Lohner Algorithm for perturbations of ODEs and differential inclusions

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Abstract

We describe a modification of the Lohner algorithm which allows us to compute rigorously bounds for solutions of differential inclusions and perturbations of ODEs.

1 Introduction

In this note we present a Lohner-type algorithm for an rigorous integration of perturbations of ODEs, which can be seen also as an algorithm for an integration of differential inclusions . This paper depends heavily on [Z1], as the proposed algorithm is just a modification running on top of the C^0-Lohner algorithm for ODEs described (after [Lo, Lo1]) there.

We study the following nonautonomous ODE

$$x'(t) = f(x(t), y(t)) \tag{1}$$

where $x \in \mathbb{R}^n$ and $y : \mathbb{R} \supset D \to \mathbb{R}^m$. Assume that we have some knowledge about $y(t)$, for example $|y(t)| < \epsilon$ for $0 \leq t \leq T$. We would like to find a rigorous enclosure for $x(t)$.

The problem of this type arises for example in the context of rigorous computation of a Poincaré map for dissipative PDEs (see [ZM, Z2] for more details). In this setting x represents the dominating modes and y is a tail of the Fourier expansion, so that (1) is complemented by the equation for y of the form $y'(t) = g(x(t), y(t))$. The proposed algorithm works, as we were able using it prove the existence of nontrivial periodic orbit for Kuramoto-Sivashinsky PDE [Z2].

The proposed algorithm can also be used to find rigorous bounds for solutions of differential inclusions

$$x' \in h(x) + \epsilon(t), \tag{2}$$

where $h : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1-vector field and $\epsilon(t) \subset \mathbb{R}^n$. We can cast (2) in the form (1) by setting $f(x, y) = h(x) + y$ and requiring that $y(t) \in \epsilon(t)$ for all t.

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Non-autonomous ODEs represent another important class of applications. While one can easily modify the Lohner algorithm to handle an nonautonomous ODE directly, it makes sense to apply the proposed Lohner-type algorithm for perturbed ODEs for (1), because only in this way we can estimate rigorously the Poincaré map on a section $\alpha(x) = 0$ (defined in terms of x only) for any initial conditions (x, t_0). This kind of algorithm shall allow to attack the question of symbolic dynamics for nonautonomous ODEs (see [CZ]) and ODEs with small delays (see [WjZ]).

The content of this note can be described as follows: in Sections 2 and 3 we derive basic equations for comparison of perturbed and unperturbed ODEs, which are the basis of our approach. In Section 4 we give a description of one step of the proposed Lohner-type algorithm. In Section 5 we describe how to estimate the trajectory of (1) between time steps which allows to compute the Poincaré map.

1.1 Basic notation

We will use the same conventions as in [Z1]. In the sequel, by arabic letters we denote single valued objects like vectors, real numbers, matrices. Quite often in this paper we will use square brackets, for example $[r]$, to denote sets. Usually this will be some set constructed in the algorithm. Sets will also be denoted by single letters, for example S, when it is clear from the context that it represents a set. In situations when we want to stress (for example in the detailed description of algorithm) that we have a set in a formula involving both single-valued objects and sets together we will rather use square bracket, hence we prefer to write $[S]$ instead of S to represent a set. From this point of view $[S]$ and S are different symbols in the alphabet used to name variables and formally speaking there is no relation between the set represented by $[S]$ and the object represented by S. Quite often in the description of the algorithm we will have a situation that both variables $[S]$ and S are used simultaneously, then usually $S \in [S]$, but this is always stated explicitly.

For a set $[S]$ by $[S]_I$ we denote the interval hull of $[S]$, i.e. the smallest product of intervals containing $[S]$. The symbol $\mathrm{hull}(x_1, \ldots, x_k)$ will denote the interval hull of intervals $x_1, \ldots, x_k$. For any interval set $[S] = [S]_I$ by $\mathrm{m}([S])$ we will denote a center point of $[S]_I$. For any interval $[a, b]$ we define a diameter by $\mathrm{diam}([a, b]) = b - a$. For an interval vector or an interval matrix $[S] = [S]_I$ by $\mathrm{diam}\,([S])$ we will denote the maximum of diameters of its components. For an interval $[x^-, x+]$ we set $right([x^-, x^+]) = x^+$ and $left([x^-, x^+]) = x^-$.

