Sharkovskii theorem for multidimensional perturbations of 1-dim maps[1]

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Abstract

We present Sharkovski theorem for multidimensional perturbations of one-dimensional maps. We show that if unperturbed one-dimensional map has a point of period n, then sufficiently close multidimensional perturbations of this map have periodic points of all periods which are allowed by Sharkovski theorem.

Keywords: fixed point index,chaos,Sharkovskii's theorem

0 Introduction

We present a multidimensional generalization of the Sharkovskii's Theorem [Sh](see also [Ste, BGMY, De, LY]) concerning the appearance of periodic points for self-maps on the real line. We address an interesting question as to what extend Sharkovskii's theorem may be carried over into multidimensional dynamics.

For a homeomorphism or orientation preserving embedding of a disc in two dimensions there exists a considerable literature concerning the forcing relations between periodic orbits (see for example [GST1, GTS2, H, Ko, Bo, Mr]) studied using topological tools (braid type), which are based on the Thurston theorem [T], which gives a canonical form for every surface homeomorphism up to isotopy. However, these topological tools appear to be applicable only in two dimensions.

In dimensions greater than two, there are a few results in this direction, for example Kloeden [K] extended Sharkovskii's theorem to the so-called triangular maps. Also related to this subject are the results concerning the order in which periodic orbits can appear when a horseshoe is created (see for example [H, St]), but in this case the maps under consideration are either restricted to the plane or some assumptions concerning derivatives must be made. We show that Sharkovskii's theorem is also true for multidimensional perturbations of one-dimensional (1D) maps. We emphasize that our result is purely topological. We neither assume the existence of derivatives nor the existence of invariant foliations as in the case of triangular maps.

Let $f : X \to X$ be a map. We say that $x \in X$ is a periodic point of period n if $f^n(x) = x$ and $f^k(x) \neq x$ for $k = 1, 2, \ldots, n-1$.

We begin with recalling Sharkovskii's theorem.

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Theorem 1 *Let the ordering of positive integers be as follows:*

$$3 \lhd 5 \lhd 7 \lhd \ldots \lhd 2 \cdot 3 \lhd 2 \cdot 5 \lhd \lhd 2 \cdot 7 \ldots \lhd 2^2 \cdot 3 \lhd 2^2 \cdot 5 \ldots \lhd 2^3 \cdot 3 \lhd 2^3 \cdot 5 \ldots \lhd 2^3 \lhd 2^2 \lhd 2 \lhd 1$$

Let $f : I \to \mathbb{R}$ be a continuous map of an interval into the real line. If $n \lhd k$ and f has a periodic point of period n then f also has a periodic point of period k.

The ordering described in Theorem 1 is called *the Sharkowskii's ordering.* We reserve the symbol '$\lhd$' for this order.

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map. Let V be a real Banach space and let a continuous decomposition $V = \mathbb{R} \oplus W$ be given. According to this decomposition we will represent elements $v \in V$ as pairs $v = (x, w)$, where $x \in \mathbb{R}$ and $w \in W$. Let $F : [0, 1] \times V \to V$ be a continuous and compact map. In the following we will use the notation F_λ for the partial map with fixed λ, so $F_\lambda(v) := F(\lambda, v)$ for $v \in V$. Suppose that $F_0(x, w) = (f(x), 0)$. We say that the maps F_λ are multidimensional perturbations of f. Let us recall that a continuous map is called *compact*, iff it maps bounded sets into relatively compact sets.

The main theorem proved in this paper is stated below.

Theorem 2 *Let f and F be as above. If $\dim V = \infty$ then we additionally assume that F is uniformly continuous and compact. Suppose f has a point of period n.*

For every $r > 0$ there exists $\lambda_0 > 0$ such that for $\lambda \le \lambda_0$
if $n \lhd m$, $m \ne n$ then F_λ has a periodic point of period m in the set $\mathbb{R} \oplus B_W(r)$, where $B_W(r)$ is an open ball in W of radius r.

The tool used in this paper to obtain periodic points is the fixed point index [D, Ch. VII, sec. 5]. The proof of Theorem 2 for odd periods consists essentially of two parts. The first part deals with all periods greater than some integer M ($> 2n$) , the second concerns periods of less than M. The result for even periods follows from that for odd ones, in a slighty more complicated way than in the one-dimensional case.

The first part, for n odd, of the proof of the existence of periodic points is based on the fixed point index theorem for composition of maps (sec. 1), properties of a nested sequence of horseshoes (sec. 2), investigations of 1D maps via covering matrices, which is contained in sec. 3 and is summarized in sec. 4.

The existence of small periods ($< M$) for n odd is proved in Section 5 by means of simple continuation. We choose appropriate periodic orbits for F_0 (1D map) and we show that their fixed point index is defined and non trivial, so it continues to nearby λ, but in this way only the existence of a finite number of periodic orbits can be obtained.

Sections 6 and 7 contain the proof of the theorem for n even.

Let us remark that assumptions concerning domains of definition for F and f can be released in an obvious way. We can assume that F is defined on the set $[0, 1] \times (I \oplus \overline{B_W(a)})$, where I is a compact segment and $a > 0$, similary f should then be defined on I only. We can extend these maps to $\tilde{F}$ and $\tilde{f}$ by composing F and f with suitable retractions. In this way no new periodic point is created. The properties such as continuity, compactness and uniform continuity are preserved by these extensions so we can apply Theorem 2 to these extended maps to obtain the same assertion for inital F and f and $r < a$.

In our theorem for the infinite-dimensional case we have assumed uniform continuity and compactness. We believe that compactness alone is sufficient, but we are unable to prove it, yet.

Unfortunately, even in dimension one, the classification of periodic points by period is very coarse. When we look for permutations determined by periodic orbit, then the classification is very fine, but the results are weaker than for periods (see, for example, [ALM]). There also exists other notions used to classify periodic orbits and forcing relations between them, namely *rotation pairs* [B] and *over-rotation pairs* [BM]. Whether the ordering induced using these notions will survive the multidimensional perturbation is an interesting question to investigate.

1 Topological theorems

First we introduce some notations which is used throughout the paper. By $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$ we denote the sets of real numbers, integers and natural numbers (including zero), respectively.

Let (X,ρ) be a metric space. For $Z \subset X$, $x \in X$ put $\rho(x,Z) = \inf\{\rho(x,y) | y \in Z\}$. $B(x,\epsilon) = \{y | \rho(x,y) < \epsilon\}$, $B(Z,\epsilon) = \{x | \rho(x,Z) < \epsilon\}$. By $\mathrm{int}(Z)$,$\overline{Z}$ and ∂Z we denote the interior, the closure and the boundary of the set Z, respectively. For a mapping F defined on $[0,1] \times Z$ we write $F_\lambda(x)$ instead of $F(\lambda,x)$.

Let $f : X \to X$ be a continuous map and $N \subset X$. By $f_{|N}$ we denote the restriction of f to the set N. The invariant part of N (with respect to f) is defined by

$$\mathrm{Inv}(N,f) = \bigcap_{i \in \mathbb{Z}} f_{|N}^{-i}(N).$$

The set of fixed points of f in N is denoted by $\mathrm{Fix}(f,N)$, hence

$$\mathrm{Fix}(f,N) := \{x \in N \quad | \quad f(x) = x\}.$$

Let $\Sigma_K := \{0,1,\ldots,K-1\}^{\mathbb{Z}}$, $\Sigma_K^+ := \{0,1,\ldots,K-1\}^{\mathbb{N}}$. Σ_K, Σ_K^+ are topological spaces with the Tichonov topology. On Σ_K, Σ_K^+ we have the shift map σ given by

$$(\sigma(c))_i = c_{i+1}.$$

Let $A = [\alpha_{ij}]$ be a $K \times K$-matrix, with entries $\alpha_{ij} \in \mathbb{R}_+ \cup \{0\}$, $i,j = 0,1,\ldots,K-1$. We define $\Sigma_A^+ \subset \Sigma_K^+$ by

$$\Sigma_A^+ := \{c = (c_i)_{i \in \mathbb{N}} \quad | \quad \alpha_{c_i c_{i+1}} > 0\}. \tag{1}$$

Obviously Σ_A^+ is invariant under σ.

The main tool used in this paper is *the fixed point index.* Below we list the basic properties of this notion. The proofs can be found in Dold's book [D, Ch. VII, sec. 5] for the finite dimensional case, the infinite-dimesional case is treated in [G].

Let W be a Banach space. Let $D \subset W$ be an open set and $f : D \to W$ be continuos and compact. Suppose that

$$\mathrm{Fix}(f,D) \quad \text{is compact.} \tag{2}$$

Then *fixed point index for f in D* is defined. We denote it by $\mathrm{I}(f,D)$.

If $\overline{D} \subset \mathrm{dom}(f)$ then (2) follows from the condition

$$\mathrm{Fix}(f,\overline{D}) \cap \partial D = \emptyset. \tag{3}$$

Let us summarize the properties of the fixed point index (see [D, Ch. VII, sec. 5] and [G]).

The index is an integer.

$$I(f, D) \in \mathbb{Z}. \tag{4}$$

Homotopy property. Let $F : [0,1] \times D \to W$ be continuous and compact. Suppose that

$$\operatorname{Fix}(F, D) := \bigcup_{\lambda \in [0,1]} \operatorname{Fix}(F_\lambda, D) \quad \text{is compact.} \tag{5}$$

Then

$$\forall \lambda \in [0,1] \quad I(F_\lambda, D) = I(F_0, D). \tag{6}$$

If $[0,1] \times \overline{D} \subset \operatorname{dom}(F)$ then (5) follows from the following condition,

$$\operatorname{Fix}(F, \overline{D}) \cap \partial D = \emptyset. \tag{7}$$

Excision property. Suppose that $C \subset D$, C is open and

$$\operatorname{Fix}(f, D) \subset C. \tag{8}$$

Then

$$I(f, C) = I(f, D). \tag{9}$$

Fixed point property.

$$\text{if} \quad I(f, D) \neq 0, \quad \text{then} \quad \exists x \in D \quad f(x) = x. \tag{10}$$

In particular,

$$I(f, \emptyset) = 0. \tag{11}$$

Fixed point index for affine maps. Suppose that W is finitely-dimensional and $f(x) = A(x - x_0) + x_0$, where A is a linear map and $x_0 \in W$. If the equation $A(x) = x$ has no non-trivial solutions and $x_0 \in \operatorname{int}(D)$, then

$$I(f, D) = \operatorname{sgn}(\det(\operatorname{Id} - A)), \quad \text{where } \mathit{Id} \text{ denotes the identity map.} \tag{12}$$

Let V, W be real Banach spaces, where $V = \mathbb{R} \oplus W$. We will often represent elements $v \in V$ as pairs $v = (x, w)$, where $x \in \mathbb{R}$ and $w \in W$. We assume that $|(x,w)|_V = \max\{|x|, |w|_W\}$. By ρ we denote a metric induced by this norm. In the following we say that $\mathbb{R} \oplus 0$ is the *horizontal direction* and the space W is the *vertical direction*. W may be a zero-dimensional space.

Let $B_W(r) := \{w \in W \mid |w| < r\}$. For a union of cylinders $P = \bigcup P_k \subset V$, where $P_k = [a_k, b_k] \times B_W(r)$ we set

$$L(P) := \bigcup \{a_k\} \times \overline{B_W(r)} \tag{13}$$
$$R(P) := \bigcup \{b_k\} \times \overline{B_W(r)} \tag{14}$$
$$V(P) := L(P) \cup R(P) \tag{15}$$
$$H(P) := \bigcup [a_k, b_k] \times \partial B_W(r) \tag{16}$$

Following our convention concerning directions we say that $L(P)$, $R(P)$, $V(P)$ and $H(P)$ are equal to the sum of the left vertical, right vertical, vertical and horizontal edges in P, respectively. If $\dim W = 0$ then $H(P) = \emptyset$.

