THE EXISTENCE OF SIMPLE CHOREOGRAPHIES FOR N-BODY PROBLEM - A COMPUTER ASSISTED PROOF

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Abstract. We consider a question of finding a periodic solution for the planar Newtonian N-body problem with equal masses, where each body is travelling along the same closed path. We provide a computer assisted proof for the following facts: the local uniqueness and the convexity of the Chenciner and Montgomery Eight, the existence (and the local uniqueness) for the Gerver's SuperEight for 4-bodies and a doubly symmetric linear chain for 6-bodies.

1. Introduction

In this paper we consider the problem of finding periodic solution to the N-body problem in which all N masses travel along a fixed curve in the plane. The N-body problem with N equal unit masses is given by a differential equation (the gravitational constant is taken equal to 1)

$$\ddot{q}_i = \sum_{j\neq i} \frac{q_j - q_i}{r_{ij}^3} \tag{1.1}$$

where $q_i \in \mathbb{R}^n$, $i = 1, ..., N$, $r_{ij} = ||q_i - q_j||$.

We consider the planar case ($n = 2$) only, we set $q_i = (x_i, y_i)$, $\dot{q}_i = (v_i, u_i)$. Using this we can express (1.1) by:

$$\begin{cases} \dot{v}_i = \sum_{j\neq i} \frac{x_j - x_i}{r_{ij}^3} \\ \dot{u}_i = \sum_{j\neq i} \frac{y_j - y_i}{r_{ij}^3} \\ \dot{x}_i = v_i \\ \dot{y}_i = u_i \end{cases} \tag{1.2}$$

Recently, this problem received a lot attention in the literature; see [M, CM, CGMS, S1, S2, S3, MR2] and papers cited there.

By a *simple choreography* [S1, S2] we mean a collision-free solution of the N-body problem in which all masses move on the same curve with a constant phase shift. This means that there exists $q : \mathbb{R} \longrightarrow \mathbb{R}^2$ a T-periodic function of time, such that the position of the k-th body ($k = 0, \ldots, N-1$) is given by $q_k(t) = q(t + k\frac{T}{N})$ and $(q_0, q_1, \ldots, q_{N-1})$ is a solution of the N-body problem. The simplest choreographies are the Langrange solutions in which the bodies are located at the vertices of a regular N-gon and move with a constant angular velocity. Another simple choreography, a figure eight curve (see Figure 1), was found numerically by C.Moore [M]. A.Chenciner and R.Montgomery [CM] gave a rigorous existence

Date: July 17, 2003.

1991 *Mathematics Subject Classification.* Primary 70F10; Secondary 37C80, 65G20 .

Key words and phrases. N-body problem, periodic orbits, choreographies, computer assisted proofs .

proof of the Eight in 2000. In December 1999, J.Gerver found orbit for $N = 4$ called the 'Super-Eight' (Figure 6). After that C.Simó found a many more simple choreographies of different shape, and with the number of bodies ranging form 4 to several hundreds (see [S, S1, CGMS] for pictures, animations and more details).

Up to now the only choreographies whose existence has been established rigorously [MR2] are the Lagrange solutions and the Eight solution. While the Lagrange solution is given analytically, the existence of the Eight was proven in [CM] using variational arguments and still there are a lot of open questions about it [Ch, MR2]. For example the uniqueness (up to obvious symmetries and rescaling) and the convexity of the lobes in the Eight. In Section 3 we give a computer assisted proof of the existence of the Eight, its local uniqueness and the convexity of the lobes.

In Sections 4, 5 and 6 we concentrate on choreographies called *doubly symmetric linear chains*. They are symmetric with respect to both coordinate axes and all self-intersection points of the curve are on the X axis. We describe a computer assisted proof of the existence (and the local uniqueness) of doubly symmetric linear chains for four (the Gerver SuperEight) and six bodies.

Proofs given in this paper are computer assisted. By this we mean that we use a computer programm to provide rigorous bounds for solutions of (1.1). Our method can be described as a variant of an interval shooting method. This approach has a long history in the interval analysis literature, were it was used in the more general context of the boundary value problem for ODEs (see for example [Ke, Lo, Lo1, Sc]).

In this paper the problem of proving the existence of a choreography is reduced to finding a zero for a suitable function. For this purpose we use the interval Newton method and the Krawczyk method [A, K, KB, Mo, N], (see Section 2). To integrate equations (1.1) we use a C^1-Lohner algorithm [ZLo]. All computations were performed on *AMD Athlon 1700XP* with 256 MB DRAM memory, with the Windows 98SE operating system. We used CAPD package [Capd] and Borland C++ 5.02 compiler. The source code of our programm is available on the first author web page[Ka]. The total computation time for 6-bodies was under 90 seconds and was considerably smaller for the Eight and the SuperEight - see Section 7 for more details.

Using an approach described in Section 4 we also proved the existence of doubly symmetric linear chains for ten and twelve bodies. Numerical data from these proofs are available on the first author web page [Ka]. Our approach failed for doubly symmetric linear chain of sixteen bodies (see Section 8 for more details).

We would like to thank Carles Simó for supplying us with the initial conditions for choreographies.

2. Two zero finding methods

The main technical tool used in this paper in order to establish the existence of solutions of equations of the form $f(x) = 0$ is *the interval Newton method*[A, Mo, N] and *the Krawczyk method*[A, K, N]. The interval Newton method was used to prove the existence of the Eight (Figure 1) and the SuperEight orbit (Figure 6). The Krawczyk method has been used for the SuperEight orbit (for comparison) and for an orbit with 6 bodies in a linear chain (Figure 8).

2.1. **Notation.** In the application of interval arithmetics to the rigorous verification of theorems single valued objects, like numbers, vectors, matrices etc are in the formulas replaced by sets containing sure bounds for them. In the sequel, we will not use any special notation to distinguish between single valued objects and sets. For a set S by $[S]$ we denote the interval hull of S, i.e. the smallest product of intervals containing S. For a set which is an interval set (i.e. can be represented as a product of intervals) we may use sometimes the square brackets to stress its interval nature. For any interval set $[S]$ by $mid([S])$ we denote a center point of $[S]$. For any interval $[a, b]$ we define a diameter by $diam[a, b] := b - a$. For an interval vector (matrix) $S = [S]$ by $diam S$ we denote a vector (matrix) of diameters of each components.

2.2. **The interval Newton method.**

Theorem 2.1. [A, N] *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 function. Let $[X] = \Pi_{i=1}^n [a_i, b_i]$, $a_i < b_i$. Assume that, the interval hull of $DF([X])$, (denoted by $[DF([X])]$) is invertible. Let $\bar{x} \in X$ and we define*

$$N(\bar{x}, [X]) = -[DF([X])]^{-1} F(\bar{x}) + \bar{x} \tag{2.1}$$

Then

0. *if* $x_1, x_2 \in [X]$ *and* $F(x_1) = F(x_2)$*, then* $x_1 = x_2$
1. *if* $N(\bar{x}, [X]) \subset [X]$*, then* $\exists! x^* \in [X]$ *such that* $F(x^*) = 0$
2. *if* $x_1 \in [X]$ *and* $F(x_1) = 0$*, then* $x_1 \in N(\bar{x}, [X])$
3. *if* $N(\bar{x}, [X]) \cap [X] = \emptyset$*, then* $F(x) \neq 0$ *for all* $x \in [X]$

The main problem with an application of the interval Newton method is that of the invertibility of $\left[\frac{\partial \Phi}{\partial x}([X])\right]$. One often can overcome this difficulty with the Krawczyk method.

2.3. **The Krawczyk method.** We assume that:

- $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a C^1 function,
- $[X] \subset \mathbb{R}^n$ is an interval set,
- $\bar{x} \in [X]$
- $C \in \mathbb{R}^{n \times n}$ is a linear isomorphism.

