Isolating segments, fixed point index and symbolic dynamics

Klaudiusz Wójcik[1], Piotr Zgliczyński[2 3]
July 7, 1999

Jagiellonian University,
Institute of Mathematics, Reymonta 4,
30-059 Kraków, Poland.
E-mail: wojcik@im.uj.edu.pl, zgliczyn@im.uj.edu.pl

Abstract

An extension of the recently introduced Srzednicki–Wójcik method for detecting chaotic dynamics in periodically forced ordinary differential equations is presented. As an application of the method we construct a topological model for the planar equation

$$z' = (1 + e^{i\kappa t}|z|^2)\overline{z}, \quad z \in \mathbb{C} \tag{1}$$

and we show by a continuation argument that the symbolic dynamics on three symbols for the topological model continues to equation (1) for $0 < \kappa \leq 0.495$.

Keywords: isolating segments, fixed point index, chaos, periodic solutions

1 Introduction

In [SW] (compare [W]) a new method for the detection of chaos in dynamical systems generated by time-periodic non-autonomous differential equations was introduced. By chaos we mean the existence of a compact invariant set such that the Poincaré map is semiconjugate to the shift on k symbols for some $k \geq 2$ and the counterimage by the semiconjugacy of any periodic point in the shift contains a periodic point of the Poincaré map (comp. [GH], [Wi]). The method presented in [SW] is based on the Ważewski Retract Theorem (see [C]) and the Lefschetz Fixed Point Theorem (see [D]). In order to apply this method we have to show the existence, in the extended phase space, of periodic isolating segments. The basic property of the periodic isolating segment is that at any point on the boundary of the segment the vector field is directed either outward or inward with respect to the segment (compare the notion of the isolating block in the Conley index theory). Actually, in all practical applications the segments are manifolds with corners. It was first observed by Roman Srzednicki in [S1], that the fixed point index of the Poincaré map inside the segment is equal to the Lefschetz number of the homeomorphism $\tilde{h}$ given by the segment.

In this paper we show that a more complete description of the dynamics may be obtained by adapting the results of [Z1] (see also [Z2]) to the context of [SW].

[1]partially supported by Foundation for Polish Science

[2]current address: CDSNS, Georgia Institute of Technology, Atlanta, GA 30322,USA

[3]supported by Polish KBN grants 2P03A 029 12, 2P03A 021 15 and NSF–NATO grant DGE–98–04459

Our modification is based on the continuation theorem (Th.10) similar in spirit to Th.2.2 in [Z1] and a construction of the topological model map. This continuation result enables us to dig deeper into the structure of the set of periodic solutions than does the Lefschetz Fixed Point Theorem used originally by Srzednicki and Wójcik in [SW]. As a result we are able to show that there exists a symbolic dynamics on three symbols for the planar system of differential equations (written using complex numbers)

$$z' = (1 + e^{i\kappa t}|z|^2)\overline{z}, \quad \text{for } 0 < \kappa \leq 0.495 \tag{2}$$

In [SW] it was shown that the above equation has symbolic dynamics on two symbols for the parameter range $0 < \kappa \leq 1/288$.

Compared to [SW] we add two new ingredients : the continuation theorem 10 and a topological model for equation (2). Our continuation theorem allows us to prove that the dynamics of the topological model continues to that of equation (2). This is a rare phenomenon in the theory of dynamical systems. Usually one cannot claim rigorously that the dynamics of the model reflects that of the system under consideration. This is for example the case for the famous Lorenz attractor (see for example [GH]).

Now, the reader may wonder how special is equation (2) and the method of isolating periodic segments developed in [SW] and in the present paper. Indeed equation (2) is a very special one, but the method of isolating periodic segments , especially the continuation theorem 10, is general and can be applied to various other situations.

Some work in this direction has been done by [W, W1], where other periodically forced ordinary differential equations leading to similar isolating segment configurations are considered. Different configurations of isolating segments leading to Smale horseshoes for Poincaré maps have been proposed by Srzednicki in [S2, S3].

The underlying idea behind the method presented is *topological hyperbolicity*, which can be explained as follows. Consider for a moment a flow φ_t induced by the vector field v. Hyperbolicity of the flow means that there is an invariant set for which there exists a splitting of the tangent space into the direction of the vector field into stable and unstable components invariant under application of $D\varphi_t$, such that we have uniform expansion in the unstable direction and contraction in the stable direction. From these assumptions one deduces a lot of interesting properties for φ, like the existence of symbolic dynamics, sensitive dependence on initial conditions etc. (see for example [GH]). The problem with hyperbolicity is that it is often very hard to verify it in the context of a concrete differential equation. The most frequently applied methods detect transversal intersection of the stable and unstable manifolds, which is sufficient to detect the existence of Smale's horseshoes. Some of these methods (based on ideas of Melnikov and Silnikov mainly) are described in detail in [GH, Wi].

In the method presented here we construct some special sets , namely periodic isolating segments and we investigate the direction of the vector field v on the boundary of an isolating segment. If our picture of v on the boundary of a periodic isolating segment is apparently realisable by the hyperbolic vector field then we can apply our method to obtain the existence of periodic points. Using this kind of reasoning we are unable to say anything about the sensitive dependence of solutions on initial conditions. Hence our results are weaker than results based on hyperbolicity, but the assumptions of the presented method are also much easier to verify.

Section 2 contains a definition of periodic isolating segment for a semiprocess. This definition is an adaptation of the definition from [S1], [SW] to the semiprocess context. Section 3 contains the description of Srzednicki and Wójcik's method and our modification of it. Section 4 contains the basic continuation result – theorem 10. In section 6 we construct a model semiprocess for isolating segments obtained for equation (2) in [SW] and we compute explicitly various fixed point indices. This together with continuation theorem 10 gives the symbolic dynamics on three symbols for equation (2).

It should be stressed that we have a good reason to state our definitions and the continuation theorem 10 for semiprocesses: a Poincaré map of our model T–periodic semiprocess constructed in section 6 is 1-dimensional, hence it is not invertible. This fact (one-dimensionality of this map) enables us to calculate the various fixed point indices of interest.

2 Semiprocesses and periodic isolating segments.

Some notation: $\mathbb{R}_+ = [0, \infty)$, ρ euclidian distance function, $B(Z, \delta)$ a ball of size δ around the set Z, $\operatorname{ind}(F, D)$ - fixed point index of map F relatively to the set D (see [D]).

We start with introducing the notion of a *local semiprocess* which formalizes the notion of a continuous family of local forward trajectories in an extended phase–space.

Definition 1 *Assume that X is a topological space and $\varphi : D \to X$ is a continuous mapping, $D \subset \mathbb{R} \times \mathbb{R}_+ \times X$ is an open set. We will denote by $\varphi_{(\sigma,t)}$ the function $\varphi(\sigma, t, \cdot)$.*

φ is called a local semiprocess *if the following conditions are satisfied*

(S1) $\forall \sigma \in \mathbb{R},\ x \in X$ *:* $\{t \in \mathbb{R}_+ : (\sigma, x, t) \in D\}$ *is an interval,*

(S2) $\forall \sigma \in \mathbb{R} : \varphi_{(\sigma,0)} = \mathrm{id}_X$

(S3) $\forall \sigma \in \mathbb{R}, \forall s, t \in \mathbb{R}_+ : \varphi_{(\sigma,s+t)} = \varphi_{(\sigma+s,t)} \circ \varphi_{(\sigma,s)}$,

If $D = \mathbb{R} \times \mathbb{R}_+ \times X$, we call φ a (global) semiprocess*. If T is a positive number such that*

(S4) $\forall \sigma, t \in \mathbb{R}_+ : \varphi_{(\sigma+T,t)} = \varphi_{(\sigma,t)}$

*we call φ a T-*periodic local semiprocess.

A local semiprocess φ on X determines a local semiflow Φ on $\mathbb{R} \times X$ by the formula

$$\Phi_t(\sigma, x) = (\sigma + t, \varphi_{(\sigma,t)}(x)). \tag{3}$$

In the sequel we will often call the first coordinate in the extended phase space $\mathbb{R} \times X$ *a time* .

Let φ be a T-periodic local semiprocess and let Φ be a local semiflow associated to φ. It follows by $(S1)$ and $(S2)$ that for every $z = (\sigma, x) \in \mathbb{R} \times X$ there is an $0 < \omega_z \leq +\infty$ such that $(\sigma, t, x) \in D$ if and only if $0 \leq t < \omega_z$. Let $x \in X$, $\sigma \in \mathbb{R}$, then *a left solution* through $z = (\sigma, x)$ is a continuous map $v : (a, 0] \to \mathbb{R} \times X$ for some $a \in [-\infty, 0)$ such that:

(I) $v(0) = z$,

(II) for all $t \in (a,0]$ and $s > 0$ with $s+t \leq 0$ it follows that $s < \omega_{v(t)}$ and $\Phi_s(v(t)) = v(t+s)$.

If $a = -\infty$ then we call v *a full left solution.* We can extend a left solution through z onto $(a,0] \cup [0,\omega_z)$ by setting $v(t) = \Phi_t((\sigma,x))$ for $0 \leq t < \omega_z$, to obtain *a solution through* z. If $a = -\infty$ and $\omega_z = +\infty$, v is called *a full solution.*

Remark 1 *The differential equation*

$$\dot{x} = f(t,x) \tag{4}$$

such that f is regular enough to guarantee the uniqueness for the solutions of the Cauchy problems associated to (4) generates a local process as follows: for $x(t_0,x_0;\cdot)$ the solution of (4) such that $x(t_0,x_0;t_0) = x_0$ we put

$$\varphi_{(t_0,\tau)}(x_0) = x(t_0,x_0;t_0+\tau).$$

If f is T-periodic with respect to t then φ is a T-periodic local process. In order to determine all T-periodic solutions of equation (4) it suffices to look for fixed points of $\varphi_{(0,T)}$ (called the Poincaré map).

2.1 Construction of semiprocess via continuous coordinate changes

Now, we describe another method of constructing semiprocesses. Suppose that we have a continuous map $C : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$. We will use the notation C_t to denote the map $C(t,\cdot)$. We will call C_t *the change of coordinates map.* Suppose that C_t is a homeomorphism for every t and the map $C_t^{-1}(x)$ is a continuous function of (t,x). Let us fix a $\sigma_0 \in \mathbb{R}$, such that $C_{\sigma_0} = \mathrm{id}$.

Suppose that we have a dynamical system $\varphi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$.

We set

$$R_\sigma = C_\sigma \circ \varphi_{\sigma-\sigma_0} \tag{5}$$

We define a new semiprocess $\Pi(C,\sigma_0,\varphi)$ (in fact this is a process)

$$\Pi(C,\sigma_0,\varphi)_{(\sigma,t)}(x) = C_{\sigma+t} \circ \varphi_{\sigma+t-\sigma_0} \circ R_\sigma^{-1}(x) \tag{6}$$

Geometrically the semiprocess $\Pi(C,\sigma_0,\varphi)$ can be described as follows. We have a family of trajectories in the extended phase space given by $t \mapsto (t, C_t \circ \varphi_{t-\sigma_0}(x_0))$. We take all trajectories of φ and then we change coordinates on every fiber $\{t\} \times \mathbb{R}^d$. From our assumptions it follows that through every point in the extended phase space pass exactly one trajectory from this family. On this we build our semiprocess. For a given point (σ,x) in the extended phase space $(\sigma_0, R_\sigma^{-1}(x))$ is a point on the distinguished fiber $\{\sigma_0\} \times \mathbb{R}^d$ from which we move according to φ for time $\sigma + t - \sigma_0$ and then we change coordinates by applying $C_{\sigma+t}$.

Proposition 2 $\Pi(C,\sigma_0,\varphi)$ *is a global semiprocess.*

Proof: We have to check only conditions (S2) and (S3). From equations (5) and (6) we obtain

$$\Pi(C,\sigma_0,\varphi)_{(\sigma,0)} = C_\sigma \circ \varphi_{\sigma-\sigma_0} \circ R_\sigma^{-1} = R_\sigma \circ R_\sigma^{-1} = \mathrm{id}. \tag{7}$$

Hence (S2) is satisfied.

$$\begin{aligned}\Pi(C,\sigma_0,\varphi)_{(\sigma+s,t)}\circ\Pi(C,\sigma_0,\varphi)_{(\sigma,s)} &=\\ C_{\sigma+s+t}\circ\varphi_{\sigma+s+t-\sigma_0}\circ R^{-1}_{\sigma+s}\circ C_{\sigma+s}\circ\varphi_{\sigma+s-\sigma_0}\circ R^{-1}_{\sigma} &=\\ C_{\sigma+s+t}\circ\varphi_{\sigma+s+t-\sigma_0}\circ[R^{-1}_{\sigma+s}\circ C_{\sigma+s}\circ\varphi_{\sigma+s-\sigma_0}]\circ R^{-1}_{\sigma} &=\\ C_{\sigma+s+t}\circ\varphi_{\sigma+s+t-\sigma_0}\circ R^{-1}_{\sigma} = \Pi(C,\sigma_0,\varphi)_{(\sigma,s+t)}&\end{aligned}$$

Hence (S3) is also satisfied. □

2.2 Concatenation of semiprocesses

Let $T>0$ be a real number and $\varphi_j:\mathbb{R}\times\mathbb{R}_+\times X\to X$ for $j=0,\ldots,n-1$ be a family of global T–periodic semiprocesses. Let us fix real numbers $0=t_0\leq t_1\leq t_2\leq t_{n-1}\leq t_n=T$. We construct now *$T$–periodic concatenation of φ_j over time intervals* $[t_j,t_{j+1}]$ denoted by $\Omega(\{(\varphi_j,t_j)\}_{j=0,\ldots,n-1},T)$. To make the notation less cumbersome in the sequel we will omit the parameters of Ω.

