Covering Relations for Multidimensional Dynamical Systems - II

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Abstract

We use a topological technique based on covering relations to prove the existence of chaotic orbits for certain Hamiltonian systems with several degrees of freedom. This paper relies on an earlier work of Zgliczyński and Gidea.

Keywords: Arnold instability; hamiltonian systems; topological transversality; transition tori.

In this paper we study a simple model of a Hamiltonian system with a partially hyperbolic fixed point of general elliptic type. The tangent space at the fixed point splits into stable, unstable and center directions. The locally defined center manifold is filled with tori invariant to the hamiltonian flow. If the unstable and stable manifolds of one of the invariant tori have a topological crossing, then, for all sufficiently close tori, their unstable and stable manifolds also have topological crossings. We use the topological method of covering relations introduced in [12] to prove the existence of shadowing orbits to every given bi-infinite sequence of such invariant tori. This type of behavior is related to Arnold's mechanism of instability. The topological crossing condition replaces the usual condition on the transversality of the stable and unstable manifolds of the tori, which can be quite difficult to check for many perturbed Hamiltonians.

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Throughout this paper, we will frequently refer to the first part of the paper [12]. In Section 1, we recall the main constructions and results from [12]. In Section 2, we briefly discuss the notion of topological crossing. In Section 3, we describe a Hamiltonian system with several degrees of freedom which serves as our model for studying Arnold instability. In Section 4 we use covering relations to prove that there exist orbits that drift from torus to torus, in a chaotic fashion.

Our development is based on ideas in [5] and [6]. The novelty here is that we give a topological argument, which is more transparent than the differentiable one, and we replace the transversality condition by a topological crossing condition (easier to check in many examples, see Remark 5). Topological treatments of Arnold instability, in the case of chains of transition tori with transverse heteroclinic connections, were also performed in [11] and [7].

1 Introduction

Denote by $B_n(c,r)$ the open ball of radius r centered at $c \in \mathbb{R}^n$. Denote by ∂Z the topological boundary, by $\operatorname{int} Z$ the interior, and by $\overline{Z}$ the closure of a set Z. Denote by bd$V$ the boundary of a manifold with boundary V. If $f : S \to \mathbb{R}^n$ is a continuous function, $D \subset \mathbb{R}^n$ is an open set contained in S, $c \in \mathbb{R}^n$, and $f^{-1}(c) \cap D$ is compact, then the local Brouwer degree of f at c in the set D can be defined. Denote by $\deg(f, D, c)$ this degree. See the Appendix of [12].

An h-set is essentially a homeomorphic copy of a multidimensional rectangle, together with a choice of expanding- and contracting-like directions.

Definition 1 *An h-set is a quadruple consisting of*

- *a compact subset N of $\mathbb{R}^n$*
- *a pair of non-negative integers $u(N), s(N)$, with $u(N) + s(N) = n$*
- *a homeomorphism $c_N : \mathbb{R}^n \to \mathbb{R}^n = \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)}$, such that*

$$c_N(N) = \overline{B_{u(N)}}(0,1) \times \overline{B_{s(N)}}(0,1).$$

With an abuse of notation, we will denote such a quadruple by N. We denote

$$\begin{aligned} N_c &= \overline{B_{u(N)}}(0,1) \times \overline{B_{s(N)}}(0,1), \\ N_c^- &= \partial \overline{B_{u(N)}}(0,1) \times \overline{B_{s(N)}}(0,1), \\ N_c^+ &= \overline{B_{u(N)}}(0,1) \times \partial \overline{B_{s(N)}}(0,1), \\ N^- &= c_N^{-1}(N_c^-), \quad N^+ = c_N^{-1}(N_c^+). \end{aligned}$$

A covering relation is, in brief, a non-trivial stretching of one h-set across another h-set.

Definition 2 *Assume that N, M are h-sets, such that $u(N) = u(M) = u$ and $s(N) = s(M) = s$. Let $f : N \to \mathbb{R}^n$ be a continuous map. Let $f_c = c_M \circ f \circ c_N^{-1} : N_c \to \mathbb{R}^u \times \mathbb{R}^s$. Let w be a nonzero integer. We say that*

$$N \overset{f,w}{\Longrightarrow} M$$

(N f-covers M with degree w) iff the following conditions are satisfied

1. *there exists a continuous homotopy $h : [0, 1] \times N_c \to \mathbb{R}^u \times \mathbb{R}^s$, such that the following conditions hold true*

$$\begin{aligned} h_0 &= f_c, & (1)\\ h([0, 1], N_c^-) \cap M_c &= \emptyset, & (2)\\ h([0, 1], N_c) \cap M_c^+ &= \emptyset. & (3) \end{aligned}$$

2. *There exists a map $A : \mathbb{R}^u \to \mathbb{R}^u$, such that*

$$\begin{aligned} h_1(p, q) &= (A(p), 0), \text{ for } p \in \overline{B_u}(0, 1) \text{ and } q \in \overline{B_s}(0, 1), & (4)\\ A(\partial B_u(0, 1)) &\subset \mathbb{R}^u \setminus \overline{B_u(0, 1)}. & (5) \end{aligned}$$

Moreover, we require that

$$\deg(A, \overline{B_u}(0, 1), 0) = w,$$

Note that in the case $u = 0$, an h-set N can cover an h-set M only with degree $w = 1$.

The following gives sufficient conditions for a covering relation.

Theorem 1 *[12, Theorem 15] Let N, M be two h-sets in $\mathbb{R}^n$, such that $u(N) = u(M) = u$ and $s(N) = s(M) = s$. Let $f : N \to \mathbb{R}^n$ be continuous. Let $f_c = c_M \circ f \circ c_N^{-1} : N_c \to \mathbb{R}^u \times \mathbb{R}^s$.*

Assume that there exists $q_0 \in \overline{B}_s(0, 1)$, such that following conditions are satisfied

1.

$$\begin{aligned} f_c(\overline{B}_u(0, 1) \times \{q_0\}) &\subset \operatorname{int}(S(M)_c^- \cup M_c), & (6)\\ f_c(N_c^-) \cap M_c &= \emptyset, & (7)\\ f_c(N_c) \cap M_c^+ &= \emptyset. & (8) \end{aligned}$$

2.1 *Case $u > 0$. We define a map $A_{q_0} : \mathbb{R}^u \to \mathbb{R}^u$ by*

$$A_{q_0}(p) = \pi_u(f_c(p, q_0)), \qquad (9)$$

where $\pi_u : \mathbb{R}^u \times \mathbb{R}^s \to \mathbb{R}^u$ is the projection onto $\mathbb{R}^u$, $\pi_u(p, q) = p$. We assume that

$$\begin{aligned} A_{q_0}(\partial B_u(0, 1)) &\subset \mathbb{R}^u \setminus \overline{B_u(0, 1)}, & (10)\\ \deg(A_{q_0}, \overline{B_u}(0, 1), 0) = w &\neq 0. & (11) \end{aligned}$$

2.2 *Case $u = 0$. We assume that*

$$f_c(N_c) \subset \operatorname{int} M_c,$$

and set $w = 1$.

Then

$$N \overset{f,w}{\Longrightarrow} M.$$

The main result on covering relations, which we will need in the sequel, provides a method for detecting orbits with prescribed trajectories and periodic orbits.