The Krawczyk operator [A, K, N] is given by

$$K(\bar{x}, [X], F) := \bar{x} - CF(\bar{x}) + (Id - C[DF([X])])([X] - \bar{x}). \tag{2.2}$$

Theorem 2.2. *1. If $x^* \in [X]$ and $F(x^*) = 0$, then $x^* \in K(\bar{x}, [X], F)$.*

2. If $K(\bar{x}, [X], F) \subset int[X]$, then there exists in [X] exactly one solution of equation $F(x) = 0$.

3. If $K(\bar{x}, [X], F) \cap [X] = \emptyset$, then $F(x) \neq 0$ for all $x \in [X]$

2.4. **The zero-finding algorithm based on Newton or Krawczyk methods.**
Theorems 2.1 and 2.2 can be used as a basis for an algorithm finding rigorous bounds for a solution of equation $F(x) = 0$. Let $T(\bar{x}, [X]) = N(\bar{x}, [X])$ if we are using the interval Newton method and $T(\bar{x}, [X]) = K(\bar{x}, [X], F)$ for the Krawczyk method based algorithm.

First, we need to have a good guess for $x^* \in \mathbb{R}^n$. For this purpose we use a nonrigorous Newton method to obtain $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$. Then we choose interval set $[X]$ which contains $\bar{x}$ and perform the following steps:

Step 1. Compute $T(\bar{x}, [X])$.
Step 2. If $T(\bar{x}, [X]) \subset [X]$, then return **success**.
Step 3. If $X \cap T(\bar{x}, [X]) = \emptyset$, then return **fail**. There are no zeroes of F in $[X]$.

Step 4. If $[X] \subset T(\bar{x}, [X])$, then modify computation parameters (for example: a time step, the order of Taylor method, size of $[X]$). Go to Step 1.

Step 5. Define a new $[X]$ by $[X] := [X] \cap T(\bar{x}, [X])$ and a new $\bar{x}$ by $\bar{x} := mid([X])$, then go to Step 1.

In practical computation it is convenient to define the maximum number of iterations allowed and return **fail** if the actual iteration count is larger.

Observe that the third assertion in both Theorems 2.1 and 2.2 can be used to exclude the existence of zero of F. This has been used by Galias [G1, G2] to find all periodic orbits up to a given period for Hénon map and Ikeda map.

3. The Eight - the existence, the local uniqueness and the convexity

The existence of the Eight has been shown in [CM] by using a mixture of symmetry and variational arguments. Here we give another existence proof and in addition we obtain the local uniqueness and the convexity of each lobe of the Eight. We follow [CM] in the use of the symmetry, but other component of the proof is different - we use the interval Newton method discussed in Section 2.

In notation we follow [CM].

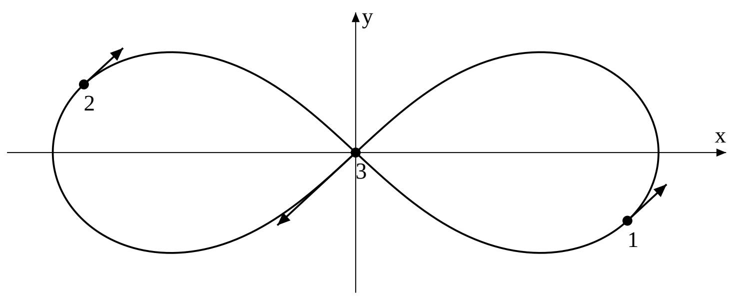

Figure 1. The Eight - the initial position

Let T be any positive real number. We define the action of the Klein group $\mathbb{Z}_2 \times \mathbb{Z}_2$ on $\mathbb{R}/T\mathbb{Z}$ and on $\mathbb{R}^2$ as follows: if σ and τ are generators, then we set

$$\sigma(t) = t + \frac{T}{2}, \ \tau(t) = -t + \frac{T}{2}, \ \sigma(x, y) = (-x, y), \ \tau(x, y) = (x, -y). \tag{3.1}$$

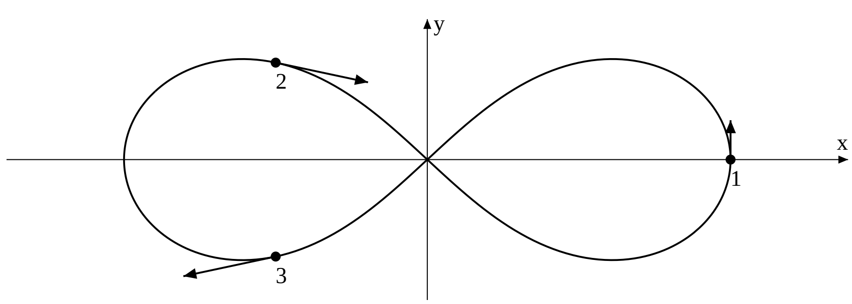

Figure 2. The Eight - the final position

For loop $q : (\mathbb{R}/T\mathbb{Z}) \longrightarrow \mathbb{R}^2$ and $i = 1, 2, 3$ we define the position for i-th body by

$$q_i(t) = q\left(t + (3-i)\cdot\frac{T}{3}\right), \tag{3.2}$$

The following theorem without the uniqueness part was proved in [CM].

Theorem 3.1. *There exists an "eight"-shaped planar loop $q : (\mathbb{R}/T\mathbb{Z}) \longrightarrow \mathbb{R}^2$ with the following properties:*

(1) *for each t,*
$$q_1(t) + q_2(t) + q_3(t) = 0;$$
(2) *q is invariant with respect to the action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on $\mathbb{R}/T\mathbb{Z}$ and on $\mathbb{R}^2$:*
$$q \circ \sigma(t) = \sigma \circ q(t) \text{ and } q \circ \tau(t) = \tau \circ q(t);$$
(3) *the loop $x : \mathbb{R}/T\mathbb{Z} \longrightarrow \mathbb{R}^6$ defined by*
$$x(t) = (q_1(t), q_2(t), q_3(t))$$
is a T-periodic solution of the planar three-body problem with equal masses.
Moreover, q is locally unique (up to obvious spatial symmetries and rescaling).

As was mentioned in the introduction the proof of Theorem 3.1 is computer assisted. The goal of next few lemmas is to transform it to the problem of solving the equation $F(x) = 0$ for a suitable F.

Remark 3.2. If conditions (1),(2),(3) are satisfied, then

a. $\dot{q}_1(t) + \dot{q}_2(t) + \dot{q}_3(t) = 0$ for each t,
b. $\dot{q} \circ \sigma(t) = \sigma \circ \dot{q}(t)$ and $\dot{q} \circ \tau(t) = \sigma \circ \dot{q}(t)$ for each t,
c. $q_3(0) = (0, 0)$ and $\dot{q}_3(0) = -2\dot{q}_1(0)$
d. $q_1(0) = -q_2(0)$ and $\dot{q}_1(0) = \dot{q}_2(0)$,
e. $q_1(T/12)$ is on the X axis and $\dot{q}_1(T/12)$ is orthogonal to the X axis,
f. $q_3(T/12) = \tau \circ q_2(T/12)$ and $\dot{q}_3(T/12) = \sigma \circ \dot{q}_2(T/12)$,

The following lemma describes the symmetry reduction for the Eight.

