On smooth dependence on initial conditions for dissipative PDEs, an ODE-type approach

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Abstract

Using elementary ODE tools we show that for a class of dissipative PDEs, including Navier-Stokes equations on the plane with periodic boundary conditions, Galerkin projections are C^1-convergent, i.e. the partial derivatives with respect to initial conditions for Galerkin projections converge giving rise to smooth dependence on initial conditions for full PDE.

1 Main result

This paper is a sequel to [Z]. The goal of this paper is to show using relatively elementary ODE methods that for a class of PDEs for which, we have the existence of compact trapping region and fast decay of Fourier coefficients, the partial derivatives with respect to initial conditions for Galerkin projections converge to the partial derivatives with respect of initial conditions of the semiflow induced by PDE. This also proves smooth dependence on initial conditions for full PDE on a trapping region (see Theorem 2 for precise statement).

Let us comment on its dependence on [Z]. An abstract part of present paper depends only on the content of Sections 4 and 5 in [Z]. An application of our main result - Theorem 2 - to the planar Navier-Stokes equations in Section 3 depends on the trapping regions, which after [MS, ES] (where the idea of trapping region for PDEs appeared for the first times in an explicite form) are given in [Z].

It should be pointed out that in literature there exist much more general results about regularity of solutions of dissipative PDEs. For example in [H] it is shown that solutions of NS in dimension two depend analytically on initial conditions. But still we believe that the simplicity of the tools used here will make this paper interesting for researches with a dynamical system theory and ODE background, especially in the context of computer assisted programm of rigorous investigation of dynamics of dissipative PDE outlined in [ZM] (see also introduction of [Z]).

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We adopt here the notations used in Section 4 and 5 in [Z]. Let H be a Hilbert space. Let $e_1, e_2, \dots$ form an orthonormal basis in H.

Let $A_n : H \to H$ denote a projection onto 1-dimensional subspace $\langle e_n \rangle$, i.e $x = \sum A_n(x) e_n$ for all $x \in H$. By X_n we will denote a space spanned by $\{e_1, \dots, e_n\}$. Let P_n denote the projection onto X_n and let $Q_n = I - P_n$.

For $x \in \mathbb{R}^n$ or $x \in H$ we set $|x|$ to be a standard (euclidean) norm ,$|x|_\infty = \max_i |x_i|$ and $|x|_1 = \sum_i |x_i|$.

We investigate the Galerkin projections of the following problem

$$x' = F(x) = L(x) + N(x), \tag{1}$$

where L is a linear operator and N is a nonlinear part of F. We assume that the basis $e_1, e_2, \dots$ of H is build from eigenvectors of L. We assume that the corresponding eigenvalues λ_k (i.e. $Le_k = \lambda_k e_k$) are ordered so that

$$\lambda_1 \geq \lambda_2 \geq \dots, \quad \text{and} \quad \lim_{k \to \infty} \lambda_k = -\infty.$$

Hence L can have only a finite number of positive eigenvalues.

Definition 1 *Let $W \subset H$ and $F : dom(F) \to H$, W is closed. We say that W and F satisfy conditions C1,C2,C3 if*

C1 *There exists $M \geq 0$, such that $P_n(W) \subset W$ for $n \geq M$*

C2 *Let $\hat{u_k} = \max_{x \in W} |A_k x|$. Then, $\hat{u} = \sum \hat{u_k} e_k \in H$. In particular, $|\hat{u}| < \infty$.*

C3 *The function $x \mapsto F(x)$ is continuous on W and $f = \sum_k f_k e_k$, given by $f_k = \max_{x \in W} |A_k F(x)|$ is in H. In particular, $|f| < \infty$.*

Definition 2 *We say that $W \subset H$ and $F = N + L$ satisfy condition* D, D1 *and* D2 *if for any i, j the function*

$$\frac{\partial N_i}{\partial x_j} : W \to \mathbb{R} \tag{2}$$

is continuous and the following condition hold

D *there exists $l \in \mathbb{R}$ such that for all $k = 1, 2, \dots$*

$$1/2 \sum_{i=1}^{\infty} \left| \frac{\partial N_k}{\partial x_i} \right| (W) + 1/2 \sum_{i=1}^{\infty} \left| \frac{\partial N_i}{\partial x_k} \right| (W) + \lambda_k \leq l \tag{3}$$