Proposition 2 *[12, Corollary 8] Let N_i, $i \in \mathbb{Z}$ be h-sets. Assume that for each $i \in \mathbb{Z}$ we have*

$$N_{i-1} \overset{f_i,w_i}{\Longrightarrow} N_i. \tag{12}$$

Then there exists a point $x \in \operatorname{int} N_0$, such that

$$f_i \circ f_{i-1} \circ \cdots \circ f_1(x) \quad \in \quad \operatorname{int} N_i, \quad i \in \mathbb{Z}. \tag{13}$$

Moreover, if $N_{i+k} = N_i$ for some $k > 0$ and all i, then the point x can be chosen so that

$$f_k \circ f_{k-1} \circ \cdots \circ f_1(x) = x. \tag{14}$$

2 Topological crossing and local Brouwer degree

We define topological crossing, following [2]. We recall that the oriented intersection number of two oriented submanifolds W^1 and W^2 of an oriented manifold W, with $\dim W^1 + \dim W^2 = \dim W$, is given by

$$I(W^1, W^2) = \sum_{x \in W^1 \cap W^2} I_x(W^1, W^2),$$

where $I_x(W^1, W^2)$ is ± 1, depending on whether the orientation induced on $T_x W^1 \oplus T_x W^2$ agrees or not with the orientation on $T_x W$ (see [9]).

Definition 3 *Let W be a n-dimensional manifold, and let W^1 and W^2 be two submanifolds of dimensions n_1 and respectively n_2 in W, such that $n_1 + n_2 = n$. We say that W^1 and W^2 cross topologically if there exist a compact embedded n_1-dimensional submanifold with boundary $V^1 \subseteq W^1$ in W, and a compact embedded n_2-dimensional submanifold with boundary $V^2 \subseteq W^2$ in W, such that*

(i) $bdV^1 \cap V^2 = bdV^2 \cap V^1 = \emptyset$,

(ii) For every $0 < \epsilon < \min\left(dist(bdV^1, V^2), dist(bdV^2, V^1)\right)$, there exists a homotopy $h : [0,1] \times V \to W$ such that

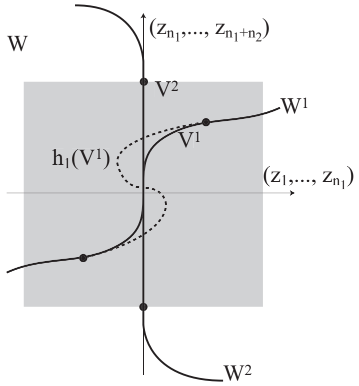

Figure 1: A topological crossing of two submanifolds.

(a) $h_0 = id$ *and* h_1 *is an embedding,*

(b) $d(h_t(p), p) < \epsilon$ *for all* $p \in V$ *and all* $t \in [0, 1]$*,*

(c) $h_1(V^1)$ *and* V^2 *are transverse submanifolds, and their oriented intersection number* $\iota := I(h_1(V^1), V^2)$ *is nonzero.*

See Figure 1. A direct consequence of this definition is that W^1 and W^2 have a non-empty intersection. It is important to notice that the intersection need not be transverse and could be of infinite order. The submanifolds V^1 and V^2 in the above definition will be referred as a *good pair* for W^1 and W^2.

Let us assume that there exist a good pair V^1, V^2, a coordinate neighborhood U of $V^1 \cap V^2$ in W, with $V_1, V_2 \subseteq U$, and local coordinates $z = (z_1, \ldots, z_{n_1}, z_{n_1+1}, \ldots, z_{n_1+n_2})$ on U such that

$$\begin{aligned} U &= B_{n_1}(0,1) \oplus B_{n_1}(0,1), \\ V^2 &= \{z \mid z_1 = \ldots = z_{n_1} = 0\}, \\ V^1 &= \{\psi(z_1, \ldots, z_{n_1}) \mid (z_1, \ldots, z_{n_1}) \in \overline{B_{n_1}}(0,1)\}, \end{aligned}$$

where $\psi = (\psi_1, \ldots, \psi_n)$ is some parametrization of V_1. See Figure 1.

Let $\pi_1 : U \to \mathbb{R}^{n_1}$ be the projection given by

$$\pi_1(z_1, \ldots, z_{n_1}, z_{n_1+1}, \ldots, z_{n_1+n_2}) = (z_1, \ldots, z_{n_1})$$

and $A : \mathbb{R}^{n_1} \to \mathbb{R}^{n_1}$ be the map given by

$$A = \pi_1 \circ \psi.$$

If $0 \notin A\,[\partial B_{n_1}(0,1)]$, then the local Brouwer degree $\deg(A, \overline{B_{n_1}}(0,1), 0)$ of A at 0 in the set $\overline{B_{n_1}}(0,1)$ is well defined.

Lemma 3 *We have* $0 \notin A\,[\partial B_{n_1}(0,1)]$*, and* $\deg(A, \overline{B_{n_1}}(0,1), 0) = \iota$*.*

Proof: The fact that $0 \notin A\left[\partial B_{n_1}(0,1)\right]$ follows from the definition of a good pair. Chose $0 < \epsilon < \min\left(\mathrm{dist}(\mathrm{bd}V^1, V^2), \mathrm{dist}(\mathrm{bd}V^2, V^1)\right)$. Let h be a homotopy as in Definition 3, which moves points by at most $\epsilon/2$. So $h_1(V^1)$ is transverse to V^2, and $h_1(V^1) \cap V^2$ consists of finitely many points $\{p_1, \ldots, p_m\}$. Let $q_i = (h_1 \circ \psi)^{-1}(p_i)$, $i = 1, \ldots, m$. Then

$$I(h_1(V^1), V^2) = \sum_{i=1}^{m} I_{q_i}(h_1 \circ \psi, V^2) = \iota.$$

Each intersection number $I_{q_i}(h_1 \circ \psi, V^2)$ is ± 1, depending on whether the orientation on $d(h_1 \circ \psi)_{q_i}(\mathbb{R}^{n_1}) \oplus T_{p_i}(V^2)$ agrees or not with the orientation on $T_{p_i}U$. Choose $0 < r_i < 1$ such that the disks

$$D_i = \{(z_1, \ldots, z_{n_1}) \mid \|(z_1, \ldots, z_{n_1}) - q_i\| \leq r_i\}$$

are mutually disjoint and contained in $\overline{B_{n_1}}(0,1)$, and each $\pi_1 \circ h_1 \circ \psi_{|D_i} : D_i \to \mathbb{R}^{n_1}$ is a diffeomorphism, as $i = 1, \ldots, m$. We obviously have

$$0 \notin (\pi_1 \circ h_1 \circ \psi)(\partial D_i), \text{ and} \tag{15}$$

$$0 \notin (\pi_1 \circ h_1 \circ \psi)(\partial B_{n_1}(0,1)). \tag{16}$$

It is easy to see that

$$I_{q_i}(h_1 \circ \psi, V^2) = \deg(\pi_1 \circ h_1 \circ \psi, D_i, 0).$$

Let $D = \bigcup_{i=1}^{m} D_i$. By the additive property of the Brouwer degree, we have $\deg(\pi_1 \circ h_1 \circ \psi, D, 0) = \iota$, and by the excision property of the Brouwer degree, we have $\deg(\pi_1 \circ h_1 \circ \psi, B_{n_1}(0,1), 0) = \iota$. ∎

Remark 4 *Note that the definition of topological crossing is restricted to submanifolds of complementary dimensions. We will later need to describe a situation in which two invariant manifolds intersect along a solution curve in a topologically crossing manner (Section 2, condition (B)). One simple way to capture this is to produce a surface of section transverse to the solution curve, such that the intersection of the stable and unstable manifolds with the surface of section are submanifolds of complementary dimensions and have a topological crossing.*

Remark 5 *In many instances, one applies a Melnikov type of method to measure the splitting of some stable and unstable manifolds, when a Hamiltonian system is perturbed. The usual way of showing that small perturbations of a completely integrable Hamiltonian system posses transverse homoclinic orbits is by verifying that some Melnikov function has a simple zero, or some Melnikov potential has a non-degenerate critical point. However, in order to conclude the existence of a topological crossing of some stable and unstable manifolds (provided that they do not coincide), one needs only to check the existence of a critical point of a Melnikov potential. If the Melnikov potentials are defined on compact manifolds, one can ensure a priori the existence of maxima or minima, but not necessarily their non-degeneracy (see [3]).*