Lemma 3.3. *Assume that $\tilde{q} : [0, \tilde{T}] \longrightarrow \mathbb{R}^6$ is a solution of the three body problem, such that $\tilde{q} = (\tilde{q}_1, \tilde{q}_2, \tilde{q}_3)$ satisfies conditions (c),(d),(e),(f) in Remark 3.2, then exists $q : (\mathbb{R}/T\mathbb{Z}) \longrightarrow \mathbb{R}^2$ satisfying conditions (1),(2),(3) in Theorem 3.1 with $T = 12\tilde{T}$.*

Proof: We define

$$\hat{q}(t) = \begin{cases} \tilde{q}_3(t) & \text{for } t \in [0, \tilde{T}] \\ \tau \circ \tilde{q}_2(2\tilde{T} - t) & \text{for } t \in [\tilde{T}, 2\tilde{T}] \\ \sigma \circ \tilde{q}_1(t - 2\tilde{T}) & \text{for } t \in [2\tilde{T}, 3\tilde{T}] \end{cases} \tag{3.3}$$

and

$$q(t) = \begin{cases} \hat{q}(t) & \text{for } t \in [0, 3\tilde{T}] \\ \tau \circ \hat{q}(\tau^{-1}(t)) & \text{for } t \in [3\tilde{T}, 6\tilde{T}] \\ \sigma \circ \hat{q}(\sigma^{-1}(t)) & \text{for } t \in [6\tilde{T}, 9\tilde{T}] \\ \sigma \circ \tau \circ \hat{q}(\tau^{-1} \circ \sigma^{-1}(t)) & \text{for } t \in [9\tilde{T}, 12\tilde{T}] \end{cases} \tag{3.4}$$

Let f_1 and f_2 be two solutions of (1.1) on intervals $[t_1, t_2]$ and $[t_2, t_3]$, respectively. If $f_1(t_2) = f_2(t_2)$ and $\dot{f}_1^-(t_2) = \dot{f}_2^+(t_2)$ then $f = \{f_1, f_2\}$ is a solution on interval

$[t_1, t_2]$. To show that $q(t)$ is a solution it is enough to show that "pieces fit together smoothly".
For example for $t = \tilde{T}$ from 3.2.f we have

$$\tilde{q}_3(\tilde{T}) = \tilde{q}_3(\frac{T}{12}) = \tau \circ \tilde{q}_2(\frac{T}{12}) = \tau \circ \tilde{q}_2(\tilde{T}) = \tau \circ \tilde{q}_2(2\tilde{T} - \tilde{T}) \tag{3.5}$$

$$\dot{\tilde{q}}_3^-(\tilde{T}) = \sigma \circ \dot{\tilde{q}}_2^+(\tilde{T}) \tag{3.6}$$

Other cases are left to reader.

Condition (2) in Theorem 3.1 follows easily from the definition of $\hat{q}(t)$ and the properties of σ and τ (for T-periodic orbit $\sigma^{-1}(t) = \sigma(t)$ and $\tau^{-1}(t) = \tau(t)$). For example, for $t \in [0, 3\tilde{T}]$ we have

$$q(\sigma(t)) = \sigma \circ \hat{q}(\sigma^{-1} \circ \sigma(t)) = \sigma \circ \hat{q}(t) = \sigma \circ q(t) \tag{3.7}$$

$$q(\tau(t)) = \tau \circ \hat{q}(\tau^{-1} \circ \tau(t)) = \tau \circ \hat{q}(t) = \tau \circ q(t) \tag{3.8}$$

We omit other cases, because the proof is very similar.

To prove condition (1) observe that, from 3.2.c and 3.2.d we obtain that $q_1(0) + q_2(0) + q_3(0) = 0$ and $\dot{q}_1(0) + \dot{q}_2(0) + \dot{q}_3(0) = 0$. This and the conservation of linear momentum implies condition (1). □

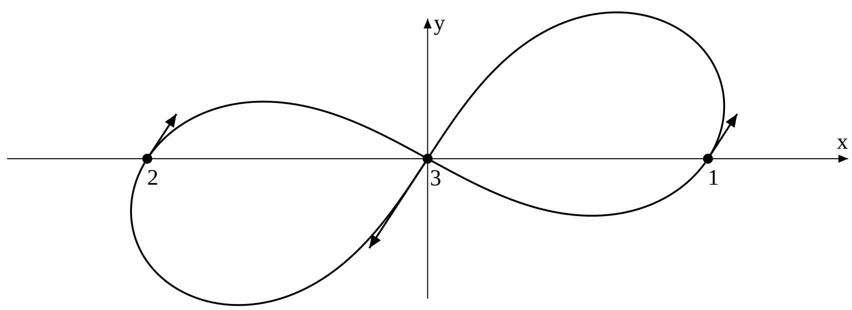

FIGURE 3. The rotated Eight - the initial position

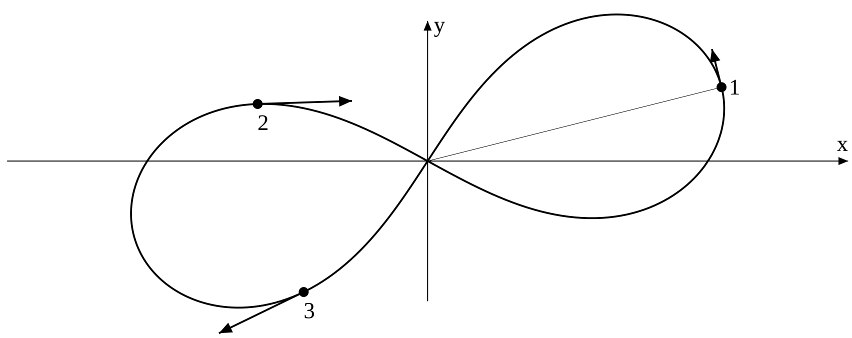

FIGURE 4. The rotated Eight - the final position

Hence to prove Theorem 3.1 it is enough to show that there exists a locally unique (up to obvious degeneracies) function satisfying assumptions of Lemma 3.3.

For this end we rewrite these assumptions as a zero finding problem to which we apply the interval Newton method in *the reduced space.*

Our original phase space is 12 dimensional, the state of bodies is given by $(q_1, q_2, q_3, \dot{q_1}, \dot{q_2}, \dot{q_3})$. The center of mass is fixed at the origin. Hence one body's position and velocity is determined by other two bodies. We start from a collinear position with the third body at the origin and with equal velocities of the first and the second body (see 3). Hence it is enough to know the position and the velocity of the first body to reconstruct initial condition of other bodies. Moreover, if we have one solution we could get another solution by a suitable rotation (both have the same shape). To remove this degeneracy we place the first body on the X axis (see Figure 3). In addition we fix the size of trajectory by setting $q_1(0) = (1, 0)$. This also fixes the period of the solution, but from the Kepler third law we can obtain a solution of any period just by rescaling. Hence the initial conditions are defined by the velocity of one body.

The *reduced space* for the Eight is two dimensional and is parameterized by the velocity of the first body. We define a map from the reduced space to the full phase space $E : \mathbb{R}^2 \longrightarrow \mathbb{R}^{12}$, which expands velocity of the first body, given by (v, u), to the initial conditions for the 3-body problem $(x_1, y_1, x_2, y_2, x_3, y_3, v_1, u_1, v_2, u_2, v_3, u_3)$ for equation (1.1)

$$E(v, u) = (1, 0, -1, 0, 0, 0, v, u, v, u, -2v, -2u).$$

For each such initial condition there exists a solution of the 3-body problem defined on some interval. To each initial configuration, following that solution, we associate, if it exists, a configuration in which for the first time the position vector of the first body is orthogonal to its velocity vector. This defines the Poincaré map $P : \mathbb{R}^{12} \supset \Omega \longrightarrow \mathbb{R}^{12}$.

Now, we define map $R : \mathbb{R}^{12} \longrightarrow \mathbb{R}^2$, by

$$R(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3) = (\|q_2 - q_1\|^2 - \|q_3 - q_1\|^2, (\dot{q}_2 - \dot{q}_3) \times q_1),$$

where by $\times$ we denote vector product, and map $\Phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by

$$\Phi = R \circ P \circ E.$$

Remark 3.4. If $R(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3) = 0$, then $\|q_2 - q_1\| = \|q_3 - q_1\|$ and if in addition $y_1 = 0$ but $x_1 \neq 0$, then $\dot{y}_2 = \dot{y}_3$. The bodies are then located as in Fig. 4 and conditions (e) and (f) in Rem. 3.2 are satisfied in a suitably rotated coordinate frame.

