelson system [W1, W2]. In all the examples listed above we are talking about the computer assisted proofs. There exist also some nontrivial applications of covering relations, not related to any computer assisted proofs, like the stability of Sharkovski order and estimates for the topological entropy for multidimensional perturbations of one-dimensional maps [MZ, ZS], the delay differential equations with small delays [WoZ] or to the Arnold diffusion [GiL, GiR].

However the topological tools are inadequate to handle the questions of local uniqueness of some periodic orbits, the sensitive dependence on initial conditions or the hyperbolicity.

The goal of the present paper to address the following question: how to effectively link the covering relations with the cone conditions of some kind, so that the covering relations will be used to prove that some dynamically interesting objects exist and the cone conditions will be used establish their properties implied by the hyperbolicity. The standard way to establish the hyperbolic behavior is usually through the cone fields in the tangent space which

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are mapped into itself by the tangent map and/or its inverse, see [T] and the references given there. In this paper we introduce a different approach, which is based on the two point Lapunov function for a map f, by which we understand the function of two variables $L(z_1, z_2)$, satisfying locally the following condition $L(f(z_1), f(z_2)) > L(z_1, z_2)$ for $z_1 \neq z_2$. For uniformly hyperbolic systems the proposed approach appears to be equivalent to the standard one, but our method does not require the uniform hyperbolicity. The proposed approach has been already applied to the study (a computer assisted proof) of the cocoon bifurcation in the Michelson system in [KWZ].

The main application of the proposed approach discussed in this paper is a new 'geometric' proof of the existence of the (un)stable manifold for a hyperbolic fixed point for a map or an ODE. The stable manifold theorem goes back to Poincaré, Hadamard and Perron, see [Ha] and the references given there, but there still appear new proofs in the literature, the recent ones are [C, HL, McS]. The interesting feature of our approach is that the whole proof in made in the phase space, is local and gives explicit bounds on the size of the (un)stable manifolds. A proof, similar in sprit, but not in the realization, has been proposed by Hartman in [Ha, Exercises 5.3 and 5.4].

Our geometric approach to the proof of the stable manifold theorem should be contrasted with the standard approach see [Ha, Hal, I70, I80, Ro, C, HL], where the problem of the existence of stable manifold is rephrased as a question of the existence of fixed point in a suitable Banach space of graphs of functions or sequences. Moreover, our approach does not require that the fixed point is hyperbolic, the essential assumption is the existence of the two-point Lapunov function. In Section 6 we analyze a non-hyperbolic example of this type.

The results about the (un)stable manifolds for hyperbolic fixed points stated and proved in this paper are weaker than those obtained using the Perron-Irwin method [I70, I80, C] as we did not get the smoothness of the invariant manifolds, in our proof we obtain only that they are Lipschitz manifolds for C^1 maps and analytic for the analytic maps. Also contrary to the results from [I70, I80, C], which are valid in Banach space, we restrict ourselves to the finite dimensional case, but it clear that our proof can be easily adapted to compact maps on the Banach space. Both, our proof and others proofs, consist of two parts, the first one is about the existence of graph of the function, which is contained in the (un)stable set of the fixed point and in the second part, it is shown that in fact this graph contains the whole local (un)stable set. It turns out, that when using our approach the conditions required for both parts of the proof are different one from another, while in the standard approach the conditions necessary for the first part might be so strong that they imply the cone conditions necessary for the second one. It turns out that this results with weaker conditions, when using our method. This is illustrated by examples in Section 7. This is especially important when the explicit estimates for the invariant manifold are required, as it is for example in the context of computer assisted proofs.

Let us finish this introduction with the short description of the paper by listing the content of its sections. Section 2 contains the definitions and theorems about the covering relations. Section 3 is about the cone conditions expressed in terms of two-point Lapunov functions and its interactions with the notion of covering relations. Section 4 contains main theorems about (un)stable sets for chains of covering relations satisfying the cone conditions. In Section 5 we apply the tools developed in previous sections to prove the (un)stable manifold theorem for hyperbolic fixed points of maps. In Section 6 we discuss a planar example with an non-hyperbolic fixed points and its (un)stable manifold. Section 7 contains some quantitative comparisons of the range of the existence of stable manifold for a map using our approach and the standard approach. In Section 8 we prove a theorem about the continuous and Lipschitz dependence of stable manifold of hyperbolic fixed point with respect to the parameters. Section 9 contains a proof that for an analytic map the (un)stable manifold of hyperbolic fixed point is analytic and depends analytically on parameters. In Section 10 we prove the stable manifold theorem for hyperbolic fixed point for ODEs. In Section 11 we prove that for a linear map our cone conditions expressed in terms of two-point Lapunov function are equivalent to the hyperbolicity.

1.1 Notation

By $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ we denote the set of natural, integer, rational, real and complex numbers, respectively. $\mathbb{Z}_-$ and $\mathbb{Z}_+$ are negative and positive integers, respectively. By S^1 we will denote a unit circle on the complex plane.

For $\mathbb{R}^n$ we will denote the norm of x by $\|x\|$ and when in some context the formula for the norm is not specified, then it means that any norm can be used. Let $x_0 \in \mathbb{R}^s$, then $B_s(x_0, r) = \{z \in \mathbb{R}^s \mid \|x_0 - z\| < r\}$ and $B_s = B_s(0, 1)$.

For $z \in \mathbb{R}^u \times \mathbb{R}^s$ we will call usually the first coordinate, x, and the second one y. Hence $z = (x, y)$, where $x \in \mathbb{R}^u$ and $y \in \mathbb{R}^s$. We will use the projection maps $\pi_x(z) = x(z) = x$ and $\pi_y(z) = y(z) = y$.

Let $z \in \mathbb{R}^n$ and $U \subset \mathbb{R}^n$ be a compact set and $f : U \to \mathbb{R}^n$ be continuous map, such that $z \notin f(\partial U)$. Then the local Brouwer degree [S] of f on U at z is defined and will be denoted by $\deg(f, U, z)$.

Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. By $\mathrm{Sp}(A)$ we denote the spectrum of A, which is the set of $\lambda \in \mathbb{C}$, such that there exists $x \in \mathbb{C}^n \setminus \{0\}$, such that $Ax = \lambda x$.

2 Covering relations, horizontal and vertical disks

The goal of this section is to recall from [GiZ, WZ] the notions of h-sets, covering relations, horizontal and vertical disks.

Definition 1 *[GiZ, Definition 1] An h-set, N, is a quadruple $(|N|, u(N), s(N), c_N)$ such that*

- $|N|$ *is a compact subset of* $\mathbb{R}^n$
- $u(N), s(N) \in \{0, 1, 2, \dots\}$ *are such that* $u(N) + s(N) = n$

- $c_N : \mathbb{R}^n \to \mathbb{R}^n = \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)}$ *is a homeomorphism such that*

$$c_N(|N|) = \overline{B_{u(N)}} \times \overline{B_{s(N)}}.$$

We set

$$\begin{aligned} \dim(N) &:= n, \\ N_c &:= \overline{B_{u(N)}} \times \overline{B_{s(N)}}, \\ N_c^- &:= \partial B_{u(N)} \times \overline{B_{s(N)}}, \\ N_c^+ &:= \overline{B_{u(N)}} \times \partial B_{s(N)}, \\ N^- &:= c_N^{-1}(N_c^-), \quad N^+ = c_N^{-1}(N_c^+). \end{aligned}$$

Hence an h-set, N, is a product of two closed balls in some coordinate system. The numbers $u(N)$ and $s(N)$ are called the nominally unstable and nominally stable dimensions, respectively. The subscript c refers to the new coordinates given by homeomorphism c_N. Observe that if $u(N) = 0$, then $N^- = \emptyset$ and if $s(N) = 0$, then $N^+ = \emptyset$. In the sequel to make notation less cumbersome we will often drop the bars in the symbol $|N|$ and we will use N to denote both the h-sets and its support.

Definition 2 *[GiZ, Definition 3] Let N be a h-set. We define a h-set N^T as follows*

- $|N^T| = |N|$
- $u(N^T) = s(N)$, $s(N^T) = u(N)$
- *We define a homeomorphism* $c_{N^T} : \mathbb{R}^n \to \mathbb{R}^n = \mathbb{R}^{u(N^T)} \times \mathbb{R}^{s(N^T)}$, *by*

$$c_{N^T}(x) = j(c_N(x)),$$

where $j : \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)} \to \mathbb{R}^{s(N)} \times \mathbb{R}^{u(N)}$ *is given by* $j(p, q) = (q, p)$.

∎

Observe that $N^{T,+} = N^-$ and $N^{T,-} = N^+$. This operation is useful in the context of inverse maps.

Definition 3 *[W2, Definition 2.2] Assume that N, M are h-sets, such that $u(N) = u(M) = u$ and let $f : N \to \mathbb{R}^{\dim(M)}$ be continuous. Let $f_c = c_M \circ f \circ c_N^{-1} : N_c \to \mathbb{R}^u \times \mathbb{R}^{s(M)}$.*

Let w be a nonzero integer. We say that

$$N \overset{f,w}{\Longrightarrow} M$$

(N f-covers M with degree w) iff the following conditions are satisfied

1. *there exists a continuous homotopy $h : [0,1] \times N_c \to \mathbb{R}^u \times \mathbb{R}^s$, such that the following conditions hold true*

$$\begin{aligned} h_0 &= f_c, & (1) \\ h([0,1], N_c^-) \cap M_c &= \emptyset, & (2) \\ h([0,1], N_c) \cap M_c^+ &= \emptyset. & (3) \end{aligned}$$

2. *If $u > 0$, then there exists a map $A : \mathbb{R}^u \to \mathbb{R}^u$, such that*

$$\begin{aligned} h_1(p,q) &= (A(p), 0), \text{ for } p \in \overline{B_u}(0,1) \text{ and } q \in \overline{B_s}(0,1), & (4) \\ A(\partial B_u(0,1)) &\subset \mathbb{R}^u \setminus \overline{B_u}(0,1). & (5) \end{aligned}$$

Moreover, we require that

$$\deg(A, \overline{B_u}(0,1), 0) = w, \tag{6}$$

Observe that in the above definition $s(N)$ and $s(M)$ can be different, this is the only difference compared to [GiZ, Definition 6].

Remark 1 *Observe, that since for any norm in $\mathbb{R}^n$ the closed unit ball is homeomorphic to $[-1,1]^n$, therefore for h-sets and covering relations we will use different norms in different contexts.*

Remark 2 *If the map A in condition 2 of Def. 3 is a linear map, then condition (5) implies, that*

$$\deg(A, \overline{B_u}(0,1), 0) = \pm 1.$$

Hence condition (6) is in this situation automatically fulfilled with $w = \pm 1$.

In fact, this is the most common situation in the applications of covering relations.

Most of the time we will not interested in the value of w in the symbol $N \overset{f,w}{\Longrightarrow} M$ and we will often drop it and write $N \overset{f}{\Longrightarrow} M$, instead. Sometimes we may even drop the symbol f and write $N \Longrightarrow M$.

Definition 4 *[GiZ, Definition 7] Assume N, M are h-sets, such that $u(N) = u(M) = u$ and $s(N) = s(M) = s$. Let $g : \mathbb{R}^n \supset \Omega \to \mathbb{R}^n$. Assume that $g^{-1} : |M| \to \mathbb{R}^n$ is well defined and continuous. We say that $N \overset{g}{\Longleftarrow} M$ (N g-backcovers M) iff $M^T \overset{g^{-1}}{\Longrightarrow} N^T$.*

Definition 5 *[WZ, Definition 10] Let N be an h-set. Let $b : \overline{B_{u(N)}} \to |N|$ be continuous and let $b_c = c_N \circ b$. We say that b is* a horizontal disk in N *if there exists a homotopy $h : [0,1] \times \overline{B_{u(N)}} \to N_c$, such that*

$$\begin{aligned} h_0 &= b_c & (7) \\ h_1(x) &= (x, 0), \qquad \text{for all } x \in \overline{B_{u(N)}} & (8) \\ h(t,x) &\in N_c^-, \qquad \text{for all } t \in [0,1] \text{ and } x \in \partial B_{u(N)} & (9) \end{aligned}$$

Definition 6 *[WZ, Definition 11] Let N be an h-set. Let $b : \overline{B_{s(N)}} \to |N|$ be continuous and let $b_c = c_N \circ b$. We say that b is* a vertical disk in N *if there exists a homotopy $h : [0,1] \times \overline{B_{s(N)}} \to N_c$, such that*

$$\begin{aligned} h_0 &= b_c \\ h_1(x) &= (0,x), \qquad \textit{for all } x \in \overline{B_{s(N)}} \\ h(t,x) &\in N_c^+, \qquad \textit{for all } t \in [0,1] \textit{ and } x \in \partial B_{s(N)}. \end{aligned} \tag{10}$$

Definition 7 *Let N be an h-set in $\mathbb{R}^n$ and b be a horizontal (vertical) disk in N.*
We will say that $x \in \mathbb{R}^n$ belongs to b, *when $b(z) = x$ for some $z \in \mathrm{dom}(b)$.*
By $|b|$ we will denote the image of b. Hence $z \in |b|$ iff z belongs to b.

The theorem below contains a slight generalization of Theorem 9 in [GiZ]

Theorem 3 *Assume N_i, $i = 0, \dots, k$, $N_k = N_0$ are h-sets and for each $i = 1, \dots, k$ we have either*

$$N_{i-1} \overset{f_i,w_i}{\Longrightarrow} N_i \tag{11}$$

or

$$N_i \subset \mathrm{dom}(f_i^{-1}) \quad \textit{and} \quad N_{i-1} \overset{f_i,w_i}{\Longleftarrow} N_i. \tag{12}$$

Then there exists a point $x \in \mathrm{int} N_0$, such that

$$f_i \circ f_{i-1} \circ \cdots \circ f_1(x) \in \mathrm{int} N_i, \quad i = 1, \dots, k \tag{13}$$

$$f_k \circ f_{k-1} \circ \cdots \circ f_1(x) = x \tag{14}$$

Proof: Under additional assumption that $s(N_i) = s$ for $i = 1, \dots, k$ this theorem was proved in [GiZ].

The situation of different $s(N_i)$ can be reduced to the previous one as follows. Let $s = \max_{i=1,\dots,k-1} s_i$.

Let us fix the norm $\|x\| = \max_i |x_i|$.

We define new h-sets $\tilde{N}_i$ and maps $\tilde{f}_i$ as follows

$$|\tilde{N}_i| = |N_i| \times [-1,1]^{s-s_i}, \quad u(\tilde{N}_i) = u(N_i), \quad s(\tilde{N}_i) = s \tag{15}$$

$$c_{\tilde{N}_i}(x,y,\tilde{y}) = (c_{N_i}(x,y),\tilde{y}), \qquad \text{where } (x,y) \in \mathbb{R}^{\dim(N_i)},\ \tilde{y} \in \mathbb{R}^{s-s_i} \tag{16}$$

For direct covering proceed as follows. Let h_i be the homotopy from the covering relation $N_{i-1} \overset{f_i}{\Longrightarrow} N_i$. We define a new homotopy $\tilde{h}_i$ and $\tilde{f}_i$ by

$$\begin{aligned} \tilde{h}_i(t,(x,y,\tilde{y}_{i-1})) &= h_i(t,(x,y)) \times \{0\}^{s-s(N_i)} \\ \tilde{f}_i(x,y,\tilde{y}_{i-1}) &= f_i(x,y) \times \{0\}^{s-s(N_i)} \end{aligned}$$

Observe that for $i = 1, \dots, k$ such that we have direct covering (11) we have

$$\tilde{N}_{i-1} \overset{\tilde{f}_i,w_i}{\Longrightarrow} \tilde{N}_i \tag{17}$$

For backcovering (i.e. (12)) by definition we know that $s(N_{i-1}) = s(N_i)$, therefore we just add $s - s(N_{i-1})$ contracting directions (which will be expanding for inverse map). We define

$$\begin{aligned}
\tilde{h}_i(t,(x,y,\tilde{y}_i)) &= (h_i(t,(x,y)), 2\tilde{y}_i) \\
\tilde{f}_i^{-1}(x,y,\tilde{y}_i) &= (f_i^{-1}(x,y), 2\tilde{y}_i).
\end{aligned}$$

The assertion now follows from Theorem 9 in [GiZ].

∎

Theorem 4 *Let $k \geq 1$. Assume N_i, $i = 0,\dots,k$, are h-sets and for each $i = 1,\dots,k$ we have either*

$$N_{i-1} \overset{f_i,w_i}{\Longrightarrow} N_i \tag{18}$$

or

$$N_i \subset \mathrm{dom}(f_i^{-1}) \quad \textit{and} \quad N_{i-1} \overset{f_i,w_i}{\Longleftarrow} N_i. \tag{19}$$

Assume that b_0 is a horizontal disk in N_0 and b_e is a vertical disk in N_k. Then there exists a point $x \in \mathrm{int}N_0$, such that

$$\begin{aligned}
x &= b_0(t), \quad \textit{for some } t \in B_{u(N_0)}(0,1) && (20)\\
f_i \circ f_{i-1} \circ \cdots \circ f_1(x) &\in \mathrm{int}N_i, \quad i = 1,\dots,k && (21)\\
f_k \circ f_{k-1} \circ \cdots \circ f_1(x) &= b_e(z), \quad \textit{for some } z \in B_{s(N_k)}(0,1) && (22)
\end{aligned}$$

Proof: Just as in the case of Theorem 3, the assertion was proved in [WZ, Thm. 4] under the assumption that $s(N_i) = s$ is independent of i.

We can reduce the current case exactly in the same way as in the proof of Theorem 3. We define $\tilde{f}_i$ and $\tilde{N}_i$ as it was done there. For disks let h_0 and h_e be the homotopies from definitions of b_0 and b_e, respectively. We define the horizontal disk $\tilde{b}_0$ and the vertical disk $\tilde{b}_e$ and their homotopies $\tilde{h}_0$ and $\tilde{h}_e$ as follows

$$\begin{aligned}
\mathrm{dom}(\tilde{b}_0) &= \mathrm{dom}(b_0), \quad \tilde{b}_0(x) = b_0(x) \times \{0\}^{s-s(N_0)} \\
\tilde{h}_0(t,x) &= h(t,x) \times \{0\}^{s-s(N_0)} \\
\mathrm{dom}(\tilde{b}_e) &= \mathrm{dom}(b_e) \times [-1,1]^{s-s(N_k)}, \qquad \tilde{b}_e(y,\tilde{y}) = (b_e(y),\tilde{y}), \\
\tilde{h}_e(t,y,\tilde{y}) &= (h_e(y),\tilde{y}).
\end{aligned}$$

Now we apply Theorem 4 from [WZ]. ∎

3 Cone conditions

The goal of this section is to introduce a method, which will allow to handle relatively easily the hyperbolic structure on h-sets. Some of this material has been already presented in [KWZ], but is included here to make this paper reasonably self-contained.

