Isolating segments, fixed point index and symbolic dynamics III.Applications.

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Abstract

We prove the existence of complicated dynamics in some class of periodic systems on the plane. The proof is based on the continuation of the considered system to the model one.

Keywords: isolating segments, fixed point index, chaos

1 Introduction

This note is a sequel to the papers [WZ, WZ1]. We present results on an existence of the chaotic behavior for dynamical systems generated by nonautonomous time periodic ordinary differential equations on the plane. To be more specific we consider here the following class of equations

$$\dot{z} = \bar{z}^n(1 + \mathrm{e}^{i\kappa t}|z|^2), \quad z \in \mathbb{C} \tag{1}$$

where $n \in \mathbb{N}$ and $\kappa > 0$ is sufficiently small.

This system is $T = \frac{2\pi}{\kappa}$ periodic. We prove that the Poincaré map for it given by the T-time shift along trajectories of (1) has symbolic dynamics on $n+2$ symbols (see theorem 7 in section 4 for a precise statement).

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The method of proof of an existence of symbolic dynamics for ordinary differential equations used here was introduced in [WZ]. It is a development of the method introduced by Srzednicki and Wojcik in their pioneering work [SW] on the equation (1) for $n = 1$. The method used here has three main ingredients:

- the notion of an periodic isolating segment
- the continuation theorem for isolating segments
- the topological model for (1)

The notion an periodic isolating segment introduced in [SW] is a modification of an isolating block from the Conley index theory (see [C]) adapted to the setting of time periodic nonautonomous ordinary differential equations. In all practical applications the isolating segments are manifolds with corners contained in the extended phase space, such that in any point on the boundary of the segment the vector field is directed either outward or inward with respect to the segment.

The two other ingredients: the continuation theorem and the model were introduced in [WZ]. It turns out that the symbolic dynamics of the model is straightforward. Now the continuation theorem allows us to rigorously continue the symbolic dynamics from the model to the differential equations under consideration. This is rather a rare phenomenon in the theory of dynamical systems. Usually one cannot claim rigorously that the dynamics of the model reflects of that of the system under consideration.

When compared to Srzednicki-Wojcik method in [SW] the method used here gives better results. For (1) using Srzednicki-Wojcik gives symbolic dynamics on two symbols only for all n (see [SW, W2, W3]), whereas we prove here that we have symbolic dynamics on $(n + 2)$ symbols.

To make the paper reasonably self-contained in sections 2 and 3 we recall basic definitions from [WZ] and we state the continuation theorem. The new material starts in section 4, where we prove the existence of symbolic dynamics for (1). In section 5 we apply our method to some time periodic Hamiltonian system on the plane and we obtain symbolic dynamics on three symbols.

2 Semiprocesses and periodic isolating segments.

We recall here the definitions and notations from [WZ]. Some notations: $\mathbb{R}_+ = [0, \infty)$, ρ a distance function, $B(Z, \delta)$ a ball of size δ around the set Z, $\mathrm{ind}\,(F, D)$ - fixed point index of map F relatively to the set D (see [D]).

We start with introducing the notion of *local semiprocess* which formalizes the notion of continuous family of local forward trajectories in an extended phase–space.

Definition 1 *Assume that X is a topological space and $\varphi : D \to X$ is a continuous mapping, $D \subset \mathbb{R} \times \mathbb{R}_+ \times X$ is an open set. We will denote by $\varphi_{(\sigma,t)}$ the function $\varphi(\sigma, t, \cdot)$.*

φ is called a local semiprocess *if the following conditions are satisfied*

(S1) $\forall \sigma \in \mathbb{R},\ x \in X : \{t \in \mathbb{R}_+ : (\sigma, x, t) \in D\}$ *is an interval,*

(S2) $\forall \sigma \in \mathbb{R} : \varphi_{(\sigma,0)} = \mathrm{id}_X$

(S3) $\forall \sigma \in \mathbb{R}, \forall s, t \in \mathbb{R}_+ : \varphi_{(\sigma,s+t)} = \varphi_{(\sigma+s,t)} \circ \varphi_{(\sigma,s)}$,

If $D = \mathbb{R} \times \mathbb{R}_+ \times X$, we call φ a (global) semiprocess. *If T is a positive number such that*

(S4) $\forall \sigma, t \in \mathbb{R}_+ : \varphi_{(\sigma+T,t)} = \varphi_{(\sigma,t)}$

we call φ a T-periodic local semiprocess.

A local semiprocess φ on X determines a local semiflow Φ on $\mathbb{R} \times X$ by the formula

$$\Phi_t(\sigma, x) = (\sigma + t, \varphi_{(\sigma,t)(x)}). \tag{2}$$

In the sequel we will often call the first coordinate in the extended phase space $\mathbb{R} \times X$ *a time* .