The following lemma, which is crucial for the proof of Theorem 3.1 was proved with a computer assistance.

Lemma 3.5. *There exists a locally unique point* $(v, u) \in \mathbb{R}^2$, *such that* $\Phi(v, u) = (0, 0)$.

Proof: We use the interval Newton method (Theorem 2.1). First we come close to zero of Φ, starting with some rough initial condition (for example from [S1]) using the non-rigorous Newton method. Once we have a good candidate $x_0 = (v_0, u_0)$, we set $[X] = [v_0 - \delta, v_0 + \delta] \times [u_0 - \delta, u_0 + \delta]$ and compute rigorously $\Phi(x_0)$ and $\frac{\partial \Phi([X])}{\partial x}$. For this purpose we use C^1-Lohner algorithm described in [ZLo]. In this computation we used the following settings: the time step $h = 0.01$ and the order $r = 7$.

Result	
$\Phi(x_0)$	([-2.107029e-06, -2.106467e-06],[2.974991e-06, 2.976034e-06])
diam $\Phi(x_0)$	(5.625889e-10, 1.042962e-09)
$\frac{\partial\Phi([X])}{\partial x}$	$\begin{bmatrix} [17.622624, 17.643043] & [1.809772, 1.827325] \\ [-24.868548, -24.848432] & [-10.056629, -10.039221] \end{bmatrix}$
$N(x_0, [X])$	$([0.347116886243943, 0.347116889993313],$ $[0.532724941587373, 0.532724949187495])$
diam $N(x_0, [X])$	(3.749369e-09,7.600121e-09)

Table 1. Data from the proof of Lemma 3.5.

It turns out that the assumption of assertion 1 in Theorem 2.1 holds for $x_0 = (0.347116768716, 0.532724944657)$ and $\delta = 10^{-6}$. Numerical data from this computation are listed in Table 1.

Moreover, from Theorem 2.1 we know that this zero is unique in the set X. □

Proof of Theorem 3.1: From Lemma 3.5 it follows that there exists $(v, u) \in \mathbb{R}^2$ such that $\Phi(v, u) = (0, 0)$. Hence there exists solution $\bar{q}(t)$ of the three body problem defined on interval $[0, \tilde{T}]$, such that for $t = 0$ all bodies are in the collinear configuration with the third body at the origin and for $t = \tilde{T}$ the bodies form an isosceles triangle (because $r_{12} = r_{13}$). By a suitable rotation of the coordinate frame we obtain a solution $\tilde{q}(t)$, such that for $t = \tilde{T}$ the first body is on X axis. From Remark 3.4 it follows that $\tilde{q}(t)$ satisfies all conditions in Lemma 3.3 and hence there exists $q(t)$ satisfying conditions (1),(2),(3).

The local uniqueness follows from the local uniqueness in Lemma 3.5. □

3.1. **Convexity of the Eight.**

Theorem 3.6. *Each lobe of the Eight is convex.*

Proof: For the proof it is enough to show that the only inflection point on the curve $q(t)$ is the origin.

From Lemma 3.5 we obtain the set $[X]$ which includes the initial condition for the Eight in the reduced space. To expand $[X]$ to the full space we set $\bar{X} = E([X])$. In the coordinate frame in which the Eight looks as in Fig. 1 and symmetries σ and τ are the reflections with respect to coordinate axes (see 3.1) we see immediately that the symmetry properties of the Eight imply that at the origin $\frac{\partial^2 y_i}{\partial x_i^2} = 0$ and $\frac{\partial^2 x_i}{\partial y_i^2} = 0$. To prove the convexity of the Eight we follow rigorously the trajectory of set $\bar{X}$ and show that the only point in which $\frac{\partial^2 y_i}{\partial x_i^2} = 0$ and $\frac{\partial^2 x_i}{\partial y_i^2} = 0$ is the origin. The same is true if we rotate the Eight, as in Figure 3, to the coordinate system in which we performed the actual computations and in which we will work for the remainder of this proof.

Let $q_i(t) = (x_i(t), y_i(t))$ be the position of i-th body. If $0 \notin \frac{\partial x_i}{\partial t}[t_{k-1}, t_k]$ then we can write y_i as a function of x_i on interval $[x_i(t_{k-1}), x_i(t_k)]$. Otherwise we try to represent x_i as a function of y_i. For small enough time steps at least one of these representations is always possible for the Eight (this is verified during rigorous computations). Below we list formulas for the derivatives of $y_i(t(x_i))$ with respect to x_i in terms of derivatives of x_i and y_i with respect to the time variable.

$$
\begin{aligned}
\frac{\partial y_i(t(x_i))}{\partial x_i} &= \frac{\partial y_i}{\partial t}\left(\frac{\partial x_i}{\partial t}\right)^{-1} \\
\frac{\partial^2 y_i(t(x_i))}{\partial x_i^2} &= \left(\frac{\partial^2 y_i}{\partial t^2} - \frac{\partial^2 x_i}{\partial t^2}\frac{\partial y_i}{\partial x_i}\right)\left(\frac{\partial x_i}{\partial t}\right)^{-2} \\
\frac{\partial^3 y_i(t(x_i))}{\partial x_i^3} &= \frac{\frac{\partial^3 y_i}{\partial t^3}\frac{\partial x_i}{\partial t} - \frac{\partial^3 x_i}{\partial t^3}\frac{\partial y_i}{\partial t} + 2\left(\frac{\partial^2 x_i}{\partial t^2}\right)^2 \frac{\partial y_i}{\partial x_i} - 2\frac{\partial^2 x_i}{\partial t^2}\frac{\partial^2 y_i}{\partial t^2}}{\left(\frac{\partial x_i}{\partial t}\right)^4} - \frac{\left(\frac{\partial^2 x_i}{\partial t^2}\frac{\partial^2 y_i}{\partial x_i^2}\right)}{\left(\frac{\partial x_i}{\partial t}\right)^2}
\end{aligned}
$$

To obtain derivatives of x_i with respect to y_i it is enough to exchange x_i and y_i variables in above formulas.

The time derivatives of x_i and y_i for each time step are computed during the execution of the C^1-Lohner algorithm.

Before we state explicitly the conditions we check we need to introduce some notation. Let $x_0 \in \mathbb{R}^{12}$ represent initial conditions (for $t = 0$) for (1.2) then by $\varphi(t, x_0)$ we denote the state of bodies (positions and velocities) at time t. Let h_k be length of k-th time step, $t_k = h_1 + \cdots + h_k$ - the total time elapsed after k steps, $[q^k] \subset \mathbb{R}^{12}$ be an interval set, such that $\varphi(t_k, \bar{X}) \subset [q^k]$ and $[Q^k] \subset \mathbb{R}^{12}$ be an interval set, such that $\varphi([t_{k-1}, t_k], \bar{X}) \subset [Q^k]$. Let us stress here, that both $[q^k]$ and $[Q^k]$ are computed during k-th step of C^1-Lohner algorithm.

For k-th step and i-th body (except the first step and the third body ($i = 3$), which starts at the origin) we check if at least one of the following conditions is true

$$0 \notin \frac{\partial y_i}{\partial t}[X_i^k] \text{ and } 0 \notin \frac{\partial^2 x_i}{\partial y_i^2}[X_i^k] \tag{3.9}$$

$$0 \notin \frac{\partial x_i}{\partial t}[Y_i^k] \text{ and } 0 \notin \frac{\partial^2 y_i}{\partial x_i^2}[Y_i^k]. \tag{3.10}$$

For first step and the third body we check if one of following conditions is satisfied

$$0 \notin \frac{\partial y_3}{\partial t}[X_3^1] \text{ and } 0 \in \frac{\partial^2 x_3}{\partial y_3^2}[X_3^1] \text{ and } 0 \notin \frac{\partial^3 x_3}{\partial y_3^3}[X_3^1] \tag{3.11}$$

$$0 \notin \frac{\partial x_3}{\partial t}[Y_3^1] \text{ and } 0 \in \frac{\partial^2 y_3}{\partial x_3^2}[Y_3^1] \text{ and } 0 \notin \frac{\partial^3 y_3}{\partial x_3^3}[Y_3^1]. \tag{3.12}$$

To verify above conditions we follow the trajectory of set $\bar{X}$ using the C^1-Lohner algorithm until reaching the section described in proof of Lemma 3.5 and for each time step and each body we verify a suitable condition. This finishes the proof. □

3.2. **Some numerical data from the proof of the convexity of the Eight.** Parameter settings for the C^1-Lohner algorithm: the time step $h = 0.01$, the order $r = 7$, 53 time steps were needed to cross the section.