3 Hamiltonian systems with topologically crossing heteroclinic connections

We are interested in the general behavior of the orbits of an autonomous Hamiltonian with d-degrees of freedom $H(q,p)$ on $\mathbb{R}^{2d}$ near a non-degenerate critical point (q_0,p_0). Letting

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

the Hamiltonian vector field for H is $X_H = J\nabla H$, where ∇ denotes the gradient, and the Hamilton equations generated by H are $(\dot{q},\dot{p}) = J\nabla H(q,p)$. The linearized system at (q_0,p_0) is given by matrix $J\mathrm{Hess}_{(q_0,p_0)}$, where $\mathrm{Hess}_{(q_0,p_0)}$ denotes the Hessian of H at (q_0,p_0). The non-degeneracy condition means that the Hessian has no zero eigenvalues. Suppose that this matrix has $m < d$ pairs of conjugate imaginary eigenvalues, and that the bilinear form $v \to \langle v, \mathrm{Hess}_{(q_0,p_0)} v\rangle$ is positive definite on the eigenspace corresponding to the imaginary eigenvalues. In particular, the linearized system at (q_0,p_0) has $n := m - d$ positive eigenvalues and $n := m - d$ negative eigenvalues. If we move to an energy level slightly above the level of (q_0,p_0), near (q_0,p_0) there exists an invariant set to the Hamiltonian flow, which is diffeomorphic to a $(2m-1)$-dimensional sphere. Moreover, this sphere is filled with m-dimensional invariant tori, and the whole energy surface is locally diffeomorphic to $S^{2m-1}\times\mathbb{R}^{2(d-m)}$. The invariant tori have $(m+d)$-dimensional stable and unstable manifolds. If the stable and unstable manifolds of one such an invariant torus intersect transversally along a homoclinic orbit, then so do the stable and unstable manifolds of sufficiently close invariant tori, and the existence of symbolic dynamics near these invariant tori can be inferred (see, for example [11]). However, the existence of transverse homoclinic and heteroclinic connections may be very hard to verify in practice. Therefore, we will assume a milder topological crossing condition on the homoclinic and heteroclinic orbits. We are interested in showing that there exist orbits which 'shadow' any sequence of such tori in any prescribed order, and that there are also periodic orbits which 'shadow' any finite collection of such tori.

In the sequel, we consider a simplified model that exhibits such a behavior. We assume that the Hamiltonian vector field X_H is linear. This allows for an explicitly description of the invariant tori and of their stable and unstable manifolds, and for a good control on the behavior of various stable and unstable leaves under homoclinic or heteroclinic excursions.

Ultimately, one is interested in more general models of Hamiltonians, and would involve normal forms or averaging methods to describe the dynamics along the invariant tori (see, for example [8] and [4]). In this paper, we only consider a simplified model since our aim is to illustrate how to construct covering relations in order to locate orbits with prescribed itineraries, in a way that could be implemented in a numerical exploration.

Let $T^m = (S^1)^m$ denote the m-torus, U a compact neighborhood of the origin in $\mathbb{R}^m$, V a compact neighborhood of the origin in $\mathbb{R}^n$. Let (θ,ρ,x,y) be a coordinate system on $T^m \times U \times V \times V$. The variables θ_i will be referred as angle variables, and the variables ρ_i as action variables. Each angle variable $\theta_i \in \mathbb{R}$ is determined up to a multiple of 2π. For any $r \in U$, let $T^m(r)$ denote the torus $T^m \times \{r\} \times \{0\} \times \{0\}$. Consider a smooth Hamiltonian function $H : \mathbb{R}^{2m+2n} \to \mathbb{R}$ which is given by

$$H(\theta, \rho, x, y) = G(\rho) + \langle x, By \rangle, \tag{17}$$

for $(\theta, \rho, x, y) \in T^m \times U \times V \times V$, where G is a smooth function and B is a positive definite symmetric matrix. The corresponding Hamilton equations on $T^m \times U \times V \times V$ are

$$\begin{aligned} \dot{x} &= Bx, & \dot{\theta}_i &= a_i(\rho), \\ \dot{y} &= -By, & \dot{\rho}_i &= 0, \end{aligned} \tag{18}$$

where $a_i = \partial G / \partial \rho_i$, $i = 1, \ldots, m$. Suppose that for some energy level $h \in \mathbb{R}$, the Hamiltonian vector field X_H defines a (global) flow on the constant energy surface $M_h := \{H = h\}$ in $\mathbb{R}^{2m+2n}$. If an invariant torus $T^m(r)$ is contained in M_h, then its unstable and stable manifolds $W^u(r)$, $W^s(r)$ are also contained in M_h. Due to (18), these invariant manifolds are locally given by

$$W^u_{loc}(r) = \{(\theta, \rho, x, y) : \rho = r, y = 0\}, \tag{19}$$
$$W^s_{loc}(r) = \{(\theta, \rho, x, y) : \rho = r, x = 0\}. \tag{20}$$

We assume that the following conditions are satisfied:

(A) *The angle variable ρ_m can be eliminated.*

Assume that there exists a smooth real valued function g defined for all $(\rho_1, \ldots, \rho_{m-1})$, x, and y with $(\rho_1, \ldots, \rho_{m-1}, 0, x, y) \in U \times V \times V$, such that $\rho_m = g(\rho_1, \ldots, \rho_{m-1}, x, y)$ whenever $G(\rho) + \langle x, By \rangle = h$.

(B) *The stable and unstable manifolds of a distinguished torus have a topologically crossing intersection within some surface of section.*

Let S be a surface of section for the flow of X_H, defined by $\{\theta_m = 0\} \times U \times V \times V$, and $S_h = S \cap M_h$ be a surface of section for the flow restricted to the energy surface M_h. Assume that for some $r \in U$ the invariant torus $T^m(r)$ is contained in the energy surface M_h, and the unstable manifold $W^u(r)$ of $T^m(r)$ crosses topologically the stable manifold $W^s(r)$ of $T^m(r)$ within S_h. This means that there exists a good pair $V^u \subseteq W^u(r) \cap S$, $V^s \subseteq W^s(r) \cap S$ in S_h. Every point of $V^u \cap V^s$ lies on some solution curve γ which only intersects S_h once. See Figure 2.

(C) *An isoenergetic non-degeneracy condition is satisfied.*

Assume that H restricted to $T^m(r) \times \{0\} \times \{0\} \times \{0\}$ satisfies $a_m(r) \neq 0$ and

$$\det \begin{bmatrix} (\partial a_i / \partial \rho_j(r))_{i,j} & (a_i(r))_i \\ {}^t(a_i(r))_i & 0 \end{bmatrix} \neq 0 \tag{21}$$

(here ^{t} denotes the transpose of a matrix).

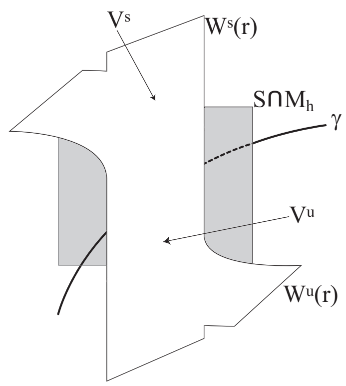

Figure 2: $W^u(r)$ topologically crossing $W^s(r)$ within the surface of section S_h.

Condition (A) implies that there exist variables $\rho' = (\rho_1, \ldots, \rho_{m-1})$ and $\theta' = (\theta_1, \ldots, \theta_{m-1})$ such that (θ', ρ', x, y) can be used as local coordinates on S_h. In these new coordinates, the energy equation becomes

$$G(\rho', g(\rho', x, y)) + \langle x, By \rangle = h.$$

We have

$$W^u_{loc}(r) \cap S = \{q \in S_h : \rho' = r', y = 0\}, \tag{22}$$

$$W^s_{loc}(r) \cap S = \{q \in S_h : \rho' = r', x = 0\}. \tag{23}$$

We can define two dynamics on S. The first dynamics is given by a Poincaré return map f defined along the flow lines contained in $T^m \times U \times V \times V$. It is obtained following the solutions of (18) one circuit around, hence

$$f(\theta, \rho, x, y) = \left(\theta + \tau(\rho)a(\rho), \rho, e^{\tau(\rho)B}x, e^{-\tau(\rho)B}y\right), \tag{24}$$

where $a(\rho) := (a_1(\rho), \ldots, a_m(\rho))$, and $\tau(\rho) = 2\pi/a_m(\rho)$.