First we extend finite sequences t_j and φ_j to infinite ones periodically as follows $t_{kn+j}=kT+t_j$, $\varphi_{kn+j}=\varphi_j$ for $k\in\mathbb{Z}$, $j=0,1,\ldots,n-1$.

Let us choose $\sigma,t\in\mathbb{R}$ and $t\geq 0$. We have to define $\Omega_{(\sigma,t)}(x)$. Observe that there exist integers s,p such that $t_{s-1}\leq\sigma\leq t_s$ and $t_p\leq\sigma+t\leq t_{p+1}$. We have two cases either $s-1=p$ or $s-1<p$.

In the first case (i.e. $s-1=p$) we set

$$\Omega_{(\sigma,t)}=\varphi_{s-1}(\sigma,t)$$

When $s-1<p$ we have

$$\sigma+t=\sigma+(t_s-\sigma)+(t_{s+1}-t_s)+\cdots+(t_p-t_{p-1})+(\sigma+t-t_p) \tag{8}$$

We set

$$\begin{aligned}\Omega_{(\sigma,t)}(x)=\varphi_{p,(t_p,\sigma+t-t_p)}\circ\varphi_{p-1,(t_{p-1},t_p-t_{p-1})}\circ\cdots\circ&\\ \varphi_{s,(t_s,t_{s+1}-t_s)}\circ\varphi_{s-1,(\sigma,t_s-\sigma)}(x)&\end{aligned}$$

This construction can be explained as follows. Let us think about a trajectory in the extended phase space. We start to move forward according to φ_{s-1}, then at time t_s we switch semiprocesses and we move according to φ_s up to time t_{s+1} and we continue to switch semiprocesses to φ_k when the time coordinate is equal to t_k. During this switching of semiprocesses first argument (denoted σ) is always taken modulo T. From this description it follows clearly that Ω is well defined , continuous and satisfies conditions **(S1),(S2),(S3),(S4)** from definition 1 (we leave the proof to the reader).

Hence we have

Proposition 3 *Ω is a global T–periodic semiprocess.*

2.3 Periodic isolating segments

We use the following notation: by $\pi_1 : \mathbb{R} \times X \to \mathbb{R}$ and $\pi_2 : \mathbb{R} \times X \to X$ we denote the projections and for a subset $Z \subset \mathbb{R} \times X$ and $t \in \mathbb{R}$ we put

$$Z_t = \{x \in X : (t, x) \in Z\}.$$

Now we are going to state the definition of the basic object in this paper, a T-periodic isolating segment, which is a modification of the notion of a periodic isolating segment over $[0, T]$ in [SW]. Notice that a T-periodic isolating segment is a T-periodic isolating block in the sense of [S1] and a T-periodic isolating segment can be easily obtained by gluing translated copies of a periodic isolating segment over $[0, T]$.

Definition 2 *We will say that a set $Z \subset \mathbb{R} \times X$ is T–periodic, iff $Z_{nT+t} = Z_t$ for every $n \in \mathbb{N}$ and $t \in \mathbb{R}$.*

Definition 3 *Let $(W, W^-) \subset \mathbb{R} \times X$ be a pair of subsets. We call W a T-*periodic isolating segment *for the T-periodic global semiprocess φ if:*

(i) *W, W^- are T–periodic*

(ii) *$(W, W^-) \cap ([0, T] \times X)$ is a pair of compact sets*

(iii) *for every $\sigma \in \mathbb{R}$, $x \in \partial W_\sigma$ there exists $\delta > 0$ such that for all $t \in (0, \delta)$ $\varphi_{(\sigma,t)}(x) \notin W_{\sigma+t}$ or $\varphi_{(\sigma,t)}(x) \in \mathrm{int} W_{\sigma+t}$,*

(iv)

$$W^- = \{(\sigma, x) \in W : \exists \delta > 0 \ \forall t \in (0, \delta) \ \varphi_{(\sigma,t)}(x) \notin W_{\sigma+t}\},$$
$$W^+ := \mathrm{cl}\,(\partial W \setminus W^-)$$

(v) *for all $z \in W^+$ and all $v : (a, 0] \to \mathbb{R} \times X$ a left solution through z there is $a \leq b < 0$ such that for all $t \in (b, 0)$ $v(t) \notin W$*

(vi) *there exists $\eta > 0$ such that for all $x \in W^-$ there exists $t > 0$ such that for all $\tau \in (0, t]$ $\Phi_\tau(x) \notin W$ and $\rho(\Phi_t(x), W) > \eta$*

Roughly speaking, W^- and W^+ are sections for the semiflows, through which trajectories leave and enter the segment W, respectively.

Recall that a topological space is an ENR iff it is homeomorphic with a retract of an open set in some Euclidean space.

Definition 4 *Let $(W, W^-) \subset \mathbb{R} \times X$ be a pair of subsets. We call W a T-*periodic regular isolating segment *for the T-periodic global semiprocess φ if W is a T-periodic isolating segment for φ and the following conditions hold*

(ii') *$(W, W^-) \cap ([0, T] \times X)$ is a pair of compact ENR's*

(vii) *there exists a T-periodic homeomorphism*

$$h : \mathbb{R} \times (W_0, W_0^-) \longrightarrow (W, W^-)$$

such that $\pi_1 = \pi_1 \circ h$.

The notion of a T-periodic regular isolating segment is a modification to the semiprocesse context of the notion of a T-periodic isolating segment over $[0,T]$ in [SW].

Lemma 4 *Let (W,W^-) be a T–periodic isolating segment. There exists $\epsilon>0$ such that for every $x\in W^-$, if for some $t>0$ $\Phi_t(x)\in W$ then $t>\epsilon$.*

Proof:

Suppose the contrary. Hence there exist the sequence $x_n\in W^-$ and $0<t_n<1/n$ for $n\in\mathbb{N}$ such that $\Phi_{t_n}(x_n)\in W$.

Since W^- is T-periodic we can assume that the sequence $x_n\to y$. Obviously $y\in W^-$.

Observe that from (vi) it follows that there exists the sequence t'_n such that

$$0<t'_n<t_n,\quad \rho(\Phi_{t'_n}(x_n),W)>\eta \tag{9}$$

Obviously $\Phi_{t'_n}\to y$. We have

$$\rho(y,W)\geq\eta>0 \tag{10}$$

But $y\in W$, hence we obtained a contradiction.

□

Definition 5 *For the periodic isolating segment W we define* exit time function $\tau_{W,\varphi}$

$$\tau_{W,\varphi}:W_0\ni x\mapsto\sup\{t\geq 0:\ \forall s\in[0,t]\,(s,\varphi(0,s,x))\in W\}\in[0,\infty]$$

By the Ważewski Retract Theorem the map $\tau_{W,\varphi}$ is continuous (compare [C]).

Definition 6 *Let W_0 be a periodic isolating segment for φ and $x\in W_0$. Let $C\subset W^-$. We will say that x* leaves W at the time t through C, *iff $t=\tau_{W,\varphi}(x)$ and $(t,\varphi(0,t,x)\in C$.*

3 Chaotic equation

We start this section with the description of the geometric method introduced in [SW]. Let $W\subset\mathbb{R}\times X$ be a T-periodic regular isolating segment for a given T-periodic process φ.

Following [SW] we define a homeomorphism

$$\tilde{h}:(W_0,W_0^-)\longrightarrow(W_T,W_T^-)=(W_0,W_0^-)$$

by $\tilde{h}(x)=\pi_2(h(T,\pi_2h^{-1}(0,x)))$ for $x\in W_0$. Geometrically, $\tilde{h}$ moves a point $x\in W_0$ to $W_T=W_0$ along the arc $h([0,T]\times\{\pi_2h^{-1}(0,x)\})$. A different choice of the homeomorphism h in (ii) leads to a map which is homotopic to $\tilde{h}$ (compare [S1]), hence the automorphism

$$\mu_W=\tilde{h}_*:H(W_0,W_0^-)\longrightarrow H(W_0,W_0^-)$$

induced by $\tilde{h}$ in singular homology, is an invariant of the segment W. Recall that its Lefschetz number is defined as

$$\mathrm{Lef}(\mu_W) = \sum_{n=0}^{\infty}(-1)^n \mathrm{tr}\ \tilde{h}_{*n}.$$

In particular, if $\mu_W = \mathrm{id}_{H(W_0,W_0^-)}$ then $\mathrm{Lef}(\mu_W)$ is equal to the Euler characteristic $\chi(W_0, W_0^-)$.

Let $\Sigma_k = \{0, 1, \ldots, k-1\}^{\mathbb{Z}}$ and $\sigma : \Sigma_k \to \Sigma_k$ be the shift map.

Definition 7 *We say that a T-periodic semiprocess φ is* Σ_k-chaotic *if there exists a compact set $I \subset M$ invariant with respect to the Poincaré map $\varphi(0,T,\cdot)$ and a continuous surjective map $g : I \to \Sigma_k$ such that:*

(j) *$\sigma \circ g = g \circ \varphi(0,T,\cdot)$, i.e. $\varphi(0,T,\cdot)$ is semiconjugate to the shift map $\sigma : \Sigma_k \to \Sigma_k$ in the set I,*

(jj) *for every n-periodic sequence $s \in \Sigma_k$ its counterimage $g^{-1}(s)$ contains at least one n-periodic point of $\varphi(0,T\cdot)$.*

We will say that equation $x' = f(t,x)$ (with f T-periodic with respect to t) is chaotic *if the local process generated by it is chaotic.*

It follows in particular that a Σ_k-chaotic equation, for every $l \in \mathbb{N}$, has a periodic solution with the basic period lT and the topological entropy of $\varphi_{(0,T)}$ is positive.

Remark 5 *Since I is compact and the set of periodic points is dense in Σ_k, condition (jj) implies that the semiconjugacy g must be a surjection.*

The following theorem was proved in [SW, Theorem 2]

Theorem 6 *Let U, W be two T-periodic regular isolating segments for the equation*

$$x' = f(t,x) \tag{11}$$

where f is T-periodic with respect to the time variable. Assume that

(a) $U \subset W$, $U_0 = W_0$, $U_0^- = W_0^-$

(b) $\mu_U = id_{H(W_0,W_0^-)} = \mu_W^2$

(c) $\mathrm{Lef}(\mu_W) \neq \chi(U_0, U_0^-)$

Then equation (11) is Σ_2-chaotic.

In [SW] the authors applied theorem 6 to the following planar nonautonomous equation of the variable $z \in \mathbb{C}$

$$z' = (1 + e^{i\kappa t}|z|^2)\overline{z} \tag{12}$$

for some $\kappa \in \mathbb{R}$. The right-hand side of eq. (12) is $2\pi/\kappa$-periodic. They proved the following theorem [SW, Th. 3]

Theorem 7 *Equation (12) in* Σ_2*-chaotic, provided* $0 < \kappa \leq 1/288$.

The aim of this paper is the following

Theorem 8 *Equation (12) is* Σ_3*-chaotic, provided* $0 < \kappa \leq 0.495$.

Now, we describe the main idea of proofs of theorems 7 and 8. In [SW] the authors constructed two periodic regular isolating segments U and W satisfying assumptions of theorem 6. In section 6 we described what the segments look like.

The symbolic dynamics is obtained in theorem 7 as follows. Let us set $T := 2\pi/\kappa$. Let

$$I_W := \{x \in W_0 | \varphi(0,t,x) \in W, \text{ for } t \in \mathbb{R} \} \tag{13}$$

The semiconjugacy map $g : I_W \to \Sigma_2$ is given by

$$g(x)_l := \begin{cases} 0, & \text{if } (t, \varphi(0,t,x)) \in U \text{ for all } t \in (lT, (l+1)T \\ 1, & \text{otherwise} \end{cases} \tag{14}$$

From this definition it follows that we have $g(x)_l = 1$, when a trajectory of $x \in I_W$ leaves a small segment U in a moment between lT and $(l+1)T$.

For the proof of theorem 8 we need to find a way to introduce another symbol. We do it as follows: when a trajectory of $x \in I_W$ leaves U it has to do that through one of two components of $U^- = U^{-1} \cup U^{+1}$, so we can define a semiconjugacy $g_3 : I_W \to \Sigma_3$ by

$$g_3(x)_l := \begin{cases} 0, & \text{if } (t, \varphi(0,t,x)) \in U \text{ for all } t \in (lT, (l+1)T) \\ 1, & \text{if } \varphi(0, lT, x) \text{ leaves } U \text{ in time less than } T \text{ through } U^{-1} \\ 2, & \text{if } \varphi(0, lT, x) \text{ leaves } U \text{ in time less than } T \text{ through } U^{+1} \end{cases} \tag{15}$$

The rest of this paper is devoted to the proof that the map g_3 is a semiconjugacy required for theorem 8.

4 Continuation theorem

Let $\varphi : \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ be a T-periodic global semiprocess.

Let U, W be two T-periodic isolating segments for φ. Assume that

$$U \subset W, \quad U_0 = W_0, \quad U_0^- = W_0^- \tag{16}$$

Let

$$U^- = \bigcup_{l=1}^{K} E(U)_l \tag{17}$$

be a decomposition of the exit set U^- into disjoint union of closed T–periodic sets $E(U)_l$. In the next section we will use the decomposition into connected components.