Let φ be a T-periodic local semiprocess and let Φ be local semiflow associated to φ. It follows by $(S1)$ and $(S2)$ that for every $z = (\sigma, x) \in \mathbb{R} \times X$ there is an $0 < \omega_z \leq +\infty$ such that $(\sigma, t, x) \in D$ if and only if $0 \leq t < \omega_z$. Let $x \in X$, $\sigma \in \mathbb{R}$, then *a left solution* through $z = (\sigma, x)$ is a continuous map $v : (a, 0] \to \mathbb{R} \times X$ for some $a \in [-\infty, 0)$ such that:

(I) $v(0) = z$,

(II) for all $t \in (a, 0]$ and $s > 0$ with $s + t \leq 0$ it follows that $s < \omega_{v(t)}$ and $\Phi_s(v(t)) = v(t + s)$.

If $a = -\infty$ then we call v *a full left solution.* We can extend a left solution through z onto $(a, 0] \cup [0, \omega_z)$ by setting $v(t) = \Phi_t((\sigma, x))$ for $0 \leq t < \omega_z$. The extended v is called *a solution* through z and if $a = -\infty$ and $\omega_z = +\infty$, it is called *a full solution.*

Remark 1 The differential equation

$$\dot{x} = f(t, x) \tag{3}$$

such that f is regular enough to guarantee the uniqueness for the solutions of the Cauchy problems associated to (3) generates a local process as follows. For $x(t_0, x_0; \cdot)$ the solution (3) such that $x(t_0, x_0; t_0) = x_0$ we put

$$\varphi_{(t_0,\tau)}(x_0) = x(t_0, x_0; t_0 + \tau).$$

If f is T-periodic with respect to t then φ is a T-periodic local process and in order to determine all T-periodic solutions of the equation (3) it suffices to look for fixed points of $\varphi_{(0,T)}$ (called the Poincaré map).

Let $T > 0$ be a real number and $\varphi_j : \mathbb{R} \times \mathbb{R}_+ \times X \to X$ for $j = 0, \ldots, n-1$ be a family of global T–periodic semiprocesses. Let us fix real numbers $0 = t_0 \leq t_1 \leq t_2 \leq t_{n-1} \leq t_n = T$. We construct now *$T$-periodic concatenation of φ_j over time intervals* $[t_j, t_{j+1}]$ denoted by $\Omega(\{(\varphi_j, t_j)\}_{j=0,\ldots,n-1}, T)$. To make notation less cumbersome in the sequel we will omit the parameters of Ω.

First we extend finite sequences t_j and φ_j to infinite ones periodically as follows $t_{kn+j} = kT + t_j$, $\varphi_{kn+j} = \varphi_j$ for $k \in \mathbb{Z}$, $j = 0, 1, \ldots, n-1$.

Let us choose $\sigma, t \in \mathbb{R}$ and $t \geq 0$. We have to define $\Omega_{(\sigma,t)}(x)$. Observe that there exist integers s, p such that $t_{s-1} \leq \sigma \leq t_s$ and $t_p \leq \sigma + t \leq t_{p+1}$. Consider two cases either $s - 1 = p$ or $s - 1 < p$.

In the first case (i.e. $s - 1 = p$) we set

$$\Omega_{(\sigma,t)} = \varphi_{s-1}(\sigma, t)$$

When $s - 1 < p$ we have

$$\sigma + t = \sigma + (t_s - \sigma) + (t_{s+1} - t_s) + \cdots + (t_p - t_{p-1}) + (\sigma + t - t_p) \tag{4}$$

We set

$$\begin{aligned}\Omega_{(\sigma,t)}(x) = \varphi_{p,(t_p, \sigma+t-t_p)} \circ \varphi_{p-1,(t_{p-1}, t_p - t_{p-1})} \circ \cdots \circ \\ \varphi_{s,(t_s, t_{s+1}-t_s)} \circ \varphi_{s-1,(\sigma, t_s - \sigma)}(x)\end{aligned}$$

We have

Proposition 2 *Ω is a global T–periodic semiprocess.*

We will use the following notation: by $\pi_1 : \mathbb{R} \times X \to \mathbb{R}$ and $\pi_2 : \mathbb{R} \times X \to X$ we denote the projections and for a subset $Z \subset \mathbb{R} \times X$ and $t \in \mathbb{R}$ we put

$$Z_t = \{x \in X : (t, x) \in Z\}.$$

Now we are going to state definition of the basic object in this paper, namely T-periodic isolating segment, which is a modification of notion of periodic isolating segment over $[0, T]$ in [SW]. Notice that a T-periodic isolating segment is a T-periodic isolating block in the sense of [S1] and a T-periodic isolating segment can be easily obtained by gluing translated copies of a periodic isolating segment over $[0, T]$.