Step 1. We start in the collinear position with the third body at the origin (at *the inflection point*). Numerical data for this case are given in Table 2. We see that

i	$\frac{\partial x_i}{\partial t}[X_i^1]$	$\frac{\partial^2 x_i}{\partial y_i^2}[X_i^1]$	$\frac{\partial^3 x_i}{\partial y_i^3}[X_i^1]$
1	[0.334402,0.347118]	[15.3592,17.9897]	[136.616,219.114]
2	[0.347116,0.360049]	[-16.5013,-14.111]	[119.951,192.562]
3	[-0.695034,-0.69385]	[-0.0682713,0.269952]	[-30.969,-26.3718]

TABLE 2. Data from the proof of Theorem 3.6, for step 1, neighborhood of deflection point.

i	$\frac{\partial x_i}{\partial t}[X_i^2]$	$\frac{\partial^2 x_i}{\partial y_i^2}[X_i^2]$	$\frac{\partial^3 x_i}{\partial y_i^3}[X_i^2]$
1	[0.32222,0.334722]	[16.7085,19.6225]	[155.007,250.021]
2	[0.35972,0.372882]	[-15.2203,-13.0177]	[105.778,171.191]
3	[-0.695669,-0.69444]	[0.126359,0.472046]	[-30.9533,-26.1203]

TABLE 3. Data from the proof of Theorem 3.6, for step 2.

i	$\frac{\partial x_i}{\partial t}[X_i^2]$	$\frac{\partial^2 x_i}{\partial y_i^2}[X_i^2]$	$\frac{\partial^3 x_i}{\partial y_i^3}[X_i^2]$
1	[-0.00287209,0.00468403]	-	-
2	[0.904939,0.922079]	[-2.55715,-2.16831]	[4.35197,10.6201]
3	[-0.919428,-0.909617]	[2.56371,3.03259]	[-3.51203,3.48564]
i	$\frac{\partial y_i}{\partial t}[Y_i^2]$	$\frac{\partial^2 y_i}{\partial x_i^2}[Y_i^2]$	$\frac{\partial^3 y_i}{\partial x_i^3}[Y_i^2]$
1	[0.480975,0.48288]	[-3.24824,-3.11737]	[-2.98453,-0.860616]

TABLE 4. Data from the proof of Theorem 3.6, for step 37.

for the third body ($i = 3$) we have $0 \in \frac{\partial^2 x_i}{\partial y_i^2}[X_i^1]$, but this derivative is monotonic (because $\frac{\partial^3 x_i}{\partial y_i^3}[X_i^1] < 0$), hence there can be only one zero of it in interval $[X_3^1]$. But from the symmetry we know that this zero is at $x_3 = 0 \in [X_3^1]$.

Steps 2-36 and 38-53 In this case all second derivatives do not contain 0. In Table 3 we list as an example data obtained in the second step.

Step 37. The first body is in the rightmost position. In this case we cannot represent y_1 as a function of x_1, hence we represent x_1 as a function of y_1 and we check condition 3.10 instead of 3.9. Table 4 contains the derivatives for this case.

4. Doubly symmetric choreographies with even number of bodies

4.1. **Symmetries.** Many of the choreographies found by Simo [S1] have at least one symmetry. In this section we introduce a notation for symmetries which will be used till the end of this paper. By S_x, S_y we denote the reflection with respect symmetry with respect to the x axis and the y axis, respectively. By S_0 we denote the reflection against the origin. These spatial symmetries act also on time variable parameterizing curves as follows.

Let T be any positive real number. We define actions of S_x, S_y and S_0 on $\mathbb{R}/T\mathbb{Z}$

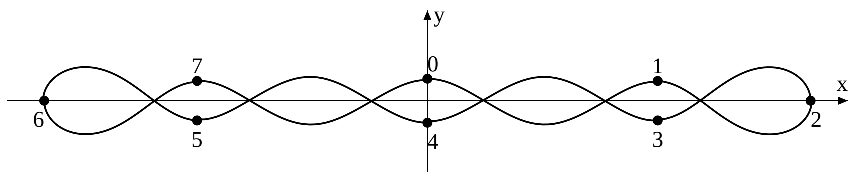

FIGURE 5. Linear chain with 8 bodies.

and on $\mathbb{R}^2$ by

$$\begin{array}{ll} S_x(t) = -t + \frac{T}{2}, & S_x(x,y) = (-x,y), \\ S_y(t) = -t, & S_y(x,y) = (x,-y), \\ S_0(t) = t + \frac{T}{2}, & S_0(x,y) = (-x,-y). \end{array}$$

It follows from these definitions that $S_0 = S_x \circ S_y = S_y \circ S_x$.

Let $q(t) : \mathbb{R}/T\mathbb{Z} \longrightarrow \mathbb{R}^2$ be a C^2 function. We say that $q(t)$ is invariant (equivariant) with respect to the action of S if $S(q(t)) = q(S(t))$ for all t. If $q(t)$ is invariant with respect to S_x (resp. S_y) then $\dot{q}(S_x(t)) = S_y(\dot{q}(t))$ (resp. $\dot{q}(S_y(t)) = S_x(\dot{q}(t))$). Hence the S_0-invariance implies that $\dot{q}(S_0(t)) = S_0(\dot{q}(t))$.

From now on we will enumerate bodies starting from 0. We set $q_i(t) = q\left(t + \frac{T}{N} \cdot i\right)$ for $i = 0, \dots, N-1$.

4.2. **Doubly symmetric choreographies with even number of bodies.** We will consider only cases with even number of bodies. Let $T = N \cdot \bar{T} > 0$. We search for a function $q(t) : \mathbb{R}/T\mathbb{Z} \longrightarrow \mathbb{R}^2$ which has following properties:

P1. for each t the origin is center of mass,

$$\sum_{i=0}^{N-1} q\left(t + \bar{T} \cdot i\right) = 0, \tag{4.1}$$

P2. $q(t)$ is invariant with respect to
  (a) S_x i.e. $q(S_x(t)) = S_x(q(t))$,
  (b) S_0 i.e. $q(S_0(t)) = S_0(q(t))$,

P3. the function $x = (q_0(t), q_2(t), \dots, q_{N-1})$, where $q_i(t) = q\left(t + \bar{T} \cdot i\right)$ for $i = 0, \dots, N-1$, is a T-periodic solution of the N-body problem (1.1).

The following Lemma, which is analogous to Lemma 3.3, gives necessary and sufficient conditions for the existence of a choreography satisfying P1,P2 and P3.