D1 *there exists $l \in \mathbb{R}$ such that for all $k = 1, 2, \dots$*

$$\sum_{i=1}^{\infty} \left| \frac{\partial N_k}{\partial x_i} \right| (W) + \lambda_k \leq l \tag{4}$$

D2 *there exists $l \in \mathbb{R}$ such that for all $k = 1, 2, \ldots$*

$$\sum_{i=1}^{\infty} \left| \frac{\partial N_i}{\partial x_k} \right| (W) + \lambda_k \leq l \tag{5}$$

The main idea behind the condition D (*D1*, *D2*) is to ensure that the Lipschitz constants of the flow induced by Galerkin projections are uniformly bounded in a standard , $||_\infty$, $| \cdot |_1$ norm, respectively (see Section 5.1 in [Z] for more details) . It is easy to see that the conjunction of *D1* and *D2* is implies D.

Definition 3 *Consider an ODE*

$$x' = f(x), \tag{6}$$

where $x \in \mathbb{R}^n$. The compact set $W \subset \mathbb{R}^n$ is called a trapping region for (6) *if for any solution, $x(t)$, of (6) if $x(0) \in W$, then $x(t) \in W$ for all $t > 0$.*

The following theorem was proved in [Z] (Thm. 13)

Theorem 1 *Assume that $W \subset H$ and F satisfy conditions* C1,C2,C3,D *and W is convex. Assume that $P_n(W)$ is a trapping region for the n-dimensional Galerkin projection of (1) for all $n > M_1$. Then*

1. **Uniform convergence and existence** *For a fixed $x_0 \in W$, let $x_n : [0, \infty] \to P_n(W)$ be a solution of $x' = P_n(F(x))$, $x(0) = P_n x_0$. Then x_n converges uniformly on compact intervals to a function $x^* : [0, \infty] \to W$, which is a solution of (1) and $x^*(0) = x_0$. The convergence of x_n on compact time intervals is uniform with respect to $x_0 \in W$.*

2. **Uniqueness within** W. *There exists only one solution of the initial value problem (1), $x(0) = x_0$ for any $x_0 \in W$, such that $x(t) \in W$ for $t > 0$.*

3. **Lipschitz constant**. *Let $x : [0, \infty] \to W$ and $y : [0, \infty] \to W$ be solutions of (1), then*
$$|y(t) - x(t)| \leq e^{lt} |x(0) - y(0)|$$

4. **Semidynamical system.** *The map $\varphi : \mathbb{R}_+ \times W \to W$, where $\varphi(\cdot, x_0)$ is a unique solution of equation (1), such that $\varphi(0, x_0) = x_0$ defines a semidynamical system on W, namely*
   - *φ is continuous*
   - $\varphi(0, x) = x$
   - $\varphi(t, \varphi(s, x)) = \varphi(t + s, x)$

Assume that all assumptions of Theorem 1 are satisfied. Let φ^n be a semidynamical system induced by the n-th Galerkin projection. From assertion 1 of Theorem 1 it follows that

$$\varphi^n(t, P_n x) \to \varphi(t, x), \tag{7}$$

and the convergence is uniform on $[0, T] \times W$ for any $T > 0$.

Now let us consider the variational matrix for φ^n given by

$$V_{ij}^n(t, x) = \frac{\partial \varphi_i^n}{\partial x_j}(t, x). \tag{8}$$

We would like to prove that also $V_{ij}^n(t, P_n x)$ converges.

To see why we expect convergence here let us remark, that V^n satisfies the following differential equation

$$\frac{dV_{ij}^n}{dt} = \lambda_i V_{ij}^n + \sum_k \frac{\partial N_i}{\partial x_k} V_{kj}^n. \tag{9}$$

Hence we can see that we have here the same strong damping as for the original equation (1). Observe that the bound for Lipschitz constant for (1) and its Galerkin projections is also a uniform bound for the norms of matrices V^n on any finite time interval. Once we have a strong damping and a-priori bounds for V^n we can use logarithmic norms to control the convergence of V^n's. In this paper we carried out this program, but we need to assume one additional condition