Definition 2 *We will say that a set $Z \subset \mathbb{R} \times X$ is T–periodic, iff $Z_{nT+t} = Z_t$ for every $n \in \mathbb{N}$ and $t \in \mathbb{R}$.*

Definition 3 *Let $(W, W^-) \subset \mathbb{R} \times X$ be a pair of subsets. We call W a T-*periodic isolating segment *for the T-periodic global semiprocess φ if:*

(i) *W, W^- are T–periodic*

(ii) *$(W, W^-) \cap ([0, T] \times X)$ is a pair of compact sets*

(iii) *for every $\sigma \in \mathbb{R}$, $x \in \partial W_\sigma$ there exists $\delta > 0$ such that for all $t \in (0, \delta)$ $\varphi_{(\sigma,t)}(x) \notin W_{\sigma+t}$ or $\varphi_{(\sigma,t)}(x) \in \mathrm{int} W_{\sigma+t}$,*

(iv)

$$W^- = \{(\sigma, x) \in W : \exists \delta > 0 \ \forall t \in (0, \delta) \ \varphi_{(\sigma,t)}(x) \notin W_{\sigma+t}\},$$
$$W^+ := \mathrm{cl}\,(\partial W \setminus W^-)$$

(v) *for all $z \in W^+$ and all $v : (a, 0] \to \mathbb{R} \times X$ a left solution through z there is $a \leq b < 0$ such that for all $t \in (b, 0)$ $v(t) \notin W$*

(vi) *there exists $\eta > 0$ such that for all $x \in W^-$ there exists $t > 0$ such that for all $\tau \in (0, t]$ $\Phi_\tau(x) \notin W$ and $\rho(\Phi_t(x), W) > \eta$*

Roughly speaking W^- and W^+ are sections for the semiflows, through which trajectories leave and enter the segment W, respectively.

Definition 4 *For the periodic isolating segment W we define* exit time function $\tau_{W,\varphi}$

$$\tau_{W,\varphi} : W_0 \ni x \mapsto \sup\{t \geq 0 : \ \forall s \in [0, t] \ \varphi_{(0,s)}(x) \in W_s\} \in [0, \infty]$$

By the Ważewski Retract Theorem the map $\tau_{W,\varphi}$ is continuous ([Wa, C]).

Definition 5 *Let W be an periodic isolating segment for φ and $x \in W_0$. Let $C \subset W^-$. We will say that x* leaves W at the time t through C, *iff $t = \tau_{W,\varphi}(x)$ and $\varphi_{(0,t)} \in C_t$.*

3 Continuation theorem

Let $\varphi : \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ be a T-periodic global semiprocess.

Let U, W be two T-periodic isolating segments for a semiprocess φ. Assume that

$$U \subset W, \quad U_0 = W_0, \quad U_0^- = W_0^- \tag{5}$$

Let

$$U^- = \bigcup_{l=1}^{K} E(U)_l \tag{6}$$

be a decomposition of the exit set U^- into disjoint union of closed T–periodic sets $E(U)_l$. In the next section we will use decomposition into connected components.

Definition 6 *For $n \in \mathbb{N}$ and $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) \in \{0, 1, \ldots, K\}^n$ and $D \subset W_0$, we define the set $D_\alpha(\varphi)$ as a set of points fulfilling the following conditions*

$$\varphi_{(0,lT)}(x) \in D, \quad \textit{for } l = 0, \ldots, n \tag{7}$$
$$(t, \varphi_{(0,t)}(x)) \in \mathrm{int}\, W, \quad \textit{for } t \in (0, nT) \tag{8}$$
$$\textit{if } \alpha_l = 0, \textit{ then} \quad (t, \varphi_{(lT,t)}(x)) \in \mathrm{int}\, U \quad \textit{for } t \in (0, T) \tag{9}$$
$$\textit{if } \alpha_l > 0, \textit{ then} \quad \varphi_{(0,lT)}(x) \textit{ leaves } U \textit{ in time less than } T \textit{ through } E(U)_{\alpha_l} \tag{10}$$

Suppose now that we have continuous family of semiprocesses $H : [0,1] \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$. We will use notation H_λ for the map with first parameter fixed, $H_\lambda(t_0, \tau, x) = H(\lambda, t_0, \tau, x)$. Let Φ^λ be a semiflow generated by the semiprocess H_λ.

The following result was proved in [WZ] (comp. [Z1, Th. 2.2])

Theorem 3 *Let H_λ be continuous family of T-periodic semiprocesses and there exists U, W - T-periodic isolating segments for semiprocess H_λ for every $\lambda \in [0,1]$, such that (5) holds.*

Assume that condition (vi) holds uniformly for all H_λ $\lambda \in [0,1]$ and isolating segments (W, W^-) and (U, U^-). By this we mean the following statement:

There exists $\eta > 0$ such that for every $\lambda \in [0,1]$ and for every $x \in W^-$ ($x \in U^-$) there exists $t > 0$ such that for $0 < \tau \leq t$ holds $\Phi_\tau(x) \notin W$ and $\rho(\Phi_t(x), W) > \eta$ (resp. $\Phi_\tau(x) \notin U$ and $\rho(\Phi_t(x), U) > \eta$).