For a set $X \subset \mathbb{R}^d$ by $\mathrm{int}\, X$ we denote an interior of X.

For $v, w \in \mathbb{R}^n$ and $A, B \in \mathbb{R}^{n\times n}$ $(n = 1, \ldots, \infty)$ we say that

$$\begin{array}{lll} v \leq w & \text{iff} & \forall i \quad v_i \leq w_i, \\ A \leq B & \text{iff} & \forall ij \quad A_{ij} \leq B_{ij}. \end{array}$$

1.2 Warning.

At the first encounter with the question of an rigorous integration of (1) one may hope that the direct application of any algorithm for rigorous integration of ODEs should be enough for (2). To this end consider a differential inclusion

$$x' \in f(x) + \epsilon(t), \qquad \epsilon(t) \in [\epsilon] = \Pi_{i=1}^{n}[-\epsilon_i, \epsilon_i]. \tag{3}$$

and a related ODE

$$x' = f(x) + \epsilon, \qquad \epsilon \in [\epsilon]. \tag{4}$$

One may naively hope that, for example, the Lohner algorithm applied to (4) with $[\epsilon]$ as an interval parameter in the definition of a constant term in $f(x)$ will give an enclosure not only for (4), but also for (3). For this to be true we need the following

Lemma 1 *Assume $x(t)$ satisfies (3) for $t \in [0, T]$.*

Then for any $t \in [0, T]$ there exists $\epsilon \in [\epsilon]$ such that $x_\epsilon(t) = x(t)$ and $x_\epsilon(0) = x(0)$, where $x_\epsilon$is a solution of (4).

The above lemma is false as shown by the following example [Se].

Consider a differential inclusion given by

$$\begin{aligned} x' &\in y + [-\epsilon, \epsilon], \\ y' &\in -x + [-\epsilon, \epsilon]. \end{aligned} \tag{5}$$

For fixed $\delta \in [-\epsilon, \epsilon]^2$ we have the following system of ODEs

$$\begin{aligned} x' &= y + \delta_1, \\ y' &= -x + \delta_2, \end{aligned} \tag{6}$$

all solutions with an initial condition in a compact set have a uniform bound independent of δ for $t > 0$, which is given by the energy integral for (6)

$$(x - \delta_2)^2 + (y + \delta_1)^2. \tag{7}$$

This is not the case for the solutions of (5) as it is clearly seen for $\epsilon(t)$ given as a resonant forcing

$$\begin{aligned} x' &= y, \\ y' &= -x + \epsilon \sin t. \end{aligned} \tag{8}$$

2 Componentwise bounds for perturbations of ODEs

2.1 One-sided Lipschitz condition

We present here some known results about component-wise bounds for systems of ODEs. What follows is a particular case of Theorem I in Section 13 in [W](see subsection IV 'The Lipschitz condition').

Consider an ordinary differential equation

$$x' = f(x), \qquad x \in \mathbb{R}^n, \quad f \in C(\mathbb{R}^n, \mathbb{R}^n) \tag{9}$$