The second dynamics is given by a Poincaré return map F defined by following orbits that approach $T^m(r)$ in the future and in the past. A key assumption in this sense is (B). It is weaker than the transversality condition required in [6] or [11]. The unstable manifold $W^u(r)$ crosses topologically the stable manifold $W^s(r)$ within S along solution curves contained in M_h. Let γ be such a solution curve. Since γ is bi-asymptotic to $T^m(r)$ (i.e. asymptotic to $T^m(r)$ in both time directions), there exists $p_0 = \gamma(t_0) \in S$ such that the backward orbit of p_0 stays in S and approaches $T^m(r)$ in backward time. There also exists $t_1 > t_0$ such that $p_1 = \gamma(t_1)$ is in S, and the forward orbit of p_1 stays in S and approaches $T^m(r)$ in forward time. Thus both p_0 and p_1 are in $W^u(r) \cap W^s(r)$. Let F be the Poincaré return map that takes p_0 into p_1. Then F maps an entire neighborhood U_0 of p_0 into a neighborhood U_1 of p_1.

Since $W^u(r)$ crosses topologically $W^s(r)$ within S, there exists a good pair $V^u \subseteq W^u(r) \cap S$, $V^s \subseteq W^s(r) \cap S$ with $p_0 \in V^u \cap V^s$. For convenience, we make an additional assumption, requiring that the 'size' of the topological crossing intersection is reasonably small.

(B′) *The topologically crossing intersection from (B) is contained in some coordinate neighborhood.*

Assume that there exists a good pair V^u, V^s for $W^s(r) \cap S$, $W^u(r) \cap S$, with $p_0 \in V^u \cap V^s$, such that $V^u \subseteq U_0$, $V^s \subseteq U_0$, $F(V_0^u) \subseteq U_1$ and $F(V_0^s) \subseteq U_1$.

The isoenergetic non-degeneracy condition (C) says that the restriction of dynamics to S_h is that of a twist map in variables (ρ', θ') (see Appendix 8 in [1]). We will need the following result.

Lemma 6 *Let $a = (a_i)_{i=1,\ldots,m}$ as before. For every $(x, y) \in V \times V$, consider the map*

$$\rho' \to (\tau a')_{(x,y)}(\rho') = \frac{2\pi}{a_m(\rho', g(\rho', x, y))}(a_1, \ldots, a_{m-1})(\rho', g(\rho', x, y)) \tag{25}$$

defined for all ρ' with $(\rho', g(\rho', x, y)) \in U$. If the isoenergetic non-degeneracy condition (21) is satisfied, then the matrix $(\partial(\tau a_i)_{(x,y)}/\partial\rho_j(r'))_{i,j=1,\ldots,m-1}$ is nonsingular and $\rho' \to (\tau a')_{(x,y)}(\rho')$ is a local diffeomorphism for ρ' in a small neighborhood of r'.

Proof: Using implicit differentiation, we obtain $\partial\rho_m/\partial\rho_j = \partial g/\partial\rho_j = -a_j/a_m$, and

$$\frac{\partial}{\partial\rho_j}\left(\frac{a_i}{a_m}\right) = \left(\frac{\partial a_i}{\partial\rho_j}\right)\frac{1}{a_m} - \left(\frac{\partial a_i}{\partial\rho_m}\right)\frac{a_j}{a_m^2} - \left(\frac{\partial a_m}{\partial\rho_j}\right)\frac{a_i}{a_m^2} + \left(\frac{\partial a_m}{\partial\rho_m}\right)\frac{a_i a_j}{a_m^3} \tag{26}$$

for all $i, j = 1, \ldots, m-1$. Condition (21) can be written as

$$\det\begin{bmatrix} \left(\frac{\partial a_i}{\partial\rho_j}\right)_{i,j} & \frac{\partial a_m}{\partial\rho_j} & (a_j)_j \\ \frac{\partial a_i}{\partial\rho_m} & \frac{\partial a_m}{\partial\rho_m} & a_m \\ {}^t(a_i)_i & a_m & 0 \end{bmatrix} \neq 0. \tag{27}$$

Performing elementary operations on the above matrix, we obtain the equivalent condition

$$\det\begin{bmatrix} \frac{1}{a_m}\left(\frac{\partial}{\partial\rho_j}\left(\frac{a_i}{a_m}\right)\right)_{i,j} & * & 0 \\ * & * & a_m \\ 0 & a_m & 0 \end{bmatrix} \neq 0, \tag{28}$$

Expanding the latter determinant proves that $(\partial(\tau a_i)_{(x,y)}/\partial\rho_j)_{i,j}$ is non-singular if and only if (21) is satisfied. ∎

4 An example

A Hamiltonian satisfying the above conditions is $H : \mathbb{R}^6 \to \mathbb{R}$, given by

$$\begin{aligned} H(x_1, y_1, x_2, y_2, x_3, y_3) &= \frac{1}{2}\left(\rho_1 + \rho_2 + \rho_2^2\right) + x_1 y_1 + \\ &+ \mu(x_1, y_1, x_2, y_2, x_3, y_3), \end{aligned}$$

where $\rho_1 = x_2^2 + y_2^2$, $\rho_2 = x_3^2 + y_3^2$, and μ is a smooth function that vanishes in some small neighborhood $B(0, \delta_1)$ of zero. Here ρ_1 and ρ_2 represents the action coordinates; the angle coordinates θ_1, θ_2 are defined by $x_2 + iy_2 = \rho_1^{1/2} e^{i\theta_1}$, and $x_3 + iy_3 = \rho_2^{1/2} e^{i\theta_2}$. In these new coordinates, the Hamilton equations near zero are

$$\begin{aligned} \dot{x}_1 &= x_1, \quad \dot{\theta}_1 = 1, \quad \dot{\theta}_2 = 1 + \rho_2, \\ \dot{y}_1 &= -y_1, \quad \dot{\rho}_1 = 0, \quad \dot{\rho}_2 = 0. \end{aligned}$$

Fix an energy level h for which the Hamiltonian flow is globally defined on M_h. One can eliminate ρ_1 from the equation $H = h$. Choose $\delta > 0$ sufficiently small such that

$$M_h \cap \{(x_1, y_1, x_2, y_2, x_3, y_3) \,|\, |x_1|, |y_1| < \delta\} \subseteq M_h \cap B(0, \delta_1).$$

It is relatively easy to show that there exists an interval $(0, R)$ such that for all $r \in (0, R)$ the equation $\rho_2 = r$ describes an invariant torus $T(r)$, whose local stable and unstable manifolds $W^s_{loc}(r)$ and $W^u_{loc}(r)$ are given by

$$\begin{aligned} W^s_{loc}(r) &= M_h \cap B(0, \delta) \cap \{(x_1, y_1, x_2, y_2, x_3, y_3) \,|\, x_1 = 0, \rho_2 = r\}, \\ W^u_{loc}(r) &= M_h \cap B(0, \delta) \cap \{(x_1, y_1, x_2, y_2, x_3, y_3) \,|\, y_1 = 0, \rho_2 = r\}. \end{aligned}$$