Definition 8 *Let $N \subset \mathbb{R}^n$ be an h-set and $Q : \mathbb{R}^n \to \mathbb{R}$ be a quadratic form*

$$Q((x, y)) = \alpha(x) - \beta(y), \qquad (x, y) \in \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)}, \tag{23}$$

where $\alpha : \mathbb{R}^{u(N)} \to \mathbb{R}$, and $\beta : \mathbb{R}^{s(N)} \to \mathbb{R}$ are positive definite quadratic forms.

The pair (N, Q) we be called an h-set with cones.

We will refer to the quadratic forms α and β as positive and negative parts of Q, respectively.

If (N, Q) is an h-set with cones, then we define a function $L_N : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$L_N(z_1, z_2) = Q(c_N(z_1) - c_N(z_2)) \tag{24}$$

Quite often we will drop Q in the symbol (N, Q) and we will say that N is an h-set with cones.

3.1 Cone conditions for horizontal and vertical disks

Definition 9 *Let (N, Q) be a h-set with cones.*

Let $b : \overline{B_u} \to |N|$ be a horizontal disk.

We will say that b satisfies the cone condition (with respect to Q) *iff for any $x_1, x_2 \in \overline{B_u}$, $x_1 \neq x_2$ holds*

$$Q(b_c(x_1) - b_c(x_2)) > 0. \tag{25}$$

Definition 10 *Let (N, Q) be a h-set with cones.*

Let $b : \overline{B_s} \to |N|$ be a vertical disk.

We will say that b satisfies the cone condition (with respect to Q) *iff for any $y_1, y_2 \in \overline{B_s}$, $y_1 \neq y_2$ holds*

$$Q(b_c(y_1) - b_c(y_2)) < 0. \tag{26}$$

Lemma 5 *Let (N, Q) be a h-set with cones and let $b : \overline{B_u} \to |N|$ be a horizontal disk satisfying the cone condition.*

Then there exists a Lipschitz function $y : \overline{B_u} \to \overline{B_s}$ such that

$$b_c(x) = (x, y(x)). \tag{27}$$

Analogously, if $b : \overline{B_s} \to |N|$ is a vertical disk satisfying the cone condition, then there exists a Lipschitz function $x : \overline{B_s} \to \overline{B_u}$

$$b_c(y) = (x(y), y)). \tag{28}$$

Proof: We will prove only the first assertion, the proof of the other one is analogous.

In the first part of this proof we will show that for any $x \in \text{int}B_{u(N)}$ there exists $z \in \text{int}B_{u(N)}$ and $y_x \in \overline{B}_{s(N)}$, such that

$$b_c(z) = (x, y_x). \tag{29}$$

For this we will use the local Brouwer degree.

In the second part using the cone condition we will show that y_x is uniquely defined and its dependence on x is Lipschitz. Then we extend the definition of $y(x)$ to $x \in \partial B_u$.

Let h be the homotopy from the definition of the horizontal disk b.

To prove (29) consider the homotopy $\pi_1 \circ h : [0,1] \times \overline{B}_{u(N)} \to \overline{B}_{u(N)}$, where $\pi_1 : \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)} \to \mathbb{R}^{u(N)}$ is a projection on the first component. Let us fix $x \in \mathrm{int} B_{u(N)}$. It is easy to see that, since $x \notin \pi_1 \circ h(t, \partial B_{u(N)})$ the local Brouwer degrees in the formula below are defined and the stated equalities are satisfied by the homotopy property of the local Brouwer degree

$$\deg(\pi_1 \circ b_c, \overline{B}_{u(N)}, x) = \deg(\pi_1 \circ h_1, \overline{B}_{u(N)}, x) = \deg(\mathrm{Id}, \overline{B}_{u(N)}, x) = 1. \quad (30)$$

This proves (29).

To prove the uniqueness of y_x, assume that there exist $z_1, z_2 \in \mathrm{int} B_{u(N)}$ and $y_1, y_2 \in \overline{B}_{s(N)}$, $y_1 \neq y_2$ such that

$$b_c(z_1) = (x, y_1), \qquad b_c(z_2) = (x, y_2). \quad (31)$$

From the cone condition for b it follows that

$$0 < Q(b_c(z_1) - b_c(z_2)) = \alpha(0) - \beta(y_1 - y_2) < 0 \quad (32)$$

which is a contradiction. Hence we have a well defined function

$$y(x) = y_x, \qquad \text{for } x \in \mathrm{int} B_{u(N)}. \quad (33)$$

Observe that from the cone condition it follows that for any $x_1, x_2 \in \mathrm{int} B_{u(N)}$, $x_1 \neq x_2$ holds

$$A\|x_1 - x_2\|^2 \geq \alpha(x_1 - x_2) > \beta(y(x_1) - y(x_2)) \geq B\|y(x_1) - y(x_2)\|^2, \quad (34)$$

where A, B are some positive constants related to quadratic forms α and β, respectively.

This proves the Lipschitz condition, which allows to continuously extend the function $y(x)$ to the boundary of $B_{u(N)}$. Observe that from the closeness of $|b|$ it follows that $(x, y(x)) \in |b|$ for $x \in \partial B_{u(N)}$. ∎

3.2 Cone conditions for maps

Definition 11 *Assume that $(N, Q_N), (M, Q_M)$ are h-sets with cones, such that $u(N) = u(M) = u$ and let $f : N \to \mathbb{R}^{\dim(M)}$ be continuous. Assume that $N \overset{f}{\Longrightarrow} M$. We say that f satisfies the cone condition (with respect to the pair (N, M)) iff for any $x_1, x_2 \in N_c$, $x_1 \neq x_2$ holds*

$$Q_M(f_c(x_1) - f_c(x_2)) > Q_N(x_1 - x_2). \quad (35)$$

Definition 12 *Assume that $(N, Q_N), (M, Q_M)$ are h-sets with cones, such that $u(N) = u(M) = u$ and $s = s(N) = s(M)$ and let $f : N \to \mathbb{R}^{u+s}$ be continuous. Assume that $N \overset{f}{\Longleftarrow} M$. We say that f satisfies the cone condition (with respect to the pair $((N, Q_N), (M, Q_M))$) iff for any $y_1, y_2 \in M_c$, $y_1 \neq y_2$ holds*

$$Q_M(y_1 - y_2) > Q_N(f_c^{-1}(y_1) - f^{-1}(y_2)). \tag{36}$$

Observe that Definition 12 is equivalent to Definition 11 applied to map f^{-1} with respect to pair $(M^T, -Q_M), (N^T, -Q_N))$.

The cone condition in Definition 11 is expressed in coordinates associated to h-sets, in the phase space it implies that

$$L_M(f(z_1), f(z_2)) > L_N(z_1, z_2), \qquad \text{for } z_1 \neq z_2,\ z_1, z_2 \in N. \tag{37}$$

Below we state and prove two basic theorems relating covering relations and the cone conditions

Theorem 6 *Assume that for $i = 0, \dots, k-1$ either*

$$N_i \overset{f_i}{\Longrightarrow} N_{i+1} \tag{38}$$

or

$$N_{i+1} \subset \text{dom}(f_i^{-1}) \quad \textit{and} \quad N_i \overset{f_i}{\Longleftarrow} N_{i+1}, \tag{39}$$

where all h-sets are h-sets with cones and f_i for $i = 0, \dots, k-1$ satisfies the cone condition.

Assume that $b : \overline{B}_{s(N_k)} \to N_k$ is a vertical disk in N_k satisfying the cone condition.

Then the set of points $z \in N_0$ satisfying the following two conditions

$$\begin{aligned} f_{i-1} \circ f_{i-2} \circ \cdots \circ f_0(z) &\in N_i, \qquad \textit{for } i = 1, \dots, k \qquad (40)\\ f_{k-1} \circ \cdots \circ f_0(z) &\in |b| \qquad (41)\end{aligned}$$

is a vertical disk satisfying the cone condition.

Proof: For the proof it is enough to consider the case of $k = 1$, only. For $k > 1$ the result follows by induction.

Without any loss of the generality we can assume that $N_0 = N_{0,c} = \overline{B}_{u(N_0)} \times \overline{B}_{s(N_0)}$, $N_1 = N_{1,c} = \overline{B}_{u(N_1)} \times \overline{B}_{s(N_1)}$, $f_0 = f_{0,c}$.

Consider a family of horizontal disks in N_0 $d_y : \overline{B}_{u(N_0)} \to N_0$ for $y \in \overline{B}_{s(N_0)}$

$$d_y(x) = (x, y). \tag{42}$$

From Theorem 4, applied to chain $N_0 \overset{f_0}{\Longrightarrow} N_1$ and disks d_y in N_0 and b in N_1 it follows that each $y \in \overline{B_{s(N_0)}}$ there exists $x \in \overline{B_{u(N_0)}}$, such that

$$f_0(x, y) \in |b|. \tag{43}$$

Let us fix $y \in \overline{B}_{s(N_0)}$. We will show that there exists only one x satisfying (43). For the proof assume the contrary, hence we have $x_1 \neq x_2$ and x_1, x_2 both satisfy (43).

Observe that $Q_{N_0}((x_1, y) - (x_2, y)) > 0$, hence from the fact that f_0 satisfies the cone condition it follows that

$$Q_{N_1}(f_0(x_1, y) - f_0(x_2, y))) > Q_{N_0}((x_1, y) - (x_2, y)) > 0. \tag{44}$$

But the above inequality is in a contradiction with the cone condition for b. Hence (43) defines a function $x(y)$ in a unique way.

It is easy to see that function $x(y)$ is continuous. Namely, from the compactness argument it follows that it is enough to prove that if we have a sequence of pairs (x_n, y_n), where $y_n \in \overline{B_s}$, $y_n \to \bar{y}$ for $n \to \infty$ and $x_n = x(y_n)$, $x_n \to \bar{x}$, then $f_0(\bar{x}, \bar{y}) \in |b|$, but this is an obvious consequence of the continuity of f_0 and the compactness of $|b|$.

Obviously, $b_0 : \overline{B_s} \to \overline{B_u} \times \overline{B_s}$ defined by $b_0(y) = (x(y), y)$ is a vertical disk in N_0. It remains to show that it satisfies the cone condition.

We will prove this by a contradiction. Assume that we have y_1 and y_2 such that

$$Q_{N_0}((x(y_1), y_1) - (x(y_2), y_2)) \geq 0, \tag{45}$$

then

$$Q_{N_1}(f_0(x(y_1), y_1) - f_0(x(y_2), y_2)) > 0, \tag{46}$$

hence the points $f_0(x(y_1), y_1)$ and $f_0(x(y_2), y_2)$ cannot both belong to b, because the cone condition is violated. ∎

Theorem 7 *Assume that for $i = 0, \ldots, k-1$ either*

$$N_i \overset{f_i}{\Longrightarrow} N_{i+1} \tag{47}$$

or

$$N_{i+1} \subset \operatorname{dom}(f_i^{-1}) \quad \textit{and} \quad N_i \overset{f_i}{\Longleftarrow} N_{i+1}, \tag{48}$$

where all h-sets are h-sets with cones and f_i for $i = 0, \ldots, k-1$ satisfies the cone condition.

Assume that $b : \overline{B}_{n(N_0)} \to N_0$ is a horizontal disk in N_0 satisfying the cone condition.

Then exists a set $Z \subset |b|$, such that for all $z \in Z$ holds

$$f_{i-1} \circ f_{i-2} \circ \cdots \circ f_0(z) \quad \in \quad N_i, \qquad \textit{for } i = 1, \ldots, k \tag{49}$$

and $f_{k-1} \circ f_{i-2} \circ \cdots \circ f_0(Z)$ a horizontal disk in N_k satisfying the cone condition.

Proof: It is enough consider $k = 1$. Consider first the case of $N_0 \overset{f_0}{\Longleftarrow} N_1$. By the definition we have $N_1^T \overset{f_0^{-1}}{\Longrightarrow} N_0^T$ and the statement follows directly from Theorem 6.

Consider now the case of direct covering $N_0 \overset{f_0}{\Longrightarrow} N_1$. Without any loss of the generality we can assume that $N_1 = N_{1,c} = \overline{B}_{u(N_1)} \times = \overline{B}_{s(N_1)}$. Then from the cone condition for this covering relation it follows that for all $z_1, z_2 \in f(N_0 \cap |b|)$, $z_1 \neq z_2$ holds

$$Q_{N_1}(z_1 - z_2) > 0. \tag{50}$$

This implies that for any $x \in \overline{B_{u(N_1)}}$ there exists at most one $y \in \mathbb{B}^{s(N_1)}$, such that $(x, y) \in f(N_0 \cap |b|) \cap N_1$. From Theorem 4 it follows that such $y = y(x)$ indeed exists. We define the horizontal disk by $x \mapsto (x, y(x))$. By (50) it satisfies the cone condition. ∎

3.3 Verification of cone conditions

Assume that (N, Q_N) and (M, Q_M) are h-sets with cones and a map $f : N \to \mathbb{R}^{\dim(M)}$ is C^1.

Observe that for $x_2 \to x_1$

$$Q_M(f_c(x_2) - f_c(x_1)) - Q_N(x_2 - x_1) \to 0. \tag{51}$$

Hence there is no chance that the cone condition can be verified rigorously on computer [N, KWZ], by direct evaluation in interval arithmetics of $Q_M(f(x_2) - f(x_1)) - Q_N(x_2 - x_1)$.

Our intention is to give a condition, which will imply the cone condition and will be verifiable on computer.

Definition 13 *Let $U \subset \mathbb{R}^n$ and let $g : U \to \mathbb{R}^n$ be a C^1 map. We define* the interval enclosure of $Dg(U)$ *by*

$$[Dg(U)] = \left\{ A \in \mathbb{R}^{n \times n} \mid \forall_{ij} A_{ij} \in \left[\inf_{x \in U} \frac{\partial g_i}{\partial x_j}(x), \sup_{x \in U} \frac{\partial g_i}{\partial x_j}(x) \right] \right\} \tag{52}$$

Let $[df_c(N_c)]$ be the interval enclosure of df_c on N_c. Observe that when $\dim(M) \neq \dim(N)$ this is not a square matrix.

Lemma 8 *Assume that for any $B \in [df_c(N_c)]$, the quadratic form*

$$V(x) = Q_M(Bx) - Q_N(x) \tag{53}$$

is positive definite, then for any $x_1, x_2 \in N_c$ such that $x_1 \neq x_2$ holds

$$Q_M(f_c(x_1) - f_c(x_2)) > Q_N(x_1 - x_2). \tag{54}$$

Proof: Let us fix x_1, x_2 in N_c. We have

$$f_c(x_2) - f_c(x_1) = \int_0^1 df_c(x_1 + t(x_2 - x_1))dt \cdot (x_2 - x_1). \tag{55}$$

Let $B = \int_0^1 df_c(x_1 + t(x_2 - x_1))dt$. Obviously $B \in [df_c]$. Hence

$$f_c(x_2) - f_c(x_1) = B(x_2 - x_1). \tag{56}$$

We have

$$Q_M(f_c(x_2) - f_c(x_1)) - Q_N(x_2 - x_1) =$$
$$Q_M(B(x_2 - x_1)) - Q_N(x_2 - x_1) = V(x_2 - x_1) > 0.$$

∎

In the light of the above lemma the verification of the cone conditions can be reduced to checking that the interval matrix corresponding to the quadratic form V for various choices of $B \in [df_c(N_c)]$ given by

$$V = [df_c(N_c)]^T Q_M [df_c(N_c)] - Q_N \tag{57}$$

is positive definite.

Observe that, since the set of positive definite matrices in an open subset of the set symmetric matrix, then if V given by (57) is positive definite, then there exist $0 < a < 1 < b$, such that

$$V = [df_c(N_c)]^T Q_M [df_c(N_c)] - (1+\epsilon) Q_N \tag{58}$$

is positive definite for $1 + \epsilon \in (a, b)$. We expect that this implies uniform hyperbolicity, see also Section 11.2 where this question is treated for linear maps.

4 Stable and unstable manifolds trough covering relations

The goal of this section and Section 5 is to prove the existence of stable and unstable manifolds for hyperbolic fixed point for maps. The proofs are topological and do not assume that the map under consideration is invertible.

We proceed as follows. In this section we prove general theorems about stable and unstable manifolds under assumption of the existence of self-covering (i.e. $N \overset{f}{\Longrightarrow} N$) satisfying the cone condition. This part does not require that the fixed (periodic) point is hyperbolic. Parts of this material appeared already in [KWZ].

In Section 5 we will show that in the neighborhood of the hyperbolic fixed point of the map we can build an h-sets, which covers itself and the cone condition holds for this relation. Then from results of Section 4 we obtain the stable and unstable manifold theorems for the fixed point under consideration.

Definition 14 *Consider the map $f : X \supset \mathrm{dom}(f) \to X$.*

Let $x \in X$. Any sequence $\{x_k\}_{k \in I}$, where $I \subset \mathbb{Z}$ is a set containing 0 and for any $l_1 < l_2 < l_3$ in $\mathbb{Z}$ if $l_1, l_3 \in I$, then $l_2 \in I$, such that

$$x_0 = x, \qquad f(x_i) = x_{i+1}, \qquad \text{for } i, i+1 \in I \tag{59}$$

will be called an orbit through x. If $I = \mathbb{Z}_-$, then we will say that $\{x_k\}_{k \in I}$ is a full backward orbit through x.

Definition 15 *Let X be a topological space and let the map $f : X \supset \mathrm{dom}(f) \to X$ be continuous.*

Let $Z \subset \mathbb{R}^n$, $x_0 \in Z$, $Z \subset \mathrm{dom}(f)$. We define

$$
\begin{aligned}
W^s_Z(z_0, f) &= \{z \mid \forall_{n\geq 0} f^n(z) \in Z, \quad \lim_{n\to\infty} f^n(z) = z_0\} \\
W^u_Z(z_0, f) &= \{z \mid \exists \ \{x_n\} \subset Z \text{ a full backward orbit through } z, \text{ such that} \\
&\qquad \lim_{n\to-\infty} x_n = z_0\} \\
W^s(z_0, f) &= \{z \mid \lim_{n\to\infty} f^n(z) = z_0\} \\
W^u(z_0, f) &= \{z \mid \exists \ \{x_n\} \text{ a full backward orbit through } z, \text{ such that} \\
&\qquad \lim_{n\to-\infty} x_n = z_0\} \\
Inv^+(Z, f) &= \{z \mid \forall_{n\geq 0} f^n(z) \in Z\} \\
Inv^-(Z, f) &= \{z \mid \exists \ \{x_n\} \subset Z \text{ a full backward orbit through } z \ \}
\end{aligned}
$$

If f is known from the context, then we will usually drop it and use $W^s(z_0)$, $W^s_Z(z_0)$ etc instead.

