How to show an existence of homoclinic trajectories using topological tools ?

Klaudiusz Wójcik [1]
Jagiellonian University, Institute of Mathematics, Reymonta 4,
30-059 Kraków, Poland. E-mail: wojcik@im.uj.edu.pl

Piotr Zgliczyński [2] [3]
Center for Dynamical Systems and Nonlinear Studies,
Georgia Institute of Technology, Atlanta, Georgia 30332, USA.
E-mail: piotrz@math.gatech.edu

Abstract

In this note we present a method combining topological data with local hyperbolic behavior to obtain homoclinic trajectories for maps and ODE's.

In this note we would like to address the following question: *Can we prove the existence of homoclinic trajectories using topological tools only ?*

At first sight the answer to this question seems to be negative, because topological tools, like the Conley index or the fixed point index, do not see what happens in very small portions of the phase space. The silent assumption behind this reasoning is that we use only a finite number of sets for which we compute various topological invariants.

1 Discrete case

The topological notion we will link with a hyperbolic behavior is the notion of the covering relation (see [Z1]), which will be denoted here by $\Longrightarrow$.

For precise definition of the covering relation the reader is refered to [Z1]. Figure 1 presents an illustration of the covering relation on the plane. Here N_0 and N_1 are two rectangles, $L(N_i)$ and $R(N_i)$ denote respectively the left and the right edge of N_i. Speaking imprecisely, $N_i \overset{f}{\Longrightarrow} N_j$ if N_i is mapped across N_j.

The basic theorem about covering relations is (see [Z1])

Theorem 1 *Given parallelograms $M_i \subset \mathbb{R}^d$ and continuous maps $f_i : M_i \to \mathbb{R}^d$, such that*

$$M_0 \overset{f_0}{\Longrightarrow} M_1 \overset{f_1}{\Longrightarrow} M_2 \overset{f_2}{\Longrightarrow} M_3 \ldots \overset{f_{n-1}}{\Longrightarrow} M_n.$$

There exists $x \in \operatorname{int} M_0$, such that $f_k \circ \ldots \circ f_1 \circ f_0(x) \in \operatorname{int} M_{k+1}$, for $k = 0, \ldots, n-1$. Moreover if $M_n = M_0$ then x may be chosen so that additionally

$$x = f_{n-1} \circ \ldots \circ f_1 \circ f_0(x).$$

Let x_0 be a hyperbolic fixed point for a C^1 map $f : \mathbb{R}^d \to \mathbb{R}^d$.

The following lemma follows easily from the linearization of f at x_0.

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[3] on leave from: Institute of Mathematics, Jagiellonian University, Cracow, Poland; e-mail: zgliczyn@im.uj.edu.pl

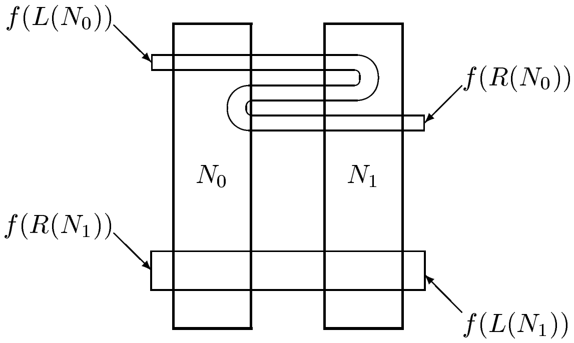

Figure 1: Topological Smale horseshoe. $N_i \overset{f}{\Longrightarrow} N_j$, $i,j = 0,1$

Lemma 2 *There exists a sequence of parallelograms N_i, $i \in \mathbb{N} \cup \{0\}$, $x_0 \in N_i$, such that*

$$N_{i+1} \subset N_i, \quad \lim_{i->\infty} diam(N_i) = 0, \quad N_{i+1} \overset{f}{\Longrightarrow} N_i, \quad N_i \overset{f}{\Longrightarrow} N_{i+1}$$

Let N_0 be as in the lemma above. Suppose that we have a loop of covering relations

$$N_0 \overset{f}{\Longrightarrow} M_1 \overset{f}{\Longrightarrow} M_2 \overset{f}{\Longrightarrow} M_2 \ldots \overset{f}{\Longrightarrow} M_k \overset{f}{\Longrightarrow} N_0 \tag{1}$$

From Theorem 1 we conclude that there exists a point $x_0 \in N_0$ such that

$$\lim_{i \to \pm\infty} f^i(x) = x_0 \quad \text{and} \quad f^i(x) \in M_i \text{ for } i = 1, \ldots, k. \tag{2}$$

The same argument applies to orbits heteroclinic to two hyperbolic periodic points.