Observe that

$$\rho(E(U)_l, E(U)_j) > 0, \quad \text{when } l \neq j \tag{18}$$

Let

$$I_W(\varphi) := \{x \in W_0 | \varphi(0,t,x) \in W, \text{ for } t \in \mathbb{R}_+ \} \tag{19}$$

For $n \in \mathbb{N}$ and $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) \in \{0, 1, \ldots, K\}^n$ and $D \subset W_0$, we define $D_\alpha(\varphi)$ as a set of points fulfilling the following conditions

$$\varphi(0, lT, x) \in D, \quad \text{for } l = 0, \ldots, n \tag{20}$$

$$(t, \varphi(0, t, x)) \in \operatorname{int} W, \quad \text{for } t \in (0, nT) \tag{21}$$

$$\text{if } \alpha_l = 0, \text{ then} \quad (t, \varphi(0, t, x)) \in \operatorname{int} U \quad \text{for } t \in (lT, (l+1)T) \tag{22}$$

$$\text{if } \alpha_l \neq 0, \text{ then} \quad \varphi(0, lT, x) \text{ leaves } U \text{ in time less than } T \text{ through } E(U)_{\alpha_l} \tag{23}$$

Lemma 9 *If D is open, then $D_\alpha(\varphi)$ is open. If D is closed, then $D_\alpha(\varphi)$ is closed.*

Proof:

Suppose first that D is open. From the continuity of $\varphi(0, lT, \cdot)$ and $\varphi(lT, t, \cdot)$ it follows that conditions (20–22) define an open subset of D. It remains to show that (23) holds for an open subset of D. Suppose that $x \in D$ is such that (20) and (23) hold. Since $\varphi(0, lT, x) \in \operatorname{int} U_0$, then the exit time $t = t_{U,\varphi}(\varphi(0, lT, x))$ belongs to the open segment $(0, T)$. From the continuity of the exit time function it follows that

$$t_{U,\varphi}(\varphi(0, lT, y)) \in (0, T).$$

for y sufficiently close to x . Hence the function

$$y \mapsto (t_{U,\varphi}(\varphi(0, lT, y)), \varphi(t_{U,\varphi}(\varphi(0, lT, y), lT, y)) \in U^- \tag{24}$$

is continuous. Observe that each set $E(U)_s$ is open in U^-, hence all points sufficiently close to x are mapped by this function into the same component $E(U)_{\alpha_l}$.

Now suppose that D is closed. $D \subset W_0$. Suppose that $x_n \to x$ and $x_n \in D_\alpha(\varphi)$. We have to show that conditions (20–23) hold for x. Condition (20) is obvious. Suppose that (21) does not hold for x, then for some $t \in (0, nT)$ we have $(t, \varphi(0, t, x)) \notin W$. But from the continuity of $\varphi(0, t, \cdot)$ it follows that $(t, \varphi(0, t, x_n)) \notin W$ for all but a finite number of the x_n's. This however contradicts (21) for x_n. The proof of (22) is analogous.

It remains to show (23). Since for all n $\varphi(0, (l+1)T, x_n) \in D \subset W_0$ it follows that the exit times of $\varphi(0, lT, x_n)$ are less than $T - \epsilon$, where ϵ is obtained from lemma 4 applied to segment (U, U^-). From this and (vi) for segment (U, U^-) it follows that $\varphi(0, lT, x)$ leaves U before time T. Suppose now that x does leave U through $E(U)_s$, where $s \neq \alpha_l$, but then the trajectories of all points sufficiently close to x, will remain near the trajectory of x, hence all but finite number of $\varphi(0, lT, x_n)$'s will leave U through $E(U)_s$, which contradicts our assumption. Hence (23) holds for x.

□

Let us define

$$\operatorname{Im}(D, \varphi, \tau) = \{\varphi(0, \tau, x) \mid x \in D,\ (t, \varphi(0, t, x)) \in W \text{ for } 0 \leq t \leq \tau\} \tag{25}$$

Suppose now that we have the continuous family of semiprocesses $H : [0,1] \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$. We will use the notation H_λ for the map with the first parameter fixed, $H_\lambda(t_0, \tau, x) = H(\lambda, t_0, \tau, x)$. Let Φ^λ be a semiflow generated by the semiprocess H_λ.

The main theorem in this section is the following (comp. [Z1, Th. 2.2])

Theorem 10 *Let H_λ be continuous family of T-periodic semiprocesses and there exists U, W - T-periodic isolating segments for semiprocess H_λ for every $\lambda \in [0,1]$, such that (16) holds.*

Assume that condition (vi) holds uniformly for all H_λ $\lambda \in [0,1]$ and isolating segments (W, W^-) and (U, U^-). By this we mean the following statement:

There exists $\eta > 0$ such that for every $\lambda \in [0,1]$ and for every $x \in W^-$ ($x \in U^-$) there exists $t > 0$ such that for $0 < \tau \leq t$ holds $\Phi^\lambda_\tau(x) \notin W$ and $\rho(\Phi^\lambda_t(x), W) > \eta$ (resp. $\Phi^\lambda_\tau(x) \notin U$ and $\rho(\Phi^\lambda_t(x), U) > \eta$).

Then for every $n > 0$ and $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) \in \{0, 1, \ldots, K\}^n$ the fixed point indices $ind(H_\lambda(0, nT), (\operatorname{int} W_0)_\alpha(H_\lambda))$ are well defined and equal (i.e. do not depend on λ).

Before we prove this theorem we first show the following

Lemma 11 *Under hypotheses of theorem 10 there exists $\delta > 0$ such that for every $\lambda \in [0,1]$ holds*

$$Im(W_0, H_\lambda, T) \cap \overline{B(W_0^+, \delta)} = \emptyset \tag{26}$$

$$\text{every point from } \overline{B(W_0^-, \delta)} \cap W_0 \text{ leaves } W \text{ in time } \tau < T \tag{27}$$

Proof:

Let $S = \{x \in W_0 : \tau_{W,H_\lambda}(x) \geq T\}$. It is easy to check that S is compact and

$$\mathrm{Im}\,(W_0, H_\lambda, T) = H_\lambda(0, T, S).$$

If $z \in H_\lambda(0,T,S)$ then z has a left trajectory contained in W, so by condition (v) from def. 3 it follows

$$H_\lambda(0, T, S) \cap W_0^+ = \emptyset,$$

hence $\rho(H_\lambda(0,T,S), W_0^+) > 0$ and $\inf_{\lambda \in [0,1]} \rho(H_\lambda(0,T,S), W_0^+) > 0$.

For the proof of (27) let us consider a continuous function

$$G : [0,1] \times W_0 \ni (\lambda, x) \to \tau_{W,H_\lambda}(x) \in [0, +\infty].$$

Since for all $\lambda \in [0,1]$ and $x \in W_0^-$ $G(\lambda, x) = 0$, it follows by the standard argument based on compactness of $[0,1]$ and W_0^- that there is an open neighborhood V of W_0^- such that for all $(\lambda, x) \in [0,1] \times V$ $G(\lambda, x) < T/2$. This completes the proof. □

Proof of theorem 10 : The proof below is an adaptation of the proof of the continuation theorem 2.2 from [Z1].

Let us choose $\beta > 0$ such that

$$\beta < \eta, \quad \text{and} \quad \beta < \operatorname{dist}(E(U)_l, E(U)_j), \text{ for } l \neq j \tag{28}$$

Let us take δ obtained from lemma 11. By eventually decreasing δ we can assume that

$$\beta/2 > \delta \tag{29}$$

Let us define

$$D := W_0 \setminus \overline{B(\partial W_0, \delta)}, \quad C := W_0 \setminus \overline{B(\partial W_0, \delta/2)} \tag{30}$$

The sets D and C are open and

$$B(D,\delta/2) \subset C, \quad B(C,\delta/2) \subset \mathrm{int}\, W_0 \tag{31}$$

Let $\lambda_0 \in [0,1]$. There exists a set Λ open in $[0,1]$, $\lambda_0 \in \Lambda$, such that for every $\lambda_1, \lambda_2 \in \Lambda$

$$\rho(H_{\lambda_1}(0,t,x), H_{\lambda_2}(0,t,x)) \leq \delta/2, \quad \text{for } 0 \leq t \leq nT,\ x \in W_0 \tag{32}$$

We show now the following

Lemma 12

$$D_\alpha(H_\lambda) \subset C_\alpha(H_{\lambda_0}) \subset (\mathrm{int}\, W_0)_\alpha(H_\lambda), \quad \textit{for } \lambda \in \Lambda \tag{33}$$

Proof: We will show only the inclusion $D_\alpha(H_\lambda) \subset C_\alpha(H_{\lambda_0})$. The proof for the second one is analogous.

Let $x \in D_\alpha(H_\lambda)$. From (20) we have $H_\lambda(0,lT,x) \in D$ for $l = 0,\ldots,n$. Hence by (31)(32) we obtain immediately that

$$H_{\lambda_0}(0,lT,x) \in C, \quad \text{for } l = 0,\ldots,n$$

To prove (21) for the semiprocess H_{λ_0} we suppose that it is not fulfilled. x leaves W in time less than nT, so we know that it has to move away from W to a distance greater than β before it enters W again (it must do it before nT from just proved (20)). Hence there exists $t < nT$ such that

$$\rho((t, H_{\lambda_0}(0,t,x)), W) \geq \beta \tag{34}$$

hence from eq. (32) and (29) it follows that

$$\rho((t, H_\lambda(0,t,x)), W) > \beta - \delta/2 > 0 \tag{35}$$

This is in a contradiction with the assumption that x does not leave W in time less than nT for the semiprocess H_λ.

The proof of (22) is analogous, one has to replace W by U in the above reasoning.

It remains to show (23) for the semiprocess H_{λ_0}. Suppose that $H_\lambda(0,lT,x)$ leaves U in time less than T. Repeating the reasoning which led to a contradiction in the proof of (21) we see that $H_{\lambda_0}(0,lT,x)$ leaves U in time less than T.

It is enough to show that both points $H_{\lambda_0}(0,lT,x)$ and $H_\lambda(0,lT,x)$ leave U through the same set $E(U)_k$.

Let us define the exit times t_λ, t_{λ_0} by

$$t_\lambda = \tau_{U,H_\lambda}(H_\lambda(0,lT,x)) \tag{36}$$
$$t_{\lambda_0} = \tau_{U,H_{\lambda_0}}(H_{\lambda_0}(0,lT,x)) \tag{37}$$

Without loss of generality we can assume that

$$t_\lambda \leq t_{\lambda_0} \tag{38}$$

We have an $s \in \{1,\ldots,K\}$

$$\Phi^\lambda_{lT+t_\lambda}(0,x) \in E(U)_s, \tag{39}$$

and we have to show that

$$\Phi^{\lambda_0}_{lT+t_{\lambda_0}}(0,x) \in E(U)_s \tag{40}$$

From the assumption of our theorem concerning η and since $\delta < \eta$ it follows that there exists $t' > 0$ such that

$$\Phi^{\lambda}_{lT+t_\lambda+t}(0,x) \notin U, \quad \text{for } t \in (0,t'] \tag{41}$$

$$\Phi^{\lambda}_{lT+t_\lambda+t}(0,x) \in B(E(U)_s, \delta/2), \quad \text{for } t \in (0,t') \tag{42}$$

$$\rho(\Phi^{\lambda}_{lT+t_\lambda+t'}(0,x), U) = \delta/2 \tag{43}$$

From conditions (43) and (32) we see

$$\Phi^{\lambda_0}_{lT+t_\lambda+t'}(0,x) \notin U \tag{44}$$

hence

$$t_\lambda \leq t_{\lambda_0} < t_\lambda + t' \tag{45}$$

From conditions (45), (42) and (32) it follows that

$$\Phi^{\lambda_0}_{lT+t_{\lambda_0}}(0,x) \in B(E(U)_s, \delta) \subset B(E(U)_s, \beta/2) \tag{46}$$

This and (28) imply (40).

This finishes the proof of (33). □

We continue the proof of theorem 10. We show now that

$$H_\lambda(0,nT,x) \neq x, \quad \text{for } x \in W_{0,\alpha}(H_\lambda) \setminus D_\alpha(H_\lambda) \tag{47}$$

Let us take $x \in W_{0,\alpha}(H_\lambda) \setminus D_\alpha(H_\lambda)$. For some $l_0 \in \{0,1,\ldots,n\}$ $H_\lambda(0,l_0T,x) \in W_0 \setminus D$. From definitions of δ and D it follows that one of the following conditions holds

$$H_\lambda(0,l_0T,x) \in \overline{B(W_0^+,\delta)} \cap W_0$$
$$H_\lambda(0,l_0T,x) \in \overline{B(W_0^-,\delta)} \cap W_0$$

In the first case, by lemma 11, $H_\lambda(0,l_0T,x) \notin \operatorname{Im}(W_0,H_\lambda,T)$, hence x cannot be a periodic point having an entire trajectory in W.

In the second case by lemma 11 $H_\lambda(0,l_0T,x)$ leaves W in time less than T, so x cannot be a periodic point having an entire trajectory in W. This finishes the proof of (47).