See [6]. Consider a surface of section S given by $\theta_1 = 0$. Assume that the function μ is chosen so that, away from the origin, there are pieces of the stable and unstable manifolds of $T(r)$ given by

$$\begin{aligned} W^s(r) &= \{(x_1, y_1, x_2, y_2, x_3, y_3) \,|\, y_1 = -(x_1 - a)^3, \rho_2 = r\}, \\ W^s(r) &= \{(x_1, y_1, x_2, y_2, x_3, y_3) \,|\, y_1 = 0, \rho_2 = r\}, \end{aligned}$$

where a is a sufficiently large positive number. It is easy to check that $W^u(r)$ and $W^s(r)$ have a topological crossing within $S \cap M_h$. Since $a_1 = 1$ and $a_2 = 1 + \rho_2$, the isoenergetic non-degeneracy condition is satisfied, since

$$\det \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 + \rho_2 \\ 1 & 1 + \rho_2 & 0 \end{bmatrix} \neq 0.$$

5 Existence of symbolic dynamics

For every $r_0 \in U$ with $T^m(r_0) \subseteq M_h$, the torus $T^m(r_0)$ is invariant to the hamiltonian flow. By condition (C), there exists a positive $\delta > 0$ such that $a_m(r_0) \neq 0$ and

$$\begin{pmatrix} (\partial a_i/\partial \rho_j(r_0))_{i,j} & (a_i(r_0))_i \\ {}^t(a_i(r_0))_i & 0 \end{pmatrix} \tag{29}$$

is nonsingular for each r with $\|r - r_0\| < \delta$. If δ is chosen small enough, condition (B) implies that $W^u(r_i)$ and $W^s(r_j)$ have non-empty topological crossing for all r_i, r_j in a δ-neighborhood of r, with $T^m(r_i), T^m(r_j) \subseteq M_h$. Moreover, due to assumption (B′), there exists a good pair V_i^u, V_i^s in U_0, such that $F(V_i^u)$, $F(V_i^s)$ is in U_1. Since we have

$$W^u_{loc}(r_i) \cap S = \{q \in S_h : \rho' = r_i', y = 0\}, \tag{30}$$
$$W^s_{loc}(r_j) \cap S = \{q \in S_h : \rho' = r_j', x = 0\}, \tag{31}$$

it results that for any point $q_{ij} \in W^u(r_i) \cap F^{-1}(W^s(r_j))$ in S_h, we have

$$y(q_{ij}) = 0 \text{ and } x(F(q_{ij})) = 0.$$

The main theorem of this paper, stated below, says that there exist symbolic dynamics in any sufficiently small neighborhood of these tori. The symbolic dynamics is described in terms of an orbit that shadows any prescribed sequence of tori. Since this sequence of tori is arbitrarily chosen, it means that there exist chaotic excursions between these tori.

Theorem 7 *There exists $\delta > 0$ such that for every bi-infinite sequence of radii $\{r_i\}_{i\in\mathbb{Z}}$ in $\mathbb{R}^m$ with $\|r_i - r\| < \delta$ for all i, and every bi-infinite sequence of positive numbers $\{\epsilon_i\}_{i\in\mathbb{Z}}$, there exist $p_i, p_i' > 0$, and an orbit $\{z_i\}_{i\in\mathbb{Z}}$ such that $z_{i+1} = f^{p_i} \circ F \circ f^{p_i'}(z_i)$ and*

$$d\left(z_i, T^m(r_i)\right) < \epsilon_i \text{ for all } i \in \mathbb{Z}.$$

Moreover, if the sequences $\{r_i\}_{i\in\mathbb{Z}}$ and $\{\epsilon_i\}_{i\in\mathbb{Z}}$ are periodic with the same period, then the orbit $\{z_i\}_{i\in\mathbb{Z}}$ can be chosen with the same period. In particular, there exist periodic orbits with arbitrarily high periods.

Proof: By Lemma 6, $\rho' \to (\tau a')_{(x,y)}(\rho')$ is a diffeomorphism on some neighborhood $V'_{(x,y)}$ of r'. By the compactness of the surface of section, one can choose such a neighborhood V' of r' independently of x and y.

Choose $\delta > 0$ sufficiently small, so that $\|r - r_i\| < \delta$ implies that $r_i' \in V'$.

Recall that U_0 and U_1 are neighborhoods of p_0 and p_1, respectively, on which the homoclinic map $F : U_0 \to U_1$ is a diffeomorphism. For each $i \in \mathbb{Z}$, let V_i^u, V_i^s be a good pair for $W^u(r_i) \cap S$ and $W^s(r_{i+1}) \cap S$ in M_h, such that V_i^u and V_i^s are both contained in U_0. In the view of the above discussion, such a pair always exists, provided δ is chosen sufficiently small. Let $K_i = V_i^u \cap V_i^s$. There exists $\hat{q}_i \in K_i$ with $y(\hat{q}_i) = 0$ and $x(F(\hat{q}_i)) = 0$. We rearrange the coordinates on S_h

as a vector u, in the order $u = (x, \theta', y, \rho')$. Let $(\hat{x}_i, \hat{\theta}'_i, 0, r'_i)$ be the coordinates of $\hat{q}_i$ in S_h. There exist $\xi_i > 0$, $\kappa_i > 0$, such that

$$K_i \subseteq B_n(\hat{x}_i, \xi_i) \times B_{m-1}(\hat{\theta}'_i, \kappa_i) \times \{0\} \times \{r'_i\}. \tag{32}$$

By (24), for every $p > 0$ we have:

$$f^{-p}(K_i) \subseteq e^{-p\tau(r_i)B} B_n(\hat{x}_i, \xi_i) \times B_{m-1}(\hat{\theta}'_i - p\tau(r_i)a'(r_i), \kappa_i) \times \{0\} \times \{r'_i\}, \tag{33}$$

where $a' = (a_1, \ldots, a_{m-1})$. There exists $p'_i > 0$ such that $\left\| e^{-p'_i\tau(r_i)B}\hat{x}_i \right\| < \epsilon_i/4$ and $e^{-p'_i\tau(r_i)B} B_n(\hat{x}_i, \xi_i) \subseteq B_n(e^{-p\tau(r_i)B}\hat{x}_i, \epsilon_i/4)$. Let $x_i = e^{-p'_i\tau(r_i)B}\hat{x}_i$ and $\theta_i = \hat{\theta}_i - p'_i\tau(r_i)a(r_i)$. Then the coordinates of $q_i = f^{-p'_i}(\hat{q}_i)$ are $(x_i, \theta'_i, 0, r'_i)$. We have

$$f^{-p'_i}(K_i) \subseteq B_n(x_i, \epsilon_i/4) \times B_{m-1}(\theta'_i, \kappa_i) \times \{0\} \times \{r'_i\}. \tag{34}$$

We have that $F'_i = F \circ f^{p'_i}$ maps an entire neighborhood $f^{-p'_i}(U_0)$ of $f^{-p'_i}(p_0)$ into the neighborhood U_1 of p_1. Let $(0, \bar{\theta}'_i, \bar{y}_i, r'_{i+1})$ be the coordinates of $F'(q_i)$ in S_h.