Definition 16 *Let $f : \mathbb{R}^n \supset \mathrm{dom}(f) \to \mathbb{R}^n$ be a continuous map.*

Loop of covering relations (for f) *is collection of h-sets N_i, $i = 0, \ldots, k$, $N_k = N_0$ and covering relations, such that for each $i = 1, \ldots, k$ we have either*

$$N_{i-1} \overset{f}{\Longrightarrow} N_i \tag{60}$$

or

$$N_i \subset \mathrm{dom}(f^{-1}) \quad \text{and} \quad N_{i-1} \overset{f}{\Longleftarrow} N_i. \tag{61}$$

k will be called the length of the loop.

Let L be a loop of covering relations, if additionally N_i are h-sets with cones Q_i, such that $Q_k = Q_0$ and each covering relation in the loop L we assume the cone condition. In this situation we will say that L satisfies cone conditions*.*

The following notation will be used for loops of covering relations $L = (N_0, N_1, \ldots, N_{k-1})$.

Definition 17 *Let $L = (N_0, N_1, \ldots, N_{k-1})$ is a loop of covering relations for f. We define*

$$S_L = N_0 \cap f^{-1}(N_1) \cap \cdots \cap f^{-(k-2)}(N_{k-2}) \cap f^{-(k-1)}(N_{k-1}). \tag{62}$$

It is easy to see that $S_{(N_0,N_1,\ldots,N_{k-1})}$ consists of points in N_0, such that

$$f^i(x) \in N_i, \qquad \text{for } i = 1, \ldots, k-1. \tag{63}$$

Lemma 9 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map.*

Assume that $L = (N_0, \ldots, N_{k-1})$ is a loop of covering relations for f satisfying the cone conditions.

Then there exists a unique $z_0 \in S_L$, such that

$$\begin{aligned} f^k(z_0) &= z_0, && (64)\\ Inv^+(S_L, f^k) &= W^s_{S_L}(z_0, f^k), && (65)\\ Inv^-(S_L, f^k) &= W^u_{S_L}(z_0, f^k). && (66) \end{aligned}$$

Proof: The existence of z_0 satisfying (64) follows directly from Theorem 3. Let us fix one such z_0.

To prove (65) it is enough to show that, if $f^{lk}(z) \in S_L$ for all $l \geq \mathbb{N}$, then $\lim_{l\to\infty} f^{lk}(z) = z_0$.

From the cone conditions for the loop L it follows that the function $V(z) = Q_{N_0}(c_{N_0}(z) - c_{N_0}(z_0))$ is a Lapunov function on S_L for f^k, i.e. is increasing on nonconstant orbits of f^k in S_L. By the standard Lapunov function argument it is easy to show that $Inv(S_L, f^k) = \{z_0\}$ and $\lim_{l\to\infty} f^{lk}(z) = z_0$. This finishes the proof of (65).

To prove (66) it is enough to show, that any backward orbit for f^k in S_L, $\{x_k\}_{k\in\mathbb{Z}_-}$ converges to z_0. But this is true by the same Lapunov function argument as in the previous paragraph. ∎

Theorem 10 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map.*

Assume that $L = (N_0, \ldots, N_{k-1})$ is a loop of covering relations for f satisfying the cone conditions.

Then there exists a unique $z_0 \in S_L$, such that $W^s_{S_L}(z_0, f^k)$ is a vertical disk in N_0 satisfying the cone condition.

Therefore, if c_{N_0} is an affine map, then $W^s_{S_L}(z_0, f^k)$ can be represented as a graph of a Lipschitz function over the nominally stable space in N_0.

Proof: First we show that for all $y \in \overline{B_s}$ there exists $x \in \overline{B_u}$, such that

$$z = c^{-1}_{N_0}(x, y) \in W^s_{S_L}(z_0). \tag{67}$$

By Lemma 9 it is equivalent to showing that

$$f^{kl}(z) \in N_0, \quad \text{for } l \in \mathbb{N}. \tag{68}$$

Consider a family of horizontal disks in N_0 $d_y : \overline{B_{u(N_0)}} \to N$ for $y \in \overline{B_{s(N_0)}}$

$$d_y(x) = (x, y). \tag{69}$$

The proof is the same for both direct- and backcovering, therefore we will just consider direct covering.

Consider an infinite chain of covering relations consisting of replicas of loop L

$$N_0 \overset{f}{\Longrightarrow} N_1 \overset{f}{\Longrightarrow} \cdots \overset{f}{\Longrightarrow} N_{k-1} \overset{f}{\Longrightarrow} N_0 \overset{f}{\Longrightarrow} \cdots N_0 \overset{f}{\Longrightarrow} \cdots \tag{70}$$

From Theorem 4 applied to d_y, an arbitrary vertical disk b_v in N_0 and finite chains $N_0 \overset{f}{\Longrightarrow} N_1 \overset{f}{\Longrightarrow} N \overset{f}{\Longrightarrow} \cdots \overset{f}{\Longrightarrow} N_0$ of increasing length using the compactness argument one can show (see [W2, Col. 3.10]) that for every $y \in \overline{B_s}$ there exists $x \in \overline{B_u}$, such that (68) holds for $z = c^{-1}_{N_0}(x, y)$.

The next step is to prove that such x is unique. Let us assume the contrary, then there exists $y \in \overline{B_s}$ and $x_1, x_2 \in \overline{B_u}$, $x_1 \neq x_2$, such that $z_i = c_{N_0}^{-1}(x_i, y)$ for $i = 1, 2$ satisfies condition (68). Observe that

$$Q_{N_0}(c_{N_0}(z_1) - c_{N_1}(z_2)) = \alpha(x_1 - x_2) > 0, \tag{71}$$

hence from the cone condition and (68) it follows that

$$Q_{N_0}(c_{N_0}(f^{lk}(z_1)) - c_{N_0}(f^{lk}(z_2)) > \alpha(x_1 - x_2), \qquad \text{for } l \in \mathbb{N}. \tag{72}$$

Passing to the limit $l \to \infty$ we obtain

$$0 = Q_{N_0}(c_{N_0}(z_0) - c_{N_0}(z_0)) = \\ \lim_{l\to\infty} Q_{N_0}(c_{N_0}(f^{lk}(z_1)) - c_{N_0}(f^{kl}(z_2))) > \alpha(x_1 - x_2) > 0.$$

This is a contradiction. Hence we have a well defined function $x(y)$ on $\overline{B_s}$.

Obviously $W^s_{S_L}(x_0, f^k) = \{c_N^{-1}(x(y), y) \mid y \in \overline{B}_s\}$. Now we prove the cone condition for $W^s_{S_L}(x_0, f^k)$. This will imply that the map $b : \overline{B}_s \to N$, given by $b(y) = c_N^{-1}(x(y), y)$ defines a vertical disk in N.

We have to check whether

$$Q_N(c_{N_0}(z_1) - c_{N_0}(z_2)) < 0, \qquad \text{for all } z_1, z_2 \in W^s_{S_L}(x_0, f^k),\ z_1 \neq z_2 \tag{73}$$

Assume that (73) is not satisfied for some $z_1, z_2 \in W^s_{S_L}(x_0, f^k)$, $z_1 \neq z_2$. We have

$$Q_{N_0}(c_{N_0}(z_1) - c_{N_0}(z_2)) \geq 0. \tag{74}$$

From the cone condition it follows that for $l > 1$ holds

$$Q_{N_0}(c_{N_0}(f^{lk}(z_1)) - c_{N_0}(f^{lk}(z_2))) > Q_{N_0}(c_{N_0}(f(z_1)) - c_{N_0}(f(z_2))) > 0.$$

Passing to the limit $l \to \infty$ we obtain

$$0 = Q_{N_0}(c_{N_0}(z_0) - c_{N_0}(z_0)) = \lim_{l\to\infty} Q_{N_0}(c_{N_0}(f^{kl}(z_1)) - c_{N_0}(f^{kl}(z_2))) > \\ Q_{N_0}(c_{N_0}(f(z_1)) - c_{N_0}(f(z_2))) > 0.$$

Which is a contradiction. This proves (73). ∎

The following remark will be used, when we will tackle the question of the analyticity of the stable manifold for analytic maps.

Remark 11 *The proof of above theorem suggests that function $x : \overline{B}_s \to \overline{B}_u$ used to parameterize $W^s_{s_L}(z_0, f^k)$ is a limit of functions $x_l : \overline{B}_s \to \overline{B}_u$ defined for $l = 1, 2, \ldots$ by implicit equation*

$$\pi_x \circ c_{N_0} \circ f^{lk} \circ c_{N_0}^{-1}(x_l(y), y) = 0. \tag{75}$$

and under the constraint

$$f^{ik} \circ c_{N_0}^{-1}(x_l(y), y) \in S_L, \qquad i = 0, \ldots, l-1. \tag{76}$$

∎

Now we would like to prove the theorem about unstable manifolds. Observe that in the case of f being non-invertible we cannot apply previous theorem to f^{-1} to obtain statement about the unstable manifold, therefore we need a different proof.

Theorem 12 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous map.*

Assume that $L = (N_0, \ldots, N_{k-1})$ is a loop of covering relations for f satisfying the cone conditions.

Then there exists a unique $z_0 \in S_L$, such that $W^u_{S_L}(z_0, f^k)$ is a horizontal disk in N_0 satisfying the cone condition.

Therefore, if c_{N_0} is an affine map, then $W^u_{S_L}(z_0, f^k)$ can be represented as a graph of a Lipschitz function over the nominally unstable space in N_0.

Proof: We will prove the theorem for the trivial loop $L = (N)$. The modifications necessary to consider loops of arbitrary length are rather obvious, see the proof of Theorem 10.

Without any loss of the generality we can assume that $N = \overline{B}_u \times \overline{B}_s$ and $c_N = id$.

We will prove that for any $x \in B_u$ there exists $y \in B_s$, such that $(x, y) \in W^u_N(x_0)$. For any $x \in \overline{B}_u$ let v_x be a vertical disk given by

$$v_x(y) = (x, y).$$

Let $h : \overline{B}_u \to \overline{B}_u \times \overline{B}_s$ be a horizontal disk given by $h(x) = (x, 0)$.

The proof is the same for both direct- and backcovering, therefore we will just consider direct covering.

Consider a chain of covering relations consisting of k replicas of $N \overset{f}{\Longrightarrow} N$. It follows from Theorem 4 it follows that there exists a finite orbit $\{w^k_{-k}, w^k_{-k+1}, \ldots, w^k_{-1}, w^k_0\}$, such that

$$\begin{aligned} w^k_{-k}, w^k_{-k+1}, \ldots, w^k_{-1}, w^k_0 &\in N \\ f(w^k_l) = w^k_{l+1}, &\qquad l = -k, \ldots, -1 \\ w^k_{-k} \in |h|, &\qquad w^k_0 \in |v_x|. \end{aligned}$$

By applying the diagonal argument we can find an infinite backward orbit $\{w_l\}_{l \in \mathbb{Z}_- \cup \{0\}}$, such that

$$w_l \in N, \qquad l = 0, -1, -2, \ldots \tag{77}$$

$$f(w_l) = w_{l-1}, \qquad l < 0 \tag{78}$$

$$w_0 \in |v_x|. \tag{79}$$

Since $V(z) = Q_N(z - z_0)$ is increasing on orbits for $z \neq z_0$ (is a Lapunov function), therefore

$$\lim_{l \to -\infty} w_l = z_0. \tag{80}$$

We have proved that

$$w_0 \in W^u_N(z_0) \cap |v_x|. \tag{81}$$

We will prove that w_0 in (81) is uniquely defined. Let p_0 also satisfies the above condition, hence there exists a backward orbit in N through p_0 $\{p_l\}_{l\in\mathbb{Z}_-\cup\{0\}}$. We have

$$Q_N(p_0 - w_0) = -\beta(y(p_0) - y(w_0)) < 0. \tag{82}$$

From the cone condition for map f it follows that the function $Q_N(p_l - w_l)$ is increasing for $l < 0$, hence

$$0 > Q_N(p_0 - w_0) > Q_N(p_l - w_l) > \lim_{l\to-\infty} Q_N(p_l - w_l) = Q_N(z_0 - z_0) = 0. \tag{83}$$

Which is a contradiction, therefore w_0 in (81) is uniquely defined.

We define a horizontal disk $d : \overline{B}_u \to \overline{B}_u \times \overline{B}_s$, by $d(x) = (x, w_0)$. From the above considerations it follows that

$$W^u_N(z_0) = |d|. \tag{84}$$

We will show that d is satisfy the cone condition (which also implies the continuity of d)

$$Q_N(w - p) > 0, \qquad \text{for all } w, p \in |d|,\ w \neq p. \tag{85}$$

Assume that (85) does not hold. Then there exists two full backward orbits $\{w_l\}, \{p_l\}$ in N through w and p and

$$Q_N(w - p) \leq 0. \tag{86}$$

We have for any $l \in \mathbb{Z}_-$

$$0 \geq Q_N(w_0 - p_0) > Q_N(w_l - p_l) > \lim_{l\to-\infty} Q_N(w_l - p_l) = Q_N(z_0 - z_0) = 0.$$

But this is a contradiction, hence (85) is satisfied. ∎

The following remark will be used, when we will tackle the question of the analyticity of the unstable manifold for analytic maps.

Remark 13 *The proof of above theorem suggests that function $y : \overline{B}_u \to \overline{B}_s$ used to parameterize $W^u_{S_L}(z_0, f^k)$ is as a limit of functions $y_l : \overline{B}_u \to \overline{B}_s$ defined for $l = 1, 2, \ldots$ by*

$$y_l(x) = \pi_y \circ c_{N_0} \circ f^{lk} \circ c^{-1}_{N_0}(x_l(x), 0), \tag{87}$$

where $x_l : \overline{B}_u \to \overline{B}_u$ for $l = 1, 2, \ldots$ is defined by implicit equation

$$\pi_x \circ c_{N_0} \circ f^{lk} \circ c^{-1}_{N_0}(x_l(x), 0) = x. \tag{88}$$

with the constraint

$$f^{ik} \circ c^{-1}_{N_0}(x_l(x), 0) \in S_L, \qquad i = 0, \ldots, l-1. \tag{89}$$

4.1 Example - multidimensional horseshoe

Assume that (N_i, Q_i) for $i = 0, 1$ are h-sets with cones. Assume that the following covering relations hold together with cone conditions

$$N_i \overset{f}{\Longrightarrow} N_j, \qquad i, j = 0, 1. \tag{90}$$

From Theorems 10 and 12 it follows that for any $\sigma = (\sigma_0, \ldots, \sigma_{k-1}) \in \{0, 1\}^k$ that there exists a unique periodic point $z_\sigma \in N_{\sigma_0}$, such that

$$f^i(z_\sigma) \in N_{\sigma_i}, i = 0, 1, \ldots, k-1 \qquad f^k(z_0) = z_0 \tag{91}$$

and local stable and unstable sets of z_σ for f^k are respectively vertical and horizontal disks in N_{σ_0}. We would like to stress here, that we have a uniform bounds for both stable and unstable manifolds for periodic orbits independent of the period, just as in the case of two-dimensional horseshoe.

As example let us consider for any $u, s \in \mathbb{N}_+$ the h-sets with cones $N_i \subset \mathbb{R}^{u+s}$, $i = 0, 1$, defined as follows. Let $u_i = u$, $s_i = s$, $p_i = ((-1)^i \cdot 2, 0, \ldots, 0)$ and $c_{N_i}^{-1}(x, y) = p_i + (x, y)$ for $(x, y) \in \mathbb{R}^u \times \mathbb{R}^s$. On N_i we define $Q_{N_i}(x, y) = x^2 - y^2$.

We define the map $f : N_0 \cup N_1 \to \mathbb{R}^{u+s}$ as follows

$$f(x, y) = (A_i(5 \cdot (x - \pi_x p_i)), 0) + p_i, \qquad \text{for } (x, y) \in N_i, \text{ for } i = 0, 1$$

where $A_i : \mathbb{R}^u \to \mathbb{R}^u$ are for $i = 0, 1$ arbitrary isometries (with respect to Euclidean metric).

Observe that $f_{|N_i}$ is a uniform expansion by the factor of 5 in the $\mathbb{R}^u \times \{0\}^s$ and retraction onto 0 in the stable direction. Observe that for any of the covering relations $N_i \overset{f}{\Longrightarrow} N_j$ the derivative is a constant linear map given by

$$df_c(x, y) = (5A_i x, 0). \tag{92}$$

From Lemma 8 it follows that the cone conditions will be satisfied if the matrix

$$\begin{bmatrix} 5A_i^T & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \cdot \begin{bmatrix} 5A_i & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} = \begin{bmatrix} 24I & 0 \\ 0 & I \end{bmatrix} \tag{93}$$

is positively defined, which is clearly the case.

Since both covering relations and being positive definite are stable with respect to small perturbations, then for sufficiently small in C^1-norm maps $h : N_0 \cup N_1 \to \mathbb{R}^{u+s}$ we obtain

$$N_i \overset{f+h}{\Longrightarrow} N_j, \qquad i, j = 0, 1 \tag{94}$$

and for all these covering relations the cone conditions are satisfied. Therefore we obtain uniform bounds for (un)stable manifolds for infinite number of periodic orbits of unbounded periods.

5 Stable and unstable manifolds for hyperbolic fixed points

In this section we apply theorems proved in Section 4 to obtain the existence of the unstable and unstable manifold for hyperbolic fixed point. The result, concerning the smoothness, is rather weak, when compared to classical results in the literature, see [HPS, I70, I80, C] and references given there, as we have only the Lipschitz condition and a suitable tangency at the fixed point. In Section 8 we will prove the continuous and Lipschitz dependence of parameters, which again are classical results.

Definition 18 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be C^1. Let $z_0 \in \mathbb{R}^n$. We say that z_0 is a* hyperbolic fixed point for f *iff $f(z_0) = z_0$ and $Sp(Df(z_0)) \cap S^1 = \emptyset$, where $Df(z_0)$ is the derivative of f at z_0.*

Theorem 14 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map. Assume that z_0 is a hyperbolic fixed point of f.*

Let $Z \subset \mathbb{R}^n$ be an open set, such that $z_0 \in Z$.

Then there exists an h-set N with cones, such that $z_0 \in \mathrm{int} N$, $N \subset Z$ and

- *$N \overset{f}{\Longrightarrow} N$ and if f is a local diffeomorphism in the neighborhood of z_0 then $N \overset{f}{\Longleftarrow} N$,*
- *$W^u_N(z_0)$ is a horizontal disk in N satisfying the cone condition*
- *$W^s_N(z_0)$ is a vertical disk in N satisfying the cone condition.*

Moreover, $W^u_N(z_0)$ can be represented as a graph of a Lipschitz function over the unstable space for the linearization of f at z_0 and tangent to it at z_0. Analogous statement is also valid for $W^s_N(z_0)$.

