MULTIDIMENSIONAL PERTURBATIONS OF ONE-DIMENSIONAL MAPS AND STABILITY OF ŠARKOVSKIĬ ORDERING

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In this paper we present Šarkovskiĭ theorem for multidimensional perturbations of one-dimensional maps. We outline a proof that if the unperturbed one-dimensional map has a point of period n, then sufficiently close multidimensional perturbations of this map have periodic points of all periods which are allowed by Šarkovskiĭ theorem. We describe also how the approach from the proof of this theorem is used to show the existence of a infinite number of periodic points for Rössler equations.

1. Introduction

The aim of this paper is to present a multidimensional generalization of Šarkovskiĭ theorem [Šarkovskiĭ, 1964](see also [Šarkovskiĭ, 1995, Stefan, 1977, Block *et al.*, 1980, Devaney, 1989, Li & Yorke, 1975]) concerning the appearance of periodic points for self-maps on the real line. The paper addresses an interesting question : To what extent can Šarkovskiĭ 's Theorem be carried over to multidimensional dynamics ? As a partial answer to this question we show that the Šarkovskiĭ ordering is stable with respect to small multidimensional perturbations.

For homeomorphisms or orientation preserving embeddings of a two-dimensional disc there exists a considerable literature concerning forcing relations between periodic orbits (see, for example, [Gambaudo *et al.*, 1989, Gambaudo *et al.*, 1990, Hall, 1994, Kolev, 1994, Boyland, 1994]) studied using topological tools (braid type), which are based on the Thurston theorem [Thurston, 1988], which gives a canonical form of every surface homeomorphism up to isotopy. But these topological tools appear to be applicable only in dimension two.

In dimension greater than two, there are a few results in this direction, for example Kloeden [Kloeden, 1979] extended Šarkovskiĭ 's Theorem to the so-called triangular maps. Related to this subject are also the results concerning the order in which periodic orbits can appear when a horseshoe is created (see for example [Hall, 1994, Holmes *et al.*, 1984, Strien, 1981]), but in this case the maps under consideration are either restricted to the plane or some assumptions concerning derivatives must be made. We show that Šarkovskiĭ 's theorem is true also for multidimensional perturbations of one-dimensional maps. We emphasize that our result is purely topological. We neither assume the existence of derivatives nor the existence of invariant foliations as in the case of triangular maps.

Let $f : X \to X$ be a map. We say that $x \in X$ is a periodic point of period n if $f^n(x) = x$ and $f^k(x) \neq x$ for $k = 1, 2, \ldots, n-1$.

We begin by recalling the theorem of Šarkovskiĭ .

Theorem 1.1. *Let the ordering of positive integers be the following:*

$$\begin{gathered} 3 \lhd 5 \lhd 7 \lhd \ldots \lhd 2 \cdot 3 \lhd 2 \cdot 5 \lhd 2 \cdot 7 \ldots \\ \lhd 2^2 \cdot 3 \lhd 2^2 \cdot 5 \lhd 2^2 \cdot 7 \lhd \ldots \\ \lhd 2^3 \cdot 3 \lhd 2^3 \cdot 5 \lhd 2^3 \cdot 7 \lhd \ldots \lhd \\ \ldots \\ \ldots \lhd 2^s \lhd 2^{s-1} \lhd \ldots \lhd 2^3 \lhd 2^2 \lhd 2 \lhd 1 \end{gathered}$$

Let $f : I \to \mathbb{R}$ be a continuous map of an interval into the real line. If $n \lhd k$ and f has a periodic point of period n then f also has a periodic point of period k.

The ordering described in Theorem 1.1 is called *the Šarkovskiĭ ordering.* We reserve the symbol '$\lhd$' for this order.

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map. Let V be a real Banach space and let a continuous decomposition $V = \mathbb{R} \oplus W$ be given. According to this decomposition we will represent elements $v \in V$ as pairs $v = (x, w)$, where $x \in \mathbb{R}$ and $w \in W$. Let $F : [0,1] \times V \to V$ be a continuous and compact map. In the sequel we will use the notation F_λ for the partial map with fixed λ, so $F_\lambda(v) := F(\lambda, v)$ for $v \in V$. Suppose that $F_0(x, w) = (f(x), 0)$. We say that the maps F_λ are *multidimensional perturbations of f*. Let us recall that a continuous map is called *compact*, iff it maps bounded sets into relatively compact sets.

The main theorem in this paper is stated below ([Zgliczyński, 1997a])

Theorem 1.2. *Let f, F be as above. If* $\dim V = \infty$ *then we additionally assume that F is uniformly continuous and compact. Suppose f has a point of period n.*

For every $r > 0$ there exists $\lambda_0 > 0$ such that for $\lambda \leq \lambda_0$ holds
if $n \lhd m$, $m \neq n$ then F_λ has a periodic point of period m in the set $\mathbb{R} \oplus B_W(r)$, where $B_W(r)$ is an open ball in W of radius r.

In Sec. 2 we present the main ideas of the proof of Theorem 1.2.

Section 3 is a description of how the ideas from the proof of Theorem 1.2 are used in practice, to obtain rigorous results for a concrete map. We consider the Poincaré map for the Rössler system of equations [Rössler, 1976].

For a suitably chosen Poincaré map for the Rössler system of equations (with parameter values $a = 5.7, b = 0.2$) the numerically observed chaotic attractor is apparently nearly one-dimensional and it has been shown by author [Zgliczyński, 1997a] that it has symbolic dynamics and points of all periods. Here we show that all those periodic points have one-dimensional origin (i.e. are continuations of a suitably chosen periodic point for a one-dimensional map, which is an approximation of the true Poincaré map).

In the last section we list other examples to which this approach was applied.