Now we need to construct some h-sets and establish the covering relations among them. Let

$$C = \overline{B_n}(0,1) \times \overline{B_{m-1}}(0,1) \times \overline{B_n}(0,1) \times \overline{B_{m-1}}(0,1) \tag{35}$$

For each 'counter' $i \in \mathbb{Z}$ and each 'shrinking factor' $0 < s_i < 1$, we define the h-set N_i by:

$$c_{N_i}^{-1} : C \to S_h, \tag{36}$$

$$c_{N_i}^{-1}(u) = (x_i, \theta'_i, 0, r'_i) + \left(\frac{\epsilon_i}{4}x, \kappa_i\theta', \frac{\epsilon_i}{4}s_i y, \frac{\epsilon_i}{4}s_i\rho'\right), \tag{37}$$

$$N_i^{-} = c_{N_i}^{-1}\left[\partial\left(\overline{B_n}(0,1) \times \overline{B_{m-1}}(0,1)\right) \times \overline{B_n}(0,1) \times \overline{B_{m-1}}(0,1)\right], \tag{38}$$

$$N_i^{+} = c_{N_i}^{-1}\left[\overline{B_n}(0,1) \times \overline{B_{m-1}}(0,1) \times \partial\left(\overline{B_n}(0,1) \times \overline{B_{m-1}}(0,1)\right)\right]. \tag{39}$$

For each $i \in \mathbb{Z}$ and each quadruple of positive real numbers $(\alpha_i, \beta_i, \gamma_i, \delta_i)$ we define the h-set M_i by

$$c_{M_i}^{-1} : C \to S_h, \tag{40}$$

$$c_{M_i}^{-1}(u) = (0, \bar{\theta}'_i, \bar{y}_i, r'_{i+1}) + (\alpha_i x, \beta_i\theta', \gamma_i y, \delta_i\rho'), \tag{41}$$

$$\begin{aligned} M_i^{-} = c_{M_i}^{-1} \big[& (\partial\overline{B_n}(0,1) \times \overline{B_{m-1}}(0,1) \times \overline{B_n}(0,1) \times \overline{B_{m-1}}(0,1)) \cup \\ & \cup (\overline{B_n}(0,1) \times \overline{B_{m-1}}(0,1) \times \overline{B_n}(0,1) \times \partial\overline{B_{m-1}}(0,1)) \big], \end{aligned} \tag{42}$$

$$M_i^{+} = c_{M_i}^{-1}\left[\overline{B_n}(0,1) \times \partial\left(\overline{B_{m-1}}(0,1) \times \partial\overline{B_n}(0,1)\right) \times \overline{B_{m-1}}(0,1)\right]. \tag{43}$$

We will apply the following two lemmas which describe the covering relations between N_i and M_i, and between M_i and N_{i+1}. Their proofs are postponed to the next section.

Lemma 8 *For every $\epsilon_i > 0$, there exist a sufficiently small s_i, sufficiently small α_i and δ_i (depending on s_i), sufficiently large β_i and γ_i (depending on s_i), such that*

$$N_i \overset{F'_i, w_i^1}{\Longrightarrow} M_i,$$

for some $w_i^1 \neq 0$.

Lemma 9 *For every sufficiently small s_{i+1}, there exists a sufficiently small $\delta_i > 0$ and a sufficiently large p_i with*

$$M_i \overset{f^{p_i}, w_i^2}{\Longrightarrow} N_{i+1},$$

for some $w_i^2 \neq 0$.

Informally speaking, F'_i stretches the (x, θ') directions of N_i along the (x, ρ') directions of M_i, due to the topological crossing condition, and f^{p_i} stretches the (x, ρ') directions of M_i along the (x, θ') directions of N_{i+1}, due to the twist condition. As we will see, the degree of the covering relation between N_i and M_i is equal to oriented intersection number of the topological crossing. This intersection number is not equal to zero, but it is not necessarily equal to ± 1. On the other hand, we will see that the covering relation between M_i and N_{i+1} has always a degree equal to ± 1. This is due to the 'twist' type of condition expressed by the isoenergetic non-degeneracy condition (C).

Remark the 'double switch' of the variables θ' and ρ', which is the key idea of the proof. It shows that the covering relations between the constructed h-sets are quite complex, and they do not reduce to a lower dimensional type of covering or to a degree ± 1 type of covering (as described in Section 2 of [12]).

We now continue with the proof of Theorem 7. Choose and fix the h-sets N_i first (this means that we choose the sequence s_i first), then choose the h-sets M_i, as in Lemma 8 and Lemma 9. We obtain

$$\ldots \Rightarrow N_i \overset{F\circ f^{p'_i}, w_i^1}{\Longrightarrow} M_i \overset{f^{p_i}, w_i^2}{\Longrightarrow} N_{i+1} \overset{F\circ f^{p'_{i+1}}, w_{i+1}^1}{\Longrightarrow} M_{i+1} \overset{f^{p_{i+1}}, w_{i+1}^2}{\Longrightarrow} N_{i+2} \Rightarrow \ldots \quad (44)$$

By Proposition 2, we have that there exists an orbit $\{z_i\}$ such that $z_i \in N_i$, for all i, $z_{i+1} = f^{p_i} \circ F \circ f^{p'_i}(z_i)$. Since $z_i \in N_i$, the x-coordinate of z_i, is at a distance of less $\epsilon_i/2$ away from the x-coordinate of q_i. By $d(q_i, T^m(r_i)) < \epsilon_i/2$, it follows that $d(z_i, T^m(r_i)) < \epsilon_i$, for all $i \in \mathbb{Z}$. The statement on periodicity also follows from Proposition 2. ∎

Remark 10 *Note that, for the class of Hamiltonian systems given by (17) and satisfying the conditions (A), (B), (C), and (B$'$), Theorem 7 confirms the so called Holmes-Marsden conjecture (see [10]), on the existence of periodic motions of arbitrarily high period, near two-way transition chains of tori.*

Remark 11 *If the sequence of tori is finite, or if the sequence $\epsilon_{i+1}/\epsilon_i$ is bounded above and below away from zero (see [6]), such a shadowing orbit persists for all sufficiently small C^1-perturbations of the Hamiltonian vector field.*

6 Proofs of Lemma 8 and Lemma 9

Proof of Lemma 8: By construction, we have that $f^{-p'_i}(K_i) \subseteq N_i$, moreover

$$f^{-p'_i}(K_i) \subseteq V_i^u \cap c_{N_i}^{-1}\left[B_n(0,1) \times B_{m-1}(0,1) \times \{0\} \times \{0\}\right]. \tag{45}$$

We also have that $F(V_i^u)$ and $F(V_i^s)$ constitutes a good pair in $U_1 = F(U_0)$, and $F(V_i^u) \cap F(V_i^s) = F(K_i)$. Since $F(V_i^s) \subseteq W^s_{loc}(r_{i+1}) \cap S_h$, it results that

$$F'_i \circ c_{N_i}^{-1}\left[\partial\left(\overline{B_n}(0,1) \times \overline{B_{m-1}}(0,1)\right) \times \{0\} \times \{0\}\right] \cap W^s_{loc}(r_{i+1}) \cap S_h = \emptyset.$$

We need to check that, for appropriate choices of the parameters, the conditions in Theorem 1 are satisfied.

Condition 1. We choose s_i sufficiently small such that

$$(F'_i \circ c_{N_i}^{-1})\left(\{x\} \times \{\theta'\} \times \overline{B_m}(0,1) \times \overline{B_{m-1}}(0,1)\right) \cap W^s_{loc}(r_{i+1}) \cap S_h = \emptyset \tag{46}$$

for all $(x,\theta') \in \partial\left(\overline{B_n}(0,1) \times \overline{B_{m-1}}(0,1)\right)$.

Then choose

$$\alpha_i, \delta_i < d\left((F'_i \circ c_{N_i}^{-1})(\{x\} \times \{\theta'\} \times \overline{B_m}(0,1) \times \overline{B_{m-1}}(0,1)), W^s_{loc}(r_{i+1}) \cap S\right)$$

for all $(x,\theta') \in \partial\left(\overline{B_m}(0,1) \times \overline{B_{m-1}}(0,1)\right)$.