Proof: Let L be a linearization of f at z_0, hence $L(z) = z_0 + df(z_0)(z - z_0)$. Let u be the dimension of the unstable manifold and s of the stable manifold of L at z_0.

Then there exists a coordinate system on $\mathbb{R}^n$ and a scalar product $(\cdot, \cdot)$ such that following holds

$$df(z_0) = \begin{bmatrix} A & 0 \\ 0 & U \end{bmatrix}, \tag{95}$$

where $A : \mathbb{R}^u \to \mathbb{R}^u$ and $U : \mathbb{R}^s \to \mathbb{R}^s$ are linear isomorphisms, such that

$$W^u(z_0, L) = \{z_0\} + \mathbb{R}^u \times \{0\}^s, \quad W^s(z_0, L) = \{z_0\} + \{0\}^u \times \mathbb{R}^s \tag{96}$$

$$\|Ax\| > \|x\|, \quad \text{for } x \in \mathbb{R}^u \setminus \{0\} \tag{97}$$

$$\|Uy\| < \|y\|, \quad \text{for } y \in \mathbb{R}^s \setminus \{0\}, \tag{98}$$

where the norms are $\|x\| = \sqrt{x^2}$ and $\|y\| = \sqrt{y^2}$. We will use these coordinates in our proof.

Observe that (97) and (98) imply that matrices $A^T A - Id$ and $Id - U^T U$ are positive definite.

For any $r > 0$ we define

$$N(r) = \{z_0\} + \overline{B}_u(0,r) \times \overline{B}_s(0,r). \tag{99}$$

We define the homotopy

$$f_\lambda(z) = (1-\lambda)f(z) + \lambda(df(z_0)(z-z_0) + z_0), \quad \text{where } \lambda \in [0,1] \text{ and } z \in \mathbb{R}^n. \tag{100}$$

It is easy to see that $f_0 = f$ and $f_1(x,y) = df(z_0)(z-z_0) + z_0$.

Let $Q((x,y)) = \alpha x^2 - \beta y^2$, where $x \in \mathbb{R}^u$ and $y \in \mathbb{R}^s$ and $\alpha > 0$, $\beta > 0$ are arbitrary positive reals.

We will need the following lemma, which will be proved after we complete the current proof.

Lemma 15 *There exists $r_0 > 0$, such that for any $0 < r \leq r_0$ for all $z_1, z_2 \in N(r_0)$, $z_1 \neq z_2$ holds*

$$Q(f_\lambda(z_1) - f_\lambda(z_2)) > Q(z_1 - z_2). \tag{101}$$

Moreover, for any $z \in N(r)$ holds

$$\begin{aligned} (\pi_x f_\lambda(z) - \pi_x z_0)^2 > r, &\qquad \text{if } \|\pi_x(z-z_0)\| = r \quad (102)\\ (\pi_y f_\lambda(z) - \pi_y z_0)^2 < r, &\qquad \text{if } \|\pi_y(z-z_0)\| = r \quad (103)\end{aligned}$$

Continuation of the proof of Theorem 14: Let us fix any $r \leq r_0$, where r_0 is as in Lemma 15.

We define an h-set N with cones as follows: we set $|N| = N(r)$, $c_N(z) = \frac{1}{r}(z - z_0)$, $u(N) = u$, $s(N) = s$ and $Q_N(z') = Q(c_N^{-1}(z'))$ for $z' \in N_c$.

From Lemma 15 it follows that the following conditions are satisfied for any $\lambda \in [0,1]$

$$\begin{aligned} Q_N(f_{\lambda,c}(z_1) - f_{\lambda,c}(z_2)) &> Q_N(z_1 - z_2), \qquad z_1, z_2 \in N_c, z_1 \neq z_2 \quad (104)\\ \pi_x f_\lambda(N) &\subset \mathbb{R}^n \setminus \pi_x N = \mathbb{R}^n \setminus \overline{B}_u(\pi_x z_0, r) \quad (105)\\ \pi_y f_\lambda(N) &\subset B_s(\pi_y z_0, r) \quad (106)\end{aligned}$$

We will prove that

$$N \stackrel{f}{\Longrightarrow} N. \tag{107}$$

For this we need a suitable homotopy. We define $H : [0,1] \times N \to \mathbb{R}^{u+s}$ as follows

$$H(\lambda, z) = \begin{cases} f_{2\lambda}(z) & \text{for } \lambda \in \left[0, \frac{1}{2}\right], \\ (A(\pi_x(z-z_0)), (-2\lambda+2)U\pi_y(z-z_0)) + z_0 & \text{for } \lambda \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

Observe that

$$\begin{aligned} H_0 &= f, \qquad H_1(z) = (A(\pi_x(z-z_0)), 0) + z_0 \quad (108)\\ \pi_x H_\lambda(N) &\subset \mathbb{R}^n \setminus \pi_x N = \mathbb{R}^n \setminus \overline{B}_u(\pi_x z_0, r) \quad (109)\\ \pi_y H_\lambda(N) &\subset B_s(\pi_y z_0, r). \quad (110)\end{aligned}$$

It is immediate to check that the homotopy $h(\lambda,z)=c_N(H(\lambda,c_N^{-1}(z))$ satisfies all conditions for the covering relation $N \stackrel{f,w}{\Longrightarrow} N$, where $w=\pm 1$ due to linearity of h_1.

When $df(z_0)$ is an isomorphism, then analogous reasoning leads to $N^T \stackrel{f^{-1}}{\Longrightarrow} N^T$ (we may need to decrease further r in the construction.)

The remaining assertions, with the exception of the one the concerning the tangency to $W^{u,s}(z_0,L)$ at z_0, follow directly from Theorems 10 and 12.

To prove the tangency of $W^u(z_0,f)$ to $z_0+\mathbb{R}^u\times\{0\}^s$ at $z_0=(x_0,y_0)$ it is enough to prove that for any $\epsilon>0$, there exists $r>0$, such that for any $z=(x,y(x))\in W^u_{N(r)}(z_0,f)$ holds

$$\|y(x)-y_0\|\leq \epsilon\|x-x_0\|. \tag{111}$$

For given α,β the set $W^u_{N(r)}(z_0,f)$ for r sufficiently small is a horizontal disk satisfying the cone condition with respect to the quadratic form $Q(x,y)=\alpha x^2-\beta y^2$. Therefore we have

$$\begin{array}{rcl} Q((x,y(x))-(x_0,y_0)) & > & 0 \\ \beta\|y(x)-y_0\|^2 & < & \alpha\ \|x-x_0\|^2 \\ \|y(x)-y_0\| & < & \sqrt{\alpha/\beta}\ \|x-x_0\|, \end{array}$$

which proves (111).

The proof of the tangency for $W^s(z_0,f)$ to $z_0+\{0\}^u\times\mathbb{R}^s$ at z_0 is analogous.

∎

Proof of Lemma 15: To see that (101) is indeed satisfied for z_i close to z_0, we derive some other condition, which forces it (compare Lemma 8). For this end let Q be a symmetric matrix corresponding the quadratic form Q. Then

$$\begin{array}{r} Q(f_\lambda(z_1)-f_\lambda(z_2))-Q(z_1-z_2)= \\ (f_\lambda(z_1)-f_\lambda(z_2))^TQ(f_\lambda(z_1)-f_\lambda(z_2))-(z_1-z_2)^TQ(z_1-z_2)= \\ (z_1-z_2)^TC^TQC(z_1-z_2)-(z_1-z_2)^TQ(z_1-z_2)= \\ (z_1-z_2)^T(C^TQC-Q)(z_1-z_2), \end{array}$$

where

$$C=C(\lambda,z_1,z_2)=\int_0^1 df_\lambda(z_1+t(z_2-z_1))dt=$$
$$(1-\lambda)\int_0^1 df_\lambda(z_1+t(z_2-z_1))dt+\lambda df(z_0)$$

Observe that for $z_1,z_2\to z_0$ the matrix $C(\lambda,z_1,z_2)$ converges to $df(z_0)$ uniformly with respect to $\lambda\in[0,1]$. Therefore it is enough to show that the symmetric matrix $V=df(x_0)^TQdf(x_0)-Q$ is positive definite.

We have

$$V=\left[\begin{array}{cc} \alpha(A^TA-Id), & 0 \\ 0, & \beta(Id-U^TU) \end{array}\right]$$

Since $\alpha > 0$, $\beta > 0$ and $A^T A - Id$ and $Id - U^T U$ are positive definite, hence V is positive definite. From this is follows that there is r_0, such that (101) holds for $z_1, z_2 \in N(r_0)$, $z_1 \neq z_2$.

Now we prove condition (102). We have

$$(\pi_x f_\lambda(z) - \pi z_0)^2 = (\pi_x f_\lambda(z) - \pi_x f_\lambda(z_0))^2 = \\ (C_{11}(\pi_x z - \pi_x z_0) + C_{12}(\pi_y z - \pi_y z_0))^2, \tag{112}$$

where

$$C_{11} = C_{11}(\lambda, z_1, z_0) = \int_0^1 \frac{\partial \pi_x f_\lambda}{\partial x}(z_0 + t(z - z_0))dt = \\ \frac{\partial \pi_x f_\lambda}{\partial x}(z_0) + O(\|z - z_0\|) = A + O(\|z - z_0\|),$$
$$C_{12} = C_{12}(\lambda, z_1, z_0) = \int_0^1 \frac{\partial \pi_x f_\lambda}{\partial y}(z_0 + t(z - z_0))dt = \\ \frac{\partial \pi_x f_\lambda}{\partial y}(z_0) + O(\|z - z_0\|) = O(\|z - z_0\|).$$

Let us fix $0 < r \leq r_0$ and $\lambda \in [0, 1]$. Let $z = (x, y) \in N(r)$, $z_0 = (x_0, y_0)$ and $\|x - x_0\| = r$. We have

$$(\pi_x f_\lambda(z) - x_0)^2 = (C_{11}(x - x_0))^2 + (C_{12}(y - y_0))^2 + \\ 2(x - x_0)^T C_{11}^T C_{12}(y - y_0) \geq (1 + a - O(r))r^2 - \\ O(r)^2 r^2 - 2(\|A\| + O(r))O(r)r^2 = (1 + a - O(r))r^2$$

where $a > 0$ is such that $x^T A^T A x \geq (1 + a)x^2$. Hence (102) holds provided r_0 is small enough.

The justification of (103) is analogous. ∎

5.1 Propagation of stable and unstable manifolds of hyperbolic fixed points for a map

Assume that z_i, $i = 0, 1$ are a fixed (or periodic) points of the map $f : \mathbb{R}^n \to \mathbb{R}^n$ and that we have (N_i, Q_i) h-set with cones, such that $z_i \in N_i$ and $N_i \overset{f}{\Longrightarrow} N_i$. Assume that we would like to show that $W^u(z_0, f)$ and $W^s(z_1, f)$ intersect transversally.

Theorems 10 and 12 give us information of pieces of $W^u(z_0, f)$ and $W^s(z_1, f)$ in terms of N_0 and N_1, respectively. Usually the sizes of N_0 and N_1 are relatively small and we need to be able to get information of much larger pieces of $W^u(z_0, f)$ and $W^s(z_1, f)$. Using the tools developed in previous sections this can be achieved as follows.

First we need some approximate heteroclinic orbit, i.e. a sequence of points $v_0, v_1, \ldots, v_K$, such that $f(v_i) \approx v_{i+1}$, for $i = 0, \ldots, K - 1$ and v_0 close to z_0 and v_K is close to z_1. Next step is find h-sets with cones (M_i, Q_{M_i}) such that,

$v_i \in \mathrm{int} M_i$, the following covering relations are satisfied together with cone conditions

$$N_0 \overset{f}{\Longrightarrow} N_0 \overset{f}{\Longrightarrow} M_0 \overset{f}{\Longrightarrow} M_1 \overset{f}{\Longrightarrow} \cdots \overset{f}{\Longrightarrow} M_K \overset{f}{\Longrightarrow} N_1 \overset{f}{\Longrightarrow} N_1 \tag{113}$$

From Theorems 10 and 6 it follows that $W^s(z_1, f) \cap N_0$ contains a vertical disk satisfying the cone condition. Since by Theorem 12 $W^u_{N_0}(z_0, f)$ is a horizontal disk in N_0 satisfying the cone conditions, therefore we obtain a transversal intersection of $W^s(z_1, f)$ and $W^u_{N_0}(z_0, f)$. In fact to talk about transversality we need at least structure of C^1-manifold on $W^s(z_1, f)$ and $W^u_{N_0}(z_0, f)$, which is not proved in this paper, but it is known for $f \in C^1$ from [I70, I80] and the cone conditions imply that then this intersection is indeed transversal.

The obvious question arises: how to find M_i's satisfying (113). Without the cone conditions this was discussed and successfully used in [AZ] on the example of Henon-Heiles hamiltonian, but we believe that the same discussion applies also to the cone condition.

6 Non-hyperbolic example

The goal of this section is to provide a simple example illustrating that our theorems from Section 4 to obtain stable and unstable manifolds for the fixed point, which has a nonhyperbolic linear part.

In this contexts one should mention here papers [BF, F] (and an earlier paper [Mc]), where under suitable assumptions the stable set of the fixed point has been proved, using the mixture of topological and analytic arguments in the phase space, to have a manifold structure, but the analytic part there (replacing our cone conditions expressed in terms of Lapunov function) is much more elaborate and subtle and leads to results in situations, where our approach may fail.

Consider the following map $f : \mathbb{R}^2 \to \mathbb{R}^2$

$$f(x, y) = (x + x^3, y - y^3) + P(x, y), \tag{114}$$

where $P(x, y)$ is a polynomial, such that the degree of all nonzero terms in P is at least 4.

Observe that $z_0 = (0, 0)$ is a non-hyperbolic fixed point, but a look at the dominant terms $(x + x^3, y - y^3)$, suggests that nevertheless z_0 will have a one dimensional stable and unstable manifolds tangent at z_0 to the coordinate axes.

We will prove the following theorem

Theorem 16 *Consider the map f given by (114).*
There exists an h-set N with cones, such that $z_0 \in \mathrm{int} N$, $N \subset Z$ and

- $N \overset{f}{\Longrightarrow} N$,
- $W^u_N(z_0)$ *is a horizontal disk in N satisfying the cone condition*

- *$W^s_N(z_0)$ is a vertical disk in N satisfying the cone condition.*

Moreover, $W^u_N(z_0)$ is at z_0 tangent to the line $y = 0$ and $W^s_N(z_0)$ is at z_0 tangent to the line $x = 0$.

Let us fix $\alpha > 0$, $\beta > 0$ and consider a quadratic form $Q_{\alpha,\beta} : \mathbb{R}^2 \to \mathbb{R}$

$$Q_{\alpha,\beta}(x, y) = \alpha x^2 - \beta y^2. \tag{115}$$

The first step in the proof of Theorem 16 is the following lemma showing the cone condition for small z_1, z_2.

Lemma 17 *There exists $\delta > 0$, such that if $|x_i| \leq \delta$ and $|y_i| \leq \delta$ for $i = 1, 2$, then*

$$Q_{\alpha,\beta}(f(z_1) - f(z_2)) > Q_{\alpha,\beta}(z_1 - z_2), \tag{116}$$

where $z_i = (x_i, y_i)$ for $i = 1, 2$.

Proof: Let us denote $f(z) = (f_1(z), f_2(z))$ and let us set

$$N(a, b) = a^2 + ab + b^2. \tag{117}$$

Obviously we have

$$\frac{a^2 + b^2}{2} \leq N(a, b) \leq \frac{3(a^2 + b^2)}{2} \tag{118}$$

Observe that

$$\begin{aligned} f_1(z_1) - f_1(z_2) = \\ x_1 - x_2 + (x_1^3 - x_2^3) + C_{1,1}(z_1, z_2)(x_1 - x_2) + C_{1,2}(z_1, z_2)(y_1 - y_2) = \\ (x_1 - x_2)\left(1 + N(x_1, x_2) + C_{1,1}(z_1, z_2)\right) + C_{1,2}(z_1, z_2)(y_1 - y_2), \end{aligned}$$

and

$$\begin{aligned} f_2(z_1) - f_2(z_2) = \\ y_1 - y_2 - (y_1^3 - y_2^3) + C_{2,1}(z_1, z_2)(x_1 - x_2) + C_{2,2}(y_1 - y_2) = \\ (y_1 - y_2)\left(1 - N(y_1, y_2) + C_{2,2}(z_1, z_2))\right) + C_{2,1}(z_1, z_2)(x_1 - x_2) \end{aligned}$$

where

$$\begin{aligned} C_{j,1}(z_1, z_2) &= \int_0^1 \frac{\partial P_j}{\partial x}(z_2 + t(z_1 - z_2))dt \\ C_{j,2}(z_1, z_2) &= \int_0^1 \frac{\partial P_j}{\partial y}(z_2 + t(z_1 - z_2))dt. \end{aligned}$$

It is easy to see that

$$C_{j,i}(z_1, z_2) = O(r^3), \tag{119}$$

where $r = \max_{i=1,2} |x_i|, |y_i|$.

Hence there exists constants $D_k > 0$, for $k = 1, 2, \dots$, such that for $\|z_i\|_\infty \leq r$ holds

$$
\begin{aligned}
&(f_1(z_1) - f_2(z_2))^2 - (x_1 - x_2)^2 \geq \\
&(x_1 - x_2)^2 \left(\left(1 + \frac{r^2}{2} - D_1 r^3 \right)^2 - 1 \right) - D_2 r^3 |x_1 - x_2| \cdot |y_1 - y_2| \geq \\
&(x_1 - x_2)^2 D_3 r^2 - D_4 r^3 \left((x_1 - x_2)^2 + (y_1 - y_2)^2 \right) \geq \\
&(x_1 - x_2)^2 D_3 r^2 - D_5 r^5
\end{aligned}
$$

Observe that $D_3 \approx 1/2$.

Analogously for the second coordinate of f we obtain, for some positive constants H_i and r sufficiently small

$$
\begin{aligned}
&(y_1 - y_2)^2 - (f_2(z_1) - f_2(z_2))^2 \geq \\
&(y_1 - y_2)^2 \left(1 - (1 - \frac{r^2}{2} + H_1 r^3)^2 \right) - H_2 r^3 |x_1 - x_2| \cdot |y_1 - y_2| \geq \\
&(y_1 - y_2)^2 \left(1 - (1 - H_3 r^2)^2 \right) - H_2 r^3 |x_1 - x_2| \cdot |y_1 - y_2| \geq \\
&(y_1 - y_2)^2 H_4 r^2 - H_2 r^3 |x_1 - x_2| \cdot |y_1 - y_2| \geq \\
&(y_1 - y_2)^2 H_4 r^2 - H_5 r^5
\end{aligned}
$$

Observe that $H_4 \approx 1/2$.