Then for every $n > 0$ and $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) \in \{0, 1, \ldots, K\}^n$ the fixed point indices $ind(H_\lambda(0, nT), (\operatorname{int} W_0)_\alpha(H_\lambda))$ are well defined and equal (i.e. do not depend on λ).

4 Symbolic dynamics for (1)

In this section we consider the system (1). For $z_1, z_2 \in \mathbb{C}$ by $[z_1, z_2]$ we will denote the closed segment joining z_1 and z_2.

Since for small $|z|$ the behavior of dynamics for (1) is similar to that generated by the autonomous vector field $v(z) = \overline{z^n}$, so it is convenient to consider first the flow generated by $\dot{z} = \overline{z^n}$. The origin is the unique nontrivial isolated invariant set for that local flow. One can also easily deduce the existence of an isolating block B_r (see Fig 1) (in the sense of the Conley index theory) for the origin, which is a regular $2(n+1)$-gon centered at origin and has the diameter $2r$. The exit set B_r^- consists of $n+1$ disjoint segments, one of which intersect perpendicularly the positive x-semi-axis. Note that both the sets B_r and B_r^- are invariant with respect to the rotation by the angle $\frac{2\pi}{n+1}$.

One can also check that for r small and any $\kappa > 0$ the set

$$U(n) = [0, \frac{2\pi}{\kappa}] \times B_r$$

is an isolating segment for (1) with the exit set

$$U(n)^- = [0, \frac{2\pi}{\kappa}] \times B_r^-.$$

If $|z|$ is large then the term $|z|^2 \overline{z^n} e^{i\kappa t}$ in equation (1) dominates and it follows from Remark 3.7 and Example 5.3 in [S1] that for R sufficiently large and any $\kappa > 0$ the set

$$W(n) = \{(t, (x,y)) \in [0, \frac{2\pi}{\kappa}] \times \mathbb{R}^2 : (x \cos t + y \sin t, y \cos t - x \sin t) \in B_R\} \quad (11)$$

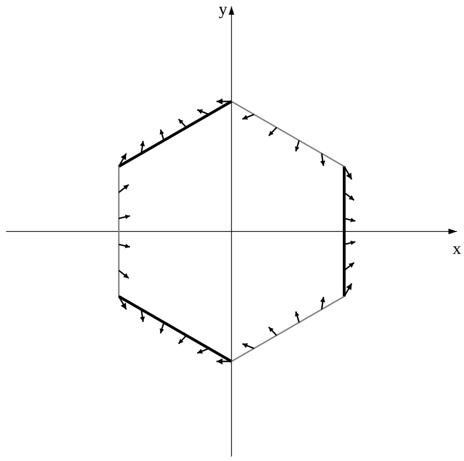

Figure 1: The isolating block B_r for $z' = \bar{z}^n$ where $n = 2$. Thick edges represent B_r^-. Arrows indicate the direction of the vector field $\bar{z}^n$ on the boundary of B_r.

is an isolating segment for (1) with the exit set

$$W(n)^- = \{(t,(x,y)) \in [0, \frac{2\pi}{\kappa}] \times \mathbb{R}^2 : (x\cos t + y\sin t, y\cos t - x\sin t) \in B_R^-\}. \tag{12}$$

When $R > \sqrt{2}r$ then $U(n) \subset W(n)$ for any $n \in \mathbb{N}$, $\kappa > 0$.

We now modify the periodic segment $U(n)$ so that, a modified segment, which we will denote again by $U(n)$, is still a periodic isolating segment for (1) for all $0 < \kappa \leq \kappa_0$, moreover we have $U(n)_0 = W(n)_0$ and $U(n)_0^- = W(n)_0^-$. The proof of this statement is given in [W3], here we will only describe how the resulting segments look like, see also figure 2.

The time t-section of the segment $U(n)$, $U(n)_t$, is a regular $2(n+1)$-gon based prism centered at the origin. The exit set $U(n)^-$ consists of $n+1$ disjoint parts. The length of edges of $U(n)_t$ decreases linearly from R to r, then stays constant and then increases linearly from r to R for $t = T$.

The segment $W(n)$, given by formulas (11) and (12), is a twisted prism with a $2(n+1)$-gon base also centered at the origin. Its time sections $W(n)_t$ are obtained by rotating the base with the angular velocity $\frac{\kappa}{n+1}$ over the time interval $[0, \frac{2\pi}{\kappa}]$. The exit set $W(n)^-$ consists of $n+1$ disjoint ribbons winding around the prism.