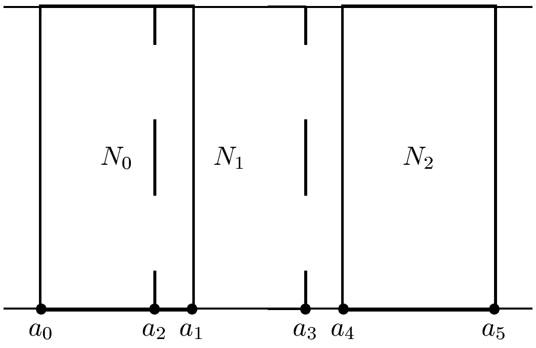

Figure 1: Sets N_0, N_1, N_2

Let us fix an $r \in \mathbb{R}$ and a sequence $a_i \in \mathbb{R}$ for $i = 0, 1, \ldots, 2K-1$, such that $a_{2i} < a_{2i+1}$. Let

$$\begin{aligned} N_i &:= [a_{2i}, a_{2i+1}] \times \overline{B_W(r)}, \quad \text{for} \quad i = 0, \ldots, K-1 && (17) \\ S_R(i) &:= (a_{2i+1}, \infty) \times \overline{B_W(r)}, \quad \text{for} \quad i = 0, \ldots, K-1 && (18) \\ S_L(i) &:= (-\infty, a_{2i}) \times \overline{B_W(r)}, \quad \text{for} \quad i = 0, \ldots, K-1 && (19) \\ N &:= N_0 \cup N_1 \cup \ldots \cup N_{K-1}. && (20) \end{aligned}$$

The sets $S_L(i)$, $S_R(i)$ are equal to the left and right components of the complement of the set N_i in the horizontal strip $(-\infty, \infty) \times \overline{B_W(r)}$, respectively.

In the following we will consider only maps which fulfill the following condition

$$\overline{f((-\infty, \infty) \times \overline{B_W(r)})} \subset (-\infty, \infty) \times B_W(r). \tag{21}$$

Definition 1 *Let $f : (-\infty, \infty) \times \overline{B_W(r)} \to V$ be a continuous map satisfying condition (21) We say that N_i* f-covers horizontally *N_j if either*

$$f(L(N_i)) \subset S_L(j) \text{ and } f(R(N_i)) \subset S_R(j) \tag{22}$$

or

$$f(L(N_i)) \subset S_R(j) \text{ and } f(R(N_i)) \subset S_L(j). \tag{23}$$

We say that N_i uniformly f-covers horizontally *N_j iff N_i* f-covers horizontally *N_j and there exists a $\delta > 0$ such that*

$$\rho(f(V(N_i)), N_j) > \delta. \tag{24}$$

One can easily see that N_i f-covers horizontally N_j if it is contained in the interior of the horizontal strip $(-\infty, \infty) \times \overline{B_W(r)}$ and the vertical edges in N_i are mapped into different sides of the set N_j.

In finite dimensions N_i f-covers horizontally N_j, iff N_i uniformly f-covers horizontally N_j, but in the infinite dimensional case these notions are not equivalent.

For maps fulfilling condition (21) we introduce *the transition matrix.*

Definition 2 *Let* $f : (-\infty,\infty) \times \overline{B_W(r)} \to V$ *be a continuous map satisfying condition (21). We define* the $K \times K$ transition matrix $T(f)$ *with entries* $T(f)[j,i]$ *defined by*

$$T(f)[j,i] := \begin{cases} 1, & \textit{if } N_i \ f\textit{-covers horizontally the set } N_j \ , \\ 0, & \textit{if } f(V(N_i)) \subset S_L(j) \quad \textit{or} \quad f(V(N_i)) \subset S_R(j), \\ \infty, & \end{cases}$$

where $i,j = 0,1,\ldots,K-1$.

It is easy to see that $T(f)[j,i]$ is finite iff

$$f(V(N_i)) \cap N_j = \emptyset, \tag{25}$$

$T(f)[j,i] = 1$ if N_j lies between the images of vertical edges of N_i and $T(f)[j,i] = 0$ if vertical edges in N_i are both mapped either to the left or to the right of N_j.

Let us fix an $n > 0$. Let $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) \in \{0,1,\ldots,K-1\}^n$.

Let $Z = \bigcup_{i=0}^{K-1} Z_k \subset V$. For an indexed family of continuous maps $f_i : (-\infty,\infty) \times \overline{B_W(r)} \to V$, $i = 1,2,\ldots,n$ define

$$Z_\alpha := \{v \in Z_{\alpha_0} | f_i \circ f_{i-1} \circ \ldots \circ f_1(v) \in Z_{\alpha_i}, \quad \text{for } i = 1,2,\ldots,n-1\}$$

and, similarly, for an indexed family of homotopies $F_i : [0,1] \times (-\infty,\infty) \times \overline{B_W(r)} \to V$, $i = 1,2,\ldots,n$, $F_{i,\lambda}$ satisfying (21), we set

$$Z_\alpha^\lambda := \{v \in Z_{\alpha_0} | F_{i,\lambda} \circ F_{i-1,\lambda} \circ \ldots \circ F_{1\lambda}(v) \in Z_{\alpha_i}, \quad \text{for } i = 1,2,\ldots,n-1\}.$$

The following theorem is a generalization of theorems about topological horseshoes by Mischaikow & Mrozek [MM, Th. 2.3] and Zgliczynski [Z].

Theorem 3 *Let* $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) \in \{0,1,\ldots,K-1\}^n$. *Let the maps* $F_i : [0,1] \times (-\infty,\infty) \times \overline{B_W(r)} \to V$, *for* $i = 1,2,\ldots,n$ *be continuous and compact, such that* $F_{i,\lambda}$ *satisfies condition (21) for every* i *and* λ. *Suppose that the following entries of the transition matrices* $T(F_{i,\lambda})[\alpha_i, \alpha_{i-1}]$ *for* $i = 1,2,\ldots,n-1$ *and* $T(F_{n,\lambda})[\alpha_0, \alpha_{n-1}]$ *are finite and do not depend on* λ *(i.e. are constant), then the fixed point index* $I(F_{n,\lambda} \circ F_{n-1,\lambda} \circ \ldots \circ F_{1,\lambda}, int N_\alpha^\lambda)$ *is well defined and constant (i.e. it does not depend on* λ*).*

Proof

For $\delta > 0$ define

$$L(N_i,\delta) := [a_{2i}, a_{2i} + \delta] \times \overline{B_W(r)} \tag{26}$$
$$R(N_i,\delta) := [a_{2i+1} - \delta, a_{2i+1}] \times \overline{B_W(r)} \tag{27}$$
$$V(N_i,\delta) := L(N_i,\delta) \cup R(N_i,\delta) \tag{28}$$
$$H(N_i,\delta) := [a_{2i}, a_{2i+1}] \times (\overline{B_W(r)} \setminus B_W(r-\delta)) \tag{29}$$
$$H(N,\delta) := H(N_0,\delta) \cup H(N_1,\delta) \cup \ldots \cup H(N_{K-1},\delta). \tag{30}$$

For $\delta < a_{2i+1} - a_{2i}$ the sets $L(N_i,\delta)$, $R(N_i,\delta)$, $H(N_i,\delta)$ are contained in N_i and are equal to the δ-thickened left vertical, right vertical and the sum of horizontal edges in N_i, respectively.

Let $Z \subset N$. We introduce the notation

$$\mathrm{Im}F := \overline{\bigcup_{i=1}^{n} F_i \circ F_{i-1} \circ \ldots \circ F_1([0,1], N)} \tag{31}$$

$$\mathrm{Fix}_\lambda(Z) := \{x \in Z \quad | \quad F_{n,\lambda} \circ F_{n-1,\lambda} \circ \ldots \circ F_{1,\lambda}(x) = x\} \tag{32}$$

It follows from the compactness of F_i that $\mathrm{Im}F$ is a compact set. It is easy to see that for $\lambda \in [0,1]$

$$\mathrm{Fix}_\lambda(N_\alpha^\lambda) \subset \mathrm{Im}F. \tag{33}$$

Obviously $\mathrm{Fix}_\lambda(N_\alpha^\lambda)$ is a compact set.

From the compactness of $\mathrm{Im}F$, the assumptions concerning $T(F_{i,\lambda})$, (21), (25) we easily conclude that there exists $\delta > 0$, such that for $i = 1, \ldots, n$ and $\lambda \in [0,1]$ the following conditions hold:

$$F_{i,\lambda}(V(N_{\alpha_{i-1}}, 2\delta) \cap \mathrm{Im}F) \cap N_{\alpha_{i \bmod n}} = \emptyset \tag{34}$$

$$H(N, 2\delta) \cap F_{i,\lambda}(N) = \emptyset. \tag{35}$$

We define

$$C_i := (a_{2i} + \delta, a_{2i+1} - \delta) \times B_W(r - \delta) \tag{36}$$

$$D_i := (a_{2i} + 2\delta, a_{2i+1} - 2\delta) \times B_W(r - 2\delta). \tag{37}$$

We have

$$B(D_i, \delta) \subset C_i, \qquad B(C_i, \delta) \subset \mathrm{Int}N_i. \tag{38}$$

Obviously the sets D_α^λ, C_α^λ are both open and the set N_α^λ is closed for $\lambda \in [0,1]$.

We will show now that for $\lambda \in [0,1]$

$$F_{n,\lambda} \circ F_{n-1,\lambda} \circ \ldots F_{1,\lambda}(x) \neq x, \quad \text{provided} \quad x \in N_\alpha^\lambda \setminus D_\alpha^\lambda. \tag{39}$$

Suppose that (39) does not hold. Then there exists an $x \in N_\alpha^\lambda \setminus D_\alpha^\lambda$ such that

$$F_{n,\lambda} \circ F_{n-1,\lambda} \circ \ldots F_{1,\lambda}(x) = x \tag{40}$$

$$F_{i_0,\lambda} \circ F_{i_0-1,\lambda} \circ \ldots F_{1,\lambda}(x) \in (N_{\alpha_{i_0}} \setminus D_{\alpha_{i_0}}), \quad \text{for certain } i_0 \; 0 \leq i_0 \leq n-1 \tag{41}$$

It follows from (35) and (37) that

$$F_{i_0,\lambda} \circ F_{i_0-1,\lambda} \circ \ldots F_{1,\lambda}(x) \in L(N_{\alpha_{i_0}}, 2\delta) \cup R(N_{\alpha i_0}, 2\delta). \tag{42}$$

It follows from this and from (34)

$$F_{i_0+1,\lambda} \circ F_{i_0,\lambda} \circ \ldots F_{1,\lambda}(x) \notin N_{\alpha_{(i_0+1) \bmod n}} \tag{43}$$

So if $i_0 < n-1$ then $x \notin N_\alpha^\lambda$ and if $i_0 = n-1$ then it follows from (43) that x cannot be a fixed point for a map $F_{n,\lambda} \circ F_{n-1,\lambda} \circ \ldots F_{1,\lambda}$. In both cases we get a contradiction, so (39) holds.

From (39) it follows immediately that

$$\mathrm{Fix}_\lambda(D_\alpha^\lambda) = \mathrm{Fix}_\lambda(C_\alpha^\lambda) = \mathrm{Fix}_\lambda(\mathrm{int}N_\alpha^\lambda) = \mathrm{Fix}_\lambda(N_\alpha^\lambda). \tag{44}$$

Let $\lambda_0 \in [0,1]$. It follows from the uniform continuity of continuous maps on compact sets that there exists an interval Λ open in $[0,1]$, $\lambda_0 \in \overline{\Lambda}$, such that for every $\lambda_1, \lambda_2 \in \overline{\Lambda}$, $i = 1, \ldots, n$ and $x \in \mathrm{Im}F$ the following holds

$$|F_{i,\lambda_1} \circ F_{i-1,\lambda_1} \circ \ldots \circ F_{1,\lambda_1}(x) - F_{i,\lambda_2} \circ F_{i-1,\lambda_2} \ldots \circ F_{1,\lambda_2}(x)| \leq \delta. \tag{45}$$

We will now show the following inclusions for $\lambda \in \overline{\Lambda}$

$$\text{Im } F \cap D_\alpha^\lambda \subset \text{Im } F \cap C_\alpha^{\lambda_0} \subset \text{Im } F \cap \text{int}(N_\alpha^\lambda). \tag{46}$$

Let $x \in \text{Im } F \cap D_\alpha^\lambda$. Then

$$F_{i,\lambda} \circ F_{i-1,\lambda} \circ \dots F_{1,\lambda}(x) \in D_{\alpha_i}, \quad \text{for } i = 1, \dots, n-1.$$

However, it follows from (38), (45) that

$$F_{i,\lambda_0} \circ F_{i-1,\lambda_0} \circ \dots F_{1,\lambda_0}(x) \in C_{\alpha_i}, \quad \text{for } i = 0, \dots, n-1.$$

Thus $x \in \text{Im } F \cap C_\alpha^{\lambda_0}$. Proof of the second inclusion is analogous.