From lemma 12 it follows that for $\lambda, \lambda_0 \in \Lambda$ all sets

$$\partial D_\alpha(H_\lambda), \partial C_\alpha(H_\lambda), \partial C_\alpha(H_{\lambda_0}), \partial W_{0\alpha}(H_\lambda) \subset W_{0\alpha}(H_\lambda) \setminus D_\alpha(H_\lambda).$$

So from (47) we see that the fixed point index for the maps $H_\lambda(0,nT,\cdot)$ relative to sets $D_\alpha(H_\lambda)$, $C_\alpha(H_\lambda)$, $C_\alpha(H_{\lambda_0})$, $(\operatorname{int} W_0)_\alpha(H_\lambda)$ [D, Ch. VII.5] is well defined.

From the excision property of the fixed point index [D, Ch.VII Th. 5.4], (47) and lemma 12 we conclude that

$$\begin{aligned}\operatorname{ind}(H_\lambda(0,nT,\cdot), D_\alpha(H_\lambda)) = \operatorname{ind}(H_\lambda(0,nT,\cdot), C_\alpha(H_{\lambda_0})) = \\ \operatorname{ind}(H_\lambda(0,nT,\cdot), (\operatorname{int} W_0)_\alpha(H_\lambda)), \quad \forall \lambda \in \Lambda.\end{aligned} \tag{48}$$

Substituting $\lambda := \lambda_0$ we derive

$$\operatorname{ind}(H_{\lambda_0}(0,nT,\cdot), D_\alpha(H_{\lambda_0})) = \operatorname{ind}(H_{\lambda_0}(0,nT,\cdot), C_\alpha(H_{\lambda_0})) \tag{49}$$

From lemma 12 and (47) it follows that

$$\forall \lambda \in \Lambda \quad \forall x \in \partial C_\alpha(H_{\lambda_0}) \quad H_\lambda(0, nT, x) \neq x$$

So from the homotopy invariance of the fixed point index [D, Ch.VII Th. 5.8] we obtain

$$\operatorname{ind}(H_\lambda(0, nT, \cdot), C_\alpha(H_{\lambda_0})) = \operatorname{ind}(H_{\lambda_0}(0, nT, \cdot), C_\alpha(H_{\lambda_0})) \quad \forall \lambda \in \Lambda. \tag{50}$$

From (48),(49),(50) we conclude

$$\operatorname{ind}(H_\lambda(0, nT, \cdot), D_\alpha(H_\lambda)) = \operatorname{ind}(H_{\lambda_0}(0, nT, \cdot), D_\alpha(H_{\lambda_0})) \quad \forall \lambda \in \Lambda. \tag{51}$$

From (51) we see that $\operatorname{ind}(H_\lambda(0, nT, \cdot), D_\alpha(H_\lambda))$ is locally constant. Hence

$$\operatorname{ind}(H_0(0, nT, \cdot), D_\alpha(H_0)) = \operatorname{ind}(H_1(0, nT, \cdot), D_\alpha(H_1)).$$

And finally it follows from (48) that

$$\operatorname{ind}(H_0(0, nT, \cdot), (\operatorname{int} W_0)_\alpha(H_0)) = \operatorname{ind}(H_1(0, nT, \cdot), (\operatorname{int} W_0)_\alpha(H_1)).$$

This finishes the proof. □

5 Isolating segments for φ

In this section we present a construction of U, W $2\pi/\kappa$-periodic isolating segments for equation (2). This is essentially the same construction as in [SW], but we introduce improvements which lead to better estimates for a chaotic parameter range.

We start with the description of U and W. W (the big segment) is a twisted prism with a square base centered at origin, with the side of the length $2R$. Its cross-sections W_t will be obtained by a rotating base with the angular velocity $\kappa/2$ over the t-interval $[0, 2\pi/\kappa]$. The segment U is a regular square-based prism with broadening ends. Its cross-sections U_t corresponding to t near the center of the interval have the side of the length $2r$ and they are broadened to the length $2R$ when t approaches to 0 or $2\pi/\kappa$. The exit sets $W_0^- = U_0^-$ consist of two components. The monodromy map $\tilde{h}$ works as follows: on U (the small segment) it is an identity map, on W it is a rotation by the angle π.

Let $R > r$, Δ be real positive numbers. We will specify the precise values later.

In this section we will use two coordinate systems in the extended phase space $\mathbb{R} \times \mathbb{R}^2$. The first one will be a standard one, the second one is given by the following coordinate change

$$z_r = x_r + iy_r = \exp(-it\kappa/2)z. \tag{52}$$

In the sequel we will use subscript r to indicate that we use this coordinate frame. We will call these coordinates *rotating coordinates.*

We have

$$\frac{dz_r}{dt} = |z_r|^2\overline{z_r} - i\kappa z_r/2 + \exp(-i\kappa t)\overline{z_r} \tag{53}$$

In real coordinates we have

$$\frac{dx_r}{dt} = (x_r^2 + y_r^2)x_r + \kappa y_r/2 + x_r\cos(\kappa t) - y_r\sin(\kappa t) \tag{54}$$

$$\frac{dy_r}{dt} = -(x_r^2 + y_r^2)y_r - \kappa x_r/2 - x_r\sin(\kappa t) - y_r\cos(\kappa t) \tag{55}$$

5.1 Segment (W, W^-)

We define the isolating segment (W, W^-)

$$W = \{(t,z) \in \mathbb{R}\times\mathbb{C} \mid |\mathrm{Re}(\exp(-it\kappa/2)z)| \leq R,\ |\mathrm{Im}(\exp(-it\kappa/2)z)| \leq R\}$$
$$W^- = \{(t,z) \in W \mid |\mathrm{Re}(\exp(-it\kappa/2)z)| = R\}$$
$$W^+ = \{(t,z) \in W \mid |\mathrm{Im}(\exp(-it\kappa/2)z)| = R\}$$

In rotating coordinates we have

$$W = \{(t, x_r, y_r) \mid |x_r| \leq R,\ |y_r| \leq R\} \tag{56}$$
$$W^- = \{(t, x_r, y_r) \mid |x_r| = R,\ |y_r| \leq R\} \tag{57}$$
$$W^+ = \{(t, x_r, y_r) \mid |x_r| \leq R,\ |y_r| = R\} \tag{58}$$

Lemma 13 *Assume* $(R^2-1)R^2 > (1/2+\kappa/4)^2$, *then*

$$\frac{d|x_r|}{dt} > 0 \quad \textit{when } |x_r| \geq R \tag{59}$$
$$\frac{d|y_r|}{dt} < 0 \quad \textit{when } |y_r| \geq R \tag{60}$$

In particular (W, W^-) *is an isolating segment.*

Proof

From the symmetry of equation (2) it follows that it is enough to prove the following

$$\frac{dx_r}{dt} > 0, \qquad \text{for } x_r \geq R \tag{61}$$
$$\frac{dy_r}{dt} < 0, \qquad \text{for } y_r \geq R \tag{62}$$

Let us fix $x_r > 0$. Observe that from (54) it follows that

$$\frac{dx_r}{dt} \geq (x_r^2+y_r^2)x_r - \kappa|y_r|/2 - x_r - |y_r| = (x_r^2+y_r^2-1)x_r - (\kappa/2+1)|y_r| \tag{63}$$

For fixed $y_r > 0$ we obtain from equation (55)

$$\frac{dy_r}{dt} \leq -(x_r^2+y_r^2)y_r + \kappa|x_r|/2 + |x_r| + y_r = -[(x_r^2+y_r^2-1)y_r - (\kappa/2+1)|x_r|] \tag{64}$$

Hence to prove (61) and (62) it is enough to show that the function

$$f(y) := y^2x - (1+\kappa/2)y + (x^2-1)x \tag{65}$$

where x, κ are fixed and positive, is positive for $y \geq 0$. Observe that f has a minimum value for $y_m := \frac{1+\kappa/2}{2x}$. We have

$$f(y) \geq f(y_m) = -\frac{(1+\kappa/2)^2}{4x} + (x^2-1)x \tag{66}$$

Hence we have an inequality

$$f(y) > 0, \quad \text{for} \quad (x^2-1)x^2 > (1/2+\kappa/4)^2 \tag{67}$$

□

5.2 Segment (U, U^-)

We set

$$\omega = \frac{R-r}{\Delta} \tag{68}$$

Let $s : \mathbb{R} \to \mathbb{R}$ be a T-periodic function such that

$$\begin{aligned} s(t) &= R - \omega t, && \text{for } t \in [0, \Delta] \\ s(t) &= r, && \text{for } t \in [\Delta, T-\Delta] \\ s(t) &= R - \omega(T-t), && \text{for } t \in [T-\Delta, T] \end{aligned} \tag{69}$$

We define

$$U = \{(t,x,y) |\, (x,y) \in [-s(t), s(t)]^2\} \tag{70}$$

$$U^- = \{(t,x,y) \in U |\, |x| = s(t)\} \tag{71}$$

$$U^+ = \{(t,x,y) \in U |\, |y| = s(t)\} \tag{72}$$

Lemma 14 *For $(x,y) \in (-\frac{1}{2^{3/4}}, \frac{1}{2^{3/4}})^2$ the following inequalities hold*

$$\frac{dx}{dt} > 0, \quad \text{when } x > 0,\ |y| \leq |x| \tag{73}$$

$$\frac{dx}{dt} < 0, \quad \text{when } x < 0,\ |y| \leq |x| \tag{74}$$

$$\frac{dy}{dt} < 0, \quad \text{when } y > 0,\ |x| \leq |y| \tag{75}$$

$$\frac{dy}{dt} > 0, \quad \text{when } y < 0,\ |x| \leq |y| \tag{76}$$

Proof From the symmetry of equation (2) it follows that it is enough to prove only (73) and (75).

To show inequality (73) we fix $x > 0$. Assume that $\cos(\kappa t) \geq 0$ then for $y \in [-x, x]$ we have

$$\begin{aligned} \frac{dx}{dt} = x + (x^2+y^2)(x\cos(\kappa t) + y\sin(\kappa t)) \geq \\ x + x^3\cos(\kappa t) - 2x^3|\sin(\kappa t)| \geq \\ x + x^3 \min_{\alpha \in [-\pi/2, \pi/2]} (\cos(\alpha) - 2|\sin(\alpha)|) = x - 2x^3 > x - 2^{3/2}x^3 \end{aligned}$$

When $\cos(\kappa t) \leq 0$ we have

$$\begin{aligned} \frac{dx}{dt} = x + (x^2+y^2)(x\cos(\kappa t) + y\sin(\kappa t)) \geq \\ x + (2x^2)(x\cos(\kappa t) - x|\sin(\kappa t)|) \geq \\ x + x^3 2 \min_{\alpha \in [\pi/2, 3\pi/2]} (\cos(\alpha) - |\sin(\alpha)|) = x - 2^{3/2}x^3 \end{aligned}$$

To obtain inequality (75) we fix $y > 0$. Assume first that $\cos(\kappa t) \geq 0$. We have

$$\begin{aligned} \frac{dy}{dt} = -y + (x^2+y^2)(x\sin(\kappa t) - y\cos(\kappa t)) \leq \\ -y + 2y^3|\sin(\kappa t)| - y^3\cos(\kappa t) = \\ = -(y + y^3(\cos(\kappa t) - 2|\sin(\kappa t)|) \leq -(y - 2y^3) < -(y - 2^{3/2}y^3) \end{aligned}$$

For $\cos(\kappa t) \leq 0$ we have

$$\frac{dy}{dt} = -y + (x^2 + y^2)(x \sin(\kappa t) - y \cos(\kappa t)) \leq$$
$$-y + (2y^2)(y|\sin(\kappa t)| - y\cos(\kappa t)) =$$
$$= -(y + 2y^3(\cos(\kappa t) - |\sin(\kappa t)|) \leq -(y - 2^{3/2}y^3)$$

To complete the proof observe that $x - 2^{3/2}x^3$ is positive when $0 < x < \frac{1}{2^{3/4}}$.

□

Observe that the above lemma guarantees that if $r < \frac{1}{2^{3/4}}$ then the boundary of U_t for $t \in [\Delta, T - \Delta]$ is built from local sections of the flow and the Lyapunov functions for those sections are just coordinate functions.

Let us define

$$\theta = \arcsin(\frac{1}{\sqrt{5}}) \tag{77}$$

Lemma 15

$$\theta < \pi/2 \tag{78}$$
$$\cos(\alpha) - 2\sin(\alpha) > 0, \quad \textit{for } \alpha \in [0, \theta) \tag{79}$$
$$\cos(\theta) - 2\sin(\theta) = 0 \tag{80}$$

Lemma 16 *Assume $\kappa\Delta \leq \theta$ and $\omega < r$. Then*

$$\frac{d(x - s(t))}{dt} > 0, \quad \textit{for } (t, x, y) \in U^-,\ x > 0,\ t \in [0, \Delta] \cup [T - \Delta, T] \tag{81}$$
$$\frac{d(x + s(t))}{dt} < 0, \quad \textit{for } (t, x, y) \in U^-,\ x < 0,\ t \in [0, \Delta] \cup [T - \Delta, T] \tag{82}$$
$$\frac{d(y - s(t))}{dt} < 0, \quad \textit{for } (t, x, y) \in U^+,\ y > 0,\ t \in [0, \Delta] \cup [T - \Delta, T] \tag{83}$$
$$\frac{d(y + s(t))}{dt} > 0, \quad \textit{for } (t, x, y) \in U^+,\ y < 0,\ t \in [0, \Delta] \cup [T - \Delta, T] \tag{84}$$

Proof We will show the lemma for $t \in [0, \Delta]$. The proof for $t \in [T - \Delta, T]$ is analogous and leads to exactly the same conditions on Δ, ω, R, r.