The following result was proved in [W3] using the Srzednicki-Wojcik method

Theorem 4 *For any $n \in \mathbb{N}$ there exists $\kappa_0 > 0$, such that for $0 < \kappa < \kappa_0$ there are a compact set $I \subset \mathbb{R}^2$, invariant with respect to the Poincaré map and a continuous surjective map $g : I \to \Sigma_2$ such that the Poincaré map is semiconjugate to the shift map in the set I. Moreover, if n is odd then for every k-periodic sequence $s \in \Sigma_2$ its preimage by semiconjugacy contains at least one*

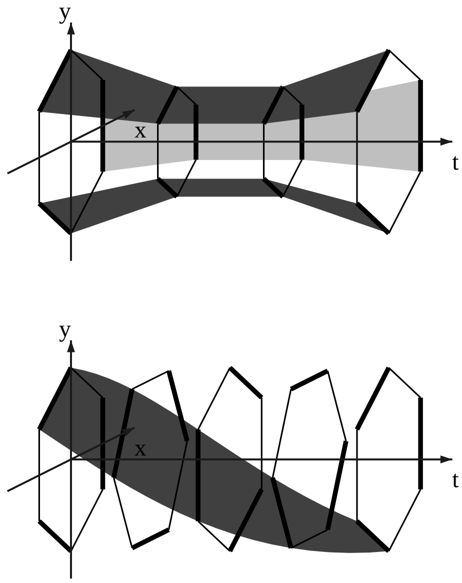

Figure 2: Isolating segments for equation (1) for $n = 2$. Above: the segment U, below the segment W. The thick lines and shaded faces are contained in the exit sets U^- and W^-. In the segment W we shaded only one part of W^-.

k-periodic point of the Poincaré map. If n is even then for any $k \in \mathbb{N}$ there is a periodic solution (1) with the principal period kT.

We will prove now a stronger result, namely the existence of symbolic dynamics on $n+2$ symbols for the same κ_0 and using the same periodic isolating segments, which where used in the proof of theorem 4.

Let $n \in \mathbb{N}$ be fixed. We construct a model semiprocess φ^M such that the sets $U = U(n)$, $W = W(n)$ are periodic isolating segments for φ^M and the Poincaré map $\varphi^M_{(0,T)}$ is essentially one-dimensional.

Put $R = \rho(0, W_0^-) > 0$. Let $h : \mathbb{C} \to \mathbb{C}$ is defined by

$$h(z) = z e^{i\frac{2\pi}{n+1}}. \tag{13}$$

Let $I_1 = [0, R] \subset \mathbb{C}$ and $I(n) = \bigcup_{k \in \{1,\ldots,n+1\}} I_k$ where $I_k = h^{k-1}(I_1)$.

Observe that each set $I(k)$ for $k = 1, \ldots, n+1$ joints 0 with the middle point of the connected component of U_0^-. We denote this connected component of U^- by $E(U)_k$. We have

$$U^- = \bigcup_{k=1}^{n+1} E(U)_k \tag{14}$$

We define a deformation retraction $r : W_0 \to I(n)$ as follows. The set W_0 is divided by $I(n)$ on $n+1$ closed parts (compare Fig. 3). Let B be one of them. We define a strong deformation retraction $r|_B : B \to B \cap I(n)$ by the following rule. The set $B \cap W_0^-$ has two components being compact intervals. We take the two lines which contain these intervals and we define C to be their intersection. The retraction is obtained by projecting $x \in B$ onto $I(n)$ along the line connecting x and C. This construction requires $n > 1$. For $n = 1$ the retraction is the projection onto x-axis along the y-direction.

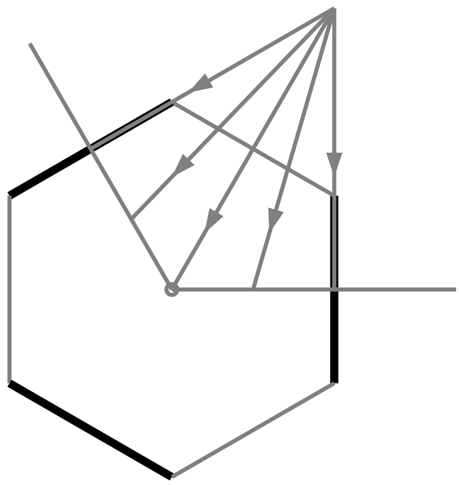

Figure 3: The retraction $r : U_0 \to I(n)$ for $n = 2$

The theorem below is a statement about the existence of the model semiprocess φ^M for periodic blocks U and W. The proof of existence of φ^M is up to details, the same as the proof of th. 20 in [WZ] φ^M is constructed from a few pieces by concatenation.

Theorem 5 *Given $n \in \mathbb{N}$, let U, W, R and the retraction r be as above.*

Then exists a semiprocess φ^M and real numbers a, b, c $0 < a < b < c < R$ with the following properties. If we define

$$J = \bigcup_{k \in \{1,\dots,n+1\}} h^{k-1}([0,a]), \tag{15}$$

$$J_k = h^{k-1}([b,c]), k \in \{1,\dots,n+1\}, \tag{16}$$

then the following assertions are true

(1) *the pairs (U, U^-), (W, W^-) are periodic isolating segments for the semiprocess φ^M,*

(2) *for $\lambda \in [0,1]$ $H_\lambda = \Omega(\{(\varphi^M, 0), (\varphi, \lambda T)\}, T)$ is a T-periodic semiprocess, (U, U^-) and (W, W^-) are periodic isolating segments for H_λ and condition (vi) holds uniformly.*