2. Main ideas of the proof of theorem 2

The tool used to obtain periodic points is *the fixed point index* (see [Dold, 1972, Granas, 1972; Ch. VII, sec. 5]). The proof of Theorem 1.2 for odd periods (n odd) has two parts:

1. All periods greater than some integer M ($> 2n$) are obtained using nested sequences of topological horseshoes for iterates of F_λ
2. Periods less than M are obtained by simple continuation. Appropriate periodic orbits for F_0 (one-dimensional map) have a nontrivial fixed point index, so it continues to nearby λ, but in this way only the existence of a finite number of periodic orbits can be obtained.

The result for even periods (n even) follows from that for odd ones, in a slighty more complicated way than in the one-dimensional case, see [Zgliczyński, 1997a] for details.

The rest of this section is devoted to the presentation of the main points of the proof of Theorem 1.2. Part 1 follows immediately from Theorem 2.6 and part 2 is obtained from Theorem 2.11.

For the full proof the reader is referred to [Zgliczyński, 1997a].

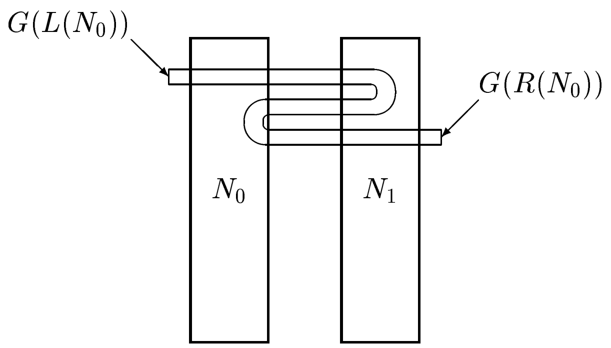

Fig. 1. $N_0 \overset{G}{\Longrightarrow} N_1$

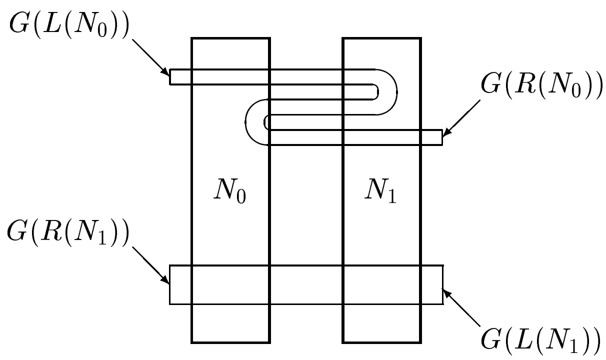

Fig. 2. Topological Smale horseshoe.

2.1. Topological horseshoes

Let us fix $r > 0$.

We will denote by $\mathcal{C}(r)$ the family of cylinders of the form $[a,b] \times \overline{B_W(r)}$.

Given $N = [a,b] \times \overline{B_W(r)} \in \mathcal{C}(r)$, we define

$$\begin{aligned} L(N) &= \{a\} \times \overline{B_W(r)}, \quad \text{left edge}, \\ R(N) &= \{b\} \times \overline{B_W(r)}, \quad \text{right edge}, \\ H(N) &= [a,b] \times \partial B_W(r), \quad \text{'horiz. boundary'} \\ S_L(N) &= (-\infty, a) \times B_W(r), \quad \text{left side of } N \\ S_R(N) &= (b, \infty) \times B_W(r), \quad \text{right side of } N \end{aligned}$$

The following notion plays a crucial role in the proof of Theorem 1.2

Definition 2.1. Let $N_0, N_1 \in \mathcal{C}(r)$. $G : V \to V$, we say N_0 **G-covers** N_1 **horizontally** (notation: $N_0 \overset{G}{\Longrightarrow} N_1$) iff

$$G(N_0) \subset (-\infty, \infty) \times B_W(r) \tag{1}$$

and one of the following two conditions hold

$$G(L(N_0)) \subset S_L(N_1), G(R(N_0)) \subset S_R(N_1) \tag{2}$$
$$G(L(N_0)) \subset S_R(N_1), G(R(N_0)) \subset S_L(N_1) \tag{3}$$

Condition (1) means that the image of N_0 under G is contained in the 'horizontal' strip defined by N_1. Conditions (2) and (3) mean that the 'vertical' edges of N_0 are mapped to different sides of N_1 (see Fig. 1).

An important yet trivial property of the above relation is its stability under small perturbations.

Theorem 2.2. *[Zgliczyński, 1997a] Suppose that*

$$N_0 \overset{G_0}{\Longrightarrow} N_1 \overset{G_1}{\Longrightarrow} N_2 \overset{G_2}{\Longrightarrow} \dots \overset{G_{l-1}}{\Longrightarrow} N_l \overset{G_l}{\Longrightarrow} N_0 \tag{4}$$

then there exists $x \in$ int N_0 such that for $i = 0, \dots, l-1$ holds

$$G_i \circ G_{i-1} \circ \dots G_0(x) \in \mathit{int}\, N_{i+1}$$

and

$$G_l \circ G_{l-1} \circ \dots G_0(x) = x$$

Idea of proof. We show that the fixed point index for the composition $G_l \circ G_{l-1} \circ \dots G_0$ over the set

$$\{x \mid G_i \circ G_{i-1} \circ \dots G_0(x) \in \text{int } N_{i+1},\ i = 0, \dots, l-1\}$$

is nontrivial. ■

Remark: The above theorem is also true when the number of 'expanding' (horizontal) directions is greater than 1, see [Zgliczyński, 1997b].

Remark: In the proof of Theorem 1.2 we will insert for G_i various powers of F_λ.