Theorem 2 *Suppose f satisfies the following Lipschitz condition*

$$f_i(x) - f_i(y) \leq \sum_j A_{ij}|x_j - y_j|, \qquad \text{if } x_i \geq y_i \tag{10}$$

for all $i = 1, \dots, n$ and $x, y \in \mathbb{R}^n$. We assume that

$$A_{ij} \geq 0, \qquad i \neq j. \tag{11}$$

Then if $\sigma : [0, \infty) \to \mathbb{R}^n$ is a solution of following linear equation

$$\sigma' = A\sigma + \delta(t), \qquad \sigma(0) = \epsilon, \tag{12}$$

the inequality

$$|x_i(t) - y_i(t)| \leq \sigma_i(t), \qquad \text{for } i = 1, \dots, n \text{ and } t \geq 0 \tag{13}$$

holds between the solution x of (9) and an approximate solution $y(t)$ for which for $i = 1, \dots, n$

$$\begin{aligned} |x_i(0) - y_i(0)| &\leq \epsilon_i \\ |y_i'(t) - f_i(y(t))| &\leq \delta_i(t). \end{aligned}$$

As a consequence one obtains the following theorem.

Theorem 3 *Assume that f in (9) is a C^1 function on its domain. Let $Z \subset dom(f)$ be a bounded closed convex set. Assume that $x : [0, T] \to Z$ is a solution of (9) and $y : [0, T] \to Z$ is a continuous function, which satisfies the following inequalities*

$$\begin{aligned} \left|\frac{d^+ y_i}{dt}(t) - f_i(y(t))\right| &\leq \delta_i, \qquad \text{for } i = 1, \dots, n \text{ and } t \in [0, T], \\ |y_i(0) - x_i(0)| &\leq \epsilon_i, \qquad \text{for } i = 1, \dots, n, \end{aligned}$$

where by $\frac{d^+ y_i}{dt}(t)$ we denote the right derivative of the function y_i.

Assume that a matrix $J \in \mathbb{R}^{n \times n}$ satisfies the following inequalities

$$J_{ij} \geq \begin{cases} \sup_{x \in Z} \left|\frac{\partial f_i}{\partial x_j}(x)\right| & \text{for } i \neq j, \\ \sup_{x \in Z} \frac{\partial f_i}{\partial x_i}(x) & \text{for } i = j. \end{cases} \tag{14}$$

Then

$$|x_i(t) - y_i(t)| \leq \left(e^{Jt}\epsilon\right)_i + \left(\int_0^t e^{J(t-s)}\delta \, ds\right)_i \tag{15}$$

For the proof it is enough to observe that from (14) it follows that (10) is satisfied.

2.2 The proof of Theorem 3

In this section we provide a proof of Theorem 3. We include it here to make it easy to verify and to make its assumptions transparent. This is not a case for Theorem 2 in [W], because there it is a consequence of other theorems obtained in a more general context, hence it is not obvious just from browsing this monograph, what the assumptions are.

We start with recalling (see [HNW, I.10]) the notion of the so-called *Dini-derivatives*

$$D^+ m(x) = \limsup_{h \to 0^+} \frac{m(x+h) - m(x)}{h}$$
$$D_+ m(x) = \liminf_{h \to 0^+} \frac{m(x+h) - m(x)}{h}$$

As an easy consequence of the triangle inequality we have

$$-|w'(x)| \leq D_+|w(x)| \leq |w'(x)|, \qquad -|w'(x)| \leq D^+|w(x)| \leq |w'(x)|, \tag{16}$$

provided that w does have the right derivative at x.

Lemma 4 *Assume that $x, f : [0,T] \to \mathbb{R}$ are continuous. If*

$$D_+ x(t) \leq f(t), \qquad \text{for } t \in [0,T]$$

then

$$x(t) \leq x(0) + \int_0^t f(s)\, ds, \qquad \text{for } t \in [0,T]$$

The same conclusion is true if D_+ is replaced by D^+.

Proof: Let us fix any $\epsilon > 0$. Obviously

$$D_+ x(t) < f(t) + \epsilon. \tag{17}$$

Let $y(t) = x(0) + \int_0^t (f(s) + \epsilon)\, ds$. We will show that $x(t) \leq y(t)$ for $t \in [0,T]$.

Assume the contrary. By taking a suitable infimum we find $t_0 \in [0,T]$, such that $x(t_0) = y(t_0)$ and $x(t_0 + h) > y(t_0 + h)$ for $h > 0$ sufficiently small.