(3)

$$Z := \{x \in W_0 : \varphi^M_{(0,t)}(x) \in W_t,\ t \in [0,T]\} = Z_J \cup \bigcup_{k=1}^{n+1} Z_{J_k}, \quad \textit{where}$$

$$Z_J := \{x \in W_0 : \varphi^M_{(0,t)}(x) \in U_t,\ t \in [0,T]\} = r^{-1}(J),$$

$$Z_{J_k} := \{x \in Z : x \textit{ leaves } U \textit{ through } E(U)_k,\} = r^{-1}(J_k)$$

(4) $r : Z \to J \cup \bigcup_{k=1}^{n+1} J_k$ *is a deformation retraction,*

(5) *there is a continuous function* $f : J \cup \bigcup_{k=1}^{n+1} J_k \to I(n)$ *symmetric with respect to* h *such that for* $k = 1, \ldots, n+1$ *holds*

$$f(0) = 0, \quad f(h^k(a)) = f(h^k(b)) = h^k(R), \quad f(h^k(c)) = h^{k+1}(R), \quad (17)$$

the restrictions of f,

$$f : [0, h^{k-1}(a)] \to [0, h^{k-1}(R)], \quad f : J_k \to [h^{k-1}(R), 0] \cup [0, h^k(R)]$$

are homeomorphisms and

$$\varphi^M_{(0,T)}(z) = f(r(z)), \quad z \in Z, \quad (18)$$

Figure 4 presents a schematic view of the Poincaré map $\varphi^M_{(0,T)}$ on $I(n)$ for $n = 2$. The gray lines above segment $[0, R]$ symbolize the image of $[0, a]$ (the lower line) and $[b, c]$ (the upper line). Obviously in fact this image is contained in $[0, R] \cup [0, h(R)]$. Arrows indicate the orientation of the image of f when going form 0 to a on the lower line and form b to c on the upper line.

We will now characterize the symbolic dynamics we for the model map. We focus on the time T Poincaré map, P_M, which according to (18) is essentially 1-dimensional. The one dimensional object here is $I(n)$.

We want now to investigate the symbolic dynamics on sets J and J_k. The symbol 0 will correspond to J and the symbol k for $k = 1, \ldots, n+1$ will correspond to J_k. To easy the notation we set $J_k = J_{(k \bmod (n+1))+1}$ for $k > n+1$. Observe that $P_M(J_k)$ covers J_k, J_{k+1} and part of J. The image of $P_M(J)$ covers J and all J_k's. But if we want to see where we can go from J_k and J under P_M^2 we need to consider the parts of J, what they cover and which part is covered by J_k.

We define the set $X \subset \Sigma_{n+2} = \{0, ..., n+1\}^{\mathbb{Z}}$, which as we will see later captures symbolic dynamics of φ^M and φ, as follows:
$c \in X$ iff the following conditions hold

(1) if $c_i = k$, $k \in \{1, ..., n+1\}$ then $c_{i+1} = 0$ or $c_{i+1} = k$ or $c_{i+1} = k(\mathrm{mod}(n+1)) + 1$,

(2) if $c_i = 0$ and for all $p < i$ $c_p = 0$ then $c_{i+1} \in \{0, ..., n+1\}$,

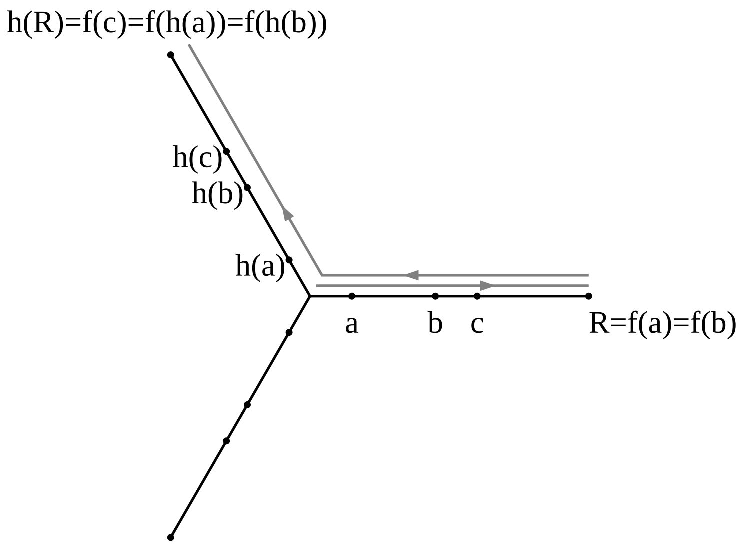

Figure 4: A schematic view of map $f = \varphi^M_{(0,T)}{}_{|I(n)}$ for $n = 2$ for angles between 0 and $2\pi/3$. See the text following theorem 5 for explanation.