Definition 4 *We say that $W \subset H$ and $F = N + L$ satisfy condition* **D3***, if for any i, j, k the function*

$$\frac{\partial^2 N_i}{\partial x_j \partial x_k} : W \to \mathbb{R} \tag{10}$$

is continuous and the following condition hold

D3 *For any $M \geq 0$, there exists $l \in \mathbb{R}$, such that for all $i = 1, \ldots$*

$$\lambda_i + \sum_k \left| \frac{\partial N_i}{\partial x_k} \right| (W) + \sum_{k,j} \left| \frac{\partial^2 N_i}{\partial x_k \partial x_j} \right| (W) \cdot |v_j| \leq l, \tag{11}$$

for any vector $\{v_j\}$, such that $\sum_j |v_j| \leq M$.

In Section 3 we show that condition **D3** holds for Navier-Stokes equations in 2D with periodic boundary conditions for the trapping regions considered in [Z]. Observe that condition **D3** implies condition **D1**.

Theorem 2 *Same assumptions as in Theorem 1 and we additionally assume that conditions* D2 *and* D3 *are satisfied. Let φ be a semidynamical system induced by (1).*

There exists a family of continuous functions $V_{ij} : [0, \infty) \times W \to \mathbb{R}$ for $i, j = 1, \ldots, \infty$, such that

convergence *For each i, j the function V_{ij}^n converges to V_{ij} uniformly on $[0, T] \times W$ for any $T > 0$*

smoothness *(smoothness of φ) For any $x, y \in W$ and any $t > 0$ we have*

$$\varphi_i(t,x) - \varphi_i(t,y) = \sum_j \int_0^1 V_{ij}(t, y + s(x-y))ds \cdot (x_j - y_j) \qquad (12)$$

equation for V *An infinite dimensional matrix $V(t,x) = \{V_{ij}(t,x)\}$ satisfies the following variational equation*

$$\frac{dV_{ij}}{dt}(t,x) = \sum_k \frac{\partial F_i}{\partial x_k}(\varphi(t,x))V_{kj}(t,x), \qquad (13)$$

in the following sense: for each (i,j) the derivative $\frac{dV_{ij}}{dt}(t,x)$ exists, the series on r.h.s. of (13) converges uniformly on $[0,T] \times W$ and equation (13) is satisfied.

The proof of this theorem is given in Section 2. In Section 3 we show that theorem applies to Navier-Stokes equation on the plane with periodic boundary conditions.

Let us remark that equation (12) says that the semiflow φ does have partial derivatives with respect to initial conditions.

2 Proof of Theorem 2.

2.1 Logarithmic norm for variational equation

Same assumptions in Theorem 2. Let $l \in \mathbb{R}$, be such that **D1** and **D2** are satisfied.

A crucial notion used in this subsection is that of *a logarithmic norm* of a square matrix. For its definition and a relation with the Lipschitz constants of the flow induced by ODE's the reader is referred to Section 5.1 in [Z] (see also [HNW]).

The goal of this section is to show that the system of equations consisting of n-th Galerkin projection and a variational equation corresponding to it

$$\frac{dx^n}{dt} = P_n F(x^n) \qquad (14)$$

$$\frac{dV^n}{dt} = \frac{\partial P_n F}{\partial x^n}(x^n) \cdot V^n \qquad (15)$$

has a uniformly (independent of n) bounded logarithmic norm for $x \in P_n W$ and therefore has uniformly bounded Lipschitz constants. We will use the following norm

$$|(x,V)|_\infty = \max\{|x|_\infty, \max_{ij} |V_{ij}|\}. \qquad (16)$$

Lemma 3 *For any n, $x \in P_n(W)$ and $t > 0$*

$$\sum_j |V^n_{ij}(t,x)| \leq e^{lt}, \quad \textit{for all } i \tag{17}$$

$$\sum_i |V^n_{ij}(t,x)| \leq e^{lt}, \quad \textit{for all } j \tag{18}$$

Proof: Observe that in the norm $|\cdot|_\infty$ the matrix norm of the square matrix $A \in \mathbb{R}^{n\times n}$ is given by

$$|A|_\infty = \max_i \sum_j |A_{ij}| \tag{19}$$

Since l bounds from above the logarithmic norm corresponding to the norm $|\cdot|_\infty$ for the n-th Galerkin projection, hence by Lemma 12 in [Z] we obtain

$$|\varphi^n(t,x) - \varphi^n(t,y)|_\infty \leq e^{lt} \tag{20}$$

Since

$$|V^n(t,x)|_\infty = |\frac{\partial \varphi^n}{\partial x}(t,x)|_\infty \leq e^{lt} \tag{21}$$

then by (19) we obtain (17).

