TOPOLOGICAL ENTROPY FOR MULTIDIMENSIONAL PERTURBATIONS OF ONE-DIMENSIONAL MAPS

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Abstract. We prove that if an interval map of positive entropy is perturbed to a compact multidimensional map then the topological entropy cannot drop down a lot if the perturbation is small.

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1. Introduction

We want to study topological entropy of multidimensional perturbations of one-dimensional maps. Since the spaces we consider are not necessarily compact, there are several possibilities for defining topological entropy. We are mainly interested in invariant compact sets, so we define topological entropy $h(f)$ of f as the supremum of topological entropies of f restricted to compact invariant sets.

A continuous map is called *compact* if it maps bounded sets into relatively compact sets (the sets with compact closures).

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map. Let V be a real Banach space and let a continuous decomposition $V = \mathbb{R} \oplus W$ be given. According to this decomposition we will represent elements $v \in V$ as pairs $v = (x, w)$, where $x \in \mathbb{R}$ and $w \in W$. By B_r we will denote the closed ball in W of radius r centered at 0, and by C_r the infinite cylinder $\mathbb{R} \times B_r$.

Let $F : [0, 1] \times C_r \to V$ be a continuous and compact map. We will use the notation F_λ for the partial map with the fixed λ, so $F_\lambda(v) := F(\lambda, v)$ for $v \in C_r$. Suppose that $F_0(x, w) = (f(x), 0)$ for each $(x, w) \in C_r$ (we will call such F_0 *one-dimensional*). In such a situation we say that the maps F_λ are *multidimensional perturbations* of f.

In this paper we prove the following theorem.

Theorem 1.1. *Let $F : [0, 1] \times C_r \to V$ be a continuous and compact map, such that F_λ are multidimensional perturbations of a continuous map $f : \mathbb{R} \to \mathbb{R}$. Then $\liminf_{\lambda\to 0} h(F_\lambda) \geq h(f)$.*

Since we work anyway with compact intervals of the real line rather than the whole line, in the above theorem we can also consider f defined on an interval and F on the corresponding set (with C_r replaced by the product of this interval and B_r).

Let us discuss a context in which our results are of special importance (apart of rather obvious situations when the space V has finite dimension). There is a class of partial differential equations for which the asymptotic dynamics is finite dimensional, i.e. is determined by a finite number of modes. This class, called *dissipative* [Hale, 1988], includes Navier-Stokes equations, Kuramoto-Sivashinsky equations and many others systems. Then we can ask an important question: *what dynamical properties of the finite dimensional model can be extended to the full PDE?* In this paper we are interested in the estimate for topological entropy in the case of a one-dimensional model.

To be more explicit, let us consider briefly the Kuramoto-Sivashinsky equations (KS) introduced in the context of phase turbulence [Kuramoto, 1980] and flame front propagation [Sivashinsky, 1977]. It is known that the time evolution for KS equations is compact (see [Hale, 1988] and references given there). Therefore, each Poincaré return map, if well defined, is compact. The numerical experiments in [Christiansen *et al.*, 1997] show that a suitably chosen Poincaré map P is essentially one dimensional and can be modeled by a one-dimensional map p. Now we build a homotopy by

$$F(\lambda, x, w) = \lambda P(x, w) + (1 - \lambda)(p(x), 0).$$

This homotopy is compact. We cannot claim that we can use Theorem 1.1 to estimate rigorously topological entropy for the Poincaré map P, because even there is no rigorous proof that the Poincaré map P studied numerically in [Christiansen *et al.*, 1997] exists. However, the ideas used in the proof of Theorem 1.1, continuation of topological horseshoes and relation between topological horseshoes and entropy, combined with recently developed rigorous numerics for KS equations (see [Zgliczyński & Mischaikow, 2000]) can be used to give rigorous estimates for topological entropy.

2. Topological Theorems

In this section we present a definition and theorems about covering relations from [Zgliczyński, 1999c] (the finite dimensional version is presented in [Zgliczyński, 1999a], [Zgliczyński, 1999b]).

We use the same notation as in the preceding section. In W we have a norm $\|\cdot\|_W$, and then in V we use the norm $\|(x,w)\|_V = \max(|x|, \|w\|_W)$. By ϱ we denote the metric in V induced by this norm. Note that W may be a zero-dimensional space.