Now we are ready to verify the cone condition

$$
\begin{aligned}
Q_{\alpha,\beta}(f(z_1) - f(z_2)) - Q(z_1 - z_2) = {} & \alpha \left((f_1(z_1) - f_1(z_2))^2 - (x_1 - x_2)^2 \right) + \\
& \beta \left((y_1 - y_2)^2 - (f_2(z_1) - f_2(z_2))^2 \right) \geq \\
& \alpha D_3 r^2 (x_1 - x_2)^2 - \alpha D_5 r^5 + \beta H_4 r^2 (y_1 - y_2)^2 - \beta H_5 r^5 \geq \\
& \min(\alpha D_3, \beta H_4) r^4 - (\alpha D_5 + \beta H_5) r^5 > 0
\end{aligned}
$$

for $r > 0$ sufficiently small. ∎

For $r > 0$ we define an h-set $N(r) \subset \mathbb{R}^2$ as follows: $u = s = 1$, $|N(r)| = [-r, r]^2$, $c_N(z) = \frac{z}{r}$.

Lemma 18 *For r sufficiently small $N(r) \overset{f}{\Longrightarrow} N(r)$.*

Proof: Since we have only one unstable direction, then from [GiZ, Thm. 16] it follows that it is enough to prove that

$$
f_1(r, y) > r, \qquad f_1(-r, y) < -r, \qquad \text{for } |y| \leq r \tag{120}
$$
$$
|f_2(x, y)| < r, \qquad \text{for } (x, y) \in N(r) \tag{121}
$$

Let r be such that, the following inequalities hold for any $(x, y) \in N(r)$

$$
|P_i(x, y)| < r^3, \quad i = 1, 2 \tag{122}
$$
$$
1 - 3y^2 + \frac{\partial P_2}{\partial y}(x, y) > 0. \tag{123}
$$

It is easy to see that (122) implies (120).

To prove (121) observe that from (123) it follows that $|f_2(x,y)|$ achieves its maximum value on $N(r)$ at $(x_0, \pm r)$. Condition (121) now follows immediately from (122). ∎

Proof of Theorem 16 Let us choose $\alpha = \beta = 1$. From the above lemmas it follows that we can take $N = N(r)$ for r sufficiently small. The statements about the existence and the cone conditions on $W_N^{u,s}(0,f)$ follow directly from Theorems 10 and 12.

The tangency of $W^{u,s}(0,f)$ to coordinate axes is obtained as in the proof of Theorem 14, because we have a freedom to choose any α and β (we may need to decrease further an r). ∎

7 Comparisons with other approaches

Usually in the literature discussing the stable manifold theorem there is not much stress on explicit bounds. But when one tries to establish the existence transversal homoclinic intersection this issue becomes very important. This issue was treated by Neumaier and Rage in [NR] for the standard map and Neumaier, Rage and Schlier in [RNS] for some hamiltonian ODE in 4D. However any detailed comparison of theirs method with the one advocated in this paper on the examples considered in papers [NR, RNS] is outside the scope of the present paper, mainly because in those papers the stress is on the propagation of the invariant manifolds and no explicit data about the size of the good neighborhood are given.

In paper [O] by Ombach the Peron-Irwin method was discussed with the stress on obtaining the possibly weakest conditions for the range of the existence of the stable and unstable manifold of the hyperbolic fixed point, as the graph of the function over stable and unstable subspace for the linearization.

Below we present some tests for two-dimensional map comparing the bounds for the stable manifold obtained using our method with the ones for the Perron-Irwin approach from paper by Ombach [O] and the version of Hartman approach from paper by Neumaier and Rage [NR]. This test shows that usually we can obtain bounds on larger set using our approach. But we should stress here that the real power of our approach is in the situation when we consider stable and unstable manifolds of periodic points of high period - see Section 4.1.

In this section we will follow the convention used in papers [O, NR] and order coordinates so that first coordinate x_s corresponds to stable directions and the second denoted by x_u is the unstable ones.

For the comparison we will use the following two-dimensional example is considered in [O]

$$F(x_s, x_u) = (F_s(x_s, x_u), F_u(x_s, x_u)) = (-0.4x_s + x_s^2 + x_u^2, 1.5x_u + x_u^3 - x_s^3). \quad (124)$$

It is easy to see that the origin point is hyperbolic fixed point for (124) with coordinate axes diagonalizing the linear part.

Let $\epsilon > 0$ and $\rho > 0$. We define $N = [-\rho, \rho] \times [-\epsilon\rho, \epsilon\rho]$. In tests reported below we will look for function $y_s : [-\rho, \rho] \to [-\epsilon\rho, \epsilon\rho]$, such that $W_N^s(0, F) = \{(x_s, y_s(x_s) \mid x_s \in [-\rho, \rho]\}$ and

$$|y_s(x_1) - y_s(x_2)| \leq L|x_1 - x_2|, \tag{125}$$

with the main objective of maximizing ρ and the secondary objective of minimizing L.

7.1 Estimates based on our method

We will treat N as an h-set with $s(N) = u(N) = 1$ with x_u being the unstable direction and x_s being the stable one. To have $N \overset{F}{\Longrightarrow} N$ it is enough to check the following conditions (see [GiZ, Thm. 16])

$$F_u(x_s, \epsilon\rho) > \epsilon\rho, \quad F_x(x_s, -\epsilon\rho) < -\epsilon\rho, \qquad \text{for all } |x_s| \leq \rho \tag{126}$$

$$|F_s(x_s, x_u)| < \rho, \qquad \text{for } (x_u, y_u) \in N. \tag{127}$$

Easy computations show that above conditions are equivalent to the set consisting from the following two conditions

$$(\epsilon \geq 1) \qquad \text{or} \qquad \rho^2 \leq \frac{\epsilon}{2(1-\epsilon^3)} \tag{128}$$

$$\rho \quad < \quad \frac{0.6}{1+\epsilon^2} \tag{129}$$

Case $\epsilon = 1$. Conditions (128) and (129) imply that $\rho < 0.3$.

To verify the cone condition for the quadratic form $Q(x_s, x_u) = x_u^2 - x_s^2$ according to Lemma 8 we have to check wether the interval matrix

$$V = [dF(N)]^T Q [dF(N)] - Q \tag{130}$$

is positive definite. A necessary and sufficient condition for this is positiveness of all main minors of V. Hence in our two-dimensional case we look for the largest ρ such that $V_{11} > 0$ and $\det(V) > 0$. This is a nonlinear condition on ρ, therefore we performed computer search for ρ. Using interval arithmetic we obtained $\rho = 0.21$, for which we have $V_{11} \subset [0.327, 1.034]$ and $\det(V) \subset [0.0377, 2.097]$.

Obviously in this case $L = 1$.

Case $\epsilon = 0.1$**.** In this case from conditions (128) and (129) we obtain that $\rho < 0.22371$.

We will try to find the quadratic form $Q(x_s, x_u) = \alpha x_u^2 - x_s^2$, where $\alpha > 0$, so that $N \overset{F}{\Longrightarrow} N$ satisfies the cone condition with respect to this form. In this case we will have the Lipschitz constant for $y_u(x_s)$ estimated by $\frac{1}{\sqrt{\alpha}}$. The goal is for a given ρ satisfying conditions (128) and (129) find the largest α so the matrix V given by (130) is positive definite. Below we list some results, the α's for which we tested the positive definiteness of V are 1 and the numbers of the form $100/2^n$ for $n = 0, \ldots, 6$.

Table 1: Computations for $\epsilon = 0.1$. α is the parameter in the quadratic form. ρ is the size of domain parameterizing the stable manifold for (124), L is the Lipschitz constant for this manifold parameterization. V_{11} and $\det(V)$ have to be positive for cone conditions to be satisfied.

ρ	α	L	V_{11}	$\det(V)$
0.22	6.25	0.4	[0.2944,1.16537]	[0.340752,9.19017]
0.21	6.25	0.4	[0.3276,1.1258]	[0.931013,8.86914]
0.2	12.5	0.2828	[0.36,1.18]	[0.408682,18.5656]
0.19	12.5	0.2828	[0.3916,1.14621]	[1.86745,18.0189]
0.18	12.5	0.2828	[0.4224,1.1165]	[3.1731,17.5381]
0.17	25	0.2	[0.4524,1.18432]	[3.3898,37.171]
0.16	25	0.2	[0.4816,1.14106]	[6.6133,35.7918]
0.15	50	0.1414	[0.51,1.21781]	[6.0094,76.3445]
0.14	50	0.1414	[0.5376,1.15847]	[13.9679,72.592]
0.13	100	0.1	[0.5644,1.23745]	[12.4131,155.001]

7.2 Estimates based on the Perron-Irwin method

First let us recall results from [O]. Consider a map

$$F(x_s, x_u) = (f_s(x_s, x_u), \mu x_u + g_u(x_s, x_u)) \tag{131}$$

where $(x_s, x_u) \in \mathbb{R}^s \times \mathbb{R}^u$ (in [O] they in fact belong to balls in Banach spaces), $\mu : \mathbb{R}^u \to \mathbb{R}^u$ is a linear expanding map and $F(0,0) = (0,0)$. On $\mathbb{R}^s \times \mathbb{R}^u$ we use the max-norm $\|(x_s, x_u)\| = \max\{\|x_s\|, \|x_u\|\}$.

Let $\rho > 0$ and $B = \overline{B}_s(0,\rho) \times \overline{B}_u(0,\rho)$. The conditions implying the existence of functions $y_s : \overline{B_s}(0,\rho) \to B_u(0,\rho)$, $y_u : \overline{B_u}(0,\rho) \to B_s(0,\rho)$, such that

$$\begin{aligned} W^s_B(0,F) &= \{(x_s, y_s(x_s)) \mid x_s \in B_s(0,\rho)\} \\ W^u_B(0,F) &= \{(y_u(x_u), x_u) \mid x_u \in B_u(0,\rho)\} \end{aligned}$$

are

$$a_s < 1, \quad (b_u + 1)\|\mu^{-1}\| < 1, \tag{132}$$

where

$$a_s = \sup\{\|Df_s(z)\| \mid z \in B\}, \quad b_u = \sup\{\|Dg_u(z)\| \mid z \in B\}. \tag{133}$$

Moreover, in this case $L = 1$.

We would like to stress here, the same condition is used to establish the existence of the graph of function being part of the (un)stable set, and then to prove that this graph is the whole local (un)stable set.

In [O] it is shown that conditions (132) for map (124) hold for $\rho < 0.15$. We will redo the computations from [O], but we add parameter ϵ in order to try to get better Lipschitz constant for $W^s(0)$. We introduce new coordinates by $(\bar{x}_s, \bar{x}_u) = (x_s, x_u/\epsilon)$. In these we coordinates (we drop bars) map F becomes

$$F(x_s, x_u) = (-0.4x_s + x_s^2 + \epsilon^2 x_u^2, 1.5x_u + \epsilon^2 x_u^3 - x_s^3/\epsilon,). \tag{134}$$

Easy computations show that

$$a_s = 0.4 + 2\rho + 2\epsilon^2\rho \tag{135}$$

$$\mu = 1.5, \quad b_u = 3\rho^2 \left(\frac{1}{\epsilon} + \epsilon^2\right). \tag{136}$$

Conditions (132) assume the following form

$$\rho < \frac{0.3}{1+\epsilon^2}, \quad \rho^2 < \frac{1}{6\left(\frac{1}{\epsilon} + \epsilon^2\right)}. \tag{137}$$

It is easy to see that for $\epsilon = 1$ we obtain $\rho < 0.15$ with $L = 1$ and for $\epsilon = 0.1$ we get $\rho < 0.129$ with $L = \epsilon = 0.1$.

Summarizing in the Perron-Irwin approach the number conditions to check (given by (132)) is smaller than in our approach, but in fact they turn out to be unnecessary strong.

7.3 The Neumaier and Rage approach

Let us start with recalling the Neumaier and Rage theorem from [NR, Thm. 1].

Theorem 19 *Let mapping $F : \mathbb{R}^n \supset \Omega \to \mathbb{R}^n$ be Lipschitz continuous with fixed point $x^* \in \Omega$, and let $A \subset \mathbb{R}^{n\times n}$ be interval matrix such that*

$$F(y) - F(x) \in A(y-x), \quad \textit{for all } x, y \in \Omega. \tag{138}$$

For some nonsingular matrix $Q \in \mathbb{R}^{n\times n}$, let

$$Q^{-1}(AQ) = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \tag{139}$$

with interval matrices B_{11}, B_{12}, B_{21} and B_{22} of sizes $p \times p$, $p \times q$, $q \times p$ and $q \times q$. respectively, where $n = p + q$. For some nonsingular matrix $C \in \mathbb{R}^{q\times q}$ and some interval matrix $L \subset \mathbb{R}^{q\times p}$, put

$$\begin{aligned} D &:= I + C(LB_{12} - B_{22}), && (140)\\ E &:= C(LB_{11} - B_{21}), && (141)\\ K &:= Q \begin{bmatrix} I \\ L \end{bmatrix}, && (142)\\ M &:= B_{11} + B_{12}L. && (143) \end{aligned}$$

If the closure conditions

$$\|D\|_p + \|C\|_q \cdot \|M\|_p \leq \beta < 1, \tag{144}$$

$$DL + E \subset L \tag{145}$$

hold for suitable norms $\|\cdot\|_p$ *in* $\mathbb{R}^p$ *and* $\|\cdot\|_q$ *in* $\mathbb{R}^q$*, then, for any subset* $\Sigma \subset \mathbb{R}^q$ *with*

$$0 \in \Sigma, Mt \subset \Sigma, \quad \textit{for } t \in \Sigma \tag{146}$$

$$x^* + Kt \subset \Omega, \quad \textit{for } t \in \Sigma \tag{147}$$

there are unique Lipschitz continuous functions $x : \Sigma \to \Omega$*,* $g : \Sigma \to \mathbb{R}^q$*, and* $\sigma : \Sigma \to \Sigma$ *such that*

$$F(x(t)) = x(\sigma(t)), \quad \textit{for } t \in \Sigma, \tag{148}$$

$$x(t) = x^+ + Q \begin{bmatrix} t \\ g(t) \end{bmatrix}, \quad \textit{for } t \in \Sigma, \tag{149}$$

$$x(0) = x^*, \ x(s) - x(t) \in K(s-t), \quad \textit{for } s,t \in \Sigma \tag{150}$$

$$g(0) = 0, \ g(s) - g(t) \in L(s-t), \quad \textit{for } s,t \in \Sigma \tag{151}$$

$$\sigma(0) = 0, \ \sigma(s) - \sigma(t) \in M(s-t), \quad \textit{for } s,t \in \Sigma \tag{152}$$

Some comments are necessary to elucidate the meaning of the above theorem. x^* is an hyperbolic fixed point with p-dimensional unstable manifold and q-dimensional stable one. For $F \in C^1$ matrix A is the interval enclosure for $df(x)$ for $x \in \Omega$. The function $x : \Sigma \to \Omega$ parameterizes the stable manifold of x^*. The matrix Q is the coordinate change diagonalizing (approximately) $dF(x^*)$. In this new coordinates the stable manifold of x^* is as graph of the function g. Condition (146) implies that B_{11} is a contraction. Conditions (150,151) are 'the cone conditions' satisfied by $W^s(x^*)$ and the Lipschitz constant for W^s is given by $\|L\|$.

Let us apply to example (124). In this case $x^* = 0$ and $Q = I$. We take $\Omega = N = [-\rho, \rho] \times [-\epsilon\rho, \epsilon\rho]$.

We have

$$A = [dF(\Omega)] = \begin{bmatrix} B_{11} = -0.4 + 2\rho[-1,1] & B_{12} = 2\epsilon\rho \\ B_{21} = -3[0,\rho^2] & B_{22} = 1.5 + 3\epsilon^2\rho^2[0,1] \end{bmatrix} \tag{153}$$

where $[dF(\Omega)]$ is the interval enclosure according to Def. 13. As suggested in [NR] we chose

$$C = \frac{2}{3} \approx B_{22}^{-1}, \quad L = [-\epsilon, \epsilon]. \tag{154}$$

This means that if assumptions of above theorem are satisfied, then ϵ is the Lipschitz constant from (125).

We compute

$$
\begin{aligned}
D &= 1 + C(LB_{12} - B_{22}) = -\epsilon^2\rho^2 + \left(\frac{4}{3}\epsilon^2\rho + \epsilon^2\rho^2\right)[-1,1] \\
E &= C(LB_{11} - B_{21}) = \rho^2 + \left(\frac{4\epsilon}{15} + \frac{4\rho\epsilon}{3} + \rho^2\right)[-1,1] \\
K &= \begin{bmatrix} 1 \\ [-\epsilon,\epsilon] \end{bmatrix} \\
M &= B_{11} + B_{12}L = -0.4 + (2\rho + 2\epsilon^2\rho)[-1,1], \\
|D| + |C|\cdot|M| &= 2\epsilon^2\rho^2 + \frac{4}{3}\epsilon^2\rho + \frac{2}{3}\left(0.4 + 2\rho + 2\epsilon^2\rho\right) = \\
&\quad \frac{4}{15} + 2\epsilon^2\rho^2 + \frac{8}{3}\epsilon^2\rho + \frac{4}{3}\rho
\end{aligned}
$$

From the above computations it follows that condition (144) is equivalent to condition

$$
2\epsilon^2\rho^2 + \frac{8}{3}\epsilon^2\rho + \frac{4}{3}\rho \leq \frac{11}{15}. \tag{155}
$$

It is easy to see that (145) that is equivalent to the following condition

$$
2\rho^2 + 2\epsilon^3\rho^2 + \frac{4}{3}\epsilon^2\rho + \frac{4}{3}\rho\epsilon \leq \frac{11}{15}\epsilon \tag{156}
$$

The condition (146) is in our case just $|M| \leq 1$ and leads to

$$
\rho + \epsilon^2\rho \leq 0.3. \tag{157}
$$

Observe that condition (147) is automatically satisfied due to our initial choices of Ω and L.

Therefore we have to satisfy three inequalities (155–157) to apply Theorem 19.

We will consider two cases $\epsilon = 1$ and $\epsilon = 0.1$.

Case $\epsilon = 1$**.** Condition (157) implies that $\rho \leq 0.15$ and it is easy to check that the remaining inequalities also hold for $\rho = 0.15$ and the Lipschitz constant is 1. We see here that we obtained considerably better result using our method in this case.

Case $\epsilon = 0.1$. In this case it turns out that condition (156) imposes that maximal possible ρ belong to the interval $(0.15, 0.16)$ and $L = 0.1$. Observe that in this case we obtained results on larger domain than for $\epsilon = 1.0$. This is not a paradox, because it turned out that our set Ω used it this setting was smaller in x_s-direction (which resulted in better bounds), but it happened that it contained the whole local stable set. Now let us compare this result with Table 1 summarizing the bounds obtained by using covering relations. We see that using our method we can obtain larger ρ (by a factor of 1.5) , but at the price of larger Lipschitz constant. For the value of ρ for which Neumaier-rage method works we get $L \approx 0.1414 > 0.1$. This also suggest that probably the cone conditions from Definition 11 are probably too strong.