We have

$$x(t_0 + h) - x(t_0) > y(t_0 + h) - y(t_0) = \int_{t_0}^{t_0+h} (f(s) + \epsilon)\, ds, \tag{18}$$

hence

$$D_+ x(t_0) \geq f(t_0) + \epsilon > f(t_0), \tag{19}$$

which contradicts our assumption.

Hence for any $\epsilon > 0$ we have

$$x(t) \leq x(0) + \epsilon t + \int_0^t f(s)\, ds. \tag{20}$$

Passing to the limit $\epsilon \to 0$ we obtain our assertion. ∎

Lemma 5 *Assume $f : \mathbb{R} \to \mathbb{R}$ is $\mathcal{C}^1$ function and $f(x) > 0$ for $x \in \mathbb{R}$. Then for any continuous function $x : I \to \mathbb{R}$, where I interval holds*

$$\begin{aligned} D_+(f(t)x(t)) &= f'(t)x(t) + f(t)D_+x(t), \\ D^+(f(t)x(t)) &= f'(t)x(t) + f(t)D^+x(t) \end{aligned}$$

Proof: We will provide the proof for D_+ only, the other case is analogous.

$$\frac{f(t+h)x(t+h) - f(t)x(t)}{h} = x(t+h)\frac{f(t+h) - f(t)}{h} + f(t)\frac{x(t+h) - x(t)}{h}$$

Continuity of x implies that the first term converges for $h \to 0$ to $x(t)f'(t)$ and positivity of $f(t)$ implies that $\liminf_{h\to 0^+} f(t)\frac{x(t+h)-x(t)}{h} = f(t)D_+x(t)$. ∎

The fundamental lemma in this section is:

Lemma 6 *Assume that $z : [0,T] \to \mathbb{R}^n$ is a continuous map with right derivatives satisfies the following equation*

$$\frac{d^+z}{dt}(t) = A(t) \cdot z(t) + \delta(t), \tag{21}$$

where $\delta : [0,T] \to \mathbb{R}^n$ and $A : [0,T] \to \mathbb{R}^{n\times n}$ are bounded functions (not necessarily continuous) and $\frac{d^+z}{dt}(t)$ denotes the right derivative of z at t.

Assume that matrix $J \in \mathbb{R}^{n\times n}$ satisfies the following inequalities

$$J_{ij} \geq \begin{cases} \sup_{t\in[0,T]} |A_{ij}(t)| & \text{for } i \neq j, \\ \sup_{t\in[0,T]} A_{ij}(t) & \text{for } i = j. \end{cases} \tag{22}$$

Then

$$|z|(t) \leq e^{Jt} \cdot |z|(0) + \int_0^t e^{J(t-s)}C\, ds, \tag{23}$$

where $C_i = \sup_{t\in[0,T]} |\delta_i(t)|$ and $|z|(t) = (|z_1(t)|, |z_2(t)|, \ldots, |z_n(t)|)$.

Proof: We define matrix $\tilde{A}(t)$ by

$$\tilde{A}_{ij}(t) = \begin{cases} A_{ij}(t) & i = j, \\ |A_{ij}(t)| & \text{otherwise.} \end{cases} \tag{24}$$

We will show that for $i = 1, \ldots, n$

$$D_+|z_i|(t) \leq \sum_j \tilde{A}_{ij}(t)|z_j(t)| + C_i. \tag{25}$$

Let us fix $t \in [0,T]$. We will consider three cases $z_i(t) > 0$, $z_i(t) < 0$ and $z_i(t) = 0$.