Example: Topological Smale horseshoe [Mischaikow & Mrozek, 1995a], [Zgliczyński, 1996]

Let $N_0, N_1 \in \mathcal{C}(r)$, int $N_0 \cap$ int $N_1 = \emptyset$. We say that a continuous map $G : V \to V$ is *a topological Smale horseshoe* (with respect to N_0, N_1) (see Fig. 2) iff

$$N_i \overset{G}{\Longrightarrow} N_j, \quad \text{for } i, j = 0, 1$$

From Theorem 2.2 we obtain easily the symbolic dynamics and all periods for G, namely

Theorem 2.3. *[Mischaikow & Mrozek, 1995a], [Zgliczyński, 1996], [Zgliczyński, 1997a] If G is a topological Smale horseshoe with respect to N_0, N_1 then for every finite sequence $(\alpha_0, \alpha_1, \ldots, \alpha_{l-1})$, where $\alpha_i \in \{0,1\}$ there exists $x \in N_{\alpha_0}$ such that*

$$G^i(x) \in int\, N_{\alpha_i}$$
$$G^l(x) = x$$

If f is a one-dim map possessing a topological horseshoe with respect to I_0, I_1 for some iterate p, then for small λ F_λ has a topological horseshoe with respect to $I_0 \times \overline{B_W(r)}$, $I_1 \times \overline{B_W(r)}$ for the same iterate p. In this way we obtain fixed points for F_λ^{kp}, for $k = 1, 2, \ldots$ (as we cannot be sure that the periods are exactly equal to kp). In the next subsection we describe a method which enables us to obtain all periods greater than some integer.

2.2. Nested sequences of topological horseshoes

Definition 2.4. Let $G : (-\infty, \infty) \times \overline{B_W(r)} \to (-\infty, \infty) \times B_W(r)$ be continuous. Let $l \in \mathbb{N}$, $k \in \mathbb{N} \cup \{\infty\}$ $0 < l < k$. Let $N_i^s \in \mathcal{C}(r)$ for $i = 0, 1$ and $s = l, \ldots, k$ such that $N_i^s \supset N_i^{s+1}$ and $\mathrm{int} N_0^l \cap \mathrm{int} N_1^l = \emptyset$.

We say that G has an (l,k)-*nested sequence of topological horseshoes*, iff for $i = 0, 1$ and $l \leq s \leq k$ N_i^s horizontally G^s-covers the sets N_0^l and N_1^l.

Theorem 2.5. *[Zgliczyński, 1997a] If G has an $(p, 2p-1)$-nested sequence of topological horseshoes, then there exists $m(p)$, such that G has all periods greater than $m(p)$.*

For example $m(2) = 6$, $m(5) = 30$. The proof of this theorem is based on Theorem 2.2 and a few arithmetical lemmas, see [Zgliczyński, 1997a].

The following theorem shows how nested sequences of topological horseshoes are used in the proof of Theorem 1.2.

Theorem 2.6. *[Zgliczyński, 1997a] Let F_λ be a multidimensional perturbation of f. Suppose that f has a periodic point of odd period (> 1). Let n be the smallest period for f which is odd and greater than 3. Then for λ small, F_λ has an $(n+1, 2n+1)$-nested sequence of topological horseshoes. Hence F_λ has all periods greater than $m(n+1)$.*

Proof Let us fix $r > 0$. In the next subsection we will show that $F_0 = f$ has an $(n+1, \infty)$-nested sequence of topological horseshoes. For small perturbations, an arbitrarily big (finite) number of F_λ-covering relations survives. So we obtain for λ small that F_λ has an $(n+1, 2n+1)$-nested sequence of topological horseshoes. From Theorem 2.5 we obtain all periods greater than $m(n+1)$.

■

2.3. Topological horseshoes for 1-dim maps with odd periods

In this subsection we demonstrate that a 1-dim map with a point of odd period greater than 1 has a nested sequence of topological horseshoes.

Theorem 2.7. *If f has a periodic point of odd period n ($n > 1$), then f has an $(n+1, \infty)$-nested sequence of topological horseshoes.*

A sketch of the proof of this theorem will be presented at the end of this subsection.

Let us fix a 1-dimensional map f. We define an f-covering relation between segments (see [Block *et al.*, 1980]).

Definition 2.8. Let $K = [a_K, b_K]$, $L = [a_L, b_L]$.

- $K \overset{+1}{\to} L$ iff $f(a_K) < f(b_K)$ and $L \subset [f(a_K), f(b_K)]$
- $K \overset{-1}{\to} L$ iff $f(a_K) > f(b_K)$ and $L \subset [f(b_K), f(a_K)]$
- $K \to L$ iff $K \overset{+1}{\to} L$ or $K \overset{-1}{\to} L$

If $K \to L$ then we say that K f-covers L.

The above definition differs from the standard one presented in [Block *et al.*, 1980] in two ways

- we require that the end-points of K are mapped to different sides of L
- we introduce a sign

Both those features will be exploited in Sec. 2.4 devoted to the continuation of individual periodic orbits.

Let us remark that the relation of *horizontal covering* is stronger than the *covering relation*: we have

$$\text{if} \quad K \overset{f}{\Longrightarrow} L, \quad \text{then} \quad K \overset{f}{\to} L$$

but there is no implication in the reverse direction (as an example, consider identity map). We need some condition which will guarantee the implication in the other direction. The following trivial lemma presents such a condition

Lemma 2.9. *Let K, L, I be closed intervals such that $L \subset \operatorname{int} I$ and $K \to I$. Then $K \overset{f}{\Longrightarrow} L$.*

Let $f : \mathbb{R} \to \mathbb{R}$ cont. Let $x_0 < x_1 < \ldots < x_{n-1}$ be a periodic orbit. We set $I_i := [x_{i-1}, x_i]$ for $i = 1, 2, \ldots, n-1$ and $I := [x_0, x_{n-1}]$.