From the symmetry of equation (2) it follows that it is enough to prove (81) and (83).

For fixed $x > 0$ and $y \in [-x, x]$ we have

$$\frac{d(x - s(t))}{dt} = x + (x^2 + y^2)(x\cos(\kappa t) + y \sin(\kappa t)) + \omega \geq$$
$$\omega + x + x^3\cos(\kappa t) - 2x^3 \sin(\kappa t) = \omega + x + x^3(\cos(\kappa t) - 2\sin(\kappa t)) \geq$$
$$\omega + x > 0$$

This proves (81).

For fixed $y > 0$ and $x \in [-y, y]$ we have

$$\frac{d(y - s(t))}{dt} = -y + (x^2 + y^2)(x \sin(\kappa t) - y\cos(\kappa t)) + \omega \leq$$
$$\omega - y + 2y^3 \sin(\kappa t) - y^3 \cos(\kappa t) = \omega - y - y^3(\cos(\kappa t) - 2\sin(\kappa t)) \leq$$
$$\omega - y \leq \omega - r < 0$$

This proves (83).

□

Hence from the above lemma it follows that $t \in [0, \Delta] \cup [T - \Delta, T]$ boundaries of U_t are sections.

So we have proved the following

Lemma 17 *Assume that*

$$r < \tfrac{1}{2^{3/4}} \tag{85}$$

$$\omega = \tfrac{R-r}{\Delta} < r \tag{86}$$

$$\kappa\Delta < \arcsin(\tfrac{1}{\sqrt{5}}) \tag{87}$$

then (U, U^-) *is an isolating segment for* ϕ.

Lemma 18 *Assume that*

$$\Delta\kappa(\sin(\kappa\Delta/2) + \cos(\kappa\Delta/2)) < 2(1 - \tfrac{r}{R}) \tag{88}$$

$$r\sqrt{2} < R \tag{89}$$

then $U \subset W$

Proof

Is is easy to see that (89) guarantees that $U_t \subset W_t$ for $t \in [\Delta, T - \Delta]$.

We will show that for $t \in (0, \Delta] \cup [T - \Delta, T)$ we have

$$x_m(t) = \min_{z \in W_t^-} |x(z)| > s(t) \tag{90}$$

$$y_m(t) = \min_{z \in W_t^+} |y(z)| > s(t) \tag{91}$$

Observe that those inequalities imply that

$$U_t = [-s(t), s(t)]^2 \subset [-x_m(t), x_m(t)] \times [-y_m(t), y_m(t)] \subset W_t, \quad \text{for } t \in [0, \Delta) \cup (T - \Delta, T]$$

Observe that $y_m(t) = x_m(t)$, hence it is enough to show (90) only. We will show it for $t \in (0, \Delta]$, the proof for $t \in [T - \Delta, T)$ is analogous.

Since

$$x_m(t) = R(\cos(\kappa t/2) - \sin(\kappa t/2)) \tag{92}$$

We have to prove the following inequality

$$R(\cos(\kappa t/2) - \sin(\kappa t/2)) > R - \omega t, \quad \text{for } t \in (0, \Delta] \tag{93}$$

For $t = 0$ we have equality in (93), hence it is enough to show that the derivative of the left-hand side is greater than derivative of the right hand side. So we need to show that

$$R\frac{\kappa}{2}(-\sin(\kappa t/2) - \cos(\kappa t/2)) > -\omega \tag{94}$$

Above inequality is equivalent to the following one

$$\frac{\kappa}{2}(\sin(\kappa t/2) + \cos(\kappa t/2)) < \frac{\omega}{R} = \frac{1}{\Delta}(1 - \frac{r}{R}) \tag{95}$$

which can be rewritten as

$$\Delta\kappa(\sin(\kappa t/2) + \cos(\kappa t/2)) < 2(1 - \frac{r}{R}) \tag{96}$$

This inequality is implied by (88) for $t \in (0, \Delta]$. □

5.3 Solution of inequalities required for $U \subset W$

It follows from lemmas 13, 17 and 18 that to show that $U \subset W$ and that (U, U^-), (W, W^-) are isolating segments, we have to solve the following set of inequalities

$$(R^2-1)R^2 > (1/2+\kappa/4)^2 \tag{97}$$
$$r < \tfrac{1}{2^{3/4}} \tag{98}$$
$$\omega = \tfrac{R-r}{\Delta} < r \tag{99}$$
$$\kappa\Delta < \arcsin(\tfrac{1}{\sqrt{5}}) \tag{100}$$
$$\Delta\kappa(\sin(\kappa\Delta/2)+\cos(\kappa\Delta/2)) < 2(1-\tfrac{r}{R}) \tag{101}$$
$$r\sqrt{2} < R \tag{102}$$

It is easy to check that (97) is fulfilled for

$$R \geq 1.15, \quad \kappa \leq 1/2 \tag{103}$$

It is easy to check that $\frac{1}{2^{3/4}} > 0.5946$. We set

$$r = 0.5946, \quad R = 1.15 \tag{104}$$

Observe that (102) holds for these values of r and R. Hence the set of inequalities (97–102) is reduced to the following one

$$\kappa \leq 1/2 \tag{105}$$
$$\Delta > \tfrac{R-r}{r} < 0.935 \tag{106}$$
$$\kappa\Delta < \arcsin(\tfrac{1}{\sqrt{5}}) > 0.463 \tag{107}$$
$$\Delta\kappa(\sin(\kappa\Delta/2)+\cos(\kappa\Delta/2)) < 2(1-\tfrac{r}{R}) \tag{108}$$

We set $\Delta = 0.935$. Then from (107) we obtain

$$\kappa \leq \frac{0.463}{0.935} > 0.495 \tag{109}$$

We set

$$\kappa \leq 0.495 \tag{110}$$

It remains to show that (108) is satisfied. We have

$$\Delta\kappa(\sin(\kappa\Delta/2)+\cos(\kappa\Delta/2)) \leq 0.935\cdot 0.495\cdot\sqrt{2} < 0.7 < 2\cdot 0.482 < 2(1-\frac{r}{R}) \tag{111}$$

Observe that (vi) holds for φ and segments (W, W^-), (U, U^-) is fulfilled, because the exit sets W^- and U^- are given as level sets for functions locally increasing along the process φ. Hence we have the following

Lemma 19 *Assume $\kappa \in (0, 0.495]$, then for $R = 1.15$, $r = 0.5946$, $\Delta = 0.935$ $U \subset W$ and the pairs (U, U^-) and (W, W^-) are periodic isolating segments for φ.*

6 Model map

In this section we construct a model semiprocess φ^M, a suitable homotopy connecting semiprocesses φ and φ^M and finally we prove theorem 8.

Let κ, r, R and Δ are as in lemma 19. Let $s(t)$, U and W be as in section 5.

Let U^{+1}, U^{-1} two connected components of U^-, right ($x > 0$) and left ($x < 0$) respectively.

The proof of theorem 8 is based on the following

Theorem 20 *There exists a semiprocess φ^M such that there are disjoint closed segments $J_{-1} = [-b, -a]$, $J_0 = [-c, c]$, $J_1 = [a, b]$, $J_l \subset (-R, R)$ for $l = -1, 0, 1$ and continuous function $f : J_{-1} \cup J_0 \cup J_{+1} \to [-R, R]$ such that*

$$Z := \{p \in W_0 \,|\, \varphi^M(0, t, p) \in W,\ t \in [0, T]\} = \{J_{-1} \cup J_0 \cup J_1\} \times [-R, R] \tag{112}$$

$$Z_0 := \{p \in W_0 \,|\, \varphi^M(0, t, p) \in U,\ t \in [0, T]\} = J_0 \times [-R, R] \tag{113}$$

$$Z_l := \{p \in Z \,|\, p \text{ leaves } U \text{ through } U^l \text{ in time} \le T\} = J_l \times [-R, R], \quad \text{for } l = -1, 1 \tag{114}$$

$$\varphi^M(0, T, (x, y)) = (f(x), 0), \quad \text{where } (x, y) \in Z \tag{115}$$

$$f(-x) = -f(x), \quad f(c) = R, \quad f(a) = R, \quad f(b) = -R. \tag{116}$$

The pairs (U, U^-) and (W, W^-) are periodic isolating segments for φ^M.

Consider the family of T–periodic semiprocesses for $\lambda \in [0, 1]$

$$H_\lambda = \Omega(\{(\varphi^M, 0), (\varphi, \lambda T)\}, T).$$

Then for every $\lambda \in [0, 1]$ H_λ is a semiprocess, (U, U^-) and (W, W^-) are periodic isolating segments and condition (vi) holds uniformly (see theorem 10 for precise meaning of this statement).

Proof:

We will construct our model semiprocess from a few pieces, i.e. as concatenation defined in section 2.2. We will construct it using the coordinates with a subscript M, i.e. a point (t, x, y) in the extended phase-space will be represented by $(t, x_M(t, x, y), y_M(t, x, y))$. We will specify the precise form of this change of coordinate later, as it is a part of the construction, but we assume that x_M, y_M are T-periodic in t and $x_M(0, x, y) = x$, $y_M(0, x, y) = y$. This means that at time $t = kT$, for $k \in \mathbb{Z}$ coordinates (x_M, y_M) are equal to standard coordinates (x, y) in which φ , (W, W^-) and (U, U^-) are expressed.

Let $\epsilon > 0$ and $0 < t_1 = \Delta < t_2 < t_3 < t_4 = T - \Delta < T = 2\pi/\kappa$ be such that

$$R - \epsilon > \sqrt{2}r + \epsilon \tag{117}$$

$$t_1 > \frac{R - r}{r} \tag{118}$$

$$t_4 - t_3 > \ln \frac{R}{\sqrt{2}r} \tag{119}$$

$$T - t_4 > \frac{R - r}{r} \tag{120}$$

We will describe these pieces which build one period as follows:

- $\mathcal{D}$ Decreasing of segment U, for time in $[0, t_1]$
- $\mathcal{C}$ A collapse of trajectories , for time in $[t_1, t_2]$
- $\mathcal{O}$ Rotation of segment W, for time in $[t_2, t_3]$
- $\mathcal{C}$ A collapse of trajectories , for time in $[t_3, t_4]$
- $\mathcal{D}$ Increasing of segment U, for time in $[t_4, T]$

This means that our model semiprocess will be given by

$$\varphi^M = \Omega(\{(\mathcal{D},0),(\mathcal{C},t_1),(\mathcal{O},t_2),(\mathcal{C},t_3),(\mathcal{D},t_4)\},T)$$

During this construction we preserve the symmetry $(x,y)\mapsto(-x,-y)$ which is also present in equation (12).

Let us define the sets

$$Z(t) = \{p\,|\,\varphi^M(0,s,p)\in W \quad \text{for } s\le t\} \tag{121}$$
$$Z_0(t) = \{p\,|\,\varphi^M(0,s,p)\in U \quad \text{for } s\le t\} \tag{122}$$
$$Z_l(t) = \{p\,|\,x\in Z(t), \quad p \text{ leaves } U \text{ through } U_l \text{ in time } \le t\} \tag{123}$$

It is easy to see that $Z(T)=Z$, $Z_0(T)=Z_0$ and $Z_l(T)=Z_l$ for $l=-1,1$. Let us remark that

$$Z_0(t)\subset Z_0(\tau), \quad \text{for } t\ge\tau \tag{124}$$

but analogous inclusion does not hold for sets Z_1. In fact this happens for the model semiprocess constructed below. We have

$$Z_1(t)\cap Z_1(\tau)=\emptyset, \qquad \text{when } t \text{ is small and } \tau \text{ is close to } T$$

We will present now details of this construction

1. **Decreasing of** U . For $0\le t\le t_1$ we define semiprocess $\mathcal{D}$ as the one induced by equations

$$x'_M = x_M, \quad y'_M = -y_M \tag{125}$$

So

$$\mathcal{D}(\sigma,t,(x_M,y_M)) = (e^t x_M, e^{-t}y_M) \tag{126}$$

We set

$$W_t=[-R,R]^2, \quad W_t^- = \{-R,R\}\times[-R,R]$$
$$U_t=[-s(t),s(t)]^2, \quad U_t^- = \{-s(t),s(t)\}\times[-s(t),s(t)]$$

It is easy to check that from (118) it follows that (U,U^-) is an isolating segment for $t\in[0,t_1]$

We obtain

$$Z(t_1) = [-e^{-t_1}R, e^{-t_1}R]\times[-R,R] \tag{127}$$
$$\varphi^M(0,t_1,Z(t_1)) = [-R,R]\times[-e^{-t_1}R,e^{-t_1}R] \tag{128}$$
$$Z_0(t_1) = [-e^{-t_1}r, e^{-t_1}r]\times[-R,R] \tag{129}$$
$$Z_{+1}(t_1) = [e^{-t_1}r, e^{-t_1}R]\times[-R,R] \tag{130}$$

At time t_1 our segments are

$$W_{t_1}=[-R,R]^2, \quad W_{t_1}^- = \{-R,R\}\times[-R,R]$$
$$U_{t_1}=[-r,r]^2, \quad U_{t_1}^- = \{-r,r\}\times[-r,r]$$

2. **Collapse** For $t_1\le t\le t_2$ we set $W_t=[-R,R]^2$, $U_t=[-r,r]^2$. Let $L>\frac{2\sqrt{R}}{t_2-t_1}$. The semiprocess $\mathcal{C}$ is induced by the following equations

$$x'_M = x_M, \quad y'_M = -L\,\mathrm{sgn}\,(y_M)\sqrt{|y_M|} \tag{131}$$

Observe that the above system of equations does not have a uniqueness property. We have the nonuniqueness when $y_M = 0$, but we can still define a continuous semiprocess (in fact a semidynamical system) as follows for $t \geq 0$ and $\sigma, \sigma + t \in [t_1, t_2]$

$$\mathcal{C}(\sigma, t, (x_M, y_M)) =$$
$$\begin{cases} (e^t x_M, \operatorname{sgn}(y_M)(-Lt/2 + \sqrt{|y_M|})^2), & t \leq 2\sqrt{|y_M|}/L \\ (e^t x_M, 0), & t \geq 2\sqrt{|y_M|}/L \end{cases}$$

From the assumption about L it follows that

$$Z(t_2) = [-e^{-t_2}R, e^{-t_2}R] \times [-R, R] \tag{132}$$
$$\varphi^M(0, t_2, (x_M, y_M)) = (e^{t_2}x_M, 0), \quad \text{for } (x_M, y_M) \in Z(t_2) \tag{133}$$
$$\varphi^M(0, t_2, Z(t_2)) = [-R, R] \times \{0\} \tag{134}$$
$$Z_0(t_2) = [-e^{-t_2}r, e^{-t_2}r] \times [-R, R] \tag{135}$$
$$Z_{+1}(t_2) = [e^{-t_2}r, e^{-t_2}R] \times [-R, R] \tag{136}$$

Observe that the image of $Z(t_2)$ after time t_2 is one dimensional.