The proof of (18) is analogous, we have to use the norm $|\cdot|_1$ instead. ∎

Lemma 4 *Same assumptions as in Theorem 2.*

The flow induced by (14,15) the logarithimc norm induced $|(x,V)|_\infty$ norm is uniformly bounded from above on $[0,T] \times W$ for any $T > 0$ (this bound does not depend on n).

Hence it has a Lipschitz constant in the norm $|(x,V)|_\infty$ on W for $[0,T]$, which is independent of n.

Proof: We will drop here the index n (which stands for the Galerkin projection) in the system (14,15).

Observe that

$$\frac{\partial}{\partial x_k}\left(\frac{dx_i}{dt}\right) = \frac{\partial F_i}{\partial x_k} \tag{22}$$

$$\frac{\partial}{\partial V_{ks}}\left(\frac{dx_i}{dt}\right) = 0 \tag{23}$$

$$\frac{\partial}{\partial x_k}\left(\frac{dV_{ij}}{dt}\right) = \frac{\partial}{\partial x_k}\left(\sum_s \frac{\partial F_i}{\partial x_s} V_{sj}\right) = \sum_s \frac{\partial^2 F_i}{\partial x_k \partial x_s} V_{sj} \tag{24}$$

$$\frac{\partial}{\partial V_{ks}}\left(\frac{dV_{ij}}{dt}\right) = \frac{\partial F_i}{\partial x_k}\delta_{js} \tag{25}$$

To compute the logarithmic norm for the system (14,15) we have to estimate from above the following expressions on W

$$L_i = \frac{\partial}{\partial x_i}\left(\frac{dx_i}{dt}\right) + \sum_{k\neq i}\left|\frac{\partial}{\partial x_i}\left(\frac{dx_i}{dt}\right)\right|$$

$$L_{ij} = \frac{\partial}{\partial V_{ij}}\left(\frac{dV_{ij}}{dt}\right) + \sum_k \left|\frac{\partial}{\partial x_k}\left(\frac{dV_{ij}}{dt}\right)\right| + \sum_{(ks)\neq(ij)} \left|\frac{\partial}{\partial V_{ks}}\left(\frac{dV_{ij}}{dt}\right)\right|$$

From assumption *D1* we know that $L_i \leq l$.

For L_{ij} we have

$$\begin{aligned} L_{ij} \leq \quad & \lambda_i \quad + \left|\frac{\partial N_i}{\partial x_i}\right| + \sum_{(ks)\neq(ij)} \left|\frac{\partial N_i}{\partial x_k}\right| \delta_{js} + \sum_k \sum_s \left|\frac{\partial^2 N_i}{\partial x_k \partial x_s}\right| |V_{sj}| = \\ & \lambda_i \quad + \sum_k \left|\frac{\partial N_i}{\partial x_k}\right| + \sum_k \sum_s \left|\frac{\partial^2 N_i}{\partial x_k \partial x_s}\right| |V_{sj}| \end{aligned}$$

Since for any j the sum $\sum_s |V_{sj}| < e^{lT}$, (see Lemma 3). Therefore from condition **D3** it follows that there exists a constant l_2 depending only on $M = e^{lT}$ such that

$$L_{ij} \leq l_2(e^{lT}). \tag{26}$$

This shows that the system (14,15) has a uniformly bounded logarithmic norms on W and the conclusion follows from Lemma 12 in [Z]. ∎

2.2 Convergence

Lemma 5 *Same assumptions as in Theorem 2.*

There exists a family of continuous functions $V_{ij} : [0,\infty) \times W \to \mathbb{R}$ for $i,j = 1,\ldots,\infty$, and a subsequence $\{n_k\}$ such that for each i,j the functions $V_{ij}^{n_k}$ converge to V_{ij} uniformly on $[0,T] \times W$ for any $T > 0$.