The following theorem, proved by Stefan [Stefan, 1977], describes a form of f-covering relation between segments I_j for a map with an odd period

Theorem 2.10. *[Stefan, 1977], [Block* et al.*, 1980] Suppose that x is a periodic point of period n, n is odd and $n > 1$, and f has no periodic points of odd periods smaller than n and larger than 1, then possibly after reversing the orientation we have*

$$I_{[n/2]+1} \overset{-1}{\to} I_{[n/2]+1}$$
$$I_{[n/2]+1} \overset{-1}{\to} I_{[n/2]} \overset{-1}{\to} I_{[n/2]+2} \overset{-1}{\to} I_{[n/2]-1} \overset{-1}{\to} \ldots$$
$$\ldots \overset{-1}{\to} I_2 \overset{-1}{\to} I_n \overset{-1}{\to} I_1$$
$$I_1 \overset{+1}{\to} I_{[n/2]+1} \cup I_{[n/2]+2} \cup \ldots \cup I_n$$

Figure 3 shows an example of a map with minimal nontrivial odd period equal to 5.

Proof of Theorem 2.7. We will give a proof only for $n = 5$, with the additional assumption that no period 3 exists.

From Theorem 2.10 it follows that

$$I \subset f^l(I_2), \quad I \subset f^l(I_3), \qquad \text{where } l \geq 6.$$

There exist two nested sequences of segments $I_2^l \subset I_2$, $I_3^l \subset I_3$ for $l \geq 6$, such that

$$I_i^l \xrightarrow{f^l} I, \quad \text{for } i = 2, 3,\ l \geq 6$$

We have $I_2 \cup I_3 \subset \operatorname{int} I$, so by Lemma 2.9 we obtain

$$I_i^l \overset{f^l}{\Longrightarrow} I_2 \cup I_3, \quad \text{for } i = 2, 3,\ l \geq 6$$

■

Figures 3–8 illustrate the proof of Theorem 2.7 for $n = 5$. In these pictures f_5 is piecewise linear and the permutation induced by f_5 is the one described in Theorem 2.10.

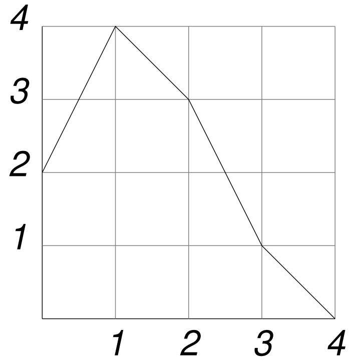

Fig. 3. Graph of f_5

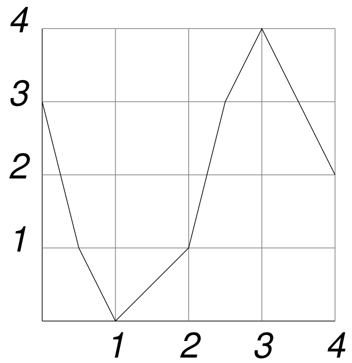

Fig. 4. Graph of f_5^2

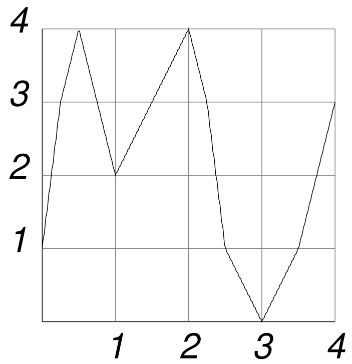

Fig. 5. Graph of f_5^3. Observe that $I \subset f_5^3(I_3)$

Remark. In the above proof we have chosen the segments I_2 and I_3 to build on them a $(6, \infty)$-nested sequence of topological horseshoes. One may hope to do better by observing that $I \subset f^l(I_3)$ and $I \subset f^l(I_1)$ already for $l = 4$, but in this case the

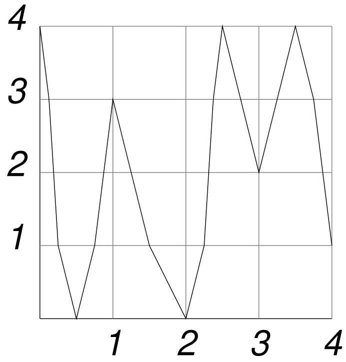

Fig. 6. Graph of f_5^4. Observe that $I \subset f_5^4(I_3)$

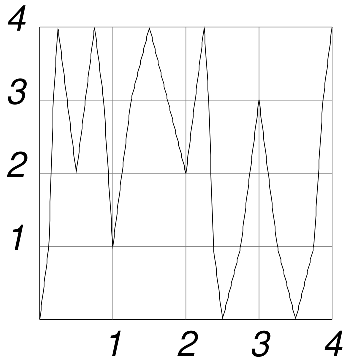

Fig. 7. Graph of f_5^5. Observe that $I \subset f_5^5(I_3)$

assumptions of Lemma 2.9 are not satisfied. We need in the proof, to build horseshoes on internal segments of periodic orbit, i.e. we need at least two intervals I_i, I_k which will be contained in the interior of I. This forces us to start the construction with a periodic orbit which has at least five points.
Remark. From the above proof one can derive the conditions for λ from the assertion of Theorem 2.6, under the assumption that $n = 5$, and no smaller odd period greater than 1 exists. By $x(v)$ we will denote the x coordinate of $v \in V$. Let us fix $r > 0$.