(3) if $c_i = 0$ and there is $p < i$, such that $c_p = k \neq 0$ and for all s $p < s \leq i$ we have $c_s = 0$, then $c_{i+1} = 0$ or $c_{i+1} = k$ or $c_{i+1} = k(\mathrm{mod}(n+1)) + 1$.

Let X_l denotes the projection of X on $0, \ldots, l-1$ coordinates. This means that if $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{l-1}) \in \{0, 1, \ldots, n+1\}^l$, then $\alpha \in X_l$ iff there exists $c \in X$ such that $\alpha_i = c_i$ for $i = 0, \ldots, l-1$.

For $\alpha \in X_l$ such that $\alpha_0 \neq 0$ we define

$$s(\alpha) = \max\{j \mid \alpha_j \neq 0\}$$

Lemma 6 *Let $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{l-1}) \in X_l$, such that $\alpha_0 \neq 0$, then there exists a closed interval $A \subset J_{\alpha_0}$ such that*

$$W_\alpha(\varphi^M) = r^{-1}(A), \quad (\mathrm{int}\, W)_\alpha(\varphi^M) = r^{-1}(\mathrm{int}\, A)$$
$$f^l(A) = J_{s(\alpha)} \cup J_{s(\alpha)+1}$$
$$f^l_{|A} \textit{is a homeomorphism}$$

Proof: An easy induction with respect to l using condition 3 from theorem 5.∎

Now we are ready to state and to prove the main theorem in this paper

Theorem 7 *For $0 < \kappa < \kappa_0$ there are a compact set $I_W \subset \mathbb{R}^2$, invariant with respect to the Poincaré map $\varphi_{(0,T)}$ and a continuous map $g : I \to \Sigma_{n+2}$ such that*

(a) $g \circ \varphi_{(0,T)} = \sigma \circ g$,

(b) $X \subset \mathrm{img}$,

(c) *if $c \in X$ is m-periodic sequence then $g^{-1}(c)$ contains m-periodic point for the Poincaré map $\varphi_{(0,T)}$.*

Proof: We define I_W by

$$I_W := \{x \in W_0 | \varphi_{(0,t)}(x) \in W_t, \text{ for } t \in \mathbb{R} \} \tag{19}$$

We define a semiconjugacy $g : I_W \to \Sigma_{n+2}$ as in [WZ] by

$$g(x)_l := \begin{cases} 0, & \text{if } (t, \varphi_{(0,t)}(x) \in U \text{ for all } t \in (lT, (l+1)T) \\ k, & \text{if } \varphi_{(0,lT)}(x) \text{ leaves } U \text{ in time less than } T \text{ through } E(U)_k \end{cases} \tag{20}$$

It is easy to see that the condition (a) holds. Observe that (b) follows from (c). Hence it is enough to prove (c).

Let $c \in X$ be a periodic sequence of period l. If $c_i = 0$ for all i then $g(0) = c$. It is easy to see that it is enough to consider $c \in X$, such that $c_0 \neq 0$. Let l be a principal period of c.

Let $\alpha = (c_0, c_1, \ldots, c_{l-1}) \in X_l$. From lemma 6 it follows that there exists a closed interval A, such that

$$(\mathrm{int}\, W)_\alpha(\varphi^M) = r^{-1}(\mathrm{int}\, A) \tag{21}$$

$$f^l(A) = J_{s(\alpha)} \cup J_{s(\alpha)+1} \tag{22}$$

$$f^l_{|A} \text{is a homeomorphism} \tag{23}$$

Observe that either $c_0 = s(\alpha)$ or $c_0 = (s(\alpha) \mathrm{mod}(n+1)) + 1$.

We have the following situation: the set $r^{-1}(A)$ is topologically a product of a segment A and another interval B, the map $f^l(r(x))$ maps $A \times B$ onto $J_{s(\alpha)} \cup J_{s(\alpha)+1}$ containing A in its interior, hence (compare [Z1, Z2])

$$\mathrm{ind}(\varphi^M_{(0,lT)}, (\mathrm{int} W_0)_\alpha(\varphi^M)) = \mathrm{ind}(f^l, \mathrm{int} A) = \pm 1 \neq 0 \tag{24}$$

From theorems 5 and 3 it follows that

$$\mathrm{ind}(\varphi_{(0,lT)}, (\mathrm{int} W_0)_\alpha(\varphi)) \neq 0. \tag{25}$$

Hence there exists $x \in (\mathrm{int} W_0)_\alpha(\varphi)$, such that $\varphi_{(0,lT)}(x) = x$. Observe that $g(x) = c$. This finishes the proof of (c). ∎

5 Time dependent Hamiltonian example on the plane.

Consider a time dependent Hamiltonian system

$$\begin{cases} \dot{x} = -\frac{\partial H}{\partial y}, \\ \dot{y} = \frac{\partial H}{\partial x}, \end{cases} \tag{26}$$

where

$$H(z,y,t) = x^3y + xy^3 + H_1(x,y,t)$$
$$H_1(x,y,t) = -\frac{1}{2}y^2\sin(\kappa t) - xy\cos(\kappa t) + \frac{1}{2}x^2\sin(\kappa t).$$