In applications of the above described method we have to find (either analytically or by C^1 computer assisted computations) a set N_0 on which f is hyperbolic. The next step is to build loops like (1), this involves only C^0 computations.

2 Continuous case

The topological notion we would like to link to hyperbolic behavior is that of the isolating segment ([SW, WZ1, WZ2]). An isolating segment for a T-periodic ordinary differential equation is a T-periodic manifold with corners in the extended phase–space , whose boundary is built from sections. We refer the reader to [SW, WZ1, WZ2] for the precise definition and the precise statement of results, which we describe below.

As an illustration we consider a planar T-periodic equation ($T = 2\pi/\kappa$)

$$z' = (1 + e^{i\kappa t}|z|^2)\overline{z}, \quad z \in \mathbb{C} \tag{3}$$

for $0 < \kappa \leq 0.495$. Here the origin belongs to a hyperbolic periodic trajectory γ.

In [WZ1] it was shown using two T–periodic isolating segments (U, U^-), (W, W^-) (see figure 2) that the equation (3) has symbolic dynamics on three symbols. The orbits corresponding to this behavior are contained in the isolating segment W. The mutual placement of U and W ($U \subset W$, $U_0 = W_0$, W_t rotates, but U_t does not) in the extended phase-space of (3) corresponds to the loop covering relations (1).

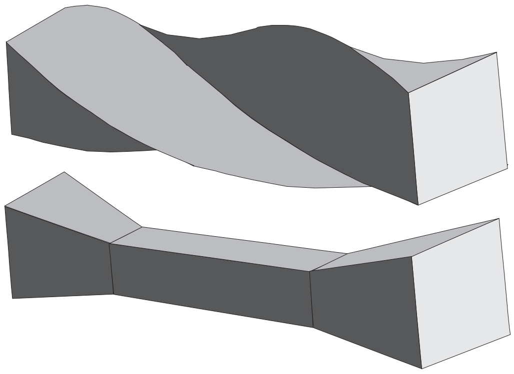

Figure 2: Segments (W, W^-) (above) and (U, U^-)(*below*). Dark faces represent exit sets W^- and U^-.

The symbolic dynamics mentioned above is introduced as follows. Let φ be a local process induced by (3) and let $U^- = U^{-1} \cup U^{+1}$ be a decomposition of U^- into connected components. We set

$$I_W = \{x \in \mathbb{R}^2 \mid (t, \varphi(0,t,x)) \in W\} \tag{4}$$

For $x \in I_W$ we define the sequence $\{g_l(x)\}_{l \in \mathbb{Z}}$ by

$$g_l(x) := \begin{cases} 0, & \text{if } (t, \varphi(0,t,x)) \in U \text{ for } t \in (lT, (l+1)T \\ -1, & \text{if } (lT, \varphi(0,lT,x)) \text{ leaves } U \text{ in time less than } T \text{ through } U^{-1} \\ 1, & \text{if } (lT, \varphi(0,lT,x)) \text{ leaves } U \text{ in time less than } T \text{ through } U^{+1} \end{cases} \tag{5}$$

Since γ is contained in U we have $g(\gamma) = (\ldots, 0, 0, \ldots)$.

To prove the existence of orbits homoclinic to γ we construct an isolating segment (Z, Z^-), such that $\gamma \subset Z$, $\lim_{|t| \to \infty} \operatorname{diam}(Z_t) = 0$ and $U_0 = Z_0$. This corresponds to lemma 2 in the discrete case.

In [WZ2], using continuation arguments applied to isolating segments built from parts of W, U and Z the following theorem is proved

Theorem 3 *For any sequence $\alpha \in \{-1, 0, 1\}^{\mathbb{Z}}$ such that $\alpha_i = 0$ for $|i| > i_0$ for some i_0, there exists a point $x \in I_W$, such that $g(x) = \alpha$ and*

$$\lim_{|t| \to \infty} \varphi(0,t,x) = 0$$

Theorem 3 says that there exists plenty of complicated orbits which are homoclinic to γ.

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