We set

$$\Delta := \min\{|x_0 - x_1|, |x_3 - x_4|\} \tag{5}$$

The conditions are

$$F_\lambda^l(I \times \overline{B_W(r)}) \subset (-\infty, \infty) \times B_W(r)$$
$$|x(F_\lambda^l(v)) - x(F_0(v))| < \Delta, \quad v \in I \times \overline{B_W(r)}$$

where $l = n+1, \ldots, 2n+1$.

When $n \neq 5$ the above conditions are valid (see [Zgliczyński, 1997a]), but we have to pick up an orbit $\{x_0 < x_1 < \ldots < x_{n-1}\}$ of minimal odd period

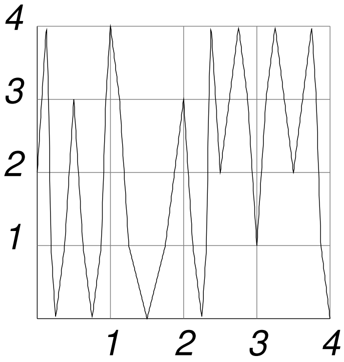

Fig. 8. Graph of f_5^6. Observe that $I \subset f_5^6(I_3)$, $I \subset f_5^6(I_2)$

greater than 3 and to change the definition of Δ as follows

$$\Delta := \min\{|x_{i-1} - x_i|, i = 1, \ldots, n-1\} \tag{6}$$

2.4. Continuation

Let $f : \mathbb{R} \to \mathbb{R}$ cont. Let $x_0 < x_1 < \ldots < x_{n-1}$ be a periodic orbit. We set $I_i := [x_{i-1}, x_i]$ for $i = 1, 2, \ldots, n-1$ and $I := [x_0, x_{n-1}]$.

In this subsection we will show the following theorem, which together with Theorem 2.6 gives us Theorem 1.2 for n odd.

Theorem 2.11. *Let F, f, V be as in Theorem 1.2. Suppose f has a point of odd period n, $n > 1$.*

For every integer $A > 0$ and every real number $r > 0$ there exists $\lambda_0 > 0$ such that for $\lambda \leq \lambda_0$ F_λ has a periodic point of period m in the set $\mathbb{R} \oplus B_W(r)$ for every m, such that $n \triangleleft m$, $m \neq n$, $m < A$.

The proof of this theorem is based on the following lemma

Lemma 2.12. *Let F, f, V be as in Theorem 1.2. Let $p \in \mathbb{N}$, $p > 0$. Suppose that there exist $i_0, i_1, i_2, \ldots, i_{p-1}$ such that*

$$I_{i_0} \overset{\epsilon_1}{\to} I_{i_1} \overset{\epsilon_2}{\to} I_{i_2} \overset{\epsilon_3}{\to} \cdots \overset{\epsilon_{p-1}}{\to} I_{i_{p-1}} \overset{\epsilon_p}{\to} I_{i_0} \tag{7}$$
$$\epsilon_1 \epsilon_2 \ldots \epsilon_p = -1 \tag{8}$$

Then for every $r > 0$ there exists $\lambda_0 > 0$ such that for $\lambda < \lambda_0$ there exists a periodic point z_λ such that

$$F_\lambda^l(z_\lambda) \in \mathit{int}\, I_{i_l} \times B_W(r), \quad l = 0, 1, \ldots, p-1$$
$$F_\lambda^p(z_\lambda) = z_\lambda$$

In the one-dimensional situation this lemma expresses the method of obtaining periodic points in the standard proof of Šarkovskii theorem (see for example [Šarkovskii , 1964], [Stefan, 1977]). In this case the assumption concerning the product of ϵ's can be dropped.

Idea of proof. See [Zgliczyński, 1997a] for full details. As in the one-dimensional case we prove that a periodic point with desired properties exists. From the assumption concerning the product of ϵ's it follows that the fixed point index for F_0^p is non-trivial (equals 1), so it continues to nearby λ. ■

For the proof of Theorem 2.11 it remains to show that all periods can be obtained from the loops of f-covering relations which fulfill the assumptions of Lemma 2.12. This is indeed the case: It follows easily from Theorem 2.10, see [Zgliczyński, 1997a] for details.

Here, we will show this only for $n = 3$. We have the following relations

$$I_2 \overset{-}{\to} I_2, \quad I_2 \overset{-}{\to} I_1$$
$$I_1 \overset{+}{\to} I_2$$

It is easy to see that by considering the loops

$$I_2 \overset{-}{\to} I_2 \overset{-}{\to} \dots \overset{-}{\to} I_2 \overset{-}{\to} I_1 \overset{+}{\to} I_2$$

we obtain all even periods.

The loops

$$I_2 \overset{-}{\to} I_2 \overset{-}{\to} \dots \overset{-}{\to} I_2 \overset{-}{\to} I_1 \overset{+}{\to} I_2 \overset{-}{\to} I_1 \overset{+}{\to} I_2$$

produce all odd periods.

3. Application to Rössler equations

The aim of this subsection is to show how the theorems about multidimensional perturbations of one-dim maps presented in the previous sections are applied to Rössler equations. The main modification in comparison with the proof of stability of the Šarkovskii ordering lies in the fact that we build a system of horizontal covering relations on segments obtained from iteration of critical point of unperturbed one-dim maps rather than from periodic point. This approach appears to be much more effective.

Before we state the main result of this section we need to recall some definitions.