Because $\text{Fix}_\lambda(C_\alpha^{\lambda_0}) \subset \text{Im}F$, we obtain from (44) and (46) for $\lambda \in \overline{\Lambda}$

$$\text{Fix}_\lambda(C_\alpha^{\lambda_0}) = \text{Fix}_\lambda(N_\alpha^\lambda). \tag{47}$$

Since $\text{Fix}_\lambda(N_\alpha^{\lambda_0})$ is a compact set and from (44) and (47) it follows that the fixed point indices of the map $F_{n,\lambda} \circ F_{n-1,\lambda} \circ \dots F_{1,\lambda}$ relative to the sets $\text{int}N_\alpha^\lambda$, $C_\alpha^{\lambda_0}$, D_α^λ are well defined and from the excision property of the fixed point index [D, Ch.VII Th. 5.4] we conclude that

$$\begin{gathered} \mathrm{I}(F_{n,\lambda} \circ F_{n-1,\lambda} \circ \dots F_{1,\lambda}, D_\alpha^\lambda) = \\ \mathrm{I}(F_{n,\lambda} \circ F_{n-1,\lambda} \circ \dots F_{1,\lambda}, C_\alpha^{\lambda_0}) = \\ \mathrm{I}(F_{n,\lambda} \circ F_{n-1,\lambda} \circ \dots F_{1,\lambda}, \text{int}N_\alpha^\lambda), \quad \forall \lambda \in \overline{\Lambda}. \end{gathered} \tag{48}$$

Substituting $\lambda := \lambda_0$ we derive

$$\mathrm{I}(F_{n,\lambda_0} \circ F_{n-1,\lambda_0} \circ \dots F_{1,\lambda_0}, D_\alpha^{\lambda_0}) = \mathrm{I}(F_{n,\lambda_0} \circ F_{n-1,\lambda_0} \circ \dots F_{1,\lambda_0}, C_\alpha^{\lambda_0}). \tag{49}$$

From (44) and (47) it follows that

$$\text{Fix}_{\overline{\Lambda}}(C_\alpha^{\lambda_0}) := \bigcup_{\lambda \in \overline{\Lambda}} \text{Fix}_\lambda(C_\alpha^{\lambda_0}) \subset C_\alpha^{\lambda_0}. \tag{50}$$

Moreover, from (44) and (47) it follows that

$$\text{Fix}_{\overline{\Lambda}}(C_\alpha^{\lambda_0}) = \bigcup_{\lambda \in \overline{\Lambda}} \text{Fix}_\lambda(N_\alpha^\lambda). \tag{51}$$

From the above condition and compactness of maps $F_{i,\lambda}$ we see that $\text{Fix}_{\overline{\Lambda}}(C_\alpha^{\lambda_0})$ is compact.

Now from the homotopy property of the fixed point index we obtain

$$\mathrm{I}(F_{n,\lambda} \circ F_{n-1,\lambda} \circ \dots F_{1,\lambda}, C_\alpha^{\lambda_0}) = \mathrm{I}(F_{n,\lambda_0} \circ F_{n-1,\lambda_0} \circ \dots F_{1,\lambda_0}, C_\alpha^{\lambda_0}), \quad \forall \lambda \in \overline{\Lambda}. \tag{52}$$

From (48),(49),(52) we conclude

$$\mathrm{I}(F_{n,\lambda} \circ F_{n-1,\lambda} \circ \dots F_{1,\lambda}, D_\alpha^\lambda) = \mathrm{I}(F_{n,\lambda_0} \circ F_{n-1,\lambda_0} \circ \dots F_{1,\lambda_0}, D_\alpha^{\lambda_0}), \quad \forall \lambda \in \overline{\Lambda}. \tag{53}$$

From the connectedness of $[0,1]$, (53) and (48) we get

$$\mathrm{I}(F_{n,\lambda} \circ F_{n-1,\lambda} \circ \dots F_{1,\lambda}, \text{Int}N_\alpha^\lambda) = \mathrm{I}(F_{n,0} \circ F_{n-1,0} \circ \dots F_{1,0}, \text{Int}N_\alpha^0), \qquad \forall \lambda \in [0,1].$$

This finishes the proof. □

The following theorem expresses the fixed point index of the composition $f_n \circ f_{n-1} \circ \dots \circ f_1$ on the set int N_α in terms of the transition matrices.

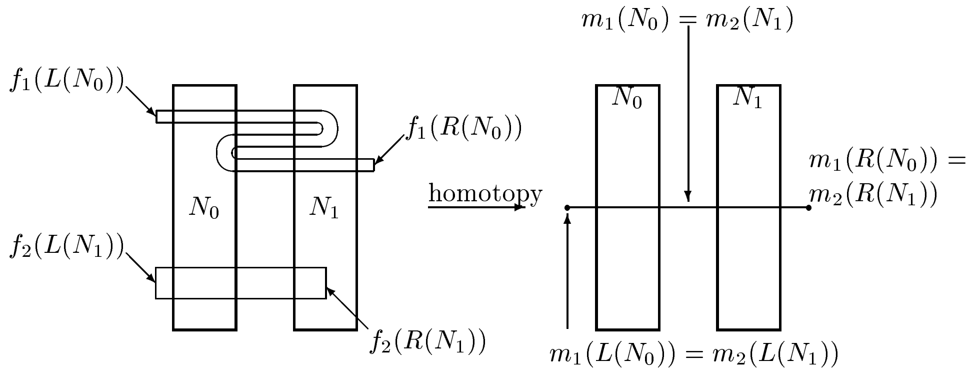

Figure 2: An example with $\alpha = (0, 1)$, nonzero index

Theorem 4 *Let $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{n-1}) \in \{0, 1, \dots, K-1\}^n$. Let the maps $f_i : (-\infty, \infty) \times \overline{B_W(r)} \to V$, for $i = 1, 2, \dots, n$ be continuous and compact, such that f_i satisfies condition (21) for every i. Suppose that the following entries of transition matrices $T(f_i)[\alpha_i, \alpha_{i-1}]$ for $i = 1, 2, \dots, n-1$ and $T(f_n)[\alpha_0, \alpha_{n-1}]$ are finite. Then*

$$I(f_n \circ f_{n-1} \circ \dots \circ f_1, \mathit{int}\ N_\alpha) =$$
$$\pm T(f_n)[\alpha_0, \alpha_{n-1}] \cdot T(f_{n-1})[\alpha_{n-1}, \alpha_{n-2}] \cdot \dots \cdot \quad T(f_2)[\alpha_2, \alpha_1] \cdot T(f_1)[\alpha_1, \alpha_0].$$

Proof

We want to connect maps f_i by homotopy with the piecewise affine one-dimensional maps m_i for which we can easily calculate the fixed point index of interest.

Let us choose $d, u \in \mathbb{R}$ such that $d < \min\{a_{2i} \mid i = 0, 1 \dots, K-1\}$ and $u > \max\{a_{2i+1} | i = 0, 1 \dots, K-1\}$. So by definition $(d, 0) \in S_L(i)$ and $(u, 0) \in S_R(i)$ for $i = 0, 1, \dots, K-1$.

We now start to define a map $m_i : (-\infty, \infty) \times \overline{B_W(r)} \to (-\infty, \infty) \times \overline{B_W(r)}$. First we define m_i on $V(N_{\alpha_{i-1}})$

$$m_i(x, w) := \begin{cases} (d, 0), & \text{if } f_i(x, w) \in S_L(\alpha_i) \\ (u, 0), & \text{if } f_i(x, w) \in S_R(\alpha_i) \end{cases} \tag{54}$$

It follows from (25) that for $(x, w) \in V(N_{\alpha_{i-1}})$ $f(x, w) \in S_L(\alpha_i) \cup S_R(\alpha_i)$, so the above formula actually defines m_i on $V(N_{\alpha_{i-1}})$.

We now extend linearly the map m_i to $N_{\alpha_{i-1}}$. So for $(x, w) \in N_{\alpha_{i-1}}$ we set

$$m_i(x, w) := (m_i(a_{2\alpha_{i-1}+1}, 0) - m_i(a_{2\alpha_{i-1}}, 0)) \frac{x - a_{2\alpha_{i-1}}}{a_{2\alpha_{i-1}+1} - a_{2\alpha_{i-1}}} + m_i(a_{2\alpha_{i-1}}, 0). \tag{55}$$

Now for $(x, w) \notin N_{\alpha_{i-1}}$ we set

$$m_i(x, w) := \begin{cases} m_i(a_{2\alpha_{i-1}}, 0), & \text{if } (x, w) \in S_L(\alpha_{i-1}) \ , \\ m_i(a_{2\alpha_{i-1}+1}, 0), & \text{if } (x, w) \in S_R(\alpha_{i-1}) \ . \end{cases} \tag{56}$$

We define the homotopy $F_i : [0, 1] \times (-\infty, \infty) \times \overline{B_W(r)} \to (-\infty, \infty) \times \overline{B_W(r)}$ as a convex combination of m_i and f_i, so

$$F_i(\lambda, v) := (1 - \lambda) f_i(v) + \lambda m_i(v) \tag{57}$$

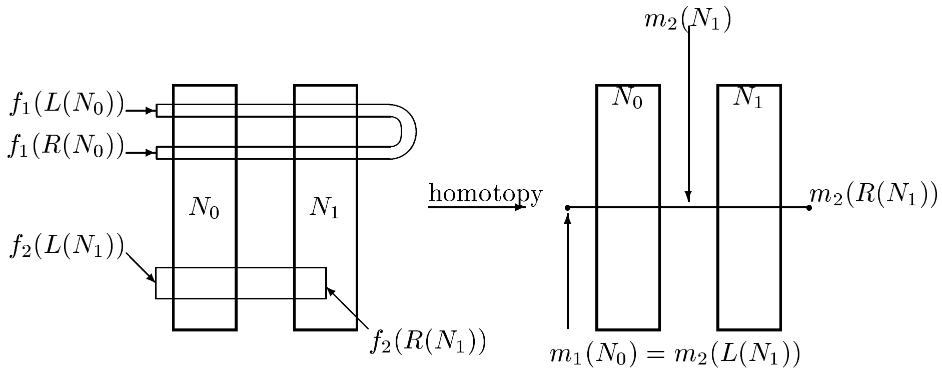

Figure 3: An example with $\alpha = (0, 1)$, zero index

It follows from the construction of m_i and F_i straightforwardly that for every $\lambda \in [0, 1]$

$$\begin{aligned} &N_{\alpha_{i-1}}\ F_{i,\lambda}\text{-covers horizontally the set } N_{\alpha_{i \bmod n}} \\ &\text{iff } N_{\alpha_{i-1}}\ f_i\text{-covers horizontally the set } N_{\alpha_{i \bmod n}}, \end{aligned} \tag{58}$$

$$\begin{aligned} N_{\alpha_{i-1}} \text{ does not } f_i\text{-cover horizontally the set } N_{\alpha_{i \bmod n}} \\ \text{iff } m_i(N_{\alpha_{i-1}}) \cap N_{\alpha_{i \bmod n}} = \emptyset. \end{aligned} \tag{59}$$

So we can apply Theorem 3 to F_i to obtain

$$I(f_n \circ f_{n-1} \circ \ldots \circ f_1, \mathrm{int} N^0_\alpha) = I(m_n \circ m_{n-1} \circ \ldots \circ m_1, \mathrm{int} N^1_\alpha). \tag{60}$$

So to finish the proof it remains to calculate the fixed point index of the superposition of the m_i's.