Proof: The proof is based on the Ascoli-Arzela Lemma. Let us fix (i,j) and $T > 0$. From Lemma 3 it follows that $V_{ij}^n(t,x) = V_{ij}^n(t, P_n x)$ is uniformly bounded on W by e^{lT}.

We would like to show now that V_{ij}^n are equicontinuous on $[0,T] \times W$. Observe that form **C2** it follows that on W the distance induced by $|\cdot|_\infty$ is topologically equivalent (induces the same topology) on W to the standard distance. We will use this distance in the remainder of the proof.

Observe that the condition **D** implies that the series $\sum_k \left|\frac{\partial N_i}{\partial x_k}\right|$ is convergent on W therefore there exists a constant $M_i(T)$ depending only on i and T, such that

$$\left|\frac{dV_{ij}^n}{dt}\right| = \left|\sum_k \frac{\partial F_i}{\partial x_k} V_{kj}^n\right| \leq \left(|\lambda_i| + \sum_k \left|\frac{\partial N_i}{\partial x_k}\right|\right) e^{lT} \leq M_i(T), \tag{27}$$

Let $L_v(T)$ be the Lipschitz constant for $V^n(t,x)$ with respect to x on $[0,T] \times W$ obtained in Lemma 4

$$\begin{aligned} |V_{ij}^n(t_1,x_1) - V_{ij}^n(t_2,x_2)| = |V_{ij}^n(t_1,P_n x_1) - V_{ij}^n(t_2,P_n x_2)| \leq \\ |V_{ij}^n(t_1,P_n x_1) - V_{ij}^n(t_1,P_n x_2)| + |V_{ij}^n(t_1,P_n x_2) - V_{ij}^n(t_2,P_n x_2)| \leq \\ L_v(T)|x_1 - x_2|_\infty + M_i(T)|t_1 - t_2| \end{aligned}$$

The conclusion follows easily from Ascoli-Arzela Lemma and a diagonal argument. ∎

Lemma 6 *Let V_{ij} be as obtained in Lemma 5. Then for any $x, y \in W$ and any $t > 0$ we have*

$$\varphi_i(t,x) - \varphi_i(t,y) = \sum_j \int_0^1 V_{ij}(t, y + s(x-y))ds \cdot (x_j - y_j). \tag{28}$$

The sum in (28) converges uniformly with respect to $x, y \in W$ on compact time intervals.

Proof: Without any loss of generality we assume that the whole sequence V^n is convergent componentwise on compacts to V.

For any n and $i \leq n$ we have

$$\varphi_i^n(t,x) - \varphi_i^n(t,y) = \sum_{j \leq n} \int_0^1 V_{ij}^n(t, P_n y + s(P_n x - P_n y))ds \cdot (x_j - y_j) \tag{29}$$

We will pass to the limit in above equation. The limit of l.h.s. is settled by Theorem 1. To consider r.h.s. let us take any $m > 0$ and introduce the following notations

$$\begin{aligned}
A &= \sum_j \int_0^1 V_{ij}(t, y + s(x-y)))ds \cdot (x_j - y_j) - \\
&\quad \sum_{j \leq n} \int_0^1 V_{ij}^n(t, P_n y + s(P_n x - P_n y))ds \cdot (x_j - y_j) \\
A_1 &= \int_0^1 \left(\sum_{j \leq m} V_{ij}(t, y + s(x-y)) \cdot (x_j - y_j) \right) ds - \\
&\quad \int_0^1 \left(\sum_{j \leq m} V_{ij}^n(t, P_n y + s(P_n x - P_n y)) \cdot (x_j - y_j) \right) ds \\
A_2 &= \sum_{j > m} \int_0^1 V_{ij}(t, y + s(x-y))ds \cdot (x_j - y_j) \\
A_3 &= \sum_{m < j \leq n} \int_0^1 V_{ij}^n(t, P_n y + s(P_n x - P_n y))ds \cdot (x_j - y_j)
\end{aligned}$$

It is enough to show that $A \to 0$ for $n \to \infty$. Obviously we have $A = A_1 + A_2 - A_3$.