A set P of the form $P = [a,b] \times B_r$ for some $a < b$ will be called a *cylinder*. We denote

$$L(P) := \{a\} \times B_r \ , \quad R(P) := \{b\} \times B_r \ , \quad V(P) := L(P) \cup R(P)$$

and

$$H(P) := [a,b] \times \partial B_r \ .$$

If $\dim W = 0$ then $H(P) = \emptyset$. Further,

$$S_L(P) := (-\infty, a) \times B_r \quad \text{and} \quad S_R(P) := (b, \infty) \times B_r \ .$$

The sets $S_L(P)$ and $S_R(P)$ are equal to the left and right components of the complement of P in C_r, respectively.

The following definition is very general. In particular, if the set A below is small, it does not describe an interesting situation. Whenever we will be using it, A will be large.

Definition 2.1. Let $f : C_r \to V$ be a continuous and compact map and let $A \subset V$. Let N and M be two cylinders. We say that N *f-covers M with respect to A* (and write $N \Rightarrow_{f,A} M$) if

$$L(N) \cap A \neq \emptyset \ , \quad R(N) \cap A \neq \emptyset \ , \quad \overline{f(N)} \cap M \subset A \ , \quad f(A \cap N) \subset C_r \ ,$$

and either

$$f(L(N) \cap A) \subset S_L(M) \quad \text{and} \quad f(R(N) \cap A) \subset S_R(M)$$

or

$$f(L(N) \cap A) \subset S_R(M) \quad \text{and} \quad f(R(N) \cap A) \subset S_L(M) \ .$$

Remark 2.2. The covering relation used in [Zgliczyński, 1999a], [Zgliczyński, 1999b] coincides with the one given above when $N \cup M \subset A$. $\square$

Remark 2.3. Our Definition 2.1 differs slightly from a similar Definition 1 of [Zgliczyński, 1999c]. The one given here does not involve any apriori given set containing $N \cup M$. Despite this differences in the definitions, the proofs of theorems about covering relations from Sec. 1 of [Zgliczyński, 1999c] do not require any changes, and hence are valid for the covering relation defined above. $\square$

The following theorem, which follows immediately from Theorem 4 of [Zgliczyński, 1999c], summarizes the most important property of covering relations: closed loop of covering relations (with respect to an appropriate set) gives rise to a periodic point realizing this loop.

Theorem 2.4. *Let $N_1, N_2, \dots, N_n$ be cylinders. Let $F_i : [0,1] \times N_{i-1} \to V$, $i = 1, \dots, n$ (where $N_0 = N_n$), be continuous and compact maps. Assume that there exist maps $f_i : \mathbb{R} \to \mathbb{R}$ such that $F_{i,0}(x,y) = (f_i(x), 0)$ for $i = 1, \dots, n$. Let*

$$A = \bigcup_i \overline{F_i([0,1] \times N_{i-1})}.$$

Suppose that for every $\lambda \in [0,1]$ and $i = 1, \dots, n$ we have $N_{i-1} \Rightarrow_{F_{i,\lambda}, A} N_i$. Then for each λ there exists $x(\lambda) \in \operatorname{int} N_0$ such that

$$F_{i,\lambda} \circ F_{i-1,\lambda} \circ \cdots \circ F_{1,\lambda}(x(\lambda)) \in \operatorname{int} N_i \quad \text{for } i = 1, \dots, n-1$$

and

$$F_{n,\lambda} \circ F_{n-1,\lambda} \circ \cdots \circ F_{1,\lambda}(x(\lambda)) = x(\lambda).$$

Definition 2.5. Let $a_0 < a_1 < \cdots < a_{2s-1}$. For $i = 0, \dots, s-1$ set $N_i = [a_{2i}, a_{2i+1}] \times B_r$ and let $N = N_0 \cup N_1 \cup \cdots \cup N_{s-1}$. A continuous compact map $f : N \to V$ will be said to have a *topological s-horseshoe* if there exists a compact homotopy $F : [0,1] \times N \to V$ such that $F_1 = f$, the map F_0 is one-dimensional, and for $A := \overline{F([0,1] \times N)}$ we have $N_i \Rightarrow_{F_\lambda, A} N_j$ for $i, j = 0, \dots, s-1$.