Lemma 4.1. *Let $N = 2n$ be the number of bodies. There exists a function $q(t)$ with properties P1, P2, P3 if and only if there are functions $q_i : [0, \bar{T}/2] \longrightarrow R^2$ for $i = 0, 1, \dots, N-1$ such as:*

(1) $q_0(0) = (0, y_0)$ *for some* $y_0 \neq 0$ *($q_0(0)$ is on Y axis),*
(2) *At time* $t_0 = 0$ *for* $i = 0, 1, \dots, N/2-1$ *we have*
  (a) $q_i(t_0) = S_x(q_{\frac{N}{2}-i}(t_0))$
  (b) $\dot{q}_i(t_0) = S_y(\dot{q}_{\frac{N}{2}-i}(t_0))$
  (c) $q_i(t_0) = S_0(q_{\frac{N}{2}+i}(t_0))$
  (d) $\dot{q}_i(t_0) = S_0(\dot{q}_{\frac{N}{2}+i}(t_0))$
(3) *At time* $t_1 = \bar{T}/2$ *for* $i = 0, 1, \dots, N/2-1$ *we have*
  (a) $q_i(t_1) = S_x(q_{\frac{N}{2}-i-1}(t_1))$
  (b) $\dot{q}_i(t_1) = S_y(\dot{q}_{\frac{N}{2}-i-1}(t_1))$

(c) $q_i(t_1) = S_0(q_{\frac{N}{2}+i}(t_1))$
(d) $\dot{q}_i(t_1) = S_0(\dot{q}_{\frac{N}{2}+i}(t_1))$

(4) $x(t) = (q_0(t), q_1(t), \ldots, q_{N-1}(t))$ *is a solution of the Newton N-body problem for* $t \in [0, \bar{T}/2]$.

From Lemma 4.1 it follows that the proof of the existence of a doubly symmetric choreography is equivalent to some boundary value problem for the N-body problem. We will now formulate this problem as a zero finding problem for a suitable map.

The original phase space for the planar N-body problem has $4N$ dimensions. It turns out that, if initial conditions satisfy all conditions in point 2 of Lemma 4.1 then it is enough to know values of only N variables to recover the rest of them. We still may obtain solutions of any period and to determine it, hence wean fix the size of curve by fixing one variable. Hence our *reduced space* is $(N-1)$-dimensional. In the next paragraph we will be more specific.

We define map $E : \mathbb{R}^{N-1} \longrightarrow \mathbb{R}^{4N}$, which expands, using symmetries from (2), initial conditions from the reduced space to the full phase space: $(x_0, y_0, \dot{x}_0, \dot{y}_0, x_1, y_1, \dot{x}_1, \dot{y}_1, \ldots, x_{N-1}, y_{N-1}, \dot{x}_{N-1}, \dot{y}_{N-1})$. We consider two cases: $N = 4k$ and $N = 4k + 2$.

For $N = 4k$ we set

$$\begin{aligned} E(\dot{x}_0 \times \prod_{i=1}^{k-1}(x_i, y_i, \dot{x}_i, \dot{y}_i) \times (x_k, \dot{y}_k)) = (0, a, \dot{x}_0, 0) \times \prod_{i=1}^{k-1}(x_i, y_i, \dot{x}_i, \dot{y}_i) \\ \times (x_k, 0, 0, \dot{y}_k) \times \prod_{i=1}^{k-1}(x_{k-i}, -y_{k-i}, -\dot{x}_{k-i}, \dot{y}_{k-i}) \\ \times (0, -a, -\dot{x}_0, 0) \times \prod_{i=1}^{k-1}(-x_i, -y_i, -\dot{x}_i, -\dot{y}_i) \\ \times (-x_k, 0, 0, -\dot{y}_k) \times \prod_{i=1}^{k-1}(-x_{k-i}, y_{k-i}, \dot{x}_{k-i}, -\dot{y}_{k-i}) \end{aligned}$$

For $N = 4k + 2$ we set

$$\begin{aligned} E(\dot{x}_0 \times \prod_{i=1}^{k}(x_i, y_i, \dot{x}_i, \dot{y}_i)) = (0, a, \dot{x}_0, 0) \times \prod_{i=1}^{k}(x_i, y_i, \dot{x}_i, \dot{y}_i) \\ \times \prod_{i=0}^{k-1}(x_{k-i}, -y_{k-i}, -\dot{x}_{k-i}, \dot{y}_{k-i}) \times (0, -a, -\dot{x}_0, 0) \\ \times \prod_{i=1}^{k}(-x_i, -y_i, -\dot{x}_i, -\dot{y}_i) \times \prod_{i=0}^{k-1}(-x_{k-i}, y_{k-i}, \dot{x}_{k-i}, -\dot{y}_{k-i}) \end{aligned}$$

In both cases a is a parameter fixing the size of the orbit.

We define the Poincaré map by requiring that

- for $N = 4k$: bodies k and $k-1$ have equal x coordinate ($x_k = x_{k-1}$),
- for $N = 4k + 2$: k-th body is on the X axis ($y_k = 0$).

This defines the Poincaré map $P : \mathbb{R}^{4N} \supset \Omega \longrightarrow \mathbb{R}^{4N}$.

We define the reduction map $R : \mathbb{R}^{4N} \longrightarrow \mathbb{R}^{N-1}$ in such way that R has zeroes in points satisfying conditions (3a) and (3b) in Lemma 4.1 and only in such points.

Observe that we don't have to worry about conditions (3c) and (3d) in Lemma 4.1, because from the properties of (1.1) if follows that if (2c) and (2d) holds, then (3c) and (3d) are satisfied for *any* t_1.
For $N = 4k$ we set

$$R\left(\prod_{i=0}^{N-1}(x_i, y_i, \dot{x}_i, \dot{y}_i)\right) = (y_k + y_{k-1}, \dot{x}_k + \dot{x}_{k-1}, \dot{y}_k - \dot{y}_{k-1})$$
$$\times \prod_{i=0}^{k-2}(x_i - x_{2k-i-1}, y_i + y_{2k-i-1}, \dot{x}_i + \dot{x}_{2k-i-1}, \dot{y}_i - \dot{y}_{2k-i-1})$$

For $N = 4k + 2$ we set

$$R\left(\prod_{i=0}^{N-1}(x_i, y_i, \dot{x}_i, \dot{y}_i)\right) = \{\dot{x}_k\} \times \prod_{i=0}^{k-1}(x_i - x_{2k-i}, y_i + y_{2k-i}, \dot{x}_i + \dot{x}_{2k-i}, \dot{y}_i - \dot{y}_{2k-i})$$

We define the map $\Phi : \mathbb{R}^{N-1} \supset E^{-1}(\Omega) \longrightarrow \mathbb{R}^{N-1}$ by

$$\Phi = R \circ P \circ E.$$

We have the following easy theorem.

Theorem 4.2. *If for some* $x \in \mathbb{R}^{N-1}$ $\Phi(x) = 0$*, then there exists a trajectory with properties P1, P2, P3.*

5. Existence of the SuperEight - the Gerver orbit

The Gerver orbit (Figure 6, 7) is a choreography with 4 bodies forming a linear chain. It's the simplest trajectory after the Eight.

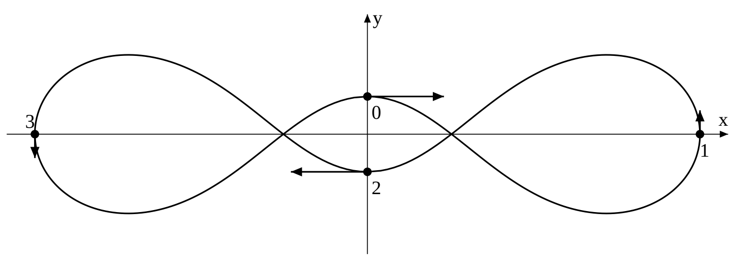

Figure 6. The Gerver orbit - the initial position

With a computer assistance we proved the following

Theorem 5.1. *The Gerver SuperEight exists and is locally unique (up to obvious symmetries and rescaling).*

We show the existence of SuperEight using an approach described in Section 4, i.e. we verify assumptions of Theorem 4.2. Below we give some details.
We set

$$E(x_1, \dot{x}_0, \dot{y}_1) = (0, a, \dot{x}_0, 0, x_1, 0, 0, \dot{y}_1, 0, -a, -\dot{x}_0, 0, -x_1, 0, 0, -\dot{y}_1), \tag{5.1}$$

where a is a parameter fixing the size of the orbit.
(5.2)
$R(x_0, y_0, \dot{x}_0, \dot{y}_0, x_1, y_1, \dot{x}_1, \dot{y}_1, x_2, y_2, \dot{x}_2, \dot{y}_2, x_3, y_3, \dot{x}_3, \dot{y}_3) = (y_1 + y_0, \dot{x}_1 + \dot{x}_0, \dot{y}_1 - \dot{y}_0),$