Let us fix an $\epsilon > 0$. Observe that $|A_2|$ and $|A_3|$ can be made less than ϵ for m big enough. Consider for example A_2, for A_3 the proof is the same.

$$|A_2| \leq \sum_{j > m} \int_0^1 |V_{ij}(t, y + s(x-y))|ds \cdot |x_j - y_j| \leq$$

$$e^{lt}\max_{j>m}|x_j-y_j|<\epsilon,\quad \text{for } m>m_0$$

The same m_0 works for A_3. Therefore we have

$$|A_2|+|A_3|<2\epsilon,\quad \text{for } m>m_0. \tag{30}$$

Consider now A_1. Let us fix an $m>m_0$. Observe that the functions

$$[0,1]\ni s\mapsto \sum_{j\le m} V_{ij}^n(t,P_ny+s(P_nx-P_ny))\cdot(x_j-y_j)$$

converge uniformly to

$$[0,1]\ni s\mapsto \sum_{j\le m} V_{ij}(t,y+s(x-y))\cdot(x_j-y_j).$$

Hence $|A_1|<\epsilon$ for n big enough.

This concludes the proof of (28). ∎

Observe that from formula (28) we obtain immediately that

$$\frac{\partial\varphi_i}{\partial x_j}(t,x)=V_{ij}(t,x) \tag{31}$$

Proof of Theorem 2: From Lemmas 5 and 6 it follows that to obtain two first assertions we have to show that indeed the whole sequence V_{ij}^n converges to V_{ij}. Assume the contrary, then by the same reasoning as in Lemma 5 we will have another limit functions $\tilde{V}_{ij}$ and for these functions Lemma 6 is also true. But then $\tilde{V}_{ij}=V_{ij}=\frac{\partial\varphi_i}{\partial x_j}$.

To prove (13) we fix (i,j), $T>0$. Observe that it is enough to prove that the sequence of continuous functions $S_n:[0,T]\times W\to\mathbb{R}$ given by

$$S_n(t,x)=\sum_{k=1}^{n}\frac{\partial F_i}{\partial x_k}(\varphi^n(t,P_nx))V_{kj}^n(t,P_nx)$$

converges uniformly to

$$S(t,x)=\sum_k \frac{\partial F_i}{\partial x_k}(\varphi(t,x))V_{kj}(t,x).$$

From this we obtain

$$V_{ij}(t,x)=\lim_{n\to\infty}V_{ij}^n(t,P_nx)=\delta_{ij}+\lim_{n\to\infty}\int_0^t S_n(s,x)ds=$$
$$\delta_{ij}+\int_0^t S(s,x)ds.$$

Since $S(t,x)$ is continuous we can differentiate the above equation with respect to t to obtain

$$\frac{dV_{ij}}{dt}(t,x) = S(t,x) = \sum_k \frac{\partial F_i}{\partial x_k}(\varphi(t,x))V_{kj}(t,x). \tag{32}$$

Now we prove that $S_n \to S$ uniformly on $[0,T] \times W$. Let us take any $n > m > i$. Then we have

$$\begin{aligned}
S(t,x) - S_n(t,x) &= S_1 + S_2 + S_3 \\
S_1 &= \lambda_i(V_{ij}(t,x) - V^n_{ij}(t,x)) + \sum_{k=1}^m \frac{\partial N_i}{\partial x_k}(\varphi(t,x))V_{kj}(t,x) - \\
&\quad \sum_{k=1}^m \frac{\partial N_i}{\partial x_k}(\varphi^n(t,x))V^n_{kj}(t,x) \\
S_2 &= \sum_{k>m} \frac{\partial N_i}{\partial x_k}(\varphi(t,x))V_{kj}(t,x) \\
S_3 &= -\sum_{k<m\le n} \frac{\partial N_i}{\partial x_k}(\varphi^n(t,P_n x))V^n_{kj}(t,x)
\end{aligned}$$

Let us fix any $\epsilon > 0$. We show that for $m > m_0$ holds: $|S_2| < \epsilon$ and $|S_3| < \epsilon$. We will do it for S_2 only, because for S_3 the proof is the same.

$$|S_2| \le \sum_{k>m} \left| \frac{\partial N_i}{\partial x_k}(\varphi(t,x)) \right| |V_{kj}(t,x)| \le \max_{k>m} \left| \frac{\partial N_i}{\partial x_k}(W) \right| e^{lt}$$

Since by condition **D** the series $\sum_k \left| \frac{\partial N_i}{\partial x_k}(W) \right|$ converges, we see that $|S_2| < \epsilon$ for $m > m_0$.