Suppose now that

$$T(f_n)[\alpha_0, \alpha_{n-1}] \cdot T(f_{n-1})[\alpha_{n-1}, \alpha_{n-2}] \cdot \ldots \cdot T(f_2)[\alpha_2, \alpha_1] \cdot T(f_1)[\alpha_1, \alpha_0] = 0.$$

Then it follows from Definition 2 that there exists $i_0 \in \{1, 2, \ldots, n\}$ such that $N_{\alpha_{i_0-1}}$ does not f_i-cover horizontally the set $N_{\alpha_{i_0 \bmod n}}$. From this and (59) we see that $N^1_\alpha = \emptyset$ if $i_0 \leq n-1$ or if $i_0 = n-1$ $m_n \circ m_{n-1} \circ \ldots \circ m_1(N^1\alpha) \cap N_{\alpha_0} = \emptyset$. So

$$I(f_n \circ f_{n-1} \circ \ldots \circ f_1, \mathrm{int} N^0_\alpha) = 0. \tag{61}$$

Now we suppose that

$$T(f_n)[\alpha_0, \alpha_{n-1}] \cdot T(f_{n-1})[\alpha_{n-1}, \alpha_{n-2}] \cdot \ldots \cdot T(f_2)[\alpha_2, \alpha_1] \cdot T(f_1)[\alpha_1, \alpha_0] = \pm 1.$$

In this case it is easy to prove that there exists $s < t$ such that $N^1_\alpha = [s, t] \times \overline{B_W(r)}$ and for $(x, w) \in N^1_\alpha$ we have

$$m_n \circ m_{n-1} \circ \ldots \circ m_1(x, w) = (Ax + B, 0),$$

where the constants $AandB$ are such that the segment with the end-points $As + B$ and $At + B$ contains in its interior segment $[s, t]$. From this it follows easily that

$$I(m_n \circ m_{n-1} \circ \ldots \circ m_1, \mathrm{int} N^1_\alpha) = \pm 1. \tag{62}$$

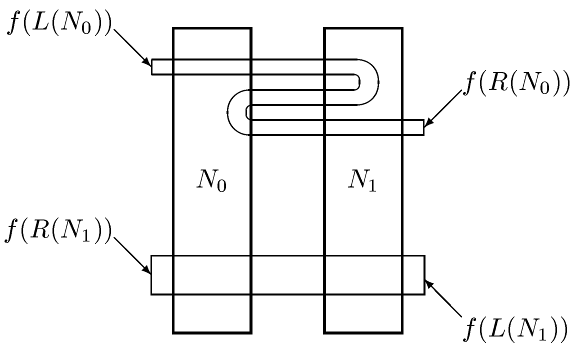

Figure 4: Topological horseshoe

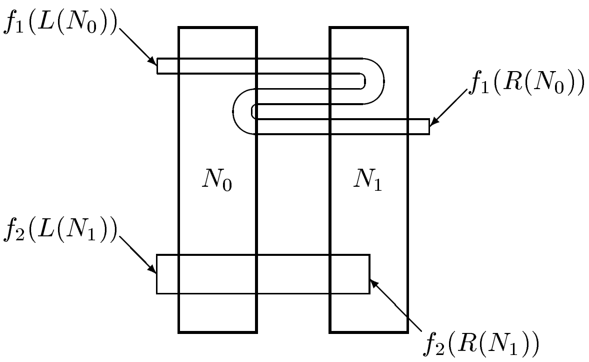

Figure 5: Topological prehorseshoe

□

We are now going to define notions of *a topological horseshoe* and *a topological prehorseshoe.*

Let $a_0 < a_1$, $a_2 < a_3$, $(a_0, a_1) \cap (a_2, a_3) = \emptyset$ and $r > 0$. $N_0 = [a_0, a_1] \times \overline{B_W(r)}$ and $N_1 = [a_2, a_3] \times \overline{B_W(r)}$. $N = N_0 \cup N_1$.

Definition 3 *A continuous and compact map $f : N \to (-\infty, \infty) \times \overline{B_W(r)}$ will be called a* (uniform) topological horseshoe *iff $\overline{f(N)} \subset (-\infty, \infty) \times B_W(r)$ (comp. condition (21)) and, for $i = 0, 1$, N_i (uniformly) f-covers horizontally sets N_0 and N_1.*

Definition 4 *A continuous and compact map $f : N \to (-\infty, \infty) \times \overline{B_W(r)}$ will be called a* (uniform) topological prehorseshoe *iff $\overline{f(N)} \subset (-\infty, \infty) \times B_W(r)$ (comp. condition (21)), N_0 (uniformly) f-covers horizontally sets N_0 and N_1, N_1 f-covers horizontally the set N_0.*

Suppose that f is a topological horseshoe (prehorseshoe). We want to show that f has a symbolic dynamics. To this end we define in a standard way a map

$\pi : \mathrm{Inv}(N, f) \to \Sigma_2^+$ by

$$\pi(x)_i := \begin{cases} 0, & \text{if } f^i(x) \in N_0 \ , \\ 1, & \text{if } f^i(x) \in N_1 \ . \end{cases} \tag{63}$$

Obviously we have a semiconjugacy

$$\sigma \circ \pi = \pi \circ f_{|\mathrm{Inv}(N,f)}.$$

Theorem 5 *(see [MM, Z]). Suppose f is a topological horseshoe. Then π is onto. Moreover π^{-1} of the periodic sequence from Σ_2^+ contains a periodic point.*

Proof

Let us note that we can extend the domain of f to $(-\infty, \infty) \times \overline{B(r)}$ in such a way that f remains compact and condition (21) holds.

From the definitions of the topological horseshoe and the transition matrix it follows immediately that

$$T(f) = \begin{bmatrix} 1 & 1 \\ 1 & 1. \end{bmatrix} \tag{64}$$

It follows from Theorem 4 that for every finite sequence $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{n-1}) \in \{0,1\}^n$ with $f_i = f$ the following holds,

$$\mathrm{I}(f^n, N_\alpha) = \pm 1. \tag{65}$$

So we see that for any periodic $\alpha \in \Sigma_2^+$ there exists $x \in N$ such that $\pi(x) = \alpha$. Since periodic sequences are dense in Σ_2^+ and $\mathrm{Inv}(N, f)$ is compact, $\pi(\mathrm{Inv}(N, f)) = \Sigma_2^+$.

□

For the topological prehorseshoe we have the following.

Theorem 6 *Let $P := \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ Assume f is a topological prehorseshoe. Then $\Sigma_P^+ \subset \sigma(Inv(N, f))$. Moreover π^{-1} of the periodic sequence from Σ_P^+ contains a periodic point.*

Proof

The proof goes essentially along the same lines as the proof of the previous theorem. The only difference is the form of the transition matrix, which in our case is

$$T(f) = \begin{bmatrix} 1 & 1 \\ 1 & \infty. \end{bmatrix} \tag{66}$$

□

Remark 7 *$W \subset W'$. Let $p : W' \to W$ be a projection onto W. Let $f : (-\infty, \infty) \times \overline{B_{W'}(r)} \to (-\infty, \infty) \times \overline{B_{W'}(r)}$ be such that condition (21) holds. Suppose that $f = p \circ f \circ p$, then $p(N_i)$ $f_{|W}$-covers horizontally $p(N_j)$ iff N_i f-covers horizontally N_j. So we see that f and $f_{|W}$ have the same transition matrix and if one of them is a topological (pre) horseshoe then the other is a topological (pre) horseshoe, too.*

Remark 8 *Suppose that we have a continuous and compact homotopy $F : [0,1] \times N \to (-\infty, \infty) \times \overline{B_W(r)}$. Suppose that F_0 is a topological (pre)horseshoe. If $\dim W < \infty$ then for λ sufficiently small F_λ is a topological (pre)horseshoe.*

If $\dim W = \infty$ and F is uniformly continuous and F_0 is a uniform topological (pre)horseshoe then for λ sufficiently small F_λ is a topological (pre)horseshoe.

2 Nested sequences of topological horseshoes

In this section we introduce the notion of nested sequences of topological horseshoes. Our aim is to show that a finite, but sufficiently long sequence of nested sequences of horseshoes guarantees the existence of all but a finite number of periods.

Definition 5 *Let $f : (-\infty, \infty) \times \overline{B_W(r)} \to (-\infty, \infty) \times B_W(r)$ be continuous and compact. Let $l \in \mathbb{N}$, $k \in \mathbb{N} \cup \{\infty\}$ $0 < l < k$. Let I_i^s for $i = 0, 1$ and $s = l, \ldots, k$ be closed intervals such that $I_i^s \supset I_i^{s+1}$ and $intI_0^l \cap intI_1^l = \emptyset$. Let $N_i^s := I_i^s \times \overline{B_W(r)}$.*

*We say that f has an (l, k)-*nested sequence of topological horseshoes (of vertical size r)*, iff for $i = 0, 1$ and $s = l, \ldots, k$ N_i^s f^s-covers horizontally the sets N_0^l and N_1^l.*

*We say that f has an (l, k)-*nested sequence of topological horseshoes with a prehorseshoe (of vertical size r) *, iff*

- *f has an (l, k)-nested sequence of topological horseshoes,*
- *there exists intervals I_i^{l-1} for $i = 0, 1$, $intI_0^{l-1} \cap intI_0^{l-1} = \emptyset$, $intI_{1-i}^{l-1} \cap intI_i^l = \emptyset$ for $i = 0, 1$ and f^{l-1} is a topological prehorseshoe with respect to sets N_0^{l-1} and N_1^{l-1}, where $N_i^{l-1} := I_i^{l-1} \times \overline{B_W(r)}$,*
- *for $i = 0, 1$ and $s = l, \ldots, k$ N_i^s f^s-covers horizontally the sets N_0^{l-1} and N_1^{l-1},*
- *N_0^{l-1} f^{l-1}-covers horizontally the sets N_0^l and N_1^l, and N_1^{l-1} f^{l-1}-covers horizontally the set N_0^l.*

We are interested in the following question: Given a nested sequence of topological horseshoes with a prehorseshoe, what is the set of numbers which are periods for some points ?

Our aim in this section is to prove the following.

Theorem 9 *Let a map $f : (-\infty, \infty) \times \overline{B_W(r)} \to (-\infty, \infty) \times B_W(r)$ be continuous and compact. Suppose f has an $(n + 1, 2n - 1)$-nested sequence of topological horseshoes with a prehorseshoe. Then there exists a constant $M(n)$ such that for every $m > M(n)$ f has a periodic point of period m.*

Before the proof of this theorem we need a few lemmas. For the rest of this section we assume that a map $f : (-\infty, \infty) \times \overline{B_W(r)} \to (-\infty, \infty) \times B_W(r)$ is continuous and compact. We also assume that f has an $(n + 1, 2n - 1)$-nested sequence of topological horseshoes with a prehorseshoe.

Lemma 10 *Let N_i^s for $i = 0, 1, \quad s = n, \ldots, 2n-1$ be the sets from the definition of an $(n, \ldots, 2n-1)$-nested sequence of topological horseshoes with a prehorseshoe. Let*

$$N_i := N_i^n \cup \ldots \cup N_i^{2n-1} \quad \textit{for } i = 0, 1. \tag{67}$$

Let $A \in \mathbb{N}$, $A \geq 2n$ and suppose

$$A = \sum_{1 \leq i \leq t} y_i, \quad \textit{where } t > 1 \textit{ and } n \leq y_i < 2n. \tag{68}$$

Let us denote

$$s_j := \sum_{1 \leq i \leq j} y_i. \tag{69}$$

Let $J \subset \mathbb{N}$, $J = \{j_1, j_2, \ldots, j_k\}$, *where* $1 \leq j_1 < j_2 < \ldots < j_k < t$ *and*

$$\textit{if } j, j+1 \in J \textit{ then } y_{j+1} > n. \tag{70}$$

Then there exists a periodic point x *such that*

$$x, f^{s_j}(x) \in \mathit{int}N_0 \quad \textit{for } 1 \leq j \leq l-1 \textit{ and } j \notin J,$$
$$f^{s_j}(x) \in \mathit{int}N_1 \quad \textit{for } j \in J,$$
$$f^A(x) = x.$$

In particular,

$$f^{s_j}(x) \neq x, \quad \textit{for } j \in J.$$

Proof

We use Theorem 4 with symbols N_i substituted by M_i. We set $K = t-1, \quad n = t$, $f_i := f^{y_i}$ for $i = 1, \ldots, t$. We define the following sets: $M_i := N_0^{y_{i+1}}$ if $i+1 \notin J$ and $M_i := N_0^{y_{i+1}}$ otherwise. For $0 \leq i < t$ we set $\alpha_i = i$. It follows from Definition 5 and (70) that for $i = 1, \ldots, t$ the set $M_{\alpha_{i-1}}$ f_i–covers horizontally the set $M_{i \bmod t}$, so

$$T(f_i)[\alpha_{i \bmod t}, \alpha_{i-1}] = 1. \tag{71}$$

From Theorem 4 we get

$$\mathrm{I}(f^A, \mathrm{int}M_\alpha) \neq 0. \tag{72}$$

Now the existence of periodic point with the required properties follows from the definition of M_α and (67).