Case $z_i(t) > 0$. In this case locally $|z_i| = z_i$, hence

$$D_+|z_i|(t) = \frac{d^+z_i}{dt}(t) = A_{ii}(t)z_i(t) + \sum_{j,j\neq i} A_{ij}(t)z_j(t) + \delta_i(t) \leq$$
$$A_{ii}(t)|z_i(t)| + \sum_{j,j\neq i} |A_{ij}(t)||z_j(t)| + C_i = \sum_j \tilde{A}_{ij}|z_j(t)| + C_i.$$

Case $z_i(t) < 0$. In this case locally $|z_i| = -z_i$, hence

$$D_+|z_i|(t) = -\frac{d^+z_i}{dt}(t) = -A_{ii}(t)z_i(t) - \sum_{j,j\neq i} A_{ij}(t)z_j(t) - \delta_i(t) \leq$$
$$A_{ii}(t)|z_i(t)| + \sum_{j,j\neq i} |A_{ij}(t)||z_j(t)| + C_i = \sum_j \tilde{A}_{ij}|z_j(t)| + C_i.$$

Case $z_i(t) = 0$.

$$D_+|z_i|(t) \leq |\frac{d^+z_i}{dt}(t)| = A_{ii}(t)|z_i(t)| + \left|\sum_{j,j\neq i} A_{ij}(t)z_j(t) + \delta_i(t)\right| \leq$$
$$A_{ii}(t)|z_i(t)| + \sum_{j,j\neq i} |A_{ij}(t)||z_j(t)| + C_i = \sum_j \tilde{A}_{ij}|z_j(t)| + C_i.$$

Let us define

$$x(t) = (x_1(t), x_2(t), \ldots, x_n(t)) = (|z_1(t)|, |z_2(t), \ldots, |z_n(t)|). \tag{26}$$

Inequality (25) can be rewritten as follows

$$D_+x(t) \leq \tilde{A}(t) \cdot x(t) + C, \tag{27}$$

moreover we know that

$$\begin{array}{rcl} \tilde{A}_{ij}(t) & \geq & 0, \quad \text{for } i \neq j \\ x & \geq & 0 \end{array}$$

We define diagonal matrix $\Lambda \in \mathbb{R}^{n\times n}$ by $\Lambda_{ii} = J_{ii}$ and matrix $B \in \mathbb{R}^{n\times n}$ by

$$B_{ij} = \begin{cases} J_{ij} & \text{for } i \neq j, \\ 0 & \text{for } i = j. \end{cases} \tag{28}$$

Obviously $B_{ij} \geq 0$ and $J = \Lambda + B$.

For any $i = 1, \ldots, n$ from (27) we obtain

$$\begin{array}{rcl} D_+x_i(t) & \leq & \Lambda_{ii}x_i(t) + (\tilde{A}_{ii}(t) - \Lambda_{ii})x_i(t) + \sum_{j,j\neq i} \tilde{A}_{ij}(t)x_j(t) + C_i \leq \\ & & \Lambda_{ii}x_i(t) + \sum_j B_{ij}x_j(t) + C_i, \\ D_+x_i(t) - \Lambda_{ii}x_i(t) & \leq & \sum_j B_{ij}x_j(t) + C_i, \end{array}$$

hence from Lemma 5 we obtain

$$D_+ \left(e^{-\Lambda_{ii} t} x_i(t)\right) \leq e^{-\Lambda_{ii} t} \left(\sum_j B_{ij} x_j(t) + C_i \right).$$

The last inequality has the following vector form

$$D_+ \left(e^{-\Lambda t} x(t)\right) \leq e^{-\Lambda t} B x(t) + e^{-\Lambda t} C \tag{29}$$

From Lemma 4 applied to each coordinate in (29) we obtain

$$e^{-\Lambda t} x(t) \leq x(0) + \int_0^t e^{-\Lambda s} \left(B x(s) + C\right) \, ds \tag{30}$$

$$x(t) \leq e^{\Lambda t} x(0) + \int_0^t e^{\Lambda(t-s)} \left(B x(s) + C\right) \, ds \tag{31}$$

Consider now the integral equation corresponding to the above inequality

$$y(t) = e^{\Lambda t} y(0) + \int_0^t e^{\Lambda(t-s)} \left(B y(s) + C\right) \, ds. \tag{32}$$