Let k be a positive integer. Let $\Sigma_k := \{0, 1, \dots, k-1\}^{\mathbb{Z}}$, $\Sigma_k^+ := \{0, 1, \dots, k-1\}^{\mathbb{N}}$. Σ_k, Σ_k^+ are topological spaces with the Tichonov weak product topology. On Σ_k, Σ_k^+ we have the shift map σ given by

$$(\sigma(c))_i = c_{i+1}$$

Let $A = [\alpha_{ij}]$ be a $k \times k$-matrix, $\alpha_{ij} \in \mathbb{R}_+ \cup \{0\}$, $i, j = 0, 1, \dots, k-1$. We define $\Sigma_A \subset \Sigma_k$ and $\Sigma_A^+ \subset \Sigma_k^+$ by

$$\Sigma_A := \{c = (c_i)_{i \in \mathbb{Z}} \mid \alpha_{c_i c_{i+1}} > 0\} \tag{9}$$
$$\Sigma_A^+ := \{c = (c_i)_{i \in \mathbb{N}} \mid \alpha_{c_i c_{i+1}} > 0\} \tag{10}$$

Obviously Σ_A^+, Σ_A are invariant under σ.

Let $F : X \to X$ be any continous map and $N \subset X$. By $F_{|N}$ we will denote the map obtained by restricting the domain of F to the set N. The maximal invariant part of N (with respect to F) is defined by

$$\mathrm{Inv}(N, F) = \bigcap_{i \in \mathbb{Z}} F_{|N}^{-i}(N).$$

The Rössler equations are given by [Rössler, 1976]

$$\begin{aligned} \dot{x} &= -(y+z) \\ \dot{y} &= x + by \\ \dot{z} &= b + z(x-a) \end{aligned} \tag{11}$$

where $a = 5.7, b = 0.2$. These are parameters values originally considered by Rössler. The flow generated by Eq.(11) exhibits a so-called strange attractor.

We will investigate the Poincaré map P generated by (11) on the section $\Theta := \{(x, y, z) | \quad x = 0, y < 0, \dot{x} > 0\}$.

The following result was proved in [Zgliczyński, 1997]

Theorem 3.1. *For all parameter values in a sufficiently small neighborhood of $(a, b) = (5.7, 0.2)$ there exists a Poincaré section $N \subset \Theta$ such that the Poincaré map P induced by Eq.(11) is well defined and continuous.*

There exists a continuous map $\pi : Inv(N, P) \to \Sigma_3$, such that

$$\pi \circ P = \sigma \circ \pi.$$

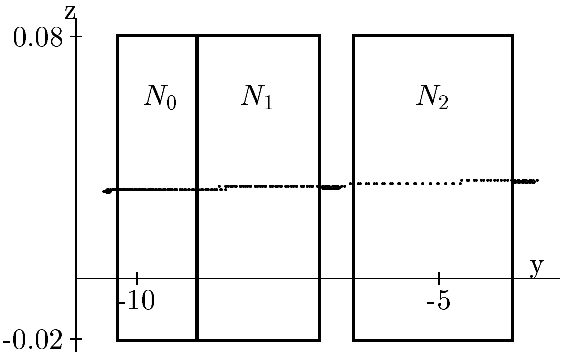

Fig. 9. Image of N for Rössler Poincaré map (dots).

$\Sigma_A \subset \pi(Inv(N, P))$, where

$$A := \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

The preimage of any periodic sequence from Σ_A contains periodic points of P.

In the above theorem $N := N_0 \cup N_1 \cup N_2$, where

$$N_0 = \{0\} \times [-10.3, -9.01] \times [-0.02, 0.08], \quad (12)$$
$$N_1 = \{0\} \times [-9, -7] \times [-0.02, 0.08], \quad (13)$$
$$N_2 = \{0\} \times [-6.4, -3.8] \times [-0.02, 0.08] \quad (14)$$

and the map π is defined by the condition

$$\pi(x) = (c_i)_{i \in \mathbb{Z}}, \quad \text{iff} \quad P^i(x) \in N_{c_i},\ i \in \mathbb{Z} \quad (15)$$

Let us remark that N_0, N_1, N_2 are cylinders of size 0.05 in the coordinate frame shifted by 0.03 in z-direction.

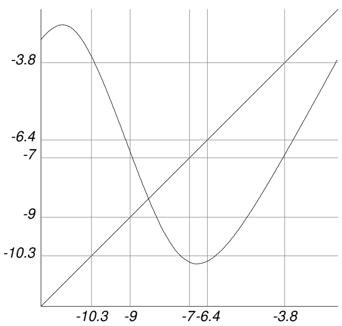

Fig. 10. $p(s)$

Figure 3 presents the image of N under P. From it one can clearly see the difference of scales in the y and z direction and manifestly strong contraction in z direction of map P. This suggests that this map can be treated as a multidimensional perturbation of a one-dimensional map p, given by

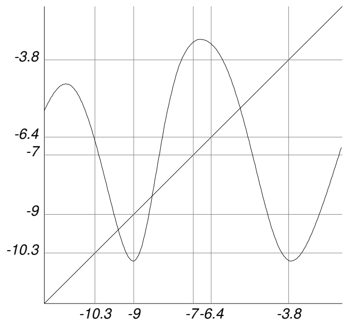

Fig. 11. $p^2(s)$

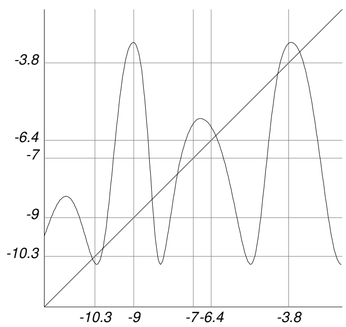

Fig. 12. $p^3(s)$

$$p(s) = P_y(0, s, 0.03) \quad (16)$$

where $s \in \mathbb{R}$, $P_y(\cdot)$ is y-coordinate of $P(\cdot)$

Figures 10 - 12 present the plot of three consecutive iterates of p.

The vertical location and size of the sets N_i are set just to enclose the entire attractor, the horizontal shape of sets N_i is 'explained' by the next lemma.