8 Dependence on parameters of invariant manifolds of hyperbolic fixed point

8.1 Continuous dependence

Theorem 20 *Let $\Lambda \subset \mathbb{R}^k$ and $V \subset \mathbb{R}^n$ be open sets. Assume that $f : \Lambda \times V \to \mathbb{R}^n$, where $\Lambda \subset \mathbb{R}^k$ be such that*

- $\forall \lambda \in \Lambda$ f_λ *is* C^1
- f *and* $\frac{\partial f}{\partial z}$ *are continuous on* $\Lambda \times \mathbb{R}^n$
- z_0 *is a hyperbolic fixed point of* f_{λ_0}.

Then there exist sets $C \subset \Lambda$ and $U \subset V$, such $(\lambda_0, z_0) \in \mathrm{int}(C \times U)$ and a continuous function $p : C \to U$, such that $p(\lambda)$ is a hyperbolic fixed point for f_λ, $p(\lambda_0) = z_0$ and $W^{u,s}_U(p(\lambda), f_\lambda)$ depend continuously on λ, for $\lambda \in C$.

The continuity of sets $W^{u,s}(p(\lambda), f_\lambda)$ with respect to $\lambda \in C$ means that they are given as graphs of some functions depending continuously on λ.

Proof: The existence of $p(\lambda)$ follows immediately from the implicit function theorem.

By proceeding as in the proof of Theorem 14, namely by using the diagonalizing coordinates for $\frac{\partial f_{\lambda_0}}{\partial z}(z_0)$ we can construct arbitrarily small h-set with cones (N, Q), $N = N(r)$, such that

$$N \overset{f_{\lambda_0}}{\Longrightarrow} N \tag{158}$$

and the interval quadratic form given by

$$V = [df_{\lambda_0,c}]^T Q [df_{\lambda_0,c}] - Q \tag{159}$$

is positive definite.

Observe that conditions (158,159) are both stable with respect to small change of map f_{λ_0}, therefore there exists a set $C \subset \Lambda$, such that $\lambda_0 \in \mathrm{int} C$ and

$$N \overset{f_\lambda}{\Longrightarrow} N \tag{160}$$

and the interval quadratic form given by

$$V = [df_{\lambda,c}]^T Q [df_{\lambda,c}] - Q \tag{161}$$

is positive definite.

Theorems 10 and 12 imply that $W^{u,s}(p(\lambda), f_\lambda)$ are horizontal or vertical disks in N, respectively.

It remains to prove the continuity of $W^{u,s}_N(p(\lambda), f_\lambda)$. From now on we will use the coordinates given by the h-set N.

Let us first consider the case of the stable manifold. From the previous reasoning it follows that there exists a function $x : C \times \overline{B}_s \to \overline{B}_u$, such that

$$z \in W^s_N(p(\lambda), f_\lambda)) \quad \text{iff} \quad z = (x(\lambda, y), y), \quad \text{for some } y \in \overline{B_s}. \tag{162}$$

We need to prove that the function $x(\lambda, x)$ is continuous with respect to both arguments. Let $(\lambda_k, y_k) \in C \times \overline{B_s}$ for $k \in \mathbb{N}$ be a sequence converging to $(\bar{\lambda}, \bar{y}) \in C \times \overline{B_s}$. Due to compactness of the range of function $x(\lambda, y)$ it is enough to show for any subsequence of $\{(\lambda_{k_i}, y_{k_i})\}$, such that $x(\lambda_{k_i}, y_{k_i})$ converges to some u, must hold that

$$(u, y) \in W^s_N(p(\lambda, f_\lambda)), \tag{163}$$

which by (162) implies that $u = x(\bar{\lambda}, \bar{y})$.

To obtain (163) observe that by passing to the limit we obtain that $f^l_{\bar{\lambda}}(u, y) \in N$ for all $l \in \mathbb{N}$. Therefore $(u, y) \in \text{Inv}^+(N, f_{\bar{\lambda}})$. From Lemma 9 it follows that (163) holds.

Now we treat the continuity of unstable manifolds. Observe first that since we don't have the invertibility of f_λ we cannot just apply the proof for the stable manifold to f^{-1}_λ.

We know that there exists a function $y : C \times \overline{B}_u \to \overline{B}_s$, such that

$$z \in W^u_N(p(\lambda), f_\lambda)) \quad \text{iff} \quad z = (x, y(\lambda, x)), \quad \text{for some } x \in \overline{B}_u. \tag{164}$$

It is enough to prove that the function $y(\lambda, x)$ is continuous with respect to both arguments. Let $(\lambda_k, x_k) \in C \times \overline{B}_u$ for $k \in \mathbb{N}$ be a sequence converging to $(\bar{\lambda}, \bar{x}) \in C \times \overline{B}_u$. Let us define $\bar{y} = y(\bar{\lambda}, \bar{x})$, $z_k = (x_k, y(\lambda_k, x_k))$.

Consider the sequence $y_k = y(\lambda_k, x_k)$ we need to show that $\lim_{k\to\infty} y_k = \bar{y}$. Observe that $y_k \in \overline{B}_s$, hence we can pick up convergent subsequences. The proof will be completed, when we show that any convergent subsequence of $\{y_k\}$ converges to $\bar{y}$.

Let $\{y_{k_n}\}$ be a subsequence of $\{y\}$ convergent to u_0. For each n there is full backward orbit of $f_{\lambda_{k_n}}$ in N through (x_{k_n}, y_{k_n}). Let us denote it by $z^l_{k_n}$ for $l \in \mathbb{Z}_-$. This means that

$$f_{\lambda_{k_n}}(z^l_{k_n}) = z^{l+1}_{k_n}, \quad l = 0, -1, -2, \ldots \qquad z^0_{k_n} = (x_{k_n}, y_{k_n}). \tag{165}$$

From the sequence $z^{-1}_{k_n}$ we can pick up a subsequence convergent to $\bar{z}^{-1}$. From the continuity f it follows that

$$f_{\bar{\lambda}}(\bar{z}^{-1}) = \bar{z} = (\bar{x}, u_0). \tag{166}$$

From this subsequence we can further pickup convergent subsequences to obtain a full backward orbit for map f_λ in N for the point $\bar{z}$.

Therefore $\bar{z} \in \text{Inv}^-(N, f_{\bar{\lambda}})$. From Lemma 9 it follows that $\bar{z} \in W^u_N(p(\bar{\lambda}), f_{\bar{\lambda}})$. Now from (164) it follows that $u_0 = \bar{y}$.

∎

8.2 The Lipschitz dependence of invariant manifolds of a hyperbolic fixed point on parameters

The goal of this subsection is to improve Theorem 20. Namely, we want to show that if the dependence on parameters is Lipschitz, then also the stable and unstable manifolds depend in the Lipschitz way on parameters. The theorem below does not contain in its statements a precise formula for the Lipschitz constant with respect to the parameter, but it can be quite easily inferred from the proof. We believe that this kind of estimates will allow to effectively implement computer assisted proofs of the existence of the homoclinic tangency for low dimensional ODEs depending on parameters.

Theorem 21 *The same assumptions as in Theorem 20 and we additionally assume that f is locally Lipschitz with respect to λ. By this we understand the following statement:*
for any compact set $C \times V \subset \Lambda \times \mathbb{R}^n$, there exists L, such that for any $\lambda \in C$ and $z \in V$ holds

$$\|f_{\lambda_1}(z) - f_{\lambda_2}(z)\| \leq L\|\lambda_1 - \lambda_2\|.$$

Then there exists sets $C \subset \Lambda$ and $U \subset V$, such $(\lambda_0, z_0) \in \operatorname{int}(C \times U)$ and a continuous function $p : C \to U$, such that $p(\lambda)$ is a hyperbolic fixed point for f_λ, $p(\lambda_0) = z_0$ and $W^s_U(p(\lambda), f_\lambda)$ depend in a Lipschitz way on λ, for $\lambda \in C$.

The Lipschitz dependence of set $W^s(p(\lambda), f_\lambda)$ with respect to $\lambda \in C$ means that it is given as a graph of some function, which satisfies the Lipschitz condition with respect to λ.

If we additionally assume that $\frac{\partial f}{\partial z}(\lambda_0, z_0)$ is invertible and the dependence of f^{-1}_λ on λ is locally Lipschitz, then the same statement is valid also for $W^u_U(p(\lambda), f_\lambda)$.

Proof: We will provide the proof for the stable manifold, only. The unstable case is obtained by considering the inverse map.

Let (N, Q) be an h-set with cones as in the proof of Theorem 20, we also assume that we use the coordinates given by h-set N. Let $C \subset \Lambda$ be as in the proof of Theorem 20.

We have a continuous function $x : C \times \overline{B}_s \to \overline{B}_u$, such

$$z \in W^s_N(p(\lambda), f_\lambda)) \quad \text{iff} \quad z = (x(\lambda, y), y), \quad \text{for some } y \in \overline{B_s}. \tag{167}$$

Moreover, from (161) if follows that there exists a constant $A > 0$, such that for $z_1, z_2 \in N$ holds

$$Q(f_\lambda(z_1) - f_\lambda(z_2)) - Q(z_1 - z_2) \geq A\|z_1 - z_2\|^2 \tag{168}$$

In fact since the positive definiteness is an open condition, it follows that for some ϵ sufficiently small holds a stronger form of (168). Namely, we have

$$Q(f_\lambda(z_1) - f_\lambda(z_2)) - (1 + \epsilon)Q(z_1 - z_2) \geq A\|z_1 - z_2\|^2 \tag{169}$$

Let us fix $\epsilon > 0$, such that (169) holds.

Let B be the bilinear form associated with Q, i.e. $Q(z) = B(z, z)$.

Observe that for $\lambda_1, \lambda_2 \in C$ and $z_1, z_2 \in N$ holds

$$\begin{aligned} Q(f_{\lambda_1}(z_1) - f_{\lambda_2}(z_2)) - (1+\epsilon)Q(z_1 - z_2) = \\ Q(f_{\lambda_1}(z_1) - f_{\lambda_1}(z_2)) - (1+\epsilon)Q(z_1 - z_2) + \\ 2B(f_{\lambda_1}(z_1) - f_{\lambda_1}(z_2), f_{\lambda_1}(z_2) - f_{\lambda_2}(z_2)) + Q(f_{\lambda_1}(z_2) - f_{\lambda_2}(z_2)) \geq \\ A\|z_1 - z_2\|^2 - 2M\|z_1 - z_2\| \cdot \|\lambda_1 - \lambda_2\| - cL^2\|\lambda_1 - \lambda_2\|^2 \end{aligned}$$

where

$$\begin{aligned} M &= \|B\| \cdot L \cdot \sup_{(\lambda,z)\in C\times N} \left\| \frac{\partial f_\lambda}{\partial z} \right\| \\ \beta(y) &\leq c\|y\|^2, \quad y \in B_s. \end{aligned}$$

In the above formula $\|B\|$ is the norm of the bilinear form B and β is the negative part of Q.

We want to show that if

$$\|\lambda_1 - \lambda_2\| \leq \Gamma\|z_1 - z_2\| \tag{170}$$

with some Γ is small enough, then

$$A\|z_1 - z_2\|^2 - 2M\|z_1 - z_2\| \cdot \|\lambda_1 - \lambda_2\| - cL^2\|\lambda_1 - \lambda_2\|^2 > 0. \tag{171}$$

Observe that (171) is implied by the following inequality

$$A\|z_1 - z_2\|^2 - 2M\Gamma\|z_1 - z_2\|^2 - cL^2\Gamma^2\|z_1 - z_2\|^2 > 0, \tag{172}$$

which is satisfied for Γ small enough. Let us fix such Γ.

We have proved that, if $\|\lambda_1 - \lambda_2\| \leq \Gamma\|z_1 - z_2\|$, then

$$Q(f_{\lambda_1}(z_1) - f_{\lambda_2}(z_2)) > (1+\epsilon)Q(z_1 - z_2). \tag{173}$$

We would like to infer from (173) that

$$Q(f^n_{\lambda_1}(z_1) - f^n_{\lambda_2}(z_2)) > (1+\epsilon)^n Q(z_1 - z_2), \tag{174}$$

but the condition (173) does not imply that $\|\lambda_1 - \lambda_2\| \leq \Gamma\|f_{\lambda_1}(z_1) - f_{\lambda_2}(z_2)\|$, therefore we cannot iterate (173).

To fix this we will use a different condition. For $\delta > 0$ we define a set $G(\delta)$ by

$$G(\delta) = \{((\lambda_1, z_1), (\lambda_2, z_2)) \in (C \times N)^2 \mid \|\lambda_1 - \lambda_2\|^2 \leq \delta Q(z_1 - z_2)\}. \tag{175}$$

Observe that if $((\lambda_1, z_1), (\lambda_2, z_2)) \in G(\delta)$, then

$$\|\lambda_1 - \lambda_2\|^2 \leq \delta Q(z_1 - z_2) \leq \delta\alpha(x_1 - x_2) \leq \delta D\|z_1 - z_2\|^2,$$

where D is a constant satisfying

$$\alpha(\pi_x z) \leq D\|z\|^2. \tag{176}$$

We set $\delta = \Gamma^2/D$.

Observe that, if $((\lambda_1, z_1), (\lambda_2, z_2)) \in G(\delta)$, then

$$Q(f_{\lambda_1}(z_1) - f_{\lambda_2}(z_2)) > (1+\epsilon)Q(z_1 - z_2) \tag{177}$$

and if $f_{\lambda_1}(z_1) \in N$ and $f_{\lambda_2}(z_2) \in N$, then

$$((\lambda_1, f_{\lambda_1}(z_1)), (\lambda_2, f_{\lambda_2}(z_2))) \in G(\delta). \tag{178}$$

Therefore by the induction argument we obtain the following

Lemma 22 *Let $((\lambda_1, z_1), (\lambda_2, z_2)) \in G(\delta)$ be such that for $j = 1, \ldots, n$ and $i = 1, 2$*

$$f^j_{\lambda_i}(z_i) \in N.$$

Then for $j = 1, \ldots, n$

$$\Big((\lambda_1, f^j_{\lambda_1}(z_1)), (\lambda_2, f^j_{\lambda_2}(z_2))\Big) \quad \in \quad G(\delta), \tag{179}$$

$$Q(f^j_{\lambda_1}(z_1) - f^j_{\lambda_2}(z_2)) \quad > \quad (1+\epsilon)^j Q(z_1 - z_2). \tag{180}$$

Lemma 23 *Let $\lambda_1 \neq \lambda_2$, $\lambda_i \in C$. Let $z_i = (x(\lambda_i, y), y) \in W^s(p(\lambda_i), f_{\lambda_i})$ for $i = 1, 2$. Then*

$$\|\lambda_1 - \lambda_2\|^2 > \delta Q(z_1 - z_2). \tag{181}$$

Proof: Assume that (181) is not satisfied for some pair $((\lambda_1, z_1), (\lambda_2, z_2))$. Let us fix this pair for the remainder of the proof. We have

$$\|\lambda_1 - \lambda_2\|^2 \leq \delta Q(z_1 - z_2), \tag{182}$$

therefore $((\lambda_1, z_1), (\lambda_2, z_2)) \in G(\delta)$ and $x(\lambda_1, y) \neq x(\lambda_2, y)$. Observe that by the definition of z_i, $f^j_{\lambda_i}(z_i) \in N$ for all j positive. From Lemma 22 it follows that for all $j > 0$

$$Q(f^j_{\lambda_1}(z_1) - f^j_{\lambda_2}(z_2)) > (1+\epsilon)^j Q(z_1 - z_2) \geq (1+\epsilon)^j \alpha(x(\lambda_1, y) - x(\lambda_2, y)) \tag{183}$$

Let us consider the limit $j \to \infty$. We have

$$Q(f^j_{\lambda_1}(z_1) - f^j_{\lambda_2}(z_2)) \to Q(p(\lambda_1) - p(\lambda_2))$$
$$(1+\epsilon)^j \alpha(x(\lambda_1, y) - x(\lambda_2, y)) \to \infty.$$

We obtain a contradiction. Therefore condition (181) is satisfied. ∎

Conclusion of the proof of Theorem 21: From Lemma 23 it follows that

$$\|\lambda_1 - \lambda_2\|^2 > \delta Q((x(\lambda_1, y), y) - (x(\lambda_2, y), y)) \geq$$
$$\delta\alpha(x(\lambda_1, y) - x(\lambda_2, y)) \geq \Gamma a \|x(\lambda_1, y) - x(\lambda_2, y)\|^2$$

where $a > 0$ is such that

$$\alpha(x) \geq a\|x\|^2. \tag{184}$$

Finally, we obtain

$$\|x(\lambda_1, y) - x(\lambda_2, y)\| < \frac{1}{\delta a}\|\lambda_1 - \lambda_2\|. \tag{185}$$

∎

9 Analyticity of (un)stable manifolds for analytic maps

The goal of this section is improve results from Section 5 and to prove that when the map f under consideration is real-analytic and x_0 is a hyperbolic fixed point, then the local stable and unstable manifolds of x_0 are real-analytic manifolds.

Theorem 24 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a real-analytic map. Assume that z_0 is a hyperbolic fixed point of f.*

Let $Z \subset \mathbb{R}^n$ be an open set, such that $z_0 \in Z$.

Then there exists an h-set N with cones, such that $z_0 \in \mathrm{int} N$, $N \subset Z$ and

- *$N \overset{f}{\Longrightarrow} N$ and if f is local diffeomorphism in the neighborhood of z_0 then $N \overset{f}{\Longleftarrow} N$,*
- *$W^u_N(z_0)$ is a horizontal disk in N satisfying the cone condition*
- *$W^s_N(z_0)$ is a vertical disk in N satisfying the cone condition.*

Moreover, $W^u_N(z_0)$ can be represented as a graph of an real-analytic function over the unstable space for the linearization of f at z_0 and tangent to it at z_0. Analogous statement is also valid for $W^s_N(z_0)$.

Proof: The bulleted part of the assertion follows directly from Theorem 14, but we will need to alter the proof of this theorem in order to get the analyticity. We will follow the proof of Theorem 14 to construct complex version of N, covering relation $N \overset{f}{\Longrightarrow} N$ and cone condition on N. Then we will start redoing proofs of Theorems 10 and 12, for stable and unstable manifolds, respectively, modifying them using observations made in Remarks 11 and 13.

Let u and s be the (real) dimensions of unstable and stable manifolds of $df(z_0)$. We start as in the proof of Theorem 14 with the linear (hence real-analytic) coordinate change. From now on we will work in these coordinates in $\mathbb{C}^n$. We will now complexify the construction of the h-set with cones (N, Q) from the proof of Theorem 14.