3. Rotation of W . Here time is in $[t_2, t_3]$.

Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be given by

$$h(s) = \begin{cases} 0, & \text{for } s \leq \sqrt{2}r + \epsilon \\ \pi, & \text{for } s \geq R - \epsilon \\ \frac{\pi}{2} + \arcsin\left(\frac{2(s - \sqrt{2}r - \epsilon)}{(R-\epsilon)-(\sqrt{2}r+\epsilon)} - 1\right), & \text{otherwise} \end{cases}$$

Observe that h is strictly increasing on $[\sqrt{2}r + \epsilon, R - \epsilon]$.

Let $\mu : \mathbb{R} \to \mathbb{R}$ be a T–periodic continuous function such that the following conditions hold

$$\mu(t) = 0, \quad \text{for } 0 \leq t \leq t_2$$
$$\mu(t) = \frac{t - t_2}{t_3 - t_2}, \quad \text{for } t_2 \leq t \leq t_3$$

We define

$$\omega(t, x_M, y_M) = \mu(t) h(\sqrt{x_M^2 + y_M^2}) \tag{137}$$

Consider the following t dependent change of coordinates

$$C_t(x_M, y_M) = \begin{bmatrix} \cos(\omega(t, x_M, y_M))x_M - \sin(\omega(t, x_M, y_M))y_M \\ \sin(\omega(t, x_M, y_M))x_M + \cos(\omega(t, x_M, y_M))y_M \end{bmatrix} \tag{138}$$

Observe that $C_{t_2} = \mathrm{id}$ and $C_t^{-1}(x_M)$ depend continuously on t and x_M.

It is clear from the definition that C_t is equal to the identity map on $[-r - \epsilon, r + \epsilon]^2$ and it is linear rotation on and outside a circle of the radius $R - \epsilon$.

Let β be a dynamical system induced by equations

$$x'_M = x_M, \quad y'_M = -y_M. \tag{139}$$

We set (see subsection 2.1 for definition of Π)

$$\mathcal{O} = \Pi(C, t_2, \beta) \tag{140}$$

We have

$$\mathcal{O}(t_2, t, (x_M, y_M)) = \begin{bmatrix} \cos(\omega(t, e^t x_M, e^{-t} y_M))e^t x_M - \sin(\omega(t, e^t x_M, e^{-t} y_M))e^{-t} y_M \\ \sin(\omega(t, e^t x_M, e^{-t} y_M))e^t x_M + \cos(\omega(t, e^t x_M, e^{-t} y_M))e^{-t} y_M \end{bmatrix}$$

From the above description it follows that W and U defined below are isolating segments.

$$W_t = C_t([-R, R]^2), \quad W_t^- = C_t(\{-R, R\} \times [-R, R])$$
$$U_t = [-r, r]^2, \quad U_t^- = \{-r, r\} \times [-r, r]$$

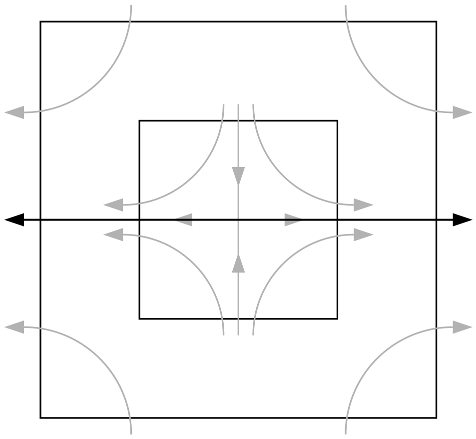

Figure 1: Angle 0

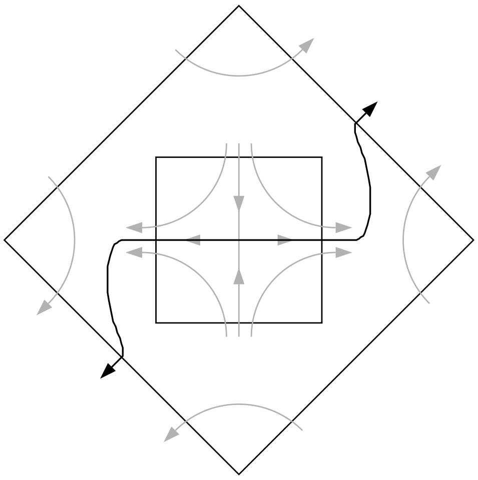

Figure 2: Angle $\pi/4$

Figures 1–5 show what happens during this evolution for times corresponding to the rotation of the big segment by multiples of $\pi/4$ (this the rotating rectangle on pictures). The fixed small square represents a set $[-r,r]^2$ which is equal to U_t for $t \in [t_1,t_4]$. By the solid line we marked the image of $Z(t_2)$, by gray lines and arrows we indicated the flow lines of the vector field at given time (these are not trajectories of our process!!).

It is easy to see that

$$Z(t_3) = [-e^{-t_3}R, e^{-t_3}R] \times [-R,R] \tag{141}$$

$$\varphi^M(0,t_3,(x_M,y_M)) =$$
$$(\cos(h(e^{t_3}|x_M|))e^{t_3}x_M, \sin(h(e^{t_3}|x_M|))e^{t_3}x_M),\ (x_M,y_M) \in Z(t_2) \tag{142}$$

$$Z_0(t_3) = [-e^{-t_3}r, e^{-t_3}r] \times [-R,R] \tag{143}$$

$$\varphi^M(0,t_3,(x_M,y_M)) = (e^{t_3}x_M, 0), \quad (x_M,y_M) \in Z_0(t_3) \tag{144}$$

$$Z_{+1}(t_3) = [e^{-t_3}r, e^{-t_3}R] \times [-R,R] \tag{145}$$

For the next stages of the construction we need the following

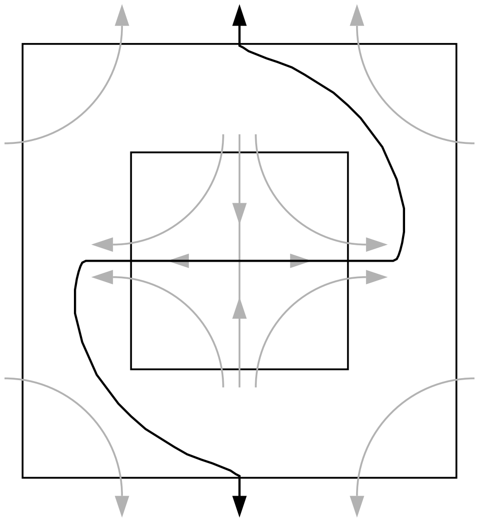

Figure 3: Angle $\pi/2$

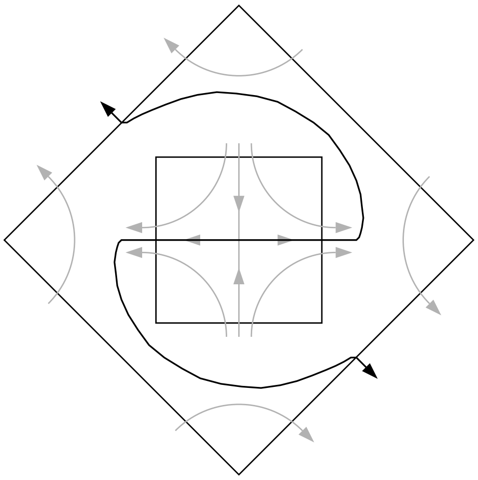

Figure 4: Angle $3\pi/4$

Lemma 21 *There exist unique positive numbers a_4,b_4 such that*

$$e^{-t_3}r < a_4 < b_4 < e^{-t_3}R \qquad (146)$$
$$\varphi^M(0,t_3,(a_4,y_M))_{x_M} = e^{-(t_4-t_3)}R, \quad \textit{for } y_M \in [-R,R] \qquad (147)$$
$$\varphi^M(0,t_3,(b_4,y_M))_{x_M} = -e^{-(t_4-t_3)}R, \quad \textit{for } y_M \in [-R,R] \qquad (148)$$
$$\textit{a function } \varphi^M(0,t_3,(x_M,0))_{x_M} \textit{ is strictly decreasing on } [a_4,b_4] \qquad (149)$$

Proof of lemma 21

Let $g(x)$ equals to the x_M–coordinate function of $\varphi^M(0,t_3,(x_M,y_M))$, (see also fig. 5).

We have

$$g(|x|) = e^{t_3}|x|, \quad \text{for } |x| \le e^{-t_3}r\sqrt{2} \qquad (150)$$
$$g(|x|) = -e^{t_3}|x|, \quad \text{for } |x| \ge e^{-t_3}R \qquad (151)$$

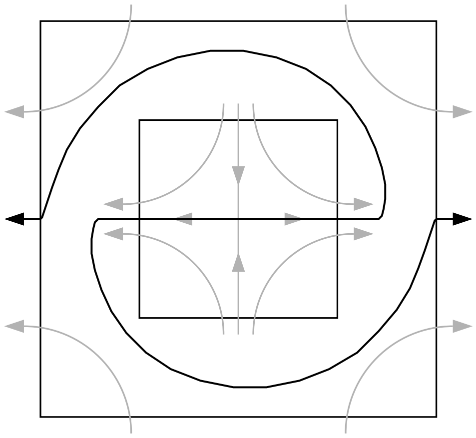

Figure 5: Angle π

It follows from (118) that

$$e^{t_3}r > e^{t_1}r > (1+t_1)r > (1+\frac{R-r}{r})r = R$$

Let us denote by $g_m = \max\{g(x), x \geq 0\}$, and $x_m = \sup\{x \geq 0, g(x) = g_m\}$. From (150) it follows that

$$g_m \geq \sqrt{2}r + \epsilon, \quad x_m \geq \sqrt{2}r + \epsilon \tag{152}$$

From (119) it follows that

$$[-e^{-(t_4-t_3)}R, e^{-(t_4-t_3)}R] \subset [-R, \sqrt{2}r] \subset [g(e^{-t_3}R), g(x_m)] \tag{153}$$

To finish the proof of (146–149) it is enough to show that g is strictly decreasing on $[x_m, e^{-t_3}R]$.

Observe that function $g_1(s) = \cos(h(s))s$ has the following properties

$$g_1(s) = s, \qquad \text{for } s \leq \sqrt{2}r + \epsilon \tag{154}$$

$$g_1(s) = -(\frac{2(s-\sqrt{2}r-\epsilon)}{(R-\epsilon)-(\sqrt{2}r+\epsilon)} - 1)s, \quad \text{for } \sqrt{2}r+\epsilon \leq s \leq R-\epsilon \tag{155}$$

$$g_1(s) = -s, \qquad \text{for } s \geq R-\epsilon \tag{156}$$

The function g_1 has a global maximum for some point $s_m \in (\sqrt{2}r+\epsilon, R-\epsilon)$ and for $s > s_m$ $g_1(s)$ is strictly decreasing. Since $g(x) = g_1(e^{t_3}x)$, the function g itself will have the same properties, i.e. it will be strictly decreasing for $x > x_m$.