Lemma 3.2. *Let $f : [\alpha, \beta] \to \mathbb{R}$ be a C^0 map, such that the following conditions are satisfied*

1. *there exists $a, b \in [\alpha, \beta]$, $c \in (\alpha, \beta)$ $a < c < b$, with $f(a) = f(b) = c$*
2. *$f(c) \in [\alpha, \beta]$*
3. *$f(c) < a < c < b < f^2(c)$*

For $\epsilon_1, \epsilon, \delta \in \mathbb{R}$ we set $I_0 := [f(c) + \epsilon, a + \delta]$, $I_1 := [a + \delta, c - \epsilon_1]$, $I_2 := [c + \epsilon_1, b + \delta]$.

There exist $\epsilon > 0$, $\epsilon_1 > 0$, δ such that the following conditions hold

$$I_0 \overset{f}{\Longrightarrow} I_2,$$
$$I_1 \overset{f}{\Longrightarrow} I_0, \quad I_1 \overset{f}{\Longrightarrow} I_1$$
$$I_2 \overset{f}{\Longrightarrow} I_0, \quad I_2 \overset{f}{\Longrightarrow} I_1$$

Proof

Relation $I_0 \overset{f}{\Longrightarrow} I_2$ is implied by inequalities

$$f(f(c)+\epsilon) > b+\delta, \tag{17}$$
$$f(a+\delta) < c+\epsilon_1 \tag{18}$$

Relations $I_1 \overset{f}{\Longrightarrow} I_0, I_1 \overset{f}{\Longrightarrow} I_1$ follow from inequalities

$$f(a+\delta) > c-\epsilon_1, \tag{19}$$
$$f(c-\epsilon_1) < f(c)+\epsilon, \tag{20}$$

Relations $I_2 \overset{f}{\Longrightarrow} I_0, I_2 \overset{f}{\Longrightarrow} I_1$ follow from inequalities

$$f(c+\epsilon_1) < f(c)+\epsilon, \tag{21}$$
$$f(b+\delta) > c-\epsilon_1 \tag{22}$$

To finish the proof we need to show that inequalities (17–22) have a solution.

Observe that (17) holds for

$$|\delta|, |\epsilon| < \gamma \tag{23}$$

We fix ϵ with $0 < \epsilon < \gamma$. Inequalities (20), (21), can be written as

$$f(c \pm \epsilon_1) < f(c)+\epsilon \tag{24}$$

which is true for ϵ_1 sufficiently small. We fix such an $\epsilon_1 > 0$.

The remaining Eqs (18), (19) and (22) can be rewritten as (remember that by assumption $f(a) = f(b) = c$)

$$f(a)-\epsilon_1 < f(a+\delta) < f(a)+\epsilon_1, \quad f(b+\delta) > f(b)-\epsilon_1$$

It is obvious that these inequalities are satisfied when $|\delta|$ is sufficiently small.

■

In our approach to the Rössler system, we choose c to be a minimum of p, which is clearly seen in Fig. (10), then from numerical simulations of p it follows that

$$p(c) \approx -10.57, \quad a \approx -9.01,\ c \approx -6.767, \tag{25}$$
$$b \approx -3.73,\ p^2(c) \approx -3.11, \tag{26}$$

Hence the candidates for the sets N_i suggested by Lemma 3.2 are given by

$$I_0 = [-10.57+\epsilon, -9.01+\delta], \tag{27}$$
$$I_1 = [-9.01+\delta, -6.767-\epsilon_1], \tag{28}$$
$$I_2 = [-6.767+\epsilon_1, -3.73+\delta] \tag{29}$$
$$N_i' = \{0\} \times I_i \times [-0.02, 0.08] \tag{30}$$

which are obviously not equal, but are close, to the choices taken in [Zgliczyński, 1997] given by formulas (12–14), where these sets are constructed by trial and error method.

The choice of critical point as c appears to be the best one for the continuation of p-covering relations to full Poincaré map P - due to the flatness of p near the c is have much freedom to adjust parameters ϵ, ϵ_1 and δ.

Using rigorous computer calculations it was shown in [Zgliczyński, 1997] that the p-covering relations obtained for the segments I_i are also true for the sets N_i.

Lemma 3.3. *For all parameter values in a sufficiently small neighborhood of $(a, b) = (5.7, 0.2)$ $N \subset Dom(P)$ and the following conditions hold*

$$N_0 \overset{P}{\Longrightarrow} N_2, \quad N_1 \overset{P}{\Longrightarrow} N_0 \cup N_1, \quad N_2 \overset{P}{\Longrightarrow} N_0 \cup N_1 \tag{31}$$

Theorem 3.1 follows now easily from Lemma 3.3 and Theorem 2.2.

4. Other applications

4.1. Hénon map

For the Hénon map [Hénon, 1976] (with classical parameter values $a = 1.4$, $b = 0.3$) Szymczak in [Szymczak, 1997] proved using Conley index that in the suitably chosen set containing almost entirely the numerically observed strange attractor there exist periodic points of all periods but 3 and 5, which fits to Theorem 1.2. The observed attractor is not one-dimensional, but the approach used in the proof of Theorem 1.2 may be used to reproduce Szymczak's result (see [Galias, 1997b]).

4.2. Lorenz system, Chua circuit

The Lorenz equations and the Chua circuit are systems of ODE's in three dimensions, which are well known for exhibiting strange attractors. Using the methods of multidimensional perturbations of one-dim maps presented in this paper, Galias & Zgliczyński [Galias & Zgliczyński, 1997] for the Lorenz eq. and Galias [Galias, 1997a] for the Chua system are able to prove (with computer assistance) the existence of symbolic dynamics (in both cases for classical parameter values).