Let us fix m such that $|S_2| + |S_3| < 2\epsilon$. Since $S_1 \to 0$ for $n \to \infty$, therefore S_n converges uniformly to S on $[0,T] \times W$. This finishes the proof. ∎

3 Application to Navier-Stokes equations with periodic boundary conditions on the plane

We use the notation from [Z]. The modes (coordinates in H) are indexed by integer vectors in $\mathbb{Z}^d$. We use as the coordinates u_k's, which are d-dimensional complex vectors. The NS equations in Fourier domain become [ES, Z]

$$\frac{du_k}{dt} = -i\sum_{k_1}(u_{k_1}|k)u_{k-k_1} - \nu k^2 u_k - ip_k k + f_k \tag{33}$$

where $\sqcap_k$ is an orthogonal projection onto subspace orthogonal to k.

It is easy to see that

$$
\begin{aligned}
\lambda_k &= -\nu|k|^2 \\
\frac{\partial N_k}{\partial u_{k_1}} &= (\cdot|k) \sqcap_k u_{k-k_1} + (u_{k-k_1}|k) \sqcap_k \\
\frac{\partial^2 N_k}{\partial u_{k_1} \partial u_l} &= 2(\cdot|k) \sqcap_k \delta_{l,k-k_1} \\
\left| \frac{\partial^2 N_k}{\partial u_{k_1} \partial u_l} \right| &\leq 2|k| \delta_{l,k-k_1},
\end{aligned}
$$

Let W be any trapping region satisfying assumptions of Theorem 22 in [Z]. To apply Theorem 2 we have to show that conditions *C1, C2, C3, D, D3* are satisfied on W. Conditions *C1, C2, C2, C3, D* are satisfied by Lemma 17 in [Z]. In fact in the proof of this lemma it was shown that also conditions *D1* and *D2* are true.

To prove *D3* observe that it was shown in [Z] (see proof of Lemma 17) that

$$
\sum_k \left| \frac{\partial N_i}{\partial x_k} \right| (W) \leq |i| C \tag{34}
$$

For the second partial derivatives term in condition *D3* we obtain

$$
\sum_k \sum_j \left| \frac{\partial^2 N_i}{\partial x_k \partial x_j} \right| |v_j| \leq 2|i| \sum_{k,j} \delta_{k,i-j} |v_j| = 2|i| \sum_j |v_j| \leq 2|i| M \tag{35}
$$

Since λ_i grows to minus infinity like $|i|^2$ while the other terms are increasing at most as $|i|$, we see that condition *D3* is satisfied and Theorem 2 applies.

4 Conclusions and outlook

It is rather obvious that the presented method should be easily applicable to any PDE for which, there exists a compact attractor and Gevrey class regularity results ([FT, K]) has been established. Here is a list of a few examples for which Theorem 2 holds

- Navier-Stokes equations with periodic boundary conditions in three dimensions with small initial data and small force (see [MS])
- Kuramoto-Sivashinsky equations in 1-D with periodic boundary conditions (see [ZM] and references given there)
- Navier-Stokes equation in dimension three with the Laplacian term replaced by $|\nabla|^\alpha$ for α big enough ($\alpha > 2.5$) [MS]

In all the above mentioned examples the nonlinearity was quadratic, which make it quite easy to check that conditions *D, D1, D2* and *D3* are satisfied. It will be an interesting task to see if the approach presented here can be applied to the 1D Ginzburg-Landau equations, which also posses a finite dimensional attractor, but the nonlinear part is given by a polynomial of degree three ([D]).

A reader should notice that the proof of C^1-convergence of Galerkin projections presented in this paper is not effective from the point of view of rigorous numerics, no error bounds are given, so it cannot be used for rigorous computer assisted study of dynamics of PDEs as the one outlined in [ZM]. We hope to treat this problem in near future.

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