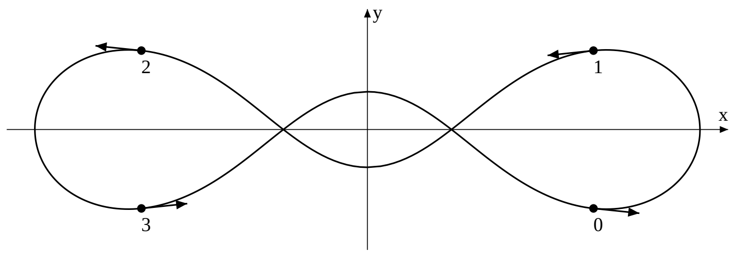

FIGURE 7. The Gerver orbit - the final position

Initial values	
$\bar{x}$	(1.382857, 1.87193510824,0.584872579881)
a	0.157029944461
$[X]$	$\bar{x} + [-10^{-7}, 10^{-7}]^3$

TABLE 5. Data from the proof of the existence of Gerver SuperEight. Initial values.

The Poincáre section S is defined by

$$S(x_0, y_0, \dot{x}_0, \dot{y}_0, x_1, y_1, \dot{x}_1, \dot{y}_1, x_2, y_2, \dot{x}_2, \dot{y}_2, x_3, y_3, \dot{x}_3, \dot{y}_3) = x_1 - x_0 = 0. \tag{5.3}$$

At first we have proved the existence of a zero of $\Phi(x)$ using the interval Newton method, but here we present data from the proof based on the Krawczyk method. In case of the Gerver orbit the choice of the method was'nt important, because if we take a time step small enough or smaller set $[X]$, then the computed matrix $\left[\frac{\partial \Phi}{\partial x}([X])\right]$ becomes invertible and the proof based on the interval Newton method goes through.

We used the C^1-Lohner algorithm with the order $r = 6$ and the time step was set to $h = 0.002$. As matrix C we used the value of $\frac{\partial \Phi(\bar{x})}{\partial x}^{-1}$ computed using a non-rigorous algorithm.

Tables 5 and 6 contain numerical data from this proof.

6. Existence of the 'Linear chain' orbit for the 6 bodies

Figure 8 displays a linear chain choreography with 6 bodies.

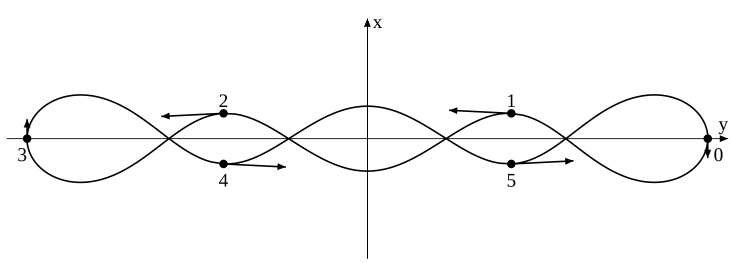

FIGURE 8. 'Linear chain' orbit for the 6 bodies - the initial position

Computed values	
C	$\begin{bmatrix} -2.15400 & 0.257911 & 0.786925 \\ -0.08163 & 0.293713 & 0.043565 \\ 0.939059 & -0.10027 & 0.158399 \end{bmatrix}$
$\Phi(\bar{x})$	$\begin{bmatrix} [-2.87020e-09, -2.26613e-09] \\ [-1.21155e-08, -1.06812e-08] \\ [-5.45542e-08, -5.10016e-08] \end{bmatrix}$
diam $\Phi(\bar{x})$	$\begin{bmatrix} 6.04064e-10 \\ 1.43432e-09 \\ 3.55268e-09 \end{bmatrix}$
$\frac{\partial\Phi}{\partial x}([X])$	$\begin{bmatrix} [-0.1664, -0.1657] & [0.39070, 0.39119] & [0.71771, 0.71790] \\ [-0.1764, -0.1750] & [3.52548, 3.52654] & [-0.0968, -0.0964] \\ [0.87189, 0.87534] & [-0.0867, -0.0842] & [1.99599, 1.99697] \end{bmatrix}$
$K(\bar{x}, [X], \Phi)$	$\begin{bmatrix} [1.382857036247056692, 1.382857041633411832] \\ [1.871935113301492981, 1.871935114053588922] \\ [0.5848725887384301769, 0.5848725902808686872] \end{bmatrix}$
diam $K(\bar{x}, [X], \Phi)$	$\begin{bmatrix} 5.386355139691545446e-09 \\ 7.520959410811656198e-10 \\ 1.54243851024915557e-09 \end{bmatrix}$

Table 6. Data from the proof of the existence of the Gerver SuperEight. Matrix C and results of computation.

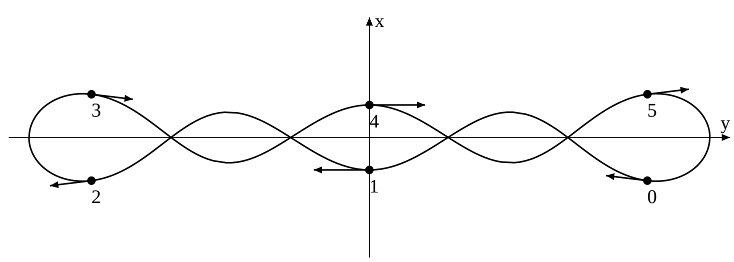

Figure 9. 'Linear chain' orbit for the 6 bodies - the final position

In this section we report about the computer assisted proof of the following

Theorem 6.1. *The linear chain for* 6 *bodies exists and is locally unique (up to obvious symmetries and rescaling).*

We prove the theorem above using an approach described in Section 4 with some minor changes. To speed up the calculation and to increase the accuracy we take into account that for all time $q_3(t) = -q_0(t)$, $q_4(t) = -q_1(t)$ and $q_5(t) =$

	Initial value
$\bar{x}$	$\begin{bmatrix} -0.635277524319 \\ 0.140342838651 \\ 0.797833002006 \\ 0.100637737317 \\ -2.03152227864 \end{bmatrix}$
a	1.887041548253914
$[X]$	$\begin{bmatrix} [-0.635277525319, -0.635277523319] \\ [0.140342837651, 0.140342839651] \\ [0.797833001006, 0.797833003006] \\ [0.100637736317, 0.100637738317] \\ [-2.03152227964, -2.03152227764] \end{bmatrix}$
diam $[X]$	$\begin{bmatrix} 2.0e-09 \\ 2.0e-09 \\ 2.0e-09 \\ 2.0e-09 \\ 2.0e-09 \end{bmatrix}$

Table 7. Data from the proof of the existence of linear chain for 6 bodies. Initial values.

$-q_2(t)$. Hence, our phase space become 12-dimensional. We use also a different time parameterization (we use a time shift of $\frac{1}{4}$ of the period). After this time shift in order to use the approach described in Section 4 we need also to interchange axes (see Fig. 8 and 9). From Lemma 4.1 we obtain, in this coordinates frame, a doubly symmetric periodic solution $q(t)$. Then it is easy to see that $\bar{q} = q(t - \frac{T}{4})$ is a solution with required symmetries in the original coordinate frame.

We set

$$E(\dot{x}_0, x_1, y_1, \dot{x}_1, \dot{y}_1) = (0, a, \dot{x}_0, 0, x_1, y_1, \dot{x}_1, \dot{y}_1, x_1, -y_1, -\dot{x}_1, \dot{y}_1), \tag{6.1}$$

where a is a parameter fixing the size of the orbit.