At last, choose β_i, γ_i sufficiently large, in order to have $F(K_i) \subseteq M_i$ and

$$c_{M_i}^{-1}\left(\overline{B_n}(0,1) \times \{\bar\theta'\} \times \{y\} \times \overline{B_{m-1}}(0,1)\right) \cap F'_i(N_i) = \emptyset \tag{47}$$

for all $(y,\theta') \in \partial\left(\overline{B_m}(0,1) \times \overline{B_{m-1}}(0,1)\right)$. It results that

$$(F'_i)(N_i^-) \cap M_i = \emptyset, \tag{48}$$

$$(F'_i)(N_i) \cap M_i^+ = \emptyset, \tag{49}$$

and

$$(F'_i)_c(\overline{B_n}(0,1) \times \overline{B_{m-1}}(0,1) \times \{0\} \times \{0\}) \subseteq \mathbb{R}^n \times B_{m-1}(0,1) \times B_n(0,1) \times \mathbb{R}^{m-1}.$$

Condition 2. Consider the neighborhood U_1 of p_1, containing the good pair $F(V_i^u)$ and $F(V_i^s) \subseteq W^s_{loc}(r_{i+1})$, and the coordinate system $(x'', \theta'', y'', \rho'')$ on U_1, defined by $x'' = (1/\alpha_i)x$, $\theta'' = (1/\beta_i)(\theta' - \bar\theta_i)$, $y'' = (1/\gamma_i)(y - \bar y_i)$, and $\rho'' = (1/\delta_i)(\rho' - r'_{i+1})$ on U_1. Note that this coordinate system is defined by the map c_{M_i}. We have

$$F(V_i^s) \subseteq \{(x'', \theta'', y'', \rho'') \,|\, x'' = 0,\ \rho'' = 0\}, \tag{50}$$

$$F(V_i^u) = \{(F'_i \circ c_{N_i}^{-1})(x, \theta', 0, 0) \,|\, (x,\theta') \in \overline{B_n}(0,1) \times \overline{B_{m-1}}(0,1)\}. \tag{51}$$

Let $A : \overline{B_n}(0,1) \times \overline{B_{m-1}}(0,1) \subseteq \mathbb{R}^{n+m-1} \to \mathbb{R}^{n+m-1}$ be the map defined by

$$A(x,\theta') = \pi_{(x'',\rho'')} \circ F'_i \circ c_{N_i}^{-1}(x, \theta', 0, 0). \tag{52}$$

According to Lemma 3, we have

$$(0,0) \notin A\left[\partial\left(\overline{B_n}(0,1) \times \overline{B_{m-1}}(0,1)\right)\right],$$

and

$$\deg(A, \overline{B_n}(0,1) \times \overline{B_{m-1}}(0,1), 0) = \iota \neq 0,$$

where ι is the oriented intersection number of $F(V_i^u)$ and $F(V_i^s)$. Note that

$$A = \pi_{(x,\theta')} \circ c_{M_i} \circ F_i' \circ c_{N_i}^{-1}, \tag{53}$$

and

$$A\left[\partial\left(\overline{B_n}(0,1) \times \overline{B_{m-1}}(0,1)\right)\right] \subseteq \mathbb{R}^{n+m-1} \setminus \left(\overline{B_n}(0,1) \times \overline{B_{m-1}}(0,1)\right),$$

provided α_i and δ_i are chosen sufficiently small.

Using Theorem 1, we conclude that

$$N_i \overset{F_i', w_i^1}{\Longrightarrow} M_i,$$

where $w_i^1 = \iota$. We emphasize that we can make the size of δ_i arbitrarily small, without changing α_i, β_i, γ_i and s_i, and still preserve the covering relation (in fact, the same is true about α_i, but we will not need that). ∎

Proof of Lemma 9: From (24), we have that f^p restricted to S_h is given by

$$f^p(u) = (e^{p\tau_{(x,y)}(\rho')B}x, \theta' + p(\tau a')_{(x,y)}(\rho'), e^{-p\tau_{(x,y)}(\rho')B}y, \rho'),$$

where $\tau_{(x,y)}(\rho') = \tau(\rho', g(\rho', x, y))$ and $a_{(x,y)}(\rho') = a(\rho', g(\rho', x, y))$.

Let (X, Θ', Y, R') denote the components $f^p \circ c_{M_i}^{-1}$, hence

$$X(u) = e^{p\tau_{(\alpha_i x, \bar{y}_i + \gamma_i y)}(r'_{i+1} + \delta_i \rho')B}\alpha_i x, \tag{54}$$

$$\Theta'(u) = \bar{\theta}_i + \beta_i\theta' + p(\tau a')_{(\alpha_i x, \bar{y}_i + \gamma_i y)}(r'_{i+1} + \delta_i\rho'), \tag{55}$$

$$Y(u) = e^{-p\tau_{(\alpha_i x, \bar{y}_i + \gamma_i y)}(r'_{i+1} + \delta_i \rho')B}(\bar{y}_i + \gamma_i y), \tag{56}$$

$$R'(u) = r'_{i+1} + \delta_i\rho'. \tag{57}$$

Recall that V' is a neighborhood of r' containing all r'_i, on which the mapping $\rho' \to (\tau a')_{(x,y)}(\rho')$ is a diffeomorphism, for all $(x, y) \in V \times V$. Choose δ_i sufficiently small such that $r'_{i+1} + \delta_i\rho'$ is in V' for all $\rho' \in \overline{B_{m-1}}(0,1)$. Fix $x = y = 0$. There exists a positive real number h such that

$$\overline{B_{m-1}}((\tau a')_{(0,\bar{y}_i)}(r'_{i+1}), h) \subseteq (\tau a')_{(0,\bar{y}_i)}\left(r'_{i+1} + \overline{B_{m-1}}(0,\delta_i)\right). \tag{58}$$

This implies that $p(\tau a')_{(0,\bar{y}_i)}(r'_{i+1} + \overline{B_{m-1}}(0,\delta_i))$ contains a ball of radius $p \cdot h$, where the size of this ball can be made arbitrarily large by choosing p large.

We need to check that the conditions in Theorem 1 are satisfied.

Condition 1. First, we need to take an appropriate number of iterates of f such that the hyperbolic contraction in the y-direction and the hyperbolic expansion in the x-direction become consistent with the desired covering relation. Choose p so that

$$x_{i+1} + \overline{B_n}\left(0, \frac{\epsilon_{i+1}}{4}\right) \subset e^{p\tau_{(\alpha_i x, \bar{y}_i + \gamma_i y)}(r'_{i+1} + \delta_i \rho')B} B_n(0, \alpha_i), \tag{59}$$

$$B_n\left(0, s_{i+1}\frac{\epsilon_{i+1}}{4}\right) \supset e^{-p\tau_{(\alpha_i x, \bar{y}_i + \gamma_i y)}(r'_{i+1} + \delta_i \rho')B} \left(\bar{y}_i + \overline{B_n}(0, \gamma_i)\right), \tag{60}$$

for all $x, y \in \overline{B_n}(0,1)$ and all $\rho' \in \overline{B_{m-1}}(0,1)$. Notice that we still have freedom to increase p.

The (θ', ρ')-coordinate section is foliated by invariant $(m-1)$-dimensional tori. There is a twist condition, due to Lemma 6 (see also Appendix 8 in [1]), that guarantees that the frequencies of rotation along these tori vary with $\rho' \in V'$. Recall that the angle variable θ' is uniquely determined up to multiples $2\pi l$ of 2π, with $l = (l_1, \ldots, l_{m-1}) \in \mathbb{Z}^{m-1}$. We need to take an appropriately large number of iterates of f so that the image of $\overline{B_{m-1}}(\bar{\theta}'_i, \beta_i) \times \overline{B_{m-1}}(r'_{i+1}, \delta_i)$ covers $\overline{B_{m-1}}(\theta'_{i+1}, \kappa_{i+1}) \times \overline{B_{m-1}}(r'_{i+1}, (s_{i+1}\epsilon_{i+1})/4)$.