From Definition 5 it follows

$$\mathrm{int}N_0 \cap \mathrm{int}N_1 = \emptyset, \tag{73}$$

so

$$f^{s_j}(x) \neq x, \quad \text{for } j \in J.$$

□

The next lemma show that numbers of the form m^k, where m is prime, can be realized as periods.

Lemma 11 *Let* m *be a prime number. If* $m > 2n$ *then* f *has a periodic point of period* m^k *for every* $k \in \mathbb{N}$*. If* $m < 2n$ *then if* $m^{k-1} > n$, f *has a periodic point of period* m^k.

Proof

Suppose $m > 2n$ and $k = 1$. We take any decomposition of $m = y_1 + \ldots y_t$, where $t \geq 2$, $n \leq y_i \leq 2n-1$ for $i = 1, \ldots, t$. We set $J = \{1\}$. From Lemma 10 we obtain x, a fixed point of f^m , such that $f^{y_1}(x) \neq x$. Hence, the period of x is equal m.

Now let $k > 1$ and $m^{k-1} > n$. We represent m^{k-1}, $m^k - m^{k-1}$ as the sum of integers from the interval $[n, 2n-1]$ and then we obtain a decomposition

$$m^k = y_1 + \ldots + y_t, \quad t \geq 2, n \leq y_i \leq 2n-1$$

such that for some j_0 $s_{j_0} = m^{k-1}$. Now we apply to f and this decomposition the results of Lemma 10 to obtain a point x such that

$$f^{m^{k-1}}(x) \neq x, \quad f^{m^k}(x) = x.$$

It is easy to see that in this case period of x is equal to m^k.

□

Now we investigate which composite natural numbers can be periods.

Let $A \in \mathbb{N}_+$. Consider a decomposition of A into prime factors. We have $A = p_1^{e_1} p_2^{e_2} \ldots p_k^{e_k}$, where $p_1 < p_2 < \ldots < p_k$ are prime numbers and $e_i \in \mathbb{N}_+$ for $i = 1, \ldots, k$. We define *the main divisors* of A by $d_i := A/p_i$ for $i = 1, 2, \ldots, k$. It is easy to see that if $m < A$ divides A then there exists a main divisor d_{i_0} of A such that m divides d_{i_0}. The order of main divisors is $d_k < d_{k-1} < \ldots < d_1$.

Definition 6 *Let A be a composite natural number. We say that A is n-*realisable *iff there exist $l_i \in \mathbb{N}$ and a permutation σ of the set $\{1, 2, \ldots, k\}$ such that*

$$\begin{gathered} l_1 d_{\sigma(1)} \geq n \\ l_i d_{\sigma(i)} - l_{i-1} d_{\sigma(i-1)} > n, \quad \textit{for } i = 2, \ldots, k \\ A - l_k d_{\sigma(k)} \geq n \end{gathered}$$

Obviously if A is n-realisable then A is also m-realisable for $m < n$.

The following lemma shows that n-realisable numbers are periods.

Lemma 12 *Let A be n-realisable. Then f has a periodic point of period A.*

Proof

Let σ and k be as in Definition 6. We can construct a decomposition of A

$$A = y_1 + y_2 + \ldots + y_t \tag{74}$$

where $n \leq y_i \leq 2n - 1$ are such that there exists $j_1 < j_2 < \ldots < j_k$,

$$s_{j_1} = l_1 d_{\sigma(1)}, \quad s_{j_2} = l_2 d_{\sigma(2)}, \ldots, s_{j_k} = l_k d_{\sigma(k)} \tag{75}$$

We set $J := \{j_1, j_2, \ldots, j_k\}$.

Let x be the periodic point obtained from lemma 10 applied to the above decomposition of A and the set J. Suppose that the period of x is less than A. So it must divide some main divisor d_s of A. Hence $f^{d_s}(x) = x$, but by Lemma 10 $f^{l_{\sigma^{-1}(s)} d_s}(x) \neq x$, so $f^{d_s}(x) \neq x$. We have a contradiction. Hence the period of x is A.

□

Our aim is to show that all sufficiently big composite numbers are n-realisable. To this end we need a few easy arithmetic lemmas.

Lemma 13 *Let $A = p_1 p_2$, where $p_1 < p_2$ are primes. If $p_2 \geq 2(n+1)$ then A is n-realisable.*

Proof

First we find a positive natural number s, such that

$$sp_1 - p_2 > n, \qquad p_1p_2 - sp_1 \geq n. \tag{76}$$

To obtain this it is enough to show that the segment of natural numbers $[p_2+n+1, p_1p_2-n]$ have at least p_1 elements, so it must contain at least one number of the form sp_1.

We define the decomposition of A by

$$A = p_2 + (sp_1 - p_2) + (A - sp_1) \tag{77}$$

We set $t = 3$, $\sigma(i) = i$. This shows that A is n-realisable.

□

The following lemma deals with the situation when A is n-realisable and we multiply it by one of its prime factors.

Lemma 14 *Let $A = p_1^{e_1}\dots p_k^{e_k}$, where $k > 1$. If A is n-realisable then $p_{i_0}A$ for $i_0 = 1,\dots,k$ is n-realisable.*

The next lemma show that a number with many prime factors is n-realisable.

Lemma 15 *Let $A = p_1^{e_1}\dots p_k^{e_k}$, where $k > 1$. If $p_1^{\Sigma_{i=1}^k e_i - 2} > n$ then A is n-realisable.*

Proof

Let us denote $E = \Sigma_{i=1}^k e_i$. First we estimate the difference between the main divisors

$$\begin{aligned} d_i - d_{i+1} = (p_{i+1} - p_i)A/(p_{i+1}p_i) \geq A/(p_{i+1}p_i) \geq p_1^{E-2} > n \\ d_k = A/p_k \geq p_1^{E-1} > n \\ A - d_1 \geq A/p_1 \geq p_1^{E-1} > n. \end{aligned}$$

We set $\sigma(i) = i$, $l_i = 1$. From the above inequalities it follows that A is n-realisable.

□

Lemma 16 *Let $A = p_1^{e_1}\cdots p_k^{e_k}$, where $k > 2$, $p_k \geq n$ and $A \geq 20/3n$. Then A is n-realisable.*

Proof

We show that there exists s such that

$$d_{k-1} \geq n \tag{78}$$

$$d_i - d_{i+1} > n, \quad \text{for i=1,...,k-2} \tag{79}$$

$$sd_k - d_1 > n, \tag{80}$$

$$A - sd_k \geq n. \tag{81}$$

It is easy to see that for $i = 1,\dots,k-2$ we have

$$d_{k-1} > p_k > n$$

$$d_i - d_{i+1} = A/p_i - A/p_{i+1} \geq p_k(p_{i+1} - p_i) > n.$$

It remains now to find s, such that conditions (80 and 81) hold. To this end it is enough to show that the segment of natural numbers $[d_1+n+1, A-n]$ has at least d_k elements. We have to show that

$$(A-n)-(d_1+n+1)+1 \geq d_k$$

$$(A-n)-(A/p_1+n+1)+1-A/p_k = A(1-1/p_1-1/p_k)-2n \geq$$
$$A(1-1/2-1/5)-2n = 3/10A-2n \geq 0.$$

Now we set $\sigma(i)=k-i$ for $i=1,\ldots,k-1$, $\sigma(k)=k$ and $l_1=l_2=\ldots=l_{k-1}=1$ and $l_k=s$. It follows from conditions (78-81) that A is n-representable.

□

Lemma 17 *For every n there exists M such that every composite number $A>M$ is n-realisable.*

Proof Let $A=p_1^{e_1}\cdots p_k^{e_k}$.

Assume that A is not n-realisable.

Suppose that $k=1$. It is easy to see that if $A/p_1>n$ than A is n-realisable, so $p_1^{e_1-1}\leq n$. Obviously $p_1\leq n$ and $e_1-1\leq \ln_2 n$, so we obtain $A\leq n^{\ln_2 n+1}$.

Suppose now that $k=2$. From Lemmas 13 and 14 it follows that $p_2<2(n+1)$. From Lemma 15 it follows that $2^{e_1+e_2-2}\leq n$. Hence

$$A\leq (2n-1)^{\ln_2 n+2}.$$

Suppose now that $k>2$. From Lemma 16 it follows that $A<20/3n$ or $p_k<n$. We have to find an upper bound for A in the case when $p_k<n$. From Lemma 15 it follows that

$$2^{\Sigma e_i-2}\leq n.$$

Hence

$$A\leq (n-1)^{\ln_2 n+2}.$$

We can take as M any number greater than $(2n-1)^{\ln_2 n+2}$.

□

Proof of Theorem 9 The assertion follows immediately from Lemmas 11, 12 and 17.

□

3 Covering matrices analysis for one dimensional maps

In this section we deal with maps defined on the real line. The aim of this section is to show that one-dimensional maps with an odd periodic point give rise to nested sequences of horseshoes.

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and assume that there exists an n-periodic point x. Let $x_0 < x_1 < \ldots < x_{n-1}$ be points from the orbit of x. With this orbit we associate a permutation $\pi(f,x)$ of the set of indices $0, 1, \ldots, n-1$ given by

$$\pi(f,x)(i) = j, \text{iff } f(x_i) = x_j. \tag{82}$$

Let $I_i := [x_{i-1}, x_i]$ for $i = 1, 2, \ldots, n-1$ and $I := [x_0, x_{n-1}]$. The segments containing x_0 or x_{n-1} will be called *external* and the remaining segments will be called *internal*. So I_1, I_{n-1} are external segments, and I_2 (for $n > 3$) is an internal segment. We define two *covering matrices* of the orbit of x by

$$D(f,x)_{ji} := \begin{cases} 1, & \text{if } f(x_{i-1}) < f(x_i) \text{ and } I_j \subset [f(x_{i-1}), f(x_i)], \\ -1, & \text{if } f(x_i) < f(x_{i-1}) \text{ and } I_j \subset [f(x_i), f(x_{i-1})], \\ 0, & \text{otherwise }, \end{cases} \tag{83}$$

$$C(f,x)_{ji} := |D(f,x)_{ji}|. \tag{84}$$

Consider the column of matrix $A(=C$ or $D)$, A_{ij}, $i = 1, \ldots, n-1$; it will be called *internal* iff it corresponds to an internal segment($1 < j < n-1$), otherwise it is called *external*.

Let us note that the covering matrices $D(f,x)$a and $C(f,x)$ are fully determined by $\pi(f,x)$. Therefore, we will also use notation $D(\pi)$ and $C(\pi)$, and when π is known from the context we write just D or C.

Obviously notions of covering matrices are not robust. Small perturbation can completely destroy the periodic orbit and the covering matrix has no sense. However, using covering matrices in a proper way will allow us to detect topological horseshoes and even more complicated behavior which is robust.

Definition 7 *Let $K = [a_K, b_K]$, $L = [a_L, b_L]$. We will use the following notation:*

- $K \overset{+}{\to} L$ *iff* $f(a_K) < f(b_K)$ *and* $L \subset [f(a_K), f(b_K)]$*;*
- $K \overset{-}{\to} L$ *iff* $f(a_K) > f(b_K)$ *and* $L \subset [f(b_K), f(a_K)]$*;*
- $K \to L$ *iff* $K \overset{+}{\to} L$ *or* $K \overset{-}{\to} L$*.*

If $K \to L$ then we say that K f-covers L.