Assume y satisfies (32), then an easy computation shows that

$$y'(t) = \Lambda e^{\Lambda t} y(0) + \Lambda \int_0^t e^{\Lambda(t-s)} \left(C + B y(s)\right) \, ds + C + B y(t) = \\ (\Lambda + B) y(t) + C.$$

Hence we have a solution for (32) given by

$$y(t) = e^{(\Lambda+B)t} y(0) + \int_0^t e^{(\Lambda+B)(t-s)} C ds. \tag{33}$$

Let us take y, such that $y(0) > x(0)$. We will show now that $y(t) > x(t)$ for $t \in [0, T]$. Let

$$t_0 = \sup\{t \in [0,T] \mid y(s) > x(s), \qquad s \in [0,t)\}. \tag{34}$$

Obviously from continuity of $y(t) - x(t)$ it follows that $t_0 > 0$. From (32) and (31) we obtain (because all matrixes and vectors in the expression on rhs are positive)

$$y(t_0) - x(t_0) \geq e^{\Lambda t_0} (y(0) - x(0)) + \int_0^{t_0} e^{\Lambda(t_0 - s)} B(y(s) - x(s)) \, ds > 0.$$

From continuity $y(t) > x(t)$ will hold for $t \in [t_0, t_0 + \delta)$ for some $\delta > 0$. This contradicts our choice of t_0.

Hence $y(t) > x(t)$. By passing to the limit $y(0) \to x(0)$ we obtain

$$x(t) \leq e^{(\Lambda+B)t} x(0) + \int_0^t e^{(\Lambda+B)(t-s)} C ds. \tag{35}$$

This finishes the proof, because $J = \Lambda + B$. ∎

Proof of Theorem 3: Let $z(t) = x(t) - y(t)$ and $\delta(t) = f(y(t)) - \frac{d^+y}{dt}(t)$. We have

$$\frac{d^+z}{dt}(t) = f(x(t)) - f(y(t)) + \delta(t) =$$

$$\int_0^1 \frac{\partial f}{\partial x}(y(t) + s(x(t) - y(t)))ds \cdot (x(t) - y(t)) + \delta(t).$$

Hence

$$\frac{d^+z}{dt}(t) = A(t) \cdot z(t) + \delta(t), \tag{36}$$

where

$$A(t) = \int_0^1 \frac{\partial f}{\partial x}\left(y(t) + s(x(t) - y(t))\right) ds. \tag{37}$$

Now the assertion of the theorem follows immediately from Lemma 4. ∎

2.3 A fundamental estimate

We rewrite here the Theorem 3 in the form suitable to use in the algorithm for a rigorous integration of (1).

Lemma 7 *Let $h > 0$. Assume that $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be C^1 and $y : [t_0, t_0 + h] \to \mathbb{R}^m$ is bounded.*

Let $[W_y] \subset \mathbb{R}^m$ be convex and such that, $y([t_0, t_0 + h]) \subset [W_y]$.

Let $y_c \in [W_y]$. Assume that $x_1, x_2 : [t_0, t_0 + h] \to \mathbb{R}^n$, both continuous, satisfy

$$\begin{aligned} x_1' &= f(x_1, y_c), \quad x_1(t_0) = x_0, & (38)\\ \frac{d^+x_2}{dt} &= f(x_2, y(t)), \quad x_2(t_0) = x_0. & (39) \end{aligned}$$

Let $[W_1] \subset [W_2] \subset \mathbb{R}^n$ be convex and compact and such that

$$x_1(t) \in [W_1], \qquad x_2(t) \in [W_2], \qquad \text{for } s \in [t_0, t_0 + h].$$