On $\mathbb{C}^u$ and $\mathbb{C}^s$ we will use euclidian norms and the scalar product given by $(w|v) = \bar{w}v$, we set

$$\begin{aligned} N^{\mathbb{C}}(r) &= \{z_0\} + \overline{B}^{\mathbb{C}}_u(0, r) \times \overline{B}^{\mathbb{C}}_s(0, r) && (186) \\ Q^{\mathbb{C}}(x, y) &= \alpha\bar{x}x - \beta\bar{y}y, \qquad \alpha, \beta \in \mathbb{R}, \alpha > 0, \beta > 0. && (187) \end{aligned}$$

where

$$B_n^{\mathbb{C}}(0,\rho) = \{x \in \mathbb{C}^n \mid \|x\| \le \rho\}.$$

In the sequel we will also use the following notation $B_n^{\mathbb{C}} = B_n^{\mathbb{C}}(0,1)$.

We will treat $N^{\mathbb{C}}(r)$ as an (real) h-set, with $s(N^{\mathbb{C}}(r)) = 2s$ and $u(N^{\mathbb{C}}(r)) = 2u$ and the map $c_{N^{\mathbb{C}}(r)}$ being the complexification of $c_{N(r)}$. With these conventions by proceeding as in the proof of Theorem 14 we obtain $r_0 > 0$ and $\epsilon > 0$, such that for any $\delta \in [-\epsilon, \epsilon]$

$$
\begin{aligned}
N^{\mathbb{C}}(r_0) &\overset{f}{\Longrightarrow} N^{\mathbb{C}}(r_0) &\quad (188)\\
Q^{\mathbb{C}}(f(z_1) - f(z_2)) &> (1+\delta)Q^{\mathbb{C}}(z_1 - z_2), \quad z_1 \neq z_2 \in N^{\mathbb{C}}(r_0), &\quad (189)\\
Q^{\mathbb{C}}(df_c(z)v) &> (1+\delta)Q^{\mathbb{C}}(v), \quad z \in N^{\mathbb{C}}(r_0), v \in \mathbb{C}^n \setminus \{0\}. &\quad (190)
\end{aligned}
$$

Now following the line of the proof of Theorem 14 we will redo proofs of Theorems 10 and 12, to obtain the analyticity of stable and instable manifolds, respectively.

To simplify the notation we set $N = N^{\mathbb{C}}(r_0)$, $Q = Q^{\mathbb{C}}$ and assume by passing to coordinates given by c_N that $z_0 = 0$, $N = \overline{B}_u^{\mathbb{C}} \times \overline{B}_s^{\mathbb{C}}$ and $f = f_c$.

Let us first focus on the stable manifold. As was suggested in Remark 11 we consider for any $l > 0$ function $x_l : \overline{B}_s^{\mathbb{C}} \to \overline{B}_u^{\mathbb{C}}$ defined for $l = 1, 2, \ldots$ by implicit equation

$$\pi_x f^l(x_l(y), y) = 0. \qquad (191)$$

and under the constraint

$$f^i(x_l(y), y) \in N, \qquad i = 0, \ldots, l-1. \qquad (192)$$

Observe that the existence for a given $y \in \overline{B}_s^{\mathbb{C}}$ of $x \in \overline{B}_u^{\mathbb{C}}$, such that

$$\pi_x f^l(x,y) = 0 \qquad \text{and} \qquad f^i(x,y) \in N, \quad i = 0, \ldots, l-1 \qquad (193)$$

follows directly from Theorem 4 applied to the chain of covering relations $N \overset{f}{\Longrightarrow} N \overset{f}{\Longrightarrow} \ldots N$ of length l, horizontal disk $d_y(x) = (x,y)$ and vertical disk $b_0(y) = (0,y)$. We will show now that such x is unique. Let $x_1, x_2 \in \overline{B}_u^{\mathbb{C}}$, such that $x_1 \neq x_2$ and $f^i(x_j, y) \in N$ for $i = 0, \ldots, l-1$. Then from (189) it follows that

$$
\begin{aligned}
\alpha\|\pi_x f^l(x_1,y) - \pi_x f^l(x_2,y)\|^2 \ge Q(f^l(x_1,y) - f^l(x_2,y)) \ge \\
Q((x_1,y) - (x_2,y)) = \alpha\|x_1 - x_2\|^2 > 0.
\end{aligned}
$$

Hence we have well defined function $x_l : \overline{B}_s^{\mathbb{C}} \to \overline{B}_u^{\mathbb{C}}$ satisfying conditions (191,192). We would like to use the implicit function theorem to prove that x_l is analytic. For this it is enough to show, that $\frac{\partial}{\partial x}\pi_x f^l(x,y)$ is an isomorphism for $(x,y) \in N$ satisfying $f^i(x,y) \in N$ for $i = 1, \ldots, l-1$. To obtain this we show that $\frac{\partial}{\partial x}\pi_x f^l(x,y) \cdot v \neq 0$ for any $v \in \mathbb{C}^u \setminus \{0\}$. Namely, from (190) it follows that

$$
\begin{aligned}
\alpha \left\| \frac{\partial}{\partial x}\pi_x f^l(x,y) \cdot v \right\|^2 = \alpha\|\pi_x(df^l(x,y)\cdot(v,0))\|^2 \ge \\
Q(df^l(x,y)\cdot(v,0)) > Q((v,0)) = \alpha\|v\|^2 > 0.
\end{aligned}
$$

From the implicit function theorem (over complex field) we obtain $x_l : \overline{B}_s^{\mathbb{C}} \to \overline{B}_u^{\mathbb{C}}$, a family of analytic functions, which are of course real on $\mathbb{R}^s$. Now we will show that they converge uniformly.

From (189) it follows that for any $l, m > 0$ holds

$$Q(f^l(x_{l+m}(y), y) - f^l(x_l(y), y)) \geq (1+\epsilon)^l Q((x_{l+m}(y) - x_l(y), 0)) = \\ (1+\epsilon)^l \alpha \|x_{l+m}(y) - x_l(y)\|^2.$$

But $f^l(x_{l+m}(y), y) \in N$ for $m \geq 0$ and Q is continuous, hence expression the right hand side of the inequality is bounded by $M = \max_{z_1, z_2 \in N} Q(z_1 - z_2)$. Therefore we obtain

$$\|x_{l+m}(y) - x_l(y)\|^2 \leq \alpha^{-1}(1+\epsilon)^{-l} M. \tag{194}$$

Hence sequence x_l satisfies the Cauchy condition, therefore it is uniformly convergent to an analytic function $x^* : \overline{B}_s^{\mathbb{C}} \to \overline{B}_u^{\mathbb{C}}$. From condition (192) it follows immediately that for any $y \in \overline{B}_s^{\mathbb{C}}$ the point $(x^*(y), y) \in \mathrm{Inv}^+(N, f)$ and we continue as in the proof of Theorem 10.

Now we treat the unstable manifold. Following Remark 13 we will investigate the convergence of functions $y_l : \overline{B}_u^{\mathbb{C}} \to \overline{B}_u^{\mathbb{C}}$ defined by the following conditions

$$\begin{aligned} \pi_x f^l(x_l(x), 0) &= x \quad &(195)\\ f^i(x_l(x), 0) &\in N, \quad \text{for } i = 0, \ldots, l \quad &(196)\\ y_l(x) &= \pi_y f^l(x_l, 0). \quad &(197) \end{aligned}$$

The existence of a point $x_l(x)$ satisfying (195,196) follows immediately from Theorem 4 applied to chain of covering relations $N \overset{f}{\Longrightarrow} N \overset{f}{\Longrightarrow} N\cdot \overset{f}{\Longrightarrow} N$ of lenght l, horizontal disk $b_0(z) = (z, 0)$ and vertical disk $b_e(y) = (y, x)$. The uniqueness is obtained as follows: for $x_1 \neq x_2$ and such that $f^i(x_j, 0) \in N$ for $j = 1, 2$ and $i = 1, \ldots, l-1$ holds

$$\alpha \|\pi_x f^l(x_1, 0) - \pi_x f^l(x_2, 0)\|^2 \geq \\ Q(f^l(x_1, 0) - f^l(x_2, 0)) > Q((x_1 - x_2), 0) = \alpha \|x_1 - x_2\|^2 > 0,$$

which proves that $x_l(x)$ is uniquely defined. We have already shown, when discussing the stable manifold, that $\frac{\partial}{\partial} \pi_x f^l(x, 0)$ is an isomorphism for $x \in \overline{B}_u^{C}$, such that $f^i(x, 0) \in N$ for $i = 1, \ldots, l$. Therefore x_l and also y_l are analytic function (real on B^u).

Now we prove that y_l converges uniformly. Let $l, m > 0$. We have

$$0 \geq -\beta \|y_l(x) - y_{l+m}(x)\|^2 = Q((x, y_l(x)) - (x, y_{l+m}(x))) = \\ Q(f^l(x_l(x), 0) - f^l(x_{l+m}(x), 0)) \geq (1-\epsilon)^l Q((x_l, 0) - f^m(x_{l+m}(x), 0))$$

and we obtain

$$\|y_l(x) - y_{l+m}(x)\|^2 \leq \beta^{-1}(1-\epsilon)^l \max_{z_1, z_2 \in N} Q(z_1 - z_2). \tag{198}$$

Therefore y_l is a Cauchy sequence converging to an analytic function $y^* : \overline{B}_u^{\mathbb{C}} \to \overline{B}_u^{\mathbb{C}}$. It is easy to see that $y^*(x) = w_0$, where w_0 is defined in (79) in the proof of Theorem 12. We continue with the proof as in Theorem 12.

∎

By combination of the reasoning contained in the proof of Theorem 20 (covering relations and cone conditions are stable with respect to C^1-perturbations) with the the proof of Theorem 24 (the stable and unstable manifolds are defined by limits of uniformly converging analytic functions) one can easily obtain the following result

Theorem 25 *Let $\Lambda \subset \mathbb{R}^k$ and $V \subset \mathbb{R}^n$ be open sets. Assume that $f : \Lambda \times V \to \mathbb{R}^n$, where $\Lambda \subset \mathbb{R}^k$, is real analytic and z_0 is a hyperbolic fixed point of f_{λ_0}.*

Then there exist sets $C \subset \Lambda$ and $U \subset V$, such $(\lambda_0, z_0) \in \mathrm{int}(C \times U)$ and an analytic function $p : C \to U$, such that $p(\lambda)$ is a hyperbolic fixed point for f_λ, $p(\lambda_0) = z_0$ and $W_U^{u,s}(p(\lambda), f_\lambda)$ depend analytically on λ, for $\lambda \in C$, which means that in suitable coordinates holds

$$W^s(p(\lambda), f_\lambda) = \{p(\lambda) + (x(\lambda, y), y) \mid y \in B_s(0, \rho_1)\} \tag{199}$$

$$W^u(p(\lambda), f_\lambda) = \{p(\lambda) + (x, y(\lambda, x)) \mid x \in B_u(0, \rho_2)\} \tag{200}$$

where $x(\lambda, y)$ and $y(\lambda, x)$ are real analytic functions.

10 The stable and unstable manifolds of hyperbolic fixed points for ODEs.

Consider an ordinary differential equation

$$z' = f(z), \qquad z \in \mathbb{R}^n, \quad f \in C^2(\mathbb{R}^n, \mathbb{R}^n). \tag{201}$$

Let us denote by $\varphi(t, p)$ the solution of (201) with the initial condition $z(0) = p$. For any $t \in \mathbb{R}$ by we define a map $\varphi(t, \cdot) : \mathbb{R}^n \to \mathbb{R}^n$ by $\varphi(t, \cdot)(x) = \varphi(t, x)$. We ignore here the question whether $\varphi(t, x)$ is defined for every (t, x), but this can be achieved by modification of f outside a large ball.

Definition 19 *Let $z_0 \in \mathbb{R}^n$. We say that z_0 is a* hyperbolic fixed point for equation (201) *iff $f(z_0) = 0$ and $Re\lambda \neq 0$ for all $\lambda \in Sp(df(z_0))$, where $Df(z_0)$ is the derivative of f at z_0 and $Re\lambda$ is the real part of λ.*

Let $Z \subset \mathbb{R}^n$, $z_0 \in Z$. We define

$$W_Z^s(z_0, \varphi) = \{z \mid \forall_{t \geq 0} \varphi(t, z) \in Z, \quad \lim_{t \to \infty} \varphi(t, z) = z_0\} \tag{202}$$

$$W_Z^u(z_0, \varphi) = \{z \mid \forall_{t \leq 0} \varphi(t, z) \in Z, \quad \lim_{t \to -\infty} \varphi(t, z) = z_0\} \tag{203}$$

$$W^s(z_0, \varphi) = \{z \mid \lim_{t \to \infty} \varphi(t, z) = z_0\} \tag{204}$$

$$W^u(z_0, \varphi) = \{z \mid \lim_{t \to -\infty} \varphi(t, x) = z_0\} \tag{205}$$

$$Inv^+(Z, \varphi) = \{z \mid \forall_{t \geq 0} \varphi(t, z) \in Z\} \tag{206}$$

$$Inv^-(Z, \varphi) = \{z \mid \forall_{t \leq 0} \varphi(t, z) \in Z\} \tag{207}$$

Sometimes, when φ is known from the context it will be dropped and we will write $W^s_Z(z_0), Inv^{\pm}(Z)$ etc.

The goal of this section is to prove the following theorem.

Theorem 26 *Assume that $z_0 = (x_0, y_0)$ is an hyperbolic fixed point for (201).*

Let $Z \subset \mathbb{R}^n$ be an open set, such that $z_0 \in Z$.

Then there exists an h-set N with cones, such that $z_0 \in N$, $N \subset Z$, $W^u_N(z_0)$ is a horizontal disk in N satisfying the cone condition and $W^s_N(z_0)$ is a vertical disk in N satisfying the cone condition.

Moreover, $W^u_N(z_0)$ can be represented as a graph of a Lipschitz function over the unstable space for the linearization of f at z_0 and tangent to it at z_0. Analogous statement is also valid for $W^s_N(z_0)$.

Proof: Consider a flow obtained from (201) by linearization

$$x' = df(z_0)(x - z_0). \tag{208}$$

Let φ_L denotes the flow induced by (208) and let u and s be the dimension of the unstable and stable manifolds for (208) at z_0. It well known that there exists a coordinate system and the scalar product $(\cdot,\cdot)$ such that following holds

$$df(z_0) = \begin{bmatrix} A & 0 \\ 0 & U \end{bmatrix} \tag{209}$$

where $A \in \mathbb{R}^{u\times u}$, $U \in \mathbb{R}^{s\times s}$, such that $A + A^T$ is positive definite and $U + U^T$ is negative definite. In this coordinate system $W^u(z_0, \varphi_L) = \{z_0\} + \mathbb{R}^u \times \{0\}^s$ and $W^s(z_0, \varphi_L) = \{z_0\} + \{0\}^u \times \mathbb{R}^s$. We will use these coordinates in our proof.

Let us fix $\alpha, \beta \in \mathbb{R}_+$. Let us define a quadratic form $Q((x, y)) = \alpha x^2 - \beta y^2$, where $x \in \mathbb{R}^u$ and $y \in \mathbb{R}^s$.

For any $\lambda \in [0, 1]$ let φ_λ be the flow induced by

$$z' = f_\lambda(z) := (1 - \lambda)f(z) + \lambda(df(z_0)(z - z_0)) \tag{210}$$

For any $r > 0$ we define $N(r)$ by

$$N(r) = \{z_0\} + \overline{B_u(0, r)} \times \overline{B_s(0, r)}. \tag{211}$$

To proceed further we need the following Lemma, which will be proved after we complete the current proof.

Lemma 27 *There exists $r_0 > 0$, such that for $\lambda \in [0, 1]$ and for any $0 < r \leq r_0$ the following conditions are satisfied.*

$$\frac{d}{dt}Q(\varphi_\lambda(t, z_1) - \varphi_\lambda(t, z_2))_{|t=0} > 0, \qquad \text{for all } z_1, z_2 \in N(r),\ z_1 \neq z_2 \tag{212}$$

$$\frac{d(\pi_x(\varphi_\lambda(t, z)) - x_0)^2}{dt}(z) > 0, \qquad z \in N(r) \text{ and } \|\pi_x(z - z_0)\| \geq \tfrac{r}{2} \tag{213}$$

$$\frac{d(\pi_y(\varphi_\lambda(t, z)) - y_0)^2}{dt}(z) < 0, \qquad z \in N(r) \text{ and } \|\pi_y(z - z_0)\| \geq \tfrac{r}{2} \tag{214}$$

Continuation of the proof of Theorem 26. Let us fix $r = r_0/2$, where r_0 is as in Lemma 27. We define the h-set N with cones as follows: we set $|N| = N(r)$, $c_N(z) = \frac{1}{r}(z - z_0)$, $u(N) = u$, $s(N) = s$ and $Q_N(z') = Q(c_N^{-1}(z'))$ for $z' \in N_c$.

Observe that from Lemma 27 it follows immediately, that in the sense of the Conley index theory [S] the pair (N, N^-) is an isolating block.

From Lemma 27 if follows that for $h > 0$ small enough the following conditions are satisfied for every $\lambda \in [0, 1]$

$$\text{if } z \in N, \text{ then } \varphi_\lambda([-h, h], z) \in N(r_0) \tag{215}$$

$$\text{if } z \in N^-, \text{ then } \varphi_\lambda((0, h], z) \notin N, \tag{216}$$

$$\text{if } z \in N^+, \text{ then } \varphi_\lambda([-h, 0), z) \notin N, \tag{217}$$

$$\text{if } z, \varphi_\lambda(h, z) \in N, \text{ then } \varphi_\lambda([0, h], z) \in N \tag{218}$$

$$\text{if } z, \varphi_\lambda(-h, z) \in N, \text{ then } \varphi_\lambda([-h, 0], z) \in N. \tag{219}$$

From Lemma 27 and condition (215) it follows that

$$Q(\varphi(h, z_1) - \varphi(h, z_2)) > Q(z_1 - z_2), \quad \text{for } z_1, z_2 \in N, z_1 \neq z_2 \tag{220}$$

We will prove that

$$N \overset{\varphi(h,\cdot)}{\Longrightarrow} N. \tag{221}$$

For the proof of (221) we need a suitable homotopy. First consider $H(\lambda, h) = \varphi_\lambda(h, \cdot)$. Obviously, $H_0 = \varphi(h, \cdot)$ and $H_1 = \varphi_L(h, \cdot)$.