Let us note that the symbol $K \to L$ has a slightly different meaning (a more restrictive one) than that used in [BGMY] or [De], as we require that the endpoints of K are mapped on different sides of the middle point of L.

There exist obvious connections between the covering matrices and the relation defined above, namely $I_i \to I_j$ if $C_{ji} = 1$, $I_i \overset{+}{\to} I_j$ if $D_{ji} = 1$ and $I_i \overset{-}{\to} I_j$ if $D_{ji} = -1$.

Now we want to examine the connection between multiplication of covering matrices and iteration of maps.

Lemma 18 *Let $p \in \mathbb{N}$, $p > 0$. Suppose that there exist $i_0, i_1, i_2, \ldots, i_{p-1}, i_p$ such that $D_{i_p i_{p-1}} D_{i_{p-1} i_{p-2}} \ldots D_{i_1 i_0} \neq 0$, then for every segment $S \subset I_p$ there exists a family of segments $J_l = [d_l, u_l] \subset I_{i_l}$ for $l = 0, 1, \ldots, p$, with $J_p = S$ such that*

$$f(int J_{l-1}) = int J_l, \quad \text{for } l = 1, \ldots, p \tag{85}$$

and for $l = 1, \ldots, p$ *either*

$$f(d_{l-1}) = d_l, \quad f(u_{l-1}) = u_l, \textit{if } D_{i_l i_{l-1}} = 1 \tag{86}$$

or

$$f(d_{l-1}) = u_l, \quad f(u_{l-1}) = d_l, \textit{if } D_{i_l i_{l-1}} = -1 \tag{87}$$

holds. Moreover, if $S \subset intI_p$ *then* $J_l \subset intI_{i_l}$.

This lemma follows immediately from the definitions of matrices D and C and the Darboux property, so we will omit the proof.

From the above lemma the following is straightforward.

Lemma 19 *Let* $p \in \mathbb{N}$, $p > 0$. *If* $C^p_{ji} > 0$, *then for every segment* $S = [b_{j-1}, b_j] \subset I_j$ *there exists segment* $J = [d, u] \subset I_i$ *such that*

$$\textit{for } l < p \ f^l(J) \subset I$$
$$f^p(J) = S$$

The end-points of segment J *are mapped by* f^p *onto the end-points of* S, *so either condition*

$$f^p(d) = b_{j-1}, \quad f^p(u) = b_j \tag{88}$$

or

$$f^p(u) = b_{j-1}, \quad f^p(d) = b_j \tag{89}$$

holds.

Lemma 20 *If* $C^p_{ji} > 0$ *and* $C^p_{li} > 0$, *where* $l > j$, *then there exists a segment* $J = [d, u] \subset I_i$ *such that*

$$I_j \cup I_{j+1} \cup \ldots \cup I_l \subset f^p(J). \tag{90}$$

The end-points of segment J *are mapped by* f^p *onto the end-points of the segment* $I_j \cup I_{j+1} \cup \ldots \cup I_l$, *so either*

$$f^p(d) = x_{j-1}, \quad f^p(u) = x_l$$
$$\textit{or}$$
$$f^p(u) = x_{j-1}, \quad f^p(d) = x_l.$$

Proof

From Lemma 19 it follows that there exist two segments $[y_1, y_2]$ and $[y_3, y_4]$ both contained in I_i such that the end–points of these segments are mapped by f^p onto end–points of segments I_j and I_l, respectively. So between points y_1, y_2, y_3, y_4 exactly one is mapped by f^p on x_{j-1} and x_l. These points give us u and d. Equation (90) follows from the Darboux property of continuous functions.

□

We now formulate in terms of covering matrices a criterion for f to have a topological (pre)horseshoe.

Definition 8 *Let* p, k, l *be positive natural numbers, such that* $1 < k, l < n - 1$, $l \neq k$.

We say that the covering matrix fulfills **criterion P for** p, k, l *iff* $C^p_{ik} > 0$ *for* $i = 1, \ldots, n-1$ *and* $C^p_{il} > 0$ *for* $i = k-1, k, k+1$.

We say that the covering matrix fulfills **criterion H for** p, k, l *iff* $C^p_{ik} > 0$ *and* $C^p_{il} > 0$ *for* $i = 1, \ldots, n-1$.

Let us note also that in order to find natural numbers k, l such that $1 < k < l < n-1$ n (the period of the orbit) must be greater than or equal to five.

Theorem 21 *Suppose C fulfills criterion H (criterion P) for p, k, l, then there exist two segments $N_0 \subset I_k$ and $N_1 \subset I_l$ such that $f^p : N_0 \cup N_1 \to \mathbb{R}$ is a topological horseshoe (prehorseshoe).*

Proof

We will prove the theorem for criterion H. For criterion P the proof is analogous.

It follows from Lemma 20 that there exists a segment $N_0 = [a_0, a_1] \subset I_k$ such that either

$$f^p(a_0) = x_0, \quad f^p(a_1) = x_{n-1} \tag{91}$$

or

$$f^p(a_1) = x_0, \quad f^p(a_2) = x_{n-1}. \tag{92}$$

Similarly there exists a segment $N_1 = [a_2, a_3] \subset I_l$ such that either

$$f^p(a_2) = x_0, \quad f^p(a_3) = x_{n-1} \tag{93}$$

or

$$f^p(a_3) = x_0, \quad f^p(a_2) = x_{n-1}. \tag{94}$$

Obviously $(a_0, a_1) \cap (a_2, a_3) = \emptyset$. It follows from conditions (91)–(94) that both sets N_0 and N_1 f^p-cover horizontally sets N_0 and N_1. So $f^p_{|N_0 \cup N_1}$ is a topological horseshoe.

□

Theorem 22 *Assume C fulfills criterion H for p, k, l, then f has a (p, ∞)-nested sequence of topological horseshoes. If additionally C fulfills criterion P for $p-1, k, l$, then f has a (p, ∞)-nested sequence of topological horseshoes with a prehorseshoe.*

Proof

First we prove the theorem for the criterion H. The sets N_0^p and N_1^p are obtained by Theorem 21. Moreover, it follows from the proof of this theorem that

$$f^p(N_0^p) = I, \quad \text{and } f^p(N_1^p) = I. \tag{95}$$

Observe that $I \subset f(I)$, so proceeding as in the proof of Lemma 19 we can find segments $J \subset I$ and $N_0^{p+1} \subset N_0^p$, $N_1^{p+1} \subset N_1^p$ such that

$$f(J) = I \tag{96}$$
$$f^p(N_0^{p+1}) = J \tag{97}$$
$$f^p(N_1^{p+1}) = J \tag{98}$$

and end–points of segments J and N_i^{p+1} mapped onto end–points of segments I and J by f and f^p, respectively. It follows from this that N_i^{p+1} f^{p+1}-covers horizontally the sets N_0^p and N_1^p. Moreover, the end–points of segments N_i^{p+1} are mapped onto the end–points of segment I by f^{p+1}. So we can continue this construction inductively.

The part concerning criterion P follows immediately from the first one.

□

Theorem 23 *Let $n = 5$. There exists $p \leq 5$ and k, l such that criterion P for p, k, l and criterion H for $p + 1, k, l$ hold.*

Proof

Let us recall that $C(f, x)$ is fully determined by $\pi(f, x)$. The permutations which are n-periodic can be represented by a n-sequence without repetition of elements $\{0, 1, \ldots, n-1\}$ with the first element equal to zero. This correspondence is given as follows. The sequence $(l_0, l_1, \ldots, l_{n-1})$ corresponds to permutation σ, $\sigma(l_i) = l_{(i+1) \bmod n}$.

So the number of possible $\pi(f, x)$ is equal to $(n-1)!$. The symmetry, which corresponds to reversing the order of points on the trajectory on x, given by $i \mapsto n - 1 - i$, reduces odd this number to $(n-1)!/2$ for n.

So for $n = 5$ this gives 12 cases to consider, which can be done by hand with some effort.

Below we have listed all possible cases. In every case we have two permutations represented as sequences, the second one is a permutation obtained from the first one by the symmetry described above. h and p are the iteration numbers for which criterion H and criterion P holds for the first time, respectively.

- $(0, 1, 2, 3, 4)$ $(0, 4, 3, 2, 1)$, $p = h = 3$
- $(0, 1, 2, 4, 3)$, $(0, 1, 4, 3, 2)$, $p = h = 2$
- $(0, 1, 3, 2, 4)$, $(0, 4, 3, 1, 2)$, $p = 3$, $h = 4$
- $(0, 1, 3, 4, 2)$, $(0, 2, 4, 3, 1)$. $p = h = 2$
- $(0, 1, 4, 2, 3)$, $(0, 2, 1, 4, 3)$. $p = 3$, $h = 4$
- $(0, 2, 1, 3, 4)$, $(0, 4, 2, 3, 1)$. $p = 2$, $h = 3$
- $(0, 2, 3, 1, 4)$, $(0, 4, 2, 1, 3)$. $p = 5$, $h = 6$
- $(0, 2, 3, 4, 1)$, $(0, 3, 4, 2, 1)$. $p = h = 3$
- $(0, 2, 4, 1, 3)$, $(0, 3, 1, 4, 2)$. $p = h = 4$
- $(0, 3, 1, 2, 4)$, $(0, 4, 1, 3, 2)$. $p = h = 2$
- $(0, 3, 2, 1, 4)$, $(0, 4, 1, 2, 3)$. $p = h = 3$
- $(0, 3, 2, 4, 1)$, $(0, 3, 4, 1, 2)$. $p = h = 3$

□

The following lemma can be easily inferred from the proof of Lemma 2.1 in [BGMY], (see also [De] page 65) or ([Ste],Theorem 2).

Lemma 24 *Suppose that x is a periodic point of period n, n is odd and $n > 1$, and f has no periodic points of odd periods smaller than n and larger than 1, then after the following renumeration of segments*

$$(J_{n-1}, J_{n-3}, \ldots, J_2, J_1, J_3, \ldots, J_{n-4}, J_{n-2}) = (I_1, I_2, \ldots, I_{n-1}), \qquad (99)$$

composed with the change of orientation permutation (if necessary) the matrix D has the following form

$$D = \begin{bmatrix} -1 & 0 & 0 & \dots & 0 & 0 & 1 \\ -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & -1 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & -1 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & -1 & 0 \end{bmatrix} \tag{100}$$

More explicitly $D_{11} = -1$, $D_{i,i+1} = -1$ for $i = 1, 2, \dots, n-2$, $D_{n-1,l} = 1$ for l odd and all remaining entries of D are equal to zero.

Proof

Let $n = 2k+1$. It follows from the proof of Lemma 2.1 in [BGMY], (see also [De] page 65 or [Ste], Th. 2) that up to orientation $\pi(f,x)$ has the following form

$$\pi(0) = k \tag{101}$$

$$\pi(i) = 2k + 1 - i, \quad \text{for i=1,...,k} \tag{102}$$

$$\pi(i) = 2k - i, \quad \text{for i=k+1,...,2k.} \tag{103}$$

The assertion of lemma follows easily from the above form of π.

□

The graphical representation of the matrix D from Lemma 24 is

$$J_1 \stackrel{-}{\rightarrow} J_1 \tag{104}$$

$$J_1 \stackrel{-}{\rightarrow} J_2 \stackrel{-}{\rightarrow} J_3 \stackrel{-}{\rightarrow} \dots \stackrel{-}{\rightarrow} J_{n-2} \tag{105}$$

$$J_{n-1} \stackrel{+}{\rightarrow} J_{2s+1}, \quad \text{for } 1 \le 2s+1 \le n-2 \tag{106}$$

Let us observe that the internal columns (required for criteria P or H) in the representation described in lemma 24 are the first $n-3$ columns and external the columns are the last two columns.