It was proved in [W2] that there exists $\kappa_0 > 0$, such that for $0 < \kappa < \kappa_0$ there are two periodic isolating segments U and W over $[0, \frac{2\pi}{\kappa}]$ such that $U \subset W$, $U_0 = W_0$ and $U_0^- = W_0^-$. The smaller segment U is a twisted prism with a square base centered at the origin. The exit set U^- consists of two disjoint ribbons $E(U)^{+1}$ and $E(U)^{-1}$ winding around the prism. The time sections U_t are obtained by rotating the base U_0 with the angular velocity $\kappa/2$ over the t-interval $[0, \frac{2\pi}{\kappa}]$. The bigger segment W is a regular prism with the same base $U_0 = W_0$ and broadening center part (see Fig. 5).

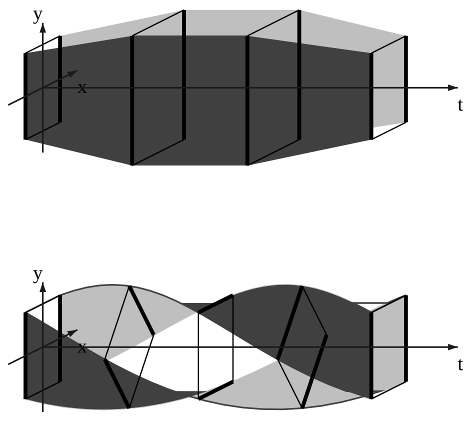

Figure 5: The isolating periodic segments for (26) Above: the segment W, below the segment U. The thick lines and shaded faces are contained in the exit sets U^- and W^-.

We see that when compared to the previous section or [WZ] we have a slightly different geometry here. There the bigger segment, W, was rotating, while here the smaller one, U, rotates. But both these situations are manifestly homeomorphic, so we can apply the methods from section 4 here.

The following was proved in [W2]

Theorem 8 *For $0 < \kappa < \kappa_0$ the Poincaré map is semiconjugate to the shift on two symbols in some compact set I and the preimage (by semiconjugacy) of any k-periodic sequence of two symbols contains a k-periodic point of the Poincaré map.*

We will prove now a stronger result, namely the existence of symbolic dynamics on 3 symbols for the same κ_0 and using the same periodic isolating segments, which where used in the proof of theorem 8 in [W2].

The proof of the next theorem is up to details, the same as the proof of th.20 in [WZ]:

Theorem 9 *Let $R = \rho(0, W_0^-) > 0$. There exists a semiprocess φ^M and $0 < a < b < c < R$, such that*

(1) *the pairs (U, U^-), (W, W^-) are periodic isolating segments for the semiprocess φ^M ,*

(2) *for $\lambda \in [0,1]$ $H_\lambda = \Omega(\{(\varphi^M, 0), (\varphi, \lambda t)\}, T)$ is a T-periodic semiprocess, (U, U^-) and (W, W^-) are periodic isolating segments for H_λ and condition (vi) hold uniformly.*

(3) *Let $J_{-1} = [-c, -b]$, $J_0 = [-a, a]$, $J_1 = [b, c]$ then*

$$Z = \{x \in W_0 : \varphi^M_{(0,t)}(x) \in W_t,\ t \in [0,T]\} = Z_{-1} \cup Z_0 \cup Z_1,$$

where

$$\begin{aligned} Z_0 &= \{x \in W_0 : \varphi^M_{(0,t)}(x) \in U_t,\ t \in [0,T]\} = J_0 \times [-R, R], \\ Z_{+1} &= \{x \in Z :\ x \text{ leaves } U \text{ through } E(U)^{+1}\ \} = J_{+1} \times [-R, R] \\ Z_{-1} &= \{x \in Z :\ x \text{ leaves } U \text{ through } E(U)^{-1}\ \} = J_{-1} \times [-R, R] \end{aligned}$$

(4) *there is a continuous function $f : J_{-1} \cup J_0 \cup J_{+1} \to [-R, R]$ such that*

$$\begin{aligned} f(0) &= 0, \\ f(-a) = f(-b) = f(c) &= R, \\ f(-c) = f(a) = f(b) &= -R, \\ \varphi^M_{(0,T)}(x, y) &= (f(x), 0) \quad (x, y) \in Z, \end{aligned}$$

Now an application of the continuation theorem (th. 3) give us as in [WZ] the following result

Theorem 10 *For $0 < \kappa < \kappa_0$ there are a compact set $I \subset \mathbb{R}^2$, invariant with respect to the Poincaré map, a surjective continuous map $g : I \to \Sigma_3$ such that the Poincaré map is semiconjugate to the shift map on three symbols by the map g. Moreover, for any k-periodic sequence $c \in \Sigma_3$, $g^{-1}(c)$ contains k-periodic point for the Poincaré map.*

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