Similar results for the Lorenz systems are obtained in the pioneering work of Mischaikow & Mrozek [Mischaikow & Mrozek, 1995b] (non-classical parameters values) and by Mischaikow, Mrozek & Szymczak in [Mischaikow *et al.*, 1997] (classical parameter values). The tool used in these papers is the Conley index.

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References

Alsedá L., Llibre J. & Misiurewicz M. [1993] *Combinatorial Dynamics and Entropy in Dimension One. (Advanced Series in Nonlinear Dynamics 5)*, World Scientific, Singapore

Block L., Guckenheimer J., Misiurewicz M. & Young L.-S. [1980] "Periodic points of one-dimensional maps," *Lecture Notes in Mathematics* **819**, 18–34.

Blokh, A. & Misiurewicz, M. [1997] "New order for periodic orbits of interval maps," *Ergod. Th. & Dynam. Sys.* **17**, 565–574.

Blokh, A. [1995] "Rotation numbers, twists and a Šarkovskiĭ –Misiurewicz–type ordering for patterns on interval," *Ergod. Th. & Dynam. Sys.* **15**, 1–14.

Boyland, P. [1994] "Topological methods in surface dynamics," *Top. and its Appl.* **58**, 223–298.

Devaney, R. [1989] *An Introduction to Chaotic Dynamical Systems* Addison-Wesley Publishing Company

Dold, A. [1972] *Lectures on Algebraic Topology* Springer-Verlag, Berlin,Heidelberg,New York

Galias, Z. [1997a] " Positive topological entropy of Chua's circuit: a computer assisted proof," *Int. J. Bifurcation and Chaos* **2**, 331–349.

Galias, Z. [1997b] "Rigorous Numerical Studies of the Existence of Periodic Orbits for the Hénon map,", *preprint*

Galias, Z. & Zgliczyński, P. [1997] "Computer assisted proof of chaos in the Lorenz system" *Physica D*, accepted

Granas, A. [1972] " The Leray-Schauder index and the fixed point theory for arbitrary ANR's," *Bull. Soc. Math. France* **100**, 209–228.

Gambaudo J.M., van Strien S. & Tresser C. [1989] "The periodic orbit structure of orientation preserving diffeomorphisms on D^2 with topological entropy zero," *Annales de l'Institut Henri Poincaré, Physique théorique* **50**, 335-356.

Gambaudo J.M., van Strien S. & Tresser C. [1990] "Vers un ordre de Sarkovskii pour les plongements du disque preservant l'orientation," *Comptes Rendus, série I* **310**, 291–294.

Hall, T. [1994] "The creation of horseshoes," *Nonlinearity* **7**, 861–924.

Holmes, P. J. & Whitley, D. C. [1984] "Bifurcations of one and two dimensional maps," *Phil. Trans. Royal Soc. London A* **311**, 43–102.

Hènon, M. [1976] "A Two-dimensional Mapping with a Strange Attractor," *Comm. Math. Phys.* **50**, 69–77.

Kolev, B. [1994] "Periodic orbits of period 3 in the disc," *Nonlinearity* **7**, 1067–1071.

Kloeden, P. E. [1979] "On Sharkovsky's cycle coexistence ordering," *Bull. Austral. Math. Soc.* **20**, 171–178.

Li, T.Y & Yorke, J. A. [1975] "Period three implies chaos," *Amer. Math. Monthly* **82**, 985–992.

Mischaikow, K. & Mrozek, M. [1995a] "Isolating neighborhoods and chaos," *JJIAM* **12**, 205–236.

Mischaikow, K. & Mrozek, M. [1995b] "Chaos in the Lorenz Equations: a Computer Assisted Proof," *Bull. of AMS* **32**, 66–72.

Mischaikow K., Mrozek M. & Szymczak A. [1997] "Chaos in the Lorenz Equations: A Computer Assisted Proof. Part III:The classical case", *preprint*

Rössler, O. E. [1976] "An Equation for Continuous Chaos," *Physics Letters* **57A**, 397–398.

Šarkovskii A. N. [1964] "Coexistence of cycles of a continuous map of a line into itself. (in Russian)," *Ukr. Math. Z.* **16**, 61–71.

Šarkovskii A. N. [1995] "Coexistence of cycles of a continuous map of a line into itself," *Int. J. of Bifurcations and Chaos* **5**, 1263–1274.

Smale, S. [1967] "Differentiable Dynamical Systems," *Bull. Amer. Math. Soc.* **73**, 747–817.

Stefan, P. [1977] "A Theorem of Sarkovskii on the Existence of Periodic Orbits of Continous Endomorphism of the Real Line," *Commun. Math. Phys.* **54**, 237–248.

van Strien, S. [1981] "On the bifurcations creating horseshoes," *Dynamical Systems and Turbulence, Lect. Notes in Math.* **898**, 316–351.

Szymczak, A. [1997] "A combinatorial procedure for finding isolating neighbourhoods and index pairs," *Proc.Royal Soc.Edinburgh*, accepted

Thurston, W.P. [1988] "On the geometry and dynamics of diffeomorphism of surfaces," *Bull. AMS* **19**, 417–431.

Zgliczyński, P. [1996] "Fixed point index for iterations of maps, topological horseshoe and chaos," *Top. Meth. Non. Anal.* **8**, 169–177.

Zgliczyński, P. [1997] "Computer assisted proof of chaos in the Hénon map and in the Rössler equations," *Nonlinearity* **10**, 243–252.

Zgliczyński, P., [1997a] " Šarkovskii 's Theorem for multidimensional perturbations of 1-dim maps" *Erg. Th & Dyn. Sys.*, accepted

Zgliczyński, P. [1997b] "On periodic points for systems of weakly coupled 1-dim maps," *preprint*