Theorem 25 *Suppose that x is a periodic point of period n, n is odd and $n \ge 5$, and f has no periodic points of odd periods smaller than n and larger than 1, then there exist positive natural numbers k, l $\quad k \neq l$, $1 < k, l < n-1$, such that criterion P holds for n, k, l and criterion H holds for $n+1, k, l$.*

Proof

We will work with the segments $J_0, \dots, J_{n-1}$ which are obtained in Lemma 24. They appear on the line (up to orientation) in the following order,

$$J_{n-1}, J_{n-3}, \dots, J_2, J_1, J_3, \dots, J_{n-4}, J_{n-2},$$

so for example $J_1 = I_{(n+1)/2}$, $J_2 = I_{(n-1)/2}$, $J_3 = I_{(n+3)/2}$, $J_{n-3} = I_2$. We set $k = (n+1)/2$, $l = 2$. (This choice of k and l means that we build our nested sequence of horseshoes inside two central segments.)

We will work in the basis defined by segments J_i, so in this basis $I_k = J_1$ and $I_l = J_{n-3}$ and segments adjacent to segment J_1 are J_2 and J_3. The matrix C is also expressed in this basis.

First we show that

$$C^s_{i1} > 0, \quad \text{for } i = 1, \dots, n-1 \text{ and } s = n, n+1 \tag{107}$$

From the form of D obtained in Lemma 24 and relation (84) between matrices C and D we see that

$$C_{i,i-1}C_{i-1,i-2}\dots C_{11}^{s-i+1} = 1 \tag{108}$$

so (107) holds.

Now we have to show that

$$C^n_{i,n-3} > 0, \quad \text{for } i = 1,2,3 \tag{109}$$

$$C^{n+1}_{i,n-3} > 0, \quad \text{for } i = 1, \dots, n-1. \tag{110}$$

From

$$C_{1,1}^{n-3}C_{1,n-1}C_{n-1,n-2}C_{n-2,n-3} = 1 \tag{111}$$

$$C_{2,1}C_{1,1}^{n-4}C_{1,n-1}C_{n-1,n-2}C_{n-2,n-3} = 1 \tag{112}$$

$$C_{3,2}C_{2,1}C_{1,1}^{n-5}C_{1,n-1}C_{n-1,n-2}C_{n-2,n-3} = 1 \tag{113}$$

we get (109). Similarly for $i = 1, \dots, n-1$

$$C_{i,i-1}\dots C_{2,1}C_{1,1}^{n-1-i}C_{1,n-1}C_{n-1,n-2}C_{n-2,n-3} = 1 \tag{114}$$

we get (110).

□

4 Existence of infinitely many periodic points for perturbations

Let W, $V = \mathbb{R} \oplus W$ be Banach spaces. Suppose that we have a continuous and compact map $F : [0,1] \times V \to V$ such that $F_0(x,w) = (f(x), 0)$, if $\dim V = \infty$ then we additionally assume that F is uniformly continuous.

Theorem 26 *Suppose f has a periodic point of odd period n, $n > 1$, where $k > 0$. There exists $p \leq \max\{n, 5\}$, such that for every $r > 0$ there exists $\lambda_0 > 0$ such that, for $\lambda < \lambda_0$, F_λ has a $(p+1, p+k)$-nested sequence of topological horseshoes with a prehorseshoe of vertical size r.*

Proof

Let p be the smallest odd period for f greater than three. It follows from Theorems 22, 23 and 25 that f as a one-dimensional map has $(p+1, \infty)$-nested sequence of topological horseshoes with a prehorseshoe. Let the segments I^s_i, where $s = p, p+1, \dots$ and $i = 0, 1$ be as in Definition 5.

Let us fix an $r > 0$. We define $N^s_i := I^s_i \times \overline{B_W(r)}$, for $i = 0, 1$ and $s = p, p+1, \dots$. We set $N^s := N^s_0 \cup N^s_1$.

It follows from uniform continuity that for λ sufficiently small we have

$$\overline{F^s_\lambda(N^p)} \subset (-\infty, \infty) \times B_W(r), \quad \text{for } s = p, p+1, \dots, p+k \tag{115}$$

Further it follows from Remark 8 that for λ sufficiently small F_λ is a $(p+1, p+k)$-nested sequence of topological horseshoes with a prehorseshoe.

□

Remark 27 *In the above theorem λ_0 is obtained as follows. Let n be the smallest odd period greater than tree for some point. Let $x_0 < x_1 < \ldots < x_{n-1}$ denote the points from this orbit. Let us denote*

$$\Delta := \min\{|x_i - x_j|, \quad \textit{where } i \neq j,\ i,j = 0,1,\ldots,n-1\ .\} \tag{116}$$

Let us fix $r > 0$. Then for $s = n, \ldots, 2n-1$ the following must hold:

$$\overline{F_\lambda^s(N^n)} \subset (-\infty,\infty) \times B_W(r), \tag{117}$$

$$|F_\lambda^s((x,y)) - F_0^s((x,y))| < \Delta, \quad \textit{for } (x,y) \in [x_0, x_{n-1}] \times \overline{B_W(r)}. \tag{118}$$

For practical purposes the procedure of establishing the range of λ is as follows:

1. take any $r > 0$;
2. find $\lambda_0 > 0$ such that for $\lambda < \lambda_0$ condition (117) holds true;
3. find $\lambda_1 \leq \lambda_0$ such that for $\lambda < \lambda_1$ condition (118) holds true.

Now we can apply theorem 9 to obtain the following.

Theorem 28 *Suppose f has a periodic point of odd period n, $n > 1$, where $k > 0$.*

There exists an integer $M(n)$ such that for every $r > 0$ there exists $\lambda_0 > 0$ such that, for $\lambda \leq \lambda_0$ and $m > M(n)$, F_λ has a periodic point of period m in the set $\mathbb{R} \oplus B_W(r)$.

Let us note that $M(n)$ is a nondecreasing function of n and $M(3) = M(5)$ as the periodic orbit, which is a skeleton for the nested sequence of horseshoes need at least to have five points.

5 Continuation of individual periodic orbits

In this section we assume that W, $V = \mathbb{R} \oplus W$ are Banach spaces. $F : [0,1] \times V \to V$ is a continuous and compact map, such that $F_0(x,w) = (f(x),0)$, if $\dim V = \infty$ then we additionally assume that F is uniformly continuous. Suppose that f has nontrivial periodic point x, so we can define segments I_i and $D(f,x)$ as in section 3.

The aim of this section is to show that many periodic orbits, which exists for f by Sharkovskii's theorem, continue for small λ.

Theorem 29 *Let $p \in \mathbb{N}$, $p > 0$. Suppose that there exist $i_0, i_1, i_2, \ldots, i_{p-1}, i_p$ where $i_0 = i_p$ such that*

$$D_{i_p i_{p-1}} D_{i_{p-1} i_{p-2}} \ldots D_{i_1 i_0} = -1.$$

Then for every $r > 0$ there exists $\lambda_0 > 0$ such that for $\lambda < \lambda_0$ there exists a periodic point z_λ such that

$$F_\lambda^l(z_\lambda) \in \mathit{Int} I_{i_l} \times B_W(r), \quad \textit{for } l = 0,1,2,\ldots,p-1 \tag{119}$$

$$F_\lambda^p(z_\lambda) = z_\lambda \tag{120}$$

Before we prove the above theorem we need one technical lemma.

Lemma 30 *Suppose that the assumptions of Theorem 29 are satisfied. Then there exists a family of segments $J_l = [d_l, u_l]$ for $l = 0, 1, \ldots, p-1$ such that*

$$J_l \subset intI_{i_l} \tag{121}$$
$$f(I_{l-1}) = I_l \tag{122}$$
$$f^p(d_0) > d_0, \qquad f^p(u_0) < u_0. \tag{123}$$

The proof of the lemma will be given after the proof of the main theorem.

Proof

Let us fix $r > 0$. It follows from uniform continuity that for λ sufficiently small we have

$$\overline{F_\lambda^s(I \times \overline{B_W(r)})} \subset (-\infty, \infty) \times B_W(r), \quad \text{for } s = 1, 2, \ldots, p \tag{124}$$

Let $J_l = [d_l, u_l]$, for $l = 0, 1, \ldots, p-1$, be a family of segments constructed in Lemma 30.

We show now that

$$\mathrm{I}(F_0^p, \mathrm{Int}J_0 \times B_W(r)) = 1. \tag{125}$$

To this end we use the homotopy property. We define the homotopy $H : [0,1] \times V \to V$ by

$$H(\lambda, (x, y)) = (1-\lambda)(f^p(x), 0) + \lambda((d_0 + u_0)/2, 0) \tag{126}$$

Let us note that it follows from (123), for $\lambda \in [0,1]$ and $y \in W$, that

$$x(H_\lambda(d_0, y)) > d_0, \qquad x(H_\lambda(u_0, y)) < u_0 \tag{127}$$

$$\mathrm{Fix}(H, J_0 \times \overline{B_W(r)}) \cap \partial(J_0 \times \overline{B_W(r)}) = \emptyset. \tag{128}$$

Therefore, by the homotopy property of the fixed point index and by the formula for index of affine maps (12) we obtain (125).

Let $\lambda_0 > 0$ be such that for $\lambda < \lambda_0$

$$x(F_\lambda^p(d_0, y)) > d_0, \quad x(F_\lambda^p(u_0, y)) < u_0, \quad \text{for } |y| \leq r. \tag{129}$$

The existence of such a λ_0 follows from (123) and the uniform continuity of F^p on the set $[0,1] \times J_0 \times B_W(r)$.

From (129) and (124) we get

$$\mathrm{Fix}(F_\lambda^p, J_0 \times \overline{B_W(r)}) \cap \partial(J_0 \times \overline{B_W(r)}) = \emptyset \tag{130}$$

for $\lambda < \lambda_0$. Therefore, we can apply again the homotopy property to F_λ. We obtain

$$\mathrm{I}(F_\lambda^p, \mathrm{int}J_{i_0} \times B_W(r)) = 1. \tag{131}$$

Now to obtain (119) we find $\lambda_1 \leq \lambda_0$ such that for $\lambda < \lambda_1$.

$$F_\lambda(J_l \times \overline{B_W(r)}) \subset \mathrm{Int}I_{l+1} \times B_W(r), \quad \text{for } 0, 1, \ldots, p-1. \tag{132}$$

Again the existence of λ_1 follows from (121), (122) and the uniform continuity of F on compacts.

□

Proof of lemma 30 Let us take $\epsilon > 0$, $2\epsilon < x_{i_0} - x_{i_0-1}$. We define the segment $S_\epsilon := [x_{i_0-1} + \epsilon, x_{i_0} - \epsilon]$. It follows from Lemma 18 applied to segment $S_\epsilon \subset \text{int} I_{i_0}$ that there exists a family of segments $J_l^\epsilon = [d_l^\epsilon, u_l^\epsilon] \subset \text{int} I_{i_l}$ for $l = 0, 1, \ldots, p$, with $J_p^\epsilon = S_\epsilon$ such that

$$f(J_{l-1}^\epsilon) = J_l^\epsilon, \quad \text{for } l = 1, \ldots, p \tag{133}$$
$$f^p(d_0^\epsilon) = x_{i_0} - \epsilon, \quad f^p(u_0^\epsilon) = x_{i_0-1} + \epsilon. \tag{134}$$

It is easy to show that for sufficiently small $\epsilon > 0$

$$f^p(d_0^\epsilon) > d_0^\epsilon, \qquad f^p(u_0^\epsilon) < u_0^\epsilon. \tag{135}$$

□

Theorem 31 *Let n be the smallest odd period for f greater than one. For every k, $n \lhd k$ for every $r > 0$ there exists $\lambda_0 > 0$ such that, for $\lambda < \lambda_0$, F_λ has a k-periodic point in the set $\mathbb{R} \oplus B_W(r)$.*

Proof

Let the segments $J_1, \ldots, J_{n-1}$ be as obtained in Lemma 24.

According to Theorem 29 it is enough to find a sequence $i_0, i_1, i_2, \ldots, i_{k-1}, i_k$, $i_0 = i_k$ such that

$$D_{i_k i_{k-1}} D_{i_{k-1} i_{k-2}} \ldots D_{i_1 i_0} = -1. \tag{136}$$

which has no smaller periodic subsequence.