First, we fix $\delta_i < (s_{i+1}\epsilon_{i+1})/4$ (we have noticed earlier that δ_i can be made arbitrarily small independently of the other quantities), and then we eventually increase p, so that

$$\left(\bar{\theta}'_i + p(\tau a')_{(0,\bar{y}_i)}(\rho') + \overline{B_{m-1}}(0, \beta_i)\right) \cap \left(\theta'_{i+1} + 2\pi l + \overline{B_{m-1}}\,(0, \kappa_{i+1})\right) = \emptyset, \tag{61}$$

for all $\rho' \in \partial B_{m-1}(r'_{i+1}, \delta_i)$, and

$$\left(\theta'_{i+1} + 2\pi l + \overline{B_{m-1}}\,(0, \kappa_{i+1})\right) \subset \left(\bar{\theta}'_i + p(\tau a')_{(0,\bar{y}_i)}(r'_{i+1} + B_{m-1}(0, \delta_i))\right), \tag{62}$$

for some $l = (l_1, \ldots, l_{m-1}) \in \mathbb{Z}^{m-1}$. Equation (62) results from (58). For simplicity, from now on we will assume that $l_i = 0$. Equation (62) will be used later, in the computation of the degree of covering.

From our choice of δ_i we obtain

$$\begin{aligned}(\Theta, R')\left(\left(\bar{\theta}'_i + \overline{B_{m-1}}(0, \beta_i)\right) \times \left(r'_{i+1} + \overline{B_{m-1}}(0, \delta_i)\right)\right) \cap \\ \cap \left(\theta'_{i+1} + \overline{B_{m-1}}\,(0, \kappa_{i+1})\right) \times \left(r'_{i+1} + \partial B_{m-1}\left(0, \frac{s_{i+1}\epsilon_{i+1}}{4}\right)\right) = \emptyset.\end{aligned} \tag{63}$$

From (61) we obtain

$$(\Theta, R')\left(\{\bar{\theta}'_i\} \times (r'_{i+1} + \overline{B_{m-1}}(0, \delta_i))\right) \subseteq \mathbb{R}^{m-1} \times \left(r'_{i+1} + B_{m-1}\left(0, \frac{s_{i+1}\epsilon_{i+1}}{4}\right)\right), \tag{64}$$

and

$$\begin{aligned}(\Theta, R')\left(\left(\bar{\theta}'_i + \overline{B_{m-1}}(0, \beta_i)\right) \times \left(r'_{i+1} + \partial B_{m-1}(0, \delta_i)\right)\right) \cap \\ \cap \left(\theta'_{i+1} + \overline{B_{m-1}}\,(0, \kappa_{i+1})\right) \times \left(r'_{i+1} + \overline{B_{m-1}}\left(0, \frac{s_{i+1}\epsilon_{i+1}}{4}\right)\right) = \emptyset.\end{aligned} \tag{65}$$

See Figure 3.

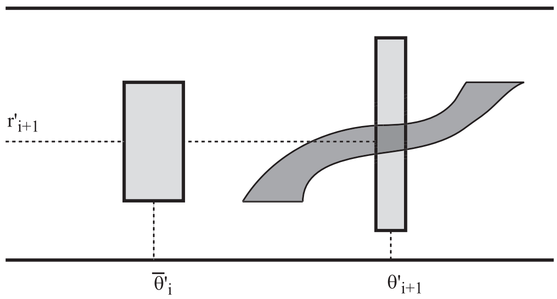

Figure 3: A schematic illustration of the effect of the twist in (θ', ρ')-coordinates.

We have that $f_c^p = c_{N_{i+1}} \circ f^p \circ c_{M_i}^{-1}$ is given by

$$f_c^p(x, \theta', y, \rho') = \tag{66}$$
$$= \left(\frac{4}{\epsilon_{i+1}} \left(e^{p\tau(r'_{i+1}+\delta_i \rho')B} \alpha_i x - x_{i+1} \right), \right.$$
$$\frac{1}{\kappa_{i+1}} \left(\beta_i \theta' + \bar{\theta}_i + p\tau(r'_{i+1} + \delta_i \rho') a'(r'_{i+1} + \delta_i \rho') - \theta'_{i+1} \right),$$
$$\left. \frac{4}{s_{i+1}\epsilon_{i+1}} \left(e^{-p\tau(r'_{i+1}+\delta_i \rho')B} (\gamma_i y + \bar{y}_i) \right), \frac{4\delta_i}{s_{i+1}\epsilon_{i+1}} \rho' \right).$$

Combine (63), (64), (65) with (59) and (60) to obtain

$$f_c^p \left(\overline{B_n}(0,1) \times \{0\} \times \{0\} \times \overline{B_{m-1}}(0,1) \right) \subseteq \tag{67}$$
$$\subseteq \mathbb{R}^n \times \mathbb{R}^{m-1} \times B_n(0,1) \times B_{m-1}(0,1),$$
$$f_c^p((M_i)_c^-) \cap (N_{i+1})_c = \emptyset, \tag{68}$$
$$f_c^p((M_i)_c) \cap (N_{i+1})_c^+ = \emptyset. \tag{69}$$

Condition 2. From (66), (59), and (61) we obtain that

$$A_{(0,0)} = \pi_{(x,\theta')} \circ f_c^p(x, 0, 0, \rho') : \mathbb{R}^{m-1+n} \to \mathbb{R}^{m-1+n},$$

satisfies

$$A_{(0,0)} \left(\overline{B_{m-1}}(0,1) \times \overline{B_n}(0,1) \right) \subset \mathbb{R}^{m-1} \times \mathbb{R}^n \setminus \left(\overline{B_{m-1}}(0,1) \times \overline{B_n}(0,1) \right). \tag{70}$$

Notice that

$$A_{(0,0)}(x, \rho') = (\pi_x \circ f_c^p(x, 0, 0, \rho'), \pi_{\theta'} \circ f_c^p(x, 0, 0, \rho')), \tag{71}$$

where

$$\pi_x \circ f_c^p(x, \rho') = \frac{4}{\epsilon_{i+1}} \left(e^{p\tau(r'_{i+1}+\delta_i \rho')B} \alpha_i x - x_{i+1} \right),$$

and

$$\pi_{\theta'} \circ f_c^p(x, \rho') = \frac{1}{\kappa_{i+1}} \left(\bar{\theta}_i + p\tau(r'_{i+1} + \delta_i \rho') a'(r'_{i+1} + \delta_i \rho') - \theta'_{i+1} \right).$$

The component $\pi_x \circ f_c^p(x, , 0, 0, \rho')$ is homotopic to an affine map $A_x(x)$ depending only on the variable x, and the component $A_{\theta'}(\rho') := \pi_{\theta'} \circ f_c^p(x, , 0, 0, \rho')$ depends only on ρ'. Here the subscripts of A_x and $A_{\theta'}$ merely indicate the component places. Hence $A_{(0,0)}(x, \rho')$ is homotopic to a product map $(A_x(x), A_{\theta'}(\rho'))$.

Since A_x is affine, from the local degree for affine maps property (see the Appendix of [12]), we have $\deg(A_x, \overline{B_n}(0,1), 0) = \pm 1$.

Let us remind the reader, that the map $A_{\theta'}(\rho') = \pi_{\theta'} \circ f_c^p(0, 0, 0, \rho')$ is a diffeomorphism onto its image, and from (62) it follows that

$$\overline{B_{m-1}}(0,1) \subset A_{\theta'}(B_{m-1}(0,1)), \tag{72}$$

Hence there exists one and only $\rho^0 \in B_{m-1}(0,1)$ such that $A'_\theta(\rho^0) = 0$. Since the local Brouwer degree of a mapping at a regular value equals the sum of the signs of the determinants of the Jacobians at the corresponding regular points (Corollary 21 in the Appendix of [12]) we have

$$\deg(A_{\theta'}, \overline{B_{m-1}}(0,1), 0) = \det D(A_{\theta'})(\rho^0) = \pm 1, \tag{73}$$

where by $D(A_{\theta'})(\rho^0)$ we denote the jacobian matrix of $A_{\theta'}$ at ρ^0. Applying the product and the homotopy properties, we obtain

$$\deg(A_{(0,0)}, \overline{B_{m-1}}(0,1) \times \overline{B_n}(0,1), 0) = \pm 1 \neq 0.$$

We conclude that

$$M_i \overset{f^p, w_i^2}{\Longrightarrow} N_{i+1},$$

where $w_i^2 = \pm 1$. ∎

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