(6.2)

$$R(x_0, y_0, \dot{x}_0, \dot{y}_0, x_1, y_1, \dot{x}_1, \dot{y}_1, x_2, y_2, \dot{x}_2, \dot{y}_2) = (\dot{x}_1, x_0 - x_2, y_0 + y_2, \dot{x}_0 + \dot{x}_2, \dot{y}_0 - \dot{y}_2).$$

The Poincáre section S is defined by

$$S(x_0, y_0, \dot{x}_0, \dot{y}_0, x_1, y_1, \dot{x}_1, \dot{y}_1, x_2, y_2, \dot{x}_2, \dot{y}_2) = y_1 = 0. \tag{6.3}$$

To find a zero of $\Phi(x)$ we use the Krawczyk method. To compute the Poincaré map we use the C^1-Lohner algorithm [ZLo] of the order $r = 9$ and the time step $h = 0.0025$ for the computation in point $\bar{x}$ and $h = 0.001$ for the computation on the set $[X]$. As matrix C we used a nonrigorusly computed point matrix $\frac{\partial \Phi(\bar{x})}{\partial x}^{-1}$. In Tables 7 and 8 we give data from these computations.

	Computed value
$\Phi(\bar{x})$	$\begin{bmatrix} [-3.1311957909658e-11, 3.156277062700585e-11] \\ [-4.528821762050939e-12, 4.574757239694804e-12] \\ [-1.063704679893362e-11, 1.051470022161993e-11] \\ [-3.084105193451592e-11, 3.117495150917193e-11] \\ [-1.203726007759087e-11, 1.193112275643671e-11] \end{bmatrix}$
$\operatorname{diam}\Phi(\bar{x})$	$\begin{bmatrix} 6.287472853666386e-11 \\ 9.103579001745743e-12 \\ 2.115174702055356e-11 \\ 6.201600344368785e-11 \\ 2.396838283402758e-11 \end{bmatrix}$
$K(\bar{x}, [X], \Phi)$	$\begin{bmatrix} [-0.6352775243616679557, -0.6352775242763283314] \\ [0.1403428386430521646, 0.1403428386590999943] \\ [0.797833001999263769, 0.797833002012834469] \\ [0.1006377372881742 5324, 0.1006377373457752189] \\ [-2.031522278710178764, -2.031522278575771612] \end{bmatrix}$
$\operatorname{diam} K(\bar{x}, [X], \Phi)$	$\begin{bmatrix} 8.533962425616437 03e-11 \\ 1.604782973174678773e-11 \\ 1.357070011920313846e-11 \\ 5.760096566387318262e-11 \\ 1.344071520748002513e-10 \end{bmatrix}$

TABLE 8. Data from the proof of the existence of linear chain for 6 bodies.

7. Some technical data

All computations were performed on *AMD Athlon 1700XP* with 256 MB DRAM memory, with Windows 98SE or Linux Red Hat 8.0 operating system. We used CAPD package[Capd] and Borland C++ 5.02 compiler (under Linux we used g++ 3.02). Our interval arithmetics (part of CAPD package) was based on the double precision reals. The source code of our programm is available at the first author web page [Ka].

In the listing below r is an order and h is a time step used in the C^1-Lohner algorithm [ZLo].

Computation times for The Eight, $h = 0.01$, $r = 7$

- in point $\bar{x}$: 1.417 sec
- for set $[X]$: 2.66 sec
- convexity : 1.15501 sec

For the proof of the existence of the Gerver solution with four bodies problem we used $r = 6$, $h = 0.002$. The computation times for both $\bar{x}$ and set $[X]$ were approximately equal to 30.5 seconds.

In the proof of the linear chain of 6-bodies

- the computation of the Poincaré map for set $[X]$ took 57.5 seconds with $h = 0.001$ and $r = 9$
- the computation for $\bar{x}$ took 23.8 seconds with $h = 0.0025$ and $r = 9$

8. Conclusions and future directions

In principle there is no theoretical limit for the number of bodies in the doubly linear chain to which our method applies. By this we mean the following if the choreography is isolated in the reduced space, then the computer assisted proof of its existence is possible based on the approach and algorithms used in this paper. This may obviously require small time steps and/or higher order Taylor method and higher precision arithmetics.

We tried to see how far we can go with the number of bodies with , double precision interval arithmetics and our code. Using the approach described in Section 4 we proved the existence of doubly symmetric linear chains for ten and twelve bodies. Data from these proofs are available on the first author web page [Ka]. Our approach failed for doubly symmetric linear chain of sixteen bodies. We skipped the fourteen bodies case, as we did not have good initial conditions. We believe that the reason for the failure for sixteen bodies was the wrapping effect [Mo, Lo] for intervals sets arising from the round-off error of the double precision arithmetic

We think that to obtain the proof for the existence of doubly symmetric linear chains with more bodies using our approach it should be enough to do one of the following modifications

- use higher precision arithmetics
- instead of the simple shooting use the multiple shooting [Lo, Lo1, Sc]. This technique corresponds to the intermediate section method used very effectively in the computer assisted proofs of chaotic behavior in the Lorenz system [GZ, MM, T]

One may also think of more radical changes (like the change of the algorithm for rigorous integration of ODEs or totaly different approach), which may eventually result in improvements over the proposed method/algorithms. Below we list some possibilities

- the Taylor model method for an integration of ODEs advocated by Berz and his coworkers (see [BMH] and references given there). This method is very slow compared to the Lohner algorithm, but it still works in cases were the Lohner algorithm fails to produce reasonable bounds.
- the shadowing technique of Stoffer-Kichgraber [SK], which is can be seen very as an efficient mixture of hyperbolic shadowing and the intermediate section method. In [SK] in the context of the planar restricted three body problem this approach has been shown much more efficient than the one based on the C^1-Lohner algorithm,
- rigorous numerics for variational methods. We believe that it is an interesting problem in itself to 'construct' the rigorous numerics for the variational approach to the N-body problem, which will turn the numerical-variational work of Simó [S2] and Nauenberg [Na] into computer assisted proofs.

Regarding the Eight: we believe that by combining the methods presented in this paper with some analytical estimates, which will make the reduced space to be searched for for the choreography compact, one should be able to answer the following open questions [Ch]

- the global uniqueness of the Eight in the class of doubly symmetric choreographies,
- the global uniqueness of the Eight in the class of choreographies with less symmetry (see [Ch, FT] for an explanation).

We hope to treat above problems in the near future.

On the other side the question of the (local) uniqueness of the Eight (up to obvious isometries and rescaling) in the class of choreographies with zero angular momentum appears to be more difficult. The problem is related to the geometric phase in the N-body problem(see [MR1] and the literature given there), which in our context can be represented as follows:

Assume that we have three bodies in the collinear configuration (with zero total linear and angular momenta). Let α_0 be a line containing all the bodies, whose velocities and positions are given by $q_i'(0)$ and $q_i(0)$ for $i = 1, 2, 3$.

Assume now that after time T the bodies are again in the collinear configuration (represented by the line $\alpha(T)$) and such that $\dot{q}_i(T) = \dot{q}_{\sigma(i)}(0)$, where σ is a cyclic permutation. Can we then claim that $\alpha_0 = \alpha_T$ and $q_i(T) = q_{\sigma(i)}(0)$?

It turns out that a similar problem appears, when one tries to use our approach to the case of the choreographies without any symmetry.

Another interesting question about the Eight is its stability. Numerical experiments of Simó [S3] show that the Eight is KAM-stable. Rigorous verification of this statement requires checking that the Eight is linearly stable (which is possible in principle using our algorithm) and that a twist condition is satisfied [SM]. The twist condition requires rigorous computations of higher derivatives of a Poincaré map, hence a development of the robust and efficient C^k-Lohner algorithm for $k > 1$ is desirable. We believe that a suitable generalization of the C^1-Lohner algorithm will do the work.

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