In a graphical representation this means the existence of f-coverings

$$I_0 \overset{\epsilon_1}{\to} I_1 \overset{\epsilon_2}{\to} I_2 \overset{\epsilon_3}{\to} \ldots \overset{\epsilon_{k-1}}{\to} I_{k-1} \overset{\epsilon_k}{\to} I_0 \tag{137}$$

with an odd number of ϵ_i-s equal to -1.

From Lemma 24 it follows that

$$J_1 \overset{-}{\to} J_1 \tag{138}$$
$$J_1 \overset{-}{\to} J_2 \overset{-}{\to} J_3 \overset{-}{\to} \ldots \overset{-}{\to} J_{n-2} \tag{139}$$
$$J_{n-1} \overset{+}{\to} J_{2s+1}, \quad \text{for } 1 \leq 2s+1 \leq n-2 \tag{140}$$

First we construct even periods. Let $k \leq n-1$. By Theorem 29 the sequences

$$J_{n-1} \overset{+}{\to} J_{n-2} \overset{-}{\to} J_{n-1} \tag{141}$$
$$J_{n-1} \overset{+}{\to} J_{n-4} \overset{-}{\to} J_{n-3} \overset{-}{\to} J_{n-2} \overset{-}{\to} J_{n-1} \tag{142}$$
$$\ldots \tag{143}$$
$$J_{n-1} \overset{+}{\to} J_1 \overset{-}{\to} J_2 \overset{-}{\to} J_3 \overset{-}{\to} \ldots \overset{-}{\to} J_{n-2} \overset{-}{\to} J_{n-1} \tag{144}$$

give us periodic points for sufficiently small λ.

Similarly, for $k > n-1$ the sequence

$$J_{n-1} \overset{+}{\to} \underbrace{J_1 \overset{-}{\to} J_1 \ldots J_1 \overset{-}{\to} J_1}_{k-n+1} \overset{-}{\to} J_2 \overset{-}{\to} J_3 \overset{-}{\to} \ldots \overset{-}{\to} J_{n-2} \overset{-}{\to} J_{n-1} \tag{145}$$

gives us k-periodic point.

Finally, for k odd the sequence

$$J_{n-1} \overset{+}{\to} J_{n-2} \overset{-}{\to} J_{n-1} \overset{+}{\to} \underbrace{J_1 \overset{-}{\to} J_1 \ldots J_1 \overset{-}{\to} J_1}_{k-n} \overset{-}{\to} J_2 \overset{-}{\to} J_3 \overset{-}{\to} \ldots \overset{-}{\to} J_{n-2} \overset{-}{\to} J_{n-1} \tag{146}$$

is good.

□

Theorem 32 *Suppose f has a point of odd period n, $n \geq 3$. Then for every $M \in \mathbb{N}$ and every $r > 0$ there exists λ_0 such that, for $\lambda \leq \lambda_0$,*
if $m \leq M$ and $n \lhd m$, $m \neq n$ then F_λ has a periodic point of period m in $\mathbb{R} \oplus B_W(r)$.

Proof Let us fix M and $r > 0$. There is only a finite number of integers such that $k \leq M$ and $n \lhd k$. It follows from the previous theorem that for any k for sufficiently small λ there exists a k-periodic point for F_λ. We can take λ small enough for all $k \leq M$.

□

Proof of Theorem 2 for n odd, $n > 1$.

From Theorem 28 existence of all periods greater than $M(n)$ follows for sufficiently small λ. The existence of the finite number of periods allowed by Sharkovski ordering follows from the above theorem.

□

6 1-dim maps with even periods

The aim of this section is to describe a form of one-dimensional maps, for which all periodic points which are not fixed points have an even period. The results of this section follow essentially from the ramification of arguments presented in [BGMY] or [Ste] for odd periods, but they are probably not stated there explicitly as they are not required for the proof of Sharkovskii's theorem in the one-dimensional case.

In this section we assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous map. Let $x_0 < x_1 < \ldots < x_{n-1}$, $n > 0$ be a periodic trajectory. As in section 3 we set $I = [x_0, x_{n-1}]$ and we define $I_i = [x_{i-1}, x_i]$. We will use notions and notation introduced in section 3.

Theorem 33 *Assume that n is even and there is in I no periodic point of period odd greater than one. Then*

$$I_{n/2} \overset{f-}{\to} I_{n/2} \tag{147}$$

$$f^{2j+1}([x_0, x_{n/2-1}]) \subset (x_{n/2-1}, \infty), \quad \textit{for } j \in \mathbb{N} \tag{148}$$

$$f^{2j}([x_0, x_{n/2-1}]) \subset (-\infty, x_{n/2}), \quad \textit{for } j \in \mathbb{N} \tag{149}$$

$$f^{2j+1}\{x_0, x_1, \ldots, x_{n/2-1}\} = \{x_{n/2}, x_{n/2+1}, \ldots, x_{n-1}\}, \quad \textit{for } j \in \mathbb{N} \tag{150}$$

$$f^{2j}\{x_0, x_1, \ldots, x_{n/2-1}\} = \{x_0, x_1, \ldots, x_{n/2-1}\}, \quad \textit{for } j \in \mathbb{N}. \tag{151}$$

The proof of the above theorem is contained in [BGMY] or [Ste].

Suppose now that $n = 2^q n'$, where $q > 0$, $n' > 0$. Suppose that f does not have on I any point of the period of the form $2^{q-1}m$ with m odd and $m > 1$. Then from the above theorem it follows that $x_0, x_1, \ldots, x_{n/2-2}$ are points from a $2^{q-1}n'$

periodic trajectory for the map f^2 and if $q-1>0$ then f^2 has no periodic point in $[x_0, x_{n/2-1}]$ of period of the form $2^{q-2}m$, where m is odd and $m>1$. If $q-1>0$ then we can continue this construction and we see that $x_0, \dots, x_{n/2^2-1}$ are points from the $2^{q-2}m$ periodic trajectory for the map f^{2^2}, and if $q-2>0$ then the map f^{2^2} has no periodic points in $[x_0, x_{2^2}-1]$ of the form $2^{q-3}m$ where m is odd and $m>1$. We can continue this induction q times. In this way we obtain the following.

Theorem 34 *Suppose that $n = 2^q n'$, where $q>0$, $n'>0$. Suppose that f does not have on I any point of the period of the form $2^{q-1}m$ with m odd and $m>1$. Then*

$$I_{n/2^{i+1}} \overset{f^{2^i}}{\longrightarrow} I_{n/2^{i+1}}, \quad \textit{for } i = 0, \dots, q-1$$
$$f^{j\cdot 2^i+s}([x_0, x_{\frac{n}{2^i}-1}]) \subset (x_{\frac{n}{2^i}-1}, \infty), \quad \textit{for } i=1,\dots,q,\ j=0,1,2\dots \textit{ and } 0<s<2^i$$
$$f^{j\cdot 2^i}([x_0, x_{\frac{n}{2^i}-1}]) \subset (-\infty, x_{\frac{n}{2^i}}), \quad \textit{for } i=1,\dots,q\ ,\ j=0,1,2\dots$$
$$f^{2^i}(\{x_0, x_1, \dots, x_{\frac{n}{2^i}-1}\}) = \{x_0, x_1, \dots, x_{\frac{n}{2^i}-1}\}, \quad \textit{for } i=1,\dots,q\ .$$

7 Proof of main theorem, even period case

Before we present a proof let us explain why we cannot not follow directly the proof of one-dimensional Sharkovskii's theorem. The situation is as follows: we have the Sharkovskii's theorem for odd periods and we consider now period of the form $n2^q$, where $n>1$ is odd. In the one-dimensional case one considers now the map $g := f^{2^q}$, obviously g has a point of odd period and in this way one obtains points x_m which are fixed points for the maps f^{m2^q}, where $n \lhd m$, but we need not consider the problem: 'What is the period of x_m', because from Sharkovskii's ordering follows that if the period is not $m2^q$, there must exist a periodic point of period $m2^q$.

In our case we need to show that the fixed points for $F_\lambda^{m2^q}$ that we find in the proof have a period exactly equal to $m2^q$.

Proof

For n odd and $n>1$ this theorem was proved in section 5.

Let $x_0 < x_1 < \dots < x_{n-1}$, $n>0$ be a periodic trajectory. As usual we set $I = [x_0, x_{n-1}]$ and we define $I_i = [x_{i-1}, x_i]$. We can assume that $n = 2^q n'$, where n' is odd, and in I there no periodic point of period $2^q m$, where m is odd and $m>1$.

Let us fix $r>0$.

Assume first that $n'>1$. We define $J := [x_0, x_{n'}]$. From Theorem 34 it follows that the set $\{x_0, x_1, \dots, x_{n'}\}$ is the periodic trajectory of the map f^{2^q} . From Theorem 2 for odd periods applied to f^{2^q} and the homotopy $F_\lambda^{2^q}$ it follows that there exists $\lambda_1 > 0$ such that for every $m \neq n'$ and $n' \lhd m$ there exists an m-periodic point $x_m \in J \times B_W(r)$ for a map $F_\lambda^{2^q}$, such that

$$F_\lambda^{j\cdot 2^q}(x_m) \in \text{ int } J \times B_W(r), \quad \text{for } j=1,2,\dots,m \tag{152}$$
$$F_\lambda^{j\cdot 2^q}(x_m) \neq x_m, \quad \text{for } j=1,\dots,m-1 \tag{153}$$
$$F_\lambda^{m\cdot 2^q}(x_m) = x_m. \tag{154}$$

It remains to show that the period of x_m with regard to the map F_λ is exactly $m2^q$ for λ sufficiently small.

From the uniform continuity of F and Theorem 34 it follows that there exists $\lambda_2 > 0$, such that

$$F_\lambda^s(J \times \overline{B_W(r)}) \cap J \times \overline{B_W(r)} = \emptyset, \quad \text{for } s = 1, 2, \ldots, 2^q - 1. \tag{155}$$

We set $\lambda_0 = \min\{\lambda_1, \lambda_2\}$. Now is easy to see that for $\lambda < \lambda_0$ the period of x_m is exactly $m2^q$. Suppose that there exists k another period for x_m, such that $k < m2^q$. k must be a number of the form $k = m'2^{q'}$. From condition (155) it follows that $q' = q$. From (153) it follows that $m' = m$. This finishes the proof of the existence of all periods of the form $2^l m$, where $l = q$ and $m > n'$ or $l > q$, when $n' > 1$.

We release now the assumption $n' > 1$. It remains to show that the numbers of the form 2^i for $i = 0, 1, \ldots, q-1$ can be realized as periods of some points in J for sufficiently small λ. It follows from Theorem 34 that for $i = 0, 1, \ldots, q-1$

$$I_{n/2^{i+1}} \overset{f^{2^i}}{\longrightarrow} I_{n/2^{i+1}} \tag{156}$$

$$f^s(I_{n/2^{i+1}}) \subset f^s([x_0, x_{n/2^i - 1}]) \subset (x_{n/2^i-1}, \infty), \quad \text{for } s = 1, \ldots, 2^i - 1. \tag{157}$$

Obviously

$$I_{n/2^{i+1}} \cap (x_{n/2^i-1}, \infty) = \emptyset, \quad \text{for } s = 1, \ldots, 2^i - 1. \tag{158}$$

From the uniform continuity of F and conditions (157) and (158) it follows that there exists $\lambda_3 > 0$ such that for $\lambda < \lambda_3$

$$F_\lambda^s(I_{n/2^{i+1}} \times \overline{B_W(r)}) \cap I_{n/2^{i+1}} \times \overline{B_W(r)} = \emptyset, \quad \text{for } s = 1, \ldots, 2^i - 1 \tag{159}$$

From the proof of Theorem 29 and condition (156) it follows that there exists $\lambda_4 > 0$, such that for $\lambda < \lambda_4$ there exists y_i a fixed point for the map $F_\lambda^{2^i}$. If $\lambda < \min\{\lambda_3, \lambda_4\}$ then from (159) it follows immediately that y_i has period 2^i.

This finishes the proof of Theorem 2. □

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