Theorem 2.6. *If f has a topological s-horseshoe then for any sequence $(\alpha_0, \alpha_1, \dots, \alpha_{n-1})$, where $\alpha_i \in \{0, \dots, s-1\}$, there exists $x \in N_{\alpha_0}$ such that $f^i(x) \in N_{\alpha_i}$ for $i = 1, \dots, n-1$ and $f^n(x) = x$. Moreover, $h(f|_N) \geq \log s$ and*

$$\lim_{n \to \infty} \frac{1}{n} \log \operatorname{Card}\{x \in N : f^n(x) = x\} \geq \log s.$$

Proof. The statements about periodic points follow immediately from Theorem 2.4. Periodic points provide also (n, ε)-separated sets of sufficient cardinalities in order to prove the statement about the entropy. According to our definition of entropy, in order to complete the proof we need to find a compact invariant set containing all periodic points whose existence is guaranteed by Theorem 2.4. We claim that the set K of all points at which all iterates of f are defined, intersected with the closure of $f(N)$, has those properties. Clearly, it contains all periodic points mentioned above and is invariant. Since f is compact (it was assumed in Definition 2.5) and N is bounded, the closure of $f(N)$ is compact. Since f is continuous and N is closed, K is closed. Thus, $K \cap f(N)$ is compact. $\blacksquare$

3. Topological Entropy for Interval Maps

In this section we recall results about relations between the topological entropy and the existence of horseshoes for interval maps and explain how to replace horseshoes by topological horseshoes.

The definition of a horseshoe for an interval map (see [Misiurewicz & Szlenk, 1980], [Alsedá *et al.*, 1993]) is the same as the definition of the topological horseshoe from the preceding section (with $\dim W = 0$), except that the intervals involved may have common endpoints. That is, instead of taking $a_0 < a_1 < \cdots < a_{2s-1}$ and $N_i = [a_{2i}, a_{2i+1}]$, we may take $b_0 < b_1 < \cdots < b_s$ and $N_i = [b_i, b_{i+1}]$. However, the way around this difficulty is well known: we remove two extreme intervals of a horseshoe and shorten the rest of them to obtain a topological horseshoe (see for instance the proof of Proposition 4.3.2 of [Alsedá *et al.*, 1993]). Hence, if an interval map f has an s-horseshoe and $s > 2$ then it has a topological $s-2$-horseshoe. Moreover, if f has an s-horseshoe then f^n has an s^n-horseshoe (see [Misiurewicz & Szlenk, 1980], [Alsedá *et al.*, 1993]) and therefore f^n has a topological $s^n - 2$-horseshoe.

The above result can be combined with the following theorem on horseshoes and entropy [Misiurewicz, 1979], [Alsedá *et al.*, 1993].

Theorem 3.1. *Let I be a closed interval and let $f : I \to I$ be continuous. If $h(f) > 0$ there exist sequences $(k_n)_{n=1}^{\infty}$ and $(s_n)_{n=1}^{\infty}$ of positive integers such that for each n the map f^{k_n} has an s_n-horseshoe and*

$$\lim_{n\to\infty} \frac{1}{k_n} \log s_n = h(f).$$

What we get is the same theorem, but with horseshoes replaced by topological horseshoes.

Theorem 3.2. *Let I be a closed interval and let $f : I \to I$ be continuous. If $h(f) > 0$ there exist sequences $(k_n)_{n=1}^{\infty}$ and $(s_n)_{n=1}^{\infty}$ of positive integers such that for each n the map f^{k_n} has a topological s_n-horseshoe and*

$$\lim_{n\to\infty} \frac{1}{k_n} \log s_n = h(f).$$

4. Proof of Theorem 1.1

We start with an auxiliary lemma. In its proof we will use a simple fact that a set K in a metric space X is relatively compact if from every sequence of points of K one can choose a subsequence convergent to a point of X.