From Lemma 27 if follows that

$$\pi_x(H([0, 1], N^-)) \subset \mathbb{R}^u \setminus \overline{B}_u(x_0, r), \tag{222}$$

$$\pi_y(H([0, 1], N)) \subset B_s(y_0, r). \tag{223}$$

Observe that the above conditions imply that

$$H([0, 1], N^-) \cap N = \emptyset, \tag{224}$$

$$H([0, 1], N) \cap N^+ = \emptyset. \tag{225}$$

We have $H_1(x, y) = (\exp(Ah)(x - x_0), \exp(Uh)(y - y_0)) + z_0$. Let us define the homotopy $G : [0, 1] \times \mathbb{R}^u \times \mathbb{R}^s \to \mathbb{R}^u \times \mathbb{R}^s$ by

$$G(\lambda, x, y) = (\exp(Ah)(x - x_0), (1 - \lambda) \exp(Uh)(y - y_0)) + z_0. \tag{226}$$

Let F be the homotopy obtained by concatenation of H and G, this means that

$$F(\lambda, z) = \begin{cases} H(2\lambda, z) & \text{for } 0 \leq \lambda \leq 1/2, \\ G(2(\lambda - 1/2), z) & \text{otherwise.} \end{cases} \tag{227}$$

It is easy to see the homotopy $F_c(\lambda, z) = c_N(F(\lambda, c_N^{-1}(z)))$ for $z \in N_c$ satisfies all conditions for the covering relation $N \overset{\varphi(h,\cdot),w}{\Longrightarrow} N$, where $w = \pm 1$ (this follows from the linearity of F_1.)

Now we apply Theorems 12 and 10 to (N,Q) and $\varphi(h,\cdot)$ to infer that $W^u_N(z_0,\varphi(h,\cdot))$ and $W^s_N(z_0,\varphi(h,\cdot))$ are horizontal and vertical disks, respectively.

To finish the proof we need to show that

$$W^u_N(z_0,\varphi(h,\cdot)) = W^u_N(z_0,\varphi) \tag{228}$$
$$W^s_N(z_0,\varphi(h,\cdot)) = W^s_N(z_0,\varphi) \tag{229}$$

Let us prove (228), the proof of (229) is analogous.

Observe first, that the inclusion $W^u_N(z_0,\varphi(h,\cdot)) \supset W^u_N(z_0,\varphi)$ is obvious. For the opposite direction let us take $z \in W^u_N(z_0,\varphi(h,\cdot))$, then from condition (219) it follows that $\varphi((-\infty,0],z) \subset N$. From Lemma 27 if follows that $V(z) = Q(z-z_0)$ is decreasing (in strong sense) as long as the orbit stays in N. Hence $\lim_{t\to-\infty}\varphi(t,z) = z_0$.

The tangency of the stable (unstable) manifolds of φ and φ_L at z_0 is obtained as in the map case - see the conclusion of the proof of Theorem 14 for more details. ∎

Proof of Lemma 27

Let us fix $\lambda \in [0,1]$. For $z_i \in \mathbb{R}^n$ and $t \in \mathbb{R}$ let $z_i(t) = \varphi_\lambda(t,z_i)$.

Let Q be a symmetric matrix corresponding the quadratic form Q. Then

$$\begin{aligned}
\frac{d}{dt}Q(z_1(t)-z_2(t))_{|t=0} &= \\
(f_\lambda(z_1)-f_\lambda(z_2))^TQ(z_1-z_2) + (z_1-z_2)^TQ(f_\lambda(z_1)-f_\lambda(z_2)) &= \\
(z_1-z_2)^TC^TQ(z_1-z_2) + (z_1-z_2)^TQC(z_1-z_2) &= \\
(z_1-z_2)^T(C^TQ+QC)(z_1-z_2),&
\end{aligned}$$

where

$$\begin{aligned}
C = C(\lambda,z_1,z_2) = \int_0^1 df_\lambda(z_1+t(z_2-z_1))dt = \\
(1-\lambda)\int_0^1 df(z_1+t(z_2-z_1))dt + \lambda df(z_0).
\end{aligned}$$

Observe that for $z_1,z_2 \to z_0$ the matrix $C(\lambda,z_1,z_2)$ converges to $df(z_0)$ uniformly with respect to $\lambda \in [0,1]$, hence it is enough to show that the symmetric matrix $df(z_0)^TQ + Qdf(z_0)$ is positive definite.

We have

$$\begin{aligned}
df(z_0)^TQ + Qdf(x_0) = \begin{bmatrix} A^T & 0 \\ 0 & U^T \end{bmatrix}\begin{bmatrix} \alpha & 0 \\ 0 & -\beta \end{bmatrix} + \\
\begin{bmatrix} \alpha & 0 \\ 0 & -\beta \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & U \end{bmatrix} = \begin{bmatrix} \alpha(A+A^T) & 0 \\ 0 & -\beta(U+U^T) \end{bmatrix}
\end{aligned}$$

Since matrices $\alpha(A+A^T)$ and $-\beta(U+U^T)$ are positive definite, then the same is true about $df(z_0)^TQ + Qdf(x_0)$.

Consider condition (213). Let $z = (x, y)$. We have for $t = 0$

$$\frac{d(\pi_x(\varphi_\lambda(t,z)) - x_0)^2}{dt} = 2(x - x_0)^T \pi_x f_\lambda(z) =$$
$$2(x - x_0)^T C_{11}(x - x_0) + 2(x - x_0)^T C_{12}(y - y_0),$$

where

$$C_{11} = C_{11}(\lambda, z, z_0) = \int_0^1 \frac{\partial \pi_x f_\lambda}{\partial x}(z_0 + t(z - z_0))dt =$$
$$\frac{\partial \pi_x f}{\partial x}(z_0) + O(\|z - z_0\|) = A + O(\|z - z_0\|),$$
$$C_{12} = C_{12}(\lambda, z, z_0) = \int_0^1 \frac{\partial \pi_x f_\lambda}{\partial y}(z_0 + t(z - z_0))dt =$$
$$\frac{\partial \pi_x f}{\partial y}(z_0) + O(\|z - z_0\|) = O(\|z - z_0\|).$$

Now let $z = (x, y) \in N(r)$ and $\|x - x_0\| \geq \frac{r}{2}$. We have for $t = 0$

$$\frac{d(\pi_x(\varphi_\lambda(t,z)) - x_0)^2}{dt} =$$
$$(x - x_0)^T(A + A^T)(x - x_0) + 2(x - x_0)^T O(r)(x - x_0) +$$
$$2(x - x_0)^T O(r)(y - y_0) \geq a(r/2)^2 - O(r)r^2 = (a/4 - O(r))r^2,$$

where $a > 0$ is such that $x^T(A + A^T)x \geq ax^2$. Hence (213) holds provided r_0 is small enough.

The justification of (214) is analogous.

∎

11 The relation with the standard notion of hyperbolicity, the linear map case

The goal of this section is to compare the cone conditions used in this paper with the notion of hyperbolicity for linear maps.

Definition 20 *Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. We say that A is* hyperbolic *iff $Sp(A) \cap S^1 = \emptyset$.*

11.1 Some examples

Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map and $Q : \mathbb{R}^n \to \mathbb{R}$ be a quadratic form.

We define the quadratic form V by

$$V(x) = Q(Ax) - Q(x). \tag{230}$$

Observe that the positive definiteness of V does not imply that A is an isomorphism (see Example 1). It is also possible to have a degenerate Q and still obtain nondegenerate V (see Example 2). However Theorem 28 shows that the positive definiteness of V implies that Q is nondegenerate.

Example 1 *Let $n = 2$ and*

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{231}$$

Then

$$V = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \tag{232}$$

is positive definitive. Observe that A is not an isomorphism.

Example 2 *Let $n = 2$ and*

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{233}$$

Then

$$V = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{234}$$

is non-degenerate, but both A and Q are singular.

11.2 Hyperbolicity

For a linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ by $W^s(A)$, $W^u(A)$, $W^c(A)$ we denote respectively the stable, unstable and central subspace for A. We have $\mathbb{R}^n = W^s(A) \oplus W^u(A) \oplus W^c(A)$.

Definition 21 *For a quadratic form $Q : \mathbb{C}^n \to \mathbb{R}$ we define two cones*

$$C^+(Q) = \{x \in \mathbb{C}^n, \quad Q(x) > 0\}$$
$$C^-(Q) = \{x \in \mathbb{C}^n, \quad Q(x) < 0\}.$$

Definition 22 *A pair of numbers $(p, q) \in \mathbb{N}^2$ is called a signature of a quadratic form $Q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ iff there exists a basis $\{e_i\}$ such that*

$$Q = \sum_{i=1}^{p} (e_i^*)^2 - \sum_{i=1}^{q} (e_{p+i}^*)^2, \tag{235}$$

where $\{e_i^*\}$ is a dual basis to $\{e_i\}$.

Theorem 28 *Assume that $A : \mathbb{R}^n \to \mathbb{R}^n$ is linear map and $Q : \mathbb{R}^n \to \mathbb{R}$ is a quadratic form of signature (n_+, n_-).*

Assume, that the quadratic form V given

$$V(x) = Q(Ax) - Q(x) \tag{236}$$

is positive definite.

Then Q is nondegenerate, A is hyperbolic and the following conditions are satisfied

$$\begin{aligned} n_+ &= \dim W^u(A), \quad n_- = \dim W^s(A) &(237)\\ W^u(A) &\subset C^+(Q), \qquad W^s(A) \subset C^-(Q) &(238)\end{aligned}$$

Proof:

It is well know that with Q we can in a unique way associate a bilinear form $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that

$$Q(u) = B(u, u). \tag{239}$$

Consider now the complexification of $\mathbb{R}^n$, Q, B and A. When complexifying B we choose the following convention

$$B(\alpha u, \beta v) = \alpha\overline{\beta}B(u, v), \qquad \alpha, \beta \in \mathbb{C}, \quad u, v \in \mathbb{C}^n \tag{240}$$

We have

$$Q(\lambda u) = |\lambda|^2 Q(u), \qquad \lambda \in \mathbb{C}, \quad u \in \mathbb{C}^n. \tag{241}$$

It is easy to see that the quadratic form V given by (236) satisfies

$$V(x) > 0, \qquad \forall x \in \mathbb{C}^n \setminus \{0\} \tag{242}$$

Let $(\lambda, v) \in \mathbb{C} \times \mathbb{C}^n$ be an eigenpair for A, i.e. $Av = \lambda v$ and $v \neq 0$.

We will show that

$$\begin{aligned} |\lambda| &\neq 1 &(243)\\ \text{if} \quad |\lambda| &> 1, \quad \text{then} \quad v \in C^+(Q) &(244)\\ \text{if} \quad |\lambda| &< 1, \quad \text{then} \quad v \in C^-(Q). &(245)\end{aligned}$$

We have

$$0 < Q(Av) - Q(v) = (|\lambda|^2 - 1)Q(v).$$

Therefore conditions (243–245) are satisfied.

Observe that we have just proved that A is hyperbolic.

Let us set

$$u = \dim(W^u(A)), \quad s = \dim(W^s(A)).$$

Let $v_1, \ldots, v_u, w_1, \ldots, w_s$ be a Jordan basis for A, such that the vectors $v_1, \ldots, v_u$ span $W^u(A)$ and $w_1, \ldots, w_s$ span $W^s(A)$. Moreover, we have

$$Av_j = \lambda_j v_j + \sum_{i<j} a_{ij} v_i, \quad j = 1, \ldots, u \tag{246}$$

$$Aw_j = \gamma_j w_j + \sum_{i<j} b_{ij} w_i, \quad j = 1, \ldots, s \tag{247}$$

where $|\lambda_j| > 1$ and $|\gamma_l| < 1$ and a_{ij}, b_{ij} are some complex numbers. Observe that A is upper triangular on $W^u(A)$ and $W^s(A)$ in this basis.

We will prove inductively that there exists a basis in $\{\tilde{v}_i\}_{i=1,\dots,dim(W^u(A))}$ in $W^u(A)$ and $\{\tilde{w}_i\}_{i=1,\dots,dim(W^s(A))}$ in $W^s(A)$, such that

$$
\begin{aligned}
B(\tilde{v}_i, \tilde{v}_j) &= 0, \quad i \neq j, \quad i,j = 1,\dots,u && (248)\\
Q(\tilde{v}_i) &= B(\tilde{v}_i, \tilde{v}_i) > 0, \quad i = 1,\dots,u && (249)\\
B(\tilde{w}_i, \tilde{w}_j) &= 0, \quad i \neq j, \quad i,j = 1,\dots,s && (250)\\
Q(\tilde{w}_i) &= B(\tilde{w}_i, \tilde{w}_i) < 0, \quad i = 1,\dots,s. && (251)
\end{aligned}
$$

To achieve this goal we apply the Gram-Schmidt orthogonalization based on the symmetric form B separately to the sets $\{v_i\}_{i=1,\dots,u}$ and $\{w_i\}_{i=1,\dots,s}$. Observe that, if successful, this procedure will result in a basis in which A preserves its upper-triangular form both in $W^u(A)$ and $W^s(A)$. The necessary condition for applicability of this procedure is that after i-th step we have

$$Q(\tilde{v}_i) \neq 0, \qquad Q(\tilde{w}_i) \neq 0. \tag{252}$$

Observe that due to (244–245) condition (252) holds for $i = 1$.

We will first provide the proof for the existence of $\{\tilde{v}_i\}$'s. Assume that we had already constructed $\tilde{v}_1, \dots, \tilde{v}_i$, such that $Q(\tilde{v}_k) > 0$ for $k = 1, \dots, i$.

From Gram-Schmidt procedure we obtain $\tilde{v}_{i+1}$. Since for any i $\lambda_i \neq 0$, therefore it is easy to see that there exists $z_{i+1} = \tilde{v}_{i+1} + \sum_{k \leq i} \alpha_k \tilde{v}_k$ such that $Az_{i+1} = \lambda_{i+1}\tilde{v}_{i+1}$.

We have

$$0 < Q(A(z_{i+1})) - Q(z_{i+1}) = |\lambda_{i+1}|^2 Q(\tilde{v}_{i+1}) - Q(\tilde{v}_{i+1}) - \sum_{k \leq i} |\alpha_k|^2 Q(\tilde{v}_k). \tag{253}$$

Since by induction assumption $Q(\tilde{v}_k) > 0$ for $k = 1, \dots, i$ therefore we obtain

$$(|\lambda_{i+1}|^2 - 1) Q(\tilde{v}_{i+1}) > 0, \tag{254}$$

which implies that $Q(\tilde{v}_{i+1}) > 0$, because $|\lambda_{i+1}| > 1$.

To handle $\tilde{w}_i$'s we need to alter a bit the above proof. Namely, for the induction step assume that we had already constructed $\tilde{w}_1, \dots, \tilde{w}_i$, such that $Q(\tilde{w}_k) < 0$ for $k = 1, \dots, i$.

From Gram-Schmidt procedure we obtain $\tilde{w}_{i+1}$. We have

$$A(\tilde{w}_{i+1}) = \gamma_{i+1}\tilde{w}_{i+1} + \sum_{k \leq i} \alpha_k \tilde{w}_k \tag{255}$$

Hence we obtain

$$0 < Q(A(\tilde{w}_{i+1})) - Q(\tilde{w}_{i+1}) = |\gamma_{i+1}|^2 Q(\tilde{w}_{i+1}) + \sum_{k \leq i} |\alpha_k|^2 Q(\tilde{w}_k) - Q(\tilde{w}_{i+1}).$$

Since by induction assumption $Q(\tilde{w}_k) < 0$ for $k = 1, \dots, i$ therefore we obtain

$$(|\gamma_{i+1}|^2 - 1) Q(\tilde{w}_{i+1}) > 0,$$

which implies that $Q(\tilde{w}_{i+1}) < 0$, because $|\gamma_{i+1}| < 1$.

Now we have to come back from complexification to $\mathbb{R}^n$. Observe that the above construction it flows that the bilinear form B is positive definite in restriction to $W^u(A)$, is negative definite on $W^s(A)$. Therefore in suitable bases in $W^u(A)$ and $W^s(A)$ has the following form

$$Q(x, y) = \sum_{i=1}^{u} x_i^2 - \sum_{j=1}^{s} y_j^2 + \sum_{i=1,j=1}^{i=u,j=s} 2a_{ij}x_i y_j \tag{256}$$

where x_i's and y_j's are coordinates on $W^u(A)$ and on $W^s(A)$, respectively.

It is easy to see that Q can in a suitable basis by written as

$$Q(x, y) = \sum_{i=1}^{u} x_i^2 - \sum_{j=1}^{s} y_j^2. \tag{257}$$

To see this consider the standard procedure bringing a quadratic form into the canonical one given by (257). The first step consist in the following transformation

$$x_1^2 + \sum_{j=1}^{s} 2a_{1j}x_1 y_j - \sum_{j=1}^{s} y_j^2 = \left(x_1 + \sum_{j=1}^{s} a_{1j} y_j \right)^2 - \sum_{j=1}^{s} (1 + a_{1j}^2) y_j^2 \tag{258}$$

We introduce new coordinates $\tilde{x}_1 = x_1 + \sum_{j=1}^{s} a_{1j} y_j$, $\tilde{x}_i = x_i$ for $i = 2, \ldots, u$ and $\tilde{y}_j = \sqrt{1 + a_{1j}^2} y_j$ for $j = 1, 2, \ldots, s$. Now our quadratic form Q has the following expression

$$Q(\tilde{x}, \tilde{y}) = \sum_{i=1}^{u} \tilde{x}_i^2 - \sum_{j=1}^{s} \tilde{y}_j^2 + \sum_{i=2,j=1}^{i=u,j=s} 2\tilde{a}_{ij}\tilde{x}_i\tilde{y}_j. \tag{259}$$

Observe that after this step we still have -1 as coefficient of terms y_i^2 and terms $x_1 y_j$ disappeared. Therefore we can remove terms $x_2 y_j$ in next step and so on, until we obtain (257) in a suitable basis.

This shows that the signature of Q is (u, s) and therefore Q is nondegenerate. ∎

Observe that, since the set of positive definite matrices in an open subset of the set symmetric matrix, then if V given by (236) is positive definite, then there exist $0 < l < 1 < k$, such that

$$V(x) = Q(Ax) - mQ(x) \tag{260}$$

is positive definite for $m \in (l, k)$. The next theorem relates the interval (l, k) and the spectrum of A.

Theorem 29 *The same assumptions as in Thm. 28. Let $0 < l < 1 < k$, be such that for any $m \in (l, k)$, the quadratic form*

$$V_m(x) = Q(Ax) - mQ(x) \tag{261}$$

is positive definite.

Then $Sp(A) \cap (\sqrt{l}, \sqrt{k}) = \emptyset$.

Proof: We complexify $\mathbb{R}^n$, Q and A as in the proof of Thm. 28.

Let (λ, v) be an eigenpair for A.

Let us fix $m \in (l, k)$. From our assumption it follows that

$$|\lambda|^2 Q(v) = Q(\lambda v) = Q(Av) > mQ(v). \tag{262}$$

Hence

$$|\lambda|^2 \neq m. \tag{263}$$

∎

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