□

Obviously we have

$$[a_4, b_4] \times [-R, R] \subset Z_{+1}(t_3) \tag{157}$$

$$([a_4, b_4] \times [-R, R]) \cap Z_0(t_3) = \emptyset \tag{158}$$

We have also

$$W_{t_3} = [-R, R]^2, \quad W^-_{t_3} = \{-R, R\} \times [-R, R]$$
$$U_{t_3} = [-r, r]^2, \quad U^-_{t_3} = \{-r, r\} \times [-r, r]$$

4. Collapse For $t_3 \leq t \leq t_4$ we define φ^M by the same equation as in point 2 of the construction. We assume that $L > \frac{2\sqrt{R}}{t_4 - t_3}$. The segments U, W are given by

$$
\begin{aligned}
W_t = [-R,R]^2, \quad W_t^- &= \{-R,R\} \times [-R,R] \\
U_t = [-r,r]^2, \quad U_t^- &= \{-r,r\} \times [-r,r]
\end{aligned}
$$

It is easy to see that after this stage

- all points $p \in W_0$ such that $|x_M(\varphi^M(0,t_3,p))| > e^{-(t_4-t_3)}R$ will leave W.
- either the image of a point from W_0 after t_4 lies on the line $y_M = 0$ or this point leaves W before time t_4

From this it follows immediately that there exists a continuous strictly decreasing function $f_1 : [a_4, b_4] \to \mathbb{R}$ such that

$$
\begin{aligned}
Z_0(t_4) &= [-e^{-t_4}r, e^{t_4}r] \times [-R,R] && (159)\\
\varphi^M(0,t_4,(x_M,y_M)) &= (e^{t_4}x_M, 0), \quad \text{for } (x_M,y_M) \in Z_0(t_4) && (160)\\
Z_{+1}(t_4) &= [a_4,b_4] \times [-R,R] && (161)\\
\varphi^M(0,t_4,(x_M,y_M)) &= (f_1(x_M), 0), \quad \text{for } (x_M,y_M) \in Z_1(t_4) && (162)\\
f_1(a_4) &= R, \quad f_1(b_4) = -R && (163)
\end{aligned}
$$

5. Increasing of the small segment U φ^M is defined by the same equation as in point 1, but this time we let the segment U to increase.

We set for $t \in [t_4, T]$

$$
\begin{aligned}
W_t = [-R,R]^2, \quad W_t^- &= \{-R,R\} \times [-R,R] \\
U_t = [-s(t),s(t)]^2, \quad U_t^- &= \{-s(t),s(t)\} \times [-s(t),s(t)]
\end{aligned}
$$

It is easy to see that there exist $0 < c < a < b$

$$0 < c = e^{-T}R < e^{-t_4}r, \quad a_4 < a < b < b_4 \qquad (164)$$

such that

$$
\begin{aligned}
Z_0(T) &= [-c,c] \times [-R,R] && (165)\\
\varphi^M(0,T,(x_M,y_M)) &= (e^T x_M, 0), \quad \text{for } (x_M,y_M) \in Z_0(T) && (166)\\
Z_{+1}(T) &= [a,b] \times [-R,R] && (167)\\
\varphi^M(0,T,(x_M,y_M)) &= (e^{T-t_4} f_1(x_M), 0), \quad \text{for } (x_M,y_M) \in Z_{+1}(T) && (168)\\
\varphi^M(0,T,(a,y_M)) &= (R,0), \quad \varphi^M(0,T,(b,y_M)) = (-R,0) && (169)
\end{aligned}
$$

To finish the construction we define $f : [-b,-a] \cup [-c,c] \cup [a,b] \to [-R,R]$ by

$$f(x) = \begin{cases} e^T x, & \text{for } x \in [-c,c] \\ e^{T-t_4} f_1(x), & \text{otherwise} \end{cases} \qquad (170)$$

At this moment we obtained the semiprocess φ^M for which the map $\varphi^M_{(0,T)}$ has the desired properties, but we do it in a different coordinate system (t, x_M, y_M). Observe that in those coordinates a pair $(U, U^-)_M$ (constructed here using (x_M, y_M) coordinates) coincides with a pair (U, U^-) defined in section 5, but this is not the case for $(W, W^-)_M$. Observe that for every t a pair (W_t, W_t^-) can be obtained from $(W_t, W_t^-)_M$ by a rotation . The rotation angle is given by a T-periodic function $\theta(t)$

$$\theta(t) = -\frac{\kappa}{2}t, \quad \text{for } t \in [0, t_2]$$
$$\theta(t) = -\frac{\kappa}{2}t + \frac{\pi(t-t_2)}{t_3 - t_2}, \quad \text{for } t \in [0, t_2]$$
$$\theta(t) = -\frac{\kappa}{2}t + \pi, \quad \text{for } t \in [T - t_3, T]$$

Hence our coordinate change from (t, x, y) to (t, x_M, y_M) shall be an identity on U and a rotation on W. From this it follows easily that all conditions from the definition of an isolating periodic segment , but (vi) , are valid for φ^M, (W, W^-) and (U, U^-) disregarding the exact form of the coordinate change function. Condition (vi) will be verified during the proof of the uniform condition (vi) for H_λ, where

$$H_\lambda = \Omega(\{(\varphi^M, 0), (\varphi, \lambda T)\}, T).$$

We recall first , that according to the definition of Ω presented in the subsection 2.2 this homotopy works in the extended phase space as follows: starting at time $\sigma = 0$ we move forward according to φ^M up to time λT . At time λT we start moving according to φ up to time T and this is repeated periodically. Observe that $H_0 = \varphi$, $H_1 = \varphi^M$ and H_λ is a semiprocess for every λ. Similarly as for φ^M it is easy to observe that all conditions of the definition of the periodic isolating segment, but (vi) are valid for H_λ and (W, W^-) and (U, U^-), independent of the exact form of the coordinate change.

Hence it remains to specify the coordinate change and to prove that condition (vi) holds uniformly for H_λ, (W, W^-) and (U, U^-).

We summarize properties of this coordinate change function in the following

Lemma 22 *There exists a homeomorphism $A : \mathbb{R}^3 \to \mathbb{R}^3$, $A(t, x, y) = (t, A_t(x, y))$ such that*

1. $A_{t+T} = A_t$

2. $A_0 = Id$

3. *circles $x^2 + y^2 = const$ are invariant under A_t*

4. *A_t is a rotation by angle $\theta(t)$ on the set $\{(x_M, y_M) \,|\, |x_M| \geq R \quad or \quad |y_M| \geq R\}$*

4'. *A_{-t} is a rotation by angle $-\theta(t)$ for (x, y) such that $(t, x, y) \notin \operatorname{int} W$*

5. *for any $\delta > 0$ A is an identity map in some open neighborhood of $U \cap ([\delta, T - \delta] + kT) \times \mathbb{R}^2$, where $k \in \mathbb{Z}$*

6. *there exists α such that for any $t \in [-\alpha, \alpha]$ if $R \geq |x| \geq s(t)$ and $|y| \leq s(t)$ then absolute values of the x-coordinate of $A_t(x, y)$ and ${A_t}^{-1}(x, y)$ is greater than $s(t)$*

We prove this lemma after completing the proof of theorem 20.

Observe that in rotating coordinates (x_r, y_r) (we have to compose the coordinate change A_t and the rotation (52)) the model semiprocess φ^M is described by the following T-periodic system of equations for $(t, x, y) \notin \operatorname{int} W$

$$x_r' = x_r \tag{171}$$

$$y_r' = \begin{cases} -y_r, & t \notin [t_1, t_2] \cup [t_3, t_4] \\ -L\operatorname{sgn}(y_r)\sqrt{|y_r|}, & t \in [t_1, t_2] \cup [t_3, t_4] \end{cases} \tag{172}$$

Equation (171) and lemma 13 show that for both semiprocesses φ^M and φ the function $|x_r|$ is increasing, when $|x_r| \geq R$. This shows that if $(\sigma, z) \in W^-$ then $|x_r(H_\lambda(\sigma,t,z))|$ is increasing for $t \geq 0$, in fact it tends to infinity. This proves that (vi) holds uniformly for H_λ and the segment (W, W^-).

It remains to show that (vi) holds uniformly for H_λ and the segment (U, U^-). Let us observe that from lemma 16 it follows that the function $|x(\varphi(\sigma,t,x_0,y_0))| - s(\sigma+t))$ is strictly increasing in some neighborhood of U^-.

Also the function $(|x_M(\varphi^M(\sigma,t,x_0,y_0))| - s(\sigma+t))$ is strictly increasing in some neighborhood of U^-. This fact can be proved as follows. Observe that from the construction of φ_M we have $x'_M = 1$ (see proof of theorem 20). From (68), (69), (99) and lemma 19 it follows that $|s'| = \omega < r = 0.5946$ for $t \in [-t_1, t_1] + kT$, $k \in \mathbb{Z}$ and $s' = 0$ otherwise.

Form the point **5** in lemma 22 it follows that for any $\delta > 0$ there exists a neighborhood of $U^- \cap ([\delta, T-\delta] \times \mathbb{R}^2)$ such that $|x_M| - s(t) = |x| - s(t)$. From this observation it follows easily that (vi) holds uniformly for H_λ and $U \cap (([\delta, T-\delta] + kT) \times \mathbb{R}^2)$, where $k \in \mathbb{Z}$, with $\eta = \eta(\delta)$ being a number such that

$$A_t = \mathrm{Id}\,, \quad \text{on } U_t + [-\eta(\delta), \eta(\delta)]^2 \tag{173}$$

It remains to show that (vi) holds uniformly when for $U \cap ([-\delta, \delta] \times \mathbb{R}^2)$, for some small $\delta > 0$.

Let us take $\delta < \alpha \leq T/2$, where α is obtained in the point **6** of lemma 22. Observe that the functions $|x| - s(t)$ and $|x_M| - s(t)$ do not coincide on any neighborhood of $U \cap (([-\delta, \delta] + kT) \times \mathbb{R}^2)$, but by taking sufficiently small δ we can assume that the following conditions are satisfied for the semiprocess φ , $|t + \sigma| \leq \delta$ and some $\gamma > 0$

$$\frac{d(|x(\varphi(\sigma,t,x_0,y_0))| - s(\sigma+t))}{dt} > 0, \quad s(\sigma) \leq |x_0| \leq R+\gamma,\ |y_0| \leq R+\gamma \tag{174}$$

$$\frac{d(|y(\varphi(\sigma,t,x_0,y_0))| - s(\sigma+t))}{dt} > 0, \quad s(\sigma) \leq |y_0| \leq R+\gamma,\ |x_0| \leq R+\gamma \tag{175}$$

$$\frac{d(|x_M(\varphi^M(\sigma,t,x_0,y_0))| - s(\sigma+t))}{dt} > 0, \quad s(\sigma) \leq |x_0| \leq R+\gamma,\ |y_0| \leq R+\gamma \tag{176}$$

$$\frac{d(|y_M(\varphi^M(\sigma,t,x_0,y_0))| - s(\sigma+t))}{dt} < 0, \quad s(\sigma) \leq |y_0| \leq R+\gamma,\ |x_0| \leq R+\gamma \tag{177}$$

By eventually decreasing δ we can also assume that

$$A_t([-R,R]^2) \subset [-R-\gamma, R+\gamma]^2, \quad \text{for } |t| \leq \delta \tag{178}$$

We will need the following

Lemma 23 *For every λ and for every $z = (\sigma, x_0, y_0) \in U^-$ such that $|\sigma| \leq \delta$ and*

$$H_\lambda(\sigma,t,x_0,y_0) \notin U_{t+\sigma}, \quad \textit{for } t > 0,\ \sigma + t \leq \delta \tag{179}$$

The proof of this lemma will be given after completing the current one.

Let us denote by Φ^λ a semidynamical system related to H_λ (see section 2), i.e.

$$\Phi^\lambda_t(\sigma,x,y) = (\sigma+t, H_\lambda(\sigma,t,x,y))$$

Form (179) it follows that

$$\Phi^\lambda_t(U^- \cap ([-\delta/2, \delta/2] \times \mathbb{R}^2)) \cap U = \emptyset, \quad \text{for } 0 < t \leq \delta/2 \text{ and } \lambda \in [0,1] \tag{180}$$

Form the compactness and continuity arguments it follows that there exists $\eta_1 > 0$ such that

$$\rho(\Phi^{\lambda}_{\delta/2}(U^- \cap ([-\delta/2, \delta/2] \times \mathbb{R}^2)), U) \geq \eta_1, \quad \text{for } \lambda \in [0,1] \tag{181}$$

From (173), (180), (181) it follows that the condition (vi) holds uniformly for $\eta = \min\{\epsilon, \eta_1, \eta(\delta/2)\}$, where ϵ was a positive real number used in the rotation part of the construction of φ^M.

□

Proof of lemma 23 Let us fix a point $(\sigma, x_0, y_0) \in U^-$, where $|\sigma| \leq \delta$.

Observe that for $\delta < T/2$ the interval $[-\delta, \delta]$ contains at most one number of the form $\lambda T + kT$, where $k \in \mathbb{Z}$.

We have the following cases

(i) - no switch $\sigma \geq 0$, $\delta \leq \lambda T$. $H_\lambda(\sigma, t, x_0, y_0) = \varphi^M(\sigma, t, x_0, y_0)$ for $t + \sigma \leq \delta$

(ii) - no switch $\sigma > \lambda T$, $H_\lambda(\sigma, t, x_0, y_0) = \varphi(\sigma, t, x_0, y_0)$, for $t + \sigma \leq \delta$

(iii) - one switch $\sigma \geq 0$, $\delta > \lambda T$. We move (σ, x_0, y_0) according to φ^M up to time $t + \sigma = \lambda T$ and then we make a switch to φ

(iv) - one switch $\sigma < 0$, $\delta < \lambda T$, $-T + \lambda T < -\delta$. First, we move according to φ and at time $t + \sigma = 0$ we switch to φ^M

(v) - two switches $\sigma < 0$, $\sigma < -T + \lambda T$. First, we move according to φ^M, then when $t + \sigma = -T + \lambda T$ we switch to φ and at time $\sigma + t = 0$ we switch back to φ^M

(vi) - two switches $\sigma < 0$, $\delta > \lambda T$. First, we move according to φ, then when $t + \sigma = 0$ we switch to φ^M and at time $\sigma + t = \lambda T$ we switch back to φ

We will deal with each case as follows:

(i) From equation (176) it follows that $H_\lambda(\sigma, t, x_0, y_0) \notin U_{\sigma+t}$ as long as $x_M(H_\lambda(\sigma, t, x_0, y_0)) \leq R$. If (σ, x_0, y_0) leaves W for $t \leq \delta - \sigma$ then it never return to W, as it was observed before.