Lemma 4.1. *Assume that $F : [0,1] \times C_r \to V$ is continuous and compact. Then F^k, defined by $F^k(\lambda, v) = (F_\lambda)^k(v)$, is also continuous and compact.*

Proof. Set $G(\lambda, v) = (\lambda, F(\lambda, v))$. Then F^k is equal to G^k (this is simply the k-th iterate of G) composed with the projection to the second coordinate. Thus, it enough to show that G is continuous and compact. Continuity is obvious. To show compactness, fix a bounded set $K \subset [0,1] \times C_r$ and choose a sequence of points (λ_n, v_n) from $G(K)$. By the compactness of F we can choose a sequence v_{n_i} convergent in V, and then we can choose a convergent sequence from the sequence λ_{n_i} (since $[0,1]$ is compact). This procedure gives us a convergent subsequence of (λ_n, v_n). Thus, $G(K)$ is relatively compact. ■

Now we can prove Theorem 1.1.

Proof of Theorem 1.1. Fix a number $a < h(f)$. From Theorem 3.2 it follows that there exist positive integers s and k such that the map f^k has a topological s-horseshoe and $(1/k)\log s > a$. Let $I_i = [a_{2i}, a_{2i+1}]$ be the intervals from the definition of the topological horseshoe. Set $N_i = I_i \times B_r$, $N = \bigcup_{i=0}^{n-1} N_i$ and $A = \overline{F(N)}$. From the compactness of F and by Lemma 4.1 it follows that A is a compact set.

Observe that we have $N_i \Rightarrow_{F_0^k, A} N_j$ for $i, j = 0, \ldots, s-1$. Since F^k is uniformly continuous on $[0,1] \times (A \cap C_r)$, the above covering relations hold if we replace F_0 by F_λ with λ sufficiently small. Now we apply Theorem 2.6 to conclude that $h(F_\lambda^k) \geq \log s$. Then $h(F_\lambda) \geq (1/k)\log s > a$. This completes the proof. ■

Remark 4.2. By Theorem 2.6, we have proved also that under the same assumptions

$$\liminf_{\lambda \to 0} \lim_{n \to \infty} \frac{1}{n} \operatorname{Card}\{x \in C_r : F_\lambda^n(x) = x\} \geq h(f).$$

□

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References

Alsedá, L., Llibre, J. & Misiurewicz, M. [1993] *Combinatorial Dynamics and Entropy in Dimension One*, Advanced Series in Nonlinear Dynamics, vol. 5 (World Scientific, Singapore).

Christiansen, F., Cvitanović, P. & Putkaradze, V. [1997] “Spatiotemporal chaos in terms of unstable recurrent patterns,” *Nonlinearity* **10**, 55–70.

Hale, J. [1988] *Assymptotic behavior of dissipative systems*, Mathematical Surveys and Monographs, vol. 25 (Amer. Math. Soc., Providence).

Kuramoto, Y. [1980] “Instability and Turbulence of Wavefronts in Reaction-Diffusion Systems,” *Prog. Theor. Phys.* **63**, 1885–1903.

Misiurewicz, M. [1979] “Horseshoes for mappings of an interval,” *Bull. Acad. Polon. Sci., Sér. Sci. Math.* **27**, 167–169.

Misiurewicz, M. & and Szlenk, W. [1980] “Entropy of piecewise monotone mappings,” *Studia Math.* **67**, 45–63.

Sivashinsky, G. I. [1977] “Nonlinear analysis of hydrodynamical instability in laminar flames – 1. Derivation of basic equations,” *Acta Astron.* **4**, 1177–1206.

Zgliczyński, P. [1999a] “Sharkovskii theorem for multidimensional perturbations of 1-dim maps,” *Ergod. Th. Dynam. Sys.* **19**, 1655–1684.

Zgliczyński, P. [1999b] “Multidimensional perturbations of one-dimensional maps and stability of Sharkovskii ordering,” *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **9**, 1867–1876.

Zgliczyński, P. [1999c] “Sharkovskii theorem for multidimensional perturbations of one-dimensional maps II,” *Topol. Methods Nonlinear Anal.* **14**, 169–182.

Zgliczyński, P. & Mischaikow, K. [2000] “Rigorous Numerics for Partial Differential Equations: the Kuramoto-Sivashinsky equation,” preprint.