(ii) From equation (174) it follows that $H_\lambda(\sigma, t, x_0, y_0) \notin U_{\sigma+t}$ as long as $x(H_\lambda(\sigma, t, x_0, y_0)) \leq R$. If (σ, x_0, y_0) leaves W for $t \leq \delta - \sigma$ then it never return to W, as it was observed before.

(iii) By reasoning like in case *(i)* we can assume that

$$H_\lambda(\sigma, t, x_0, y_0)) = \varphi^M(\sigma, t, x_0, y_0)) \in W_{t+\sigma} \setminus U_{t+\sigma}, \quad \text{for } 0 < t \leq \lambda T - \sigma \tag{182}$$

So we obtain easily from eq. (176)(177)

$$R \geq |x_M(H_\lambda(\sigma, \lambda T - \sigma, x_0, y_)))| > s(\lambda T) \tag{183}$$

$$|y_M(H_\lambda(\sigma, \lambda T - \sigma, x_0, y_0))| < s(\lambda T) \tag{184}$$

From the above conditions, point 6 of lemma 22 and (178) we obtain

$$R \geq |x(H_\lambda(\sigma, \lambda T - \sigma, x, y))| > s(\lambda T) \tag{185}$$

$$|y(H_\lambda(\sigma, \lambda T - \sigma, x, y))| \leq R + \gamma \tag{186}$$

Now from (174) it follows that $|x| - s(\sigma + t)$ is increasing for t such that $t + \sigma > \lambda T$ as long as we stay in W, hence we obtain (179).

(iv),(vi) By reasoning like in case *(i)* we can assume that up to the moment of the switch $t + \sigma \leq 0$ equation (179) holds. For $t = -\sigma$ we have

$$H_\lambda(\sigma, t, x_0, y_0)) = \varphi^M(\sigma, t, x_0, y_0)) \notin U_0 = W_0 \tag{187}$$

Hence (t, x_0, y_0) must leave W before the time of the first switch , which as it was observed in the case *(i)* gives us (179).

(v) By reasoning like in case *(iii)* we can assume that up to the time of the second switch (i.e. $\sigma + t \leq 0$) (179) holds. Hence for $t = -\sigma$ we have

$$H_\lambda(\sigma, -\sigma, x_0, y_0) \notin U_0 = W_0 \tag{188}$$

But from the considerations concerning segment (W, W^-) it follows that trajectory this never enter W, hence also U, again. This proves (179).

□

Proof of lemma 22

The inequality

$$s(t) < R(\cos(\kappa t/2) - \sin(\kappa t/2)) < R, \quad \text{for } t \in (0, \Delta] \tag{189}$$

was shown during the proof of lemma 18 (see eq. (93)).

Observe that for our choice of R, κ, ω (see lemma 19) there exists $\alpha > 0$ such that the following inequality holds for $0 < t < \alpha$

$$R(\cos(\kappa t/2) - \sin(\kappa t/2)) > \frac{s(t)(1 + \sin(\kappa t/2))}{\cos(\kappa t/2)} \tag{190}$$

This can be shown easily by observing that l.h.s. equals to r.h.s for $t = 0$. Observe that derivatives at $t = 0$ of left and right sides of inequality (190) are $-\kappa t/2$ and $-\omega + \kappa R/2$ respectively.

It is easy to check that

$$-\frac{\kappa R}{2} > -\omega + \frac{\kappa R}{2} \tag{191}$$

Hence (190) holds for t sufficiently small.

Let $p(t)$ be a T-periodic continuous function such that

$$s(t) < p(t) < R(\cos(\kappa t/2) - \sin(\kappa t/2)) < R, \quad \text{for } t \in (0, \Delta] \tag{192}$$

$$p(t) > \frac{s(t)(1 + \sin(\kappa t/2))}{\cos(\kappa t/2)}, \quad \text{when } 0 < t \leq \alpha \tag{193}$$

$$p(t) > \sqrt{2}r > s(t), \quad \text{for } t \in [\Delta, T - \Delta] \tag{194}$$

$$p(0) = R \tag{195}$$

$$p(t) = p(T - t) \tag{196}$$

Let h be a function with the following properties

$$h(t, (x, y)) \geq 0 \tag{197}$$

$$h(t, (x, y)) = 0 \quad \text{for } (x, y) \in [-p(t), p(t)]^2 \tag{198}$$

$$h(t, (x, y)) = 1 \quad \text{if } \max(|x|, |y) \geq R(\cos(\kappa t/2) - \sin(\kappa t/2), \quad 0 < t \leq \Delta \tag{199}$$

$$h(t, (x, y)) = h(T - t, (x, y)), \quad h(t, (x, y)) = h(t, (-x, -y)) \tag{200}$$

$$h(t + T, (x, y)) = h(t, (x, y)) \tag{201}$$

Obviously h must be discontinuous for $t = kT$ for $k \in \mathbb{Z}$. We assume that it is smooth at every other point.

The maps A are given as the $\tau = 1$ shift along the trajectory of the following system of differential equations

$$\begin{aligned} \frac{dx}{d\tau} &= -\theta(t)h(t,x,y)y \\ \frac{dy}{d\tau} &= \theta(t)h(t,x,y)x \\ \frac{dt}{d\tau} &= 0 \end{aligned} \tag{202}$$

It is easy to see that $A(t,x,y) = (t, A_t(x,y))$. Observe that for small t, A_t is close to identity. We can set $A_0 = \mathrm{Id}$ to obtain a continuous function defined for any t.

Observe that A_t is a homeomorphism, with A_t^{-1} given as time $\tau = -1$ map. Assertions 1–5 follow easily from the construction of A. It remains to show that assertion 6 holds. We show it for $x > 0$ and A_t. The proof for other cases is analogous.

Let us take $0 < t \leq \alpha$. Let $C(x,y)$ be a circle passing through the point (x,y). Let us take $x_0 \geq R$. Every point from this circle is rotated clockwise by A_t around the origin by an angle less than $\kappa t/2$. Hence we have

$$\begin{aligned} A_t(\{(x,y) \in C(x_0, s(t)) \cap [s(t), R] \times [-s(t), s(t)]\}) \subset \\ \{(x,y) \mid x \geq x_0 \cos(\kappa t/2) - s(t)\sin(\kappa t/2)\} \end{aligned}$$

For $x_0 \geq \frac{s(t)(1+\sin(\kappa t/2))}{\cos(\kappa t/2)}$ we have

$$A_t(\{(x,y) \in C(x_0, s(t)) \cap [s(t), R] \times [-s(t), s(t)]\}) \subset \{(x,y) \mid x \geq s(t)\} \tag{203}$$

From (193) and definition of A_t it follows that

$$A_t(x,y) = (x,y), \quad 0 < t \leq \alpha, \quad \max\{|x|, |y|\} \leq \frac{s(t)(1+\sin(\kappa t/2))}{\cos(\kappa t/2)} \tag{204}$$

Consider a point $(x_0, y_0) \in [s(t), R] \times [-s(t), s(t)]$. Suppose that assertion 6 is not true for this point. Let x_1 be such that

$$(x_1, -s(t)) \in C(x_0, y_0) \tag{205}$$

Since A_t is a shift along the trajectory, which leaves $C(x_0, y_0)$ invariant we have

$$x(A_t(x_1, -s(t))) < x(A_t(x_0, y_0)) \leq s(t) \tag{206}$$

then from (203) it follows that

$$x_1 < \frac{s(t)(1+\sin(\kappa t/2))}{\cos(\kappa t/2)} \tag{207}$$

Hence we have

$$A_t(x_1, -s(t)) \neq (x_1, -s(t)) \tag{208}$$

This is in contradiction with equation (204). □

Lemma 24 *Let φ^M be as in theorem 20.*

Then for every $(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) \in \{-1, 0, 1\}^n$ the fixed point index $ind(\varphi^M_{(0,nT)}, (\mathrm{int}\, W_0)_\alpha)$ is nontrivial.

Proof: The proof is based on the results about TS–maps presented in [Z2] (sections 1 and 2), (see also [GZ]).

To apply directly results from [Z2] we need to introduce new notations.

$$\begin{aligned} N_l &= J_l \times [-R, R] \quad \text{for } l = -1, 0, 1 \\ E_{-1} = (-\infty, -b) \times [-R, R], \quad E_0 &= (-a, -c) \times [-R, R], \\ E_1 = (c, a) \times [-R, R], \quad E_2 &= (b, \infty). \\ E_l' &= E_l \quad \text{for } l = -1, \dots, 2 \\ h_l : [0, 1] \times E_l' \leftarrow E_l', \quad h_l(t, x) &= x \quad \text{for } l = -1, \dots, 2 \\ E' &= E_{-1} \cup \dots E_2 \\ N &= N_{-1} \cup N_0 \cup N_1 \end{aligned}$$

For a rectangle $P = [x, y] \times [w, z]$ following [Z2] we define

$$\begin{aligned} L(P) &= \{x\} \times [w, z] \\ R(P) &= \{y\} \times [w, z]. \end{aligned}$$

So $L(P)$, $R(P)$ are respectively left and right vertical edges of P.

To simplify the notation we set

$$M = \varphi^M_{(0,T)}$$

It is easy to see that from the assumptions about $\varphi^M_{(0,T)}$ it follows that

$$M(N) \subset \mathbb{R} \times \{0\} \subset E' \cup N \tag{209}$$
$$M(L(N_j)) \subset E_2, \quad M(R(N_j)) \subset E_{-1}, \quad \text{for } j = -1, 1 \tag{210}$$
$$M(L(N_0)) \subset E_{-1}, \quad M(R(N_0)) \subset E_2 \tag{211}$$

Hence according to [Z2] (def. 1) M is a TS–map (relatively to the sets N, E, E') and its transition matrix $A(M)$ is full, i.e $A(M)_{lj} = 1$ for $l, j = -1, 0, 1$.

The assertion of lemma follows immediately from the proof of theorem 1 in [Z2], where the required fixed point indices were calculated.

□

Proof of theorem 8: Let φ denote the local semiprocess generated by equation (12). We start with the observation that after suitable modifications outside some large ball we can assume that the semiprocesses φ is globally defined.

In order to apply theorem 10 let us consider the following homotopy between φ and the model semiprocess φ^M

$$H_\lambda = \Omega(\{(\varphi^M, 0), (\varphi, \lambda T)\}, T).$$

Obviously $H_0 = \varphi$ and $H_1 = \varphi^M$.

Now theorems 10 and 20 and lemma 24 imply that all fixed point indices for periodic points with prescribed periodic sequence of symbols are nontrivial. Hence $g_3(I_W)$ contains all periodic sequences from Σ_3. But the set of all periodic sequences is dense in Σ_3, so $g_3(I_W) = \overline{g_3(I_w)} = \Sigma_3$. □

References

[C] C. C. Conley *Isolated Invariant Sets and the Morse Index.* 1978. CBMS vol. 38, Amer. Math. Soc., Providence

[D] A. Dold *Lectures on Algebraic Topology.* 1972. Springer-Verlag, Berlin,Heidelberg,New York

[GH] J. Guckenheimer, P. Holmes *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields* Springer-Verlag, New York-Heidelberg-Berlin,

[GZ] Z. Galias & P. Zgliczyński *Computer assisted proof of chaos in the Lorenz system*, Physica D, 115, 1998,165–188

[S1] R. Srzednicki, *Periodic and bounded solutions in blocks for time-periodic nonautonomuous ordinary differential equations.* Nonlin. Analysis, TMA., 1994, 22, 707–737

[S2] R. Srzednicki, *On detection of chaotic dynamics in ordinary differential equations*, Nonlinear Anal. TMA, 1997, 30 No. 8, 4927-4935.

[S3] R. Srzednicki, *On geometric detection of periodic solutions and chaos,* Proceedings of the Conference "Nonlinear Analysis and Boundary Value Problems", CISM Udine, October 2–6, 1995, CISM Courses and Lectures Vol. 371, Springer-Verlag, Wien, New York 1996, 197-209.

[SW] R. Srzednicki & K. Wójcik, *A geometric method for detecting chaotic dynamics*, J. Diff. Eq., 1997, 135, 66–82

[Wi] S. Wiggins *Global Bifurcation and Chaos. Analytical Methods.* Springer-Verlag, New York-Heidelberg-Berlin, 1998

[W] K. Wójcik *Isolating segments and symbolic dynamics.*, Nonlinear Anal. TMA, 1998, 33, 575-591

[W1] K. Wójcik, *Chaos in nonautonomous periodic Hamiltonian systems on the plane*, Int. J. of Bifurcations and Chaos, to appear

[Z1] P. Zgliczyński *Fixed point index for iterations of maps, topological horseshoe and chaos.*, TMNA, 1996, Vol. 8, No. 1, 169–177

[Z2] P. Zgliczyński *Computer assisted proof of chaos in the Hénon map and in the Rössler equations* Nonlinearity, 1997, Vol. 10, No. 1, 243–252