Then the following inequality holds for $t \in [t_0, t_0 + h]$ and $i = 1, \ldots, n$

$$|x_{1,i}(t) - x_{2,i}(t)| \leq \left(\int_{t_0}^t e^{J(t-s)} C \, ds \right)_i, \tag{40}$$

where

$$\begin{aligned} [\delta] &= \{f(x, y_c) - f(x, y) \mid x \in [W_1], y \in [W_y]\}, \\ C_i &\geq \sup |[\delta_i]|, \quad i = 1, \ldots, n \\ J_{ij} &\geq \begin{cases} \sup \frac{\partial f_i}{\partial x_j}([W_2], [W_y]) & \text{if } i = j, \\ \sup \left| \frac{\partial f_i}{\partial x_j}([W_2], [W_y]) \right| & \text{if } i \neq j. \end{cases} \end{aligned}$$

Proof: We have

$$\frac{d^+}{dt}(x_1 - x_2)(t) = f(x_1(t), y_c) - f(x_2(t), y(t)) =$$
$$f(x_1(t), y_c) - f(x_1(t), y(t)) + f(x_1(t), y(t)) - f(x_2(t), y(t)) =$$
$$\delta(t) + A(t) \cdot (x_1(t) - x_2(t)),$$

where $\delta(t) \in [\delta]$ and

$$A_{ij}(t) = \int_0^1 \frac{\partial f_i}{\partial x_j}\,(x_2(t) + s(x_1(t) - x_2(t)), y(t))\, ds.$$

Moreover both functions $\delta : [t_0, t_0 + h] \to \mathbb{R}^n$ and $A : [t_0, t_0 + h] \to \mathbb{R}^{n \times n}$ are bounded.

We apply Lemma 6 to the function $x_1 - x_2$ to obtain (40). ∎

3 An approach based on the logarithmic norm

We derive here an alternative to the approach outlined in Section 2 and summarized in Lemma 7. Here the difference between the solutions of an ODE and its perturbation will be quantified by one number - the distance between them.

Lemma 8 *Let $h_0 > 0$. Assume that $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be C^1 and $y : [t_0, t_0 + h_0] \to \mathbb{R}^m$ is C^1.*

Let $[W_y] \subset \mathbb{R}^m$ be convex and such that, $y([t_0, t_0 + h_0]) \subset [W_y]$.

Let $y_c \in [W_y]$. Assume that $x_1, x_2 : [t_0, t_0 + h_0] \to \mathbb{R}^n$ satisfy

$$x_1' = f(x_1, y_c), \quad x_1(t_0) = x_0, \tag{41}$$
$$x_2' = f(x_2, y(t)), \quad x_2(t_0) = x_0. \tag{42}$$

Let $[W_1] \subset [W_2] \subset \mathbb{R}^n$ be convex and compact and such that

$$x_1(t) \in [W_1], \qquad x_2(t) \in [W_2], \qquad \text{for } s \in [t_0, t_0 + h_0].$$

Then for any $t \in [0, h_0]$ holds

$$\|x_2(t_0 + t) - x_1(t_0 + t)\| \leq \exp(lt) \int_{t_0}^{t_0+t} \exp(-ls)\|[\delta]\| ds = \frac{\|[\delta]\|}{l}(\exp(lt) - 1)$$

where $l = \mathrm{right}\left(\mu(\frac{\partial f}{\partial x}([W_2], y_c))\right)$, and μ is the logarithmic norm of the matrix (see [HNW] for the definition) and

$$[\delta] = \{f(x, y_c) - f(x, y) \mid x \in [W_1], y \in [W_y]\}.$$

Proof: We follow closely [HNW], where this lemma was essentially proven in a slightly different context. Observe first that from our assumptions it follows that both x_1 and x_2 are C^2 functions. Let $m(t) = \|x_1(t) - x_2(t)\|$. Then we have for any $t \in [t_0, t_0 + h_0]$ and $h > 0$

$$\begin{aligned} m(t+h) = \|x_1(t+h) - x_2(t+h)\| \leq \\ \|x_1(t) - x_2(t) + h(f(x_1(t), y_c) - f(x_2(t), y(t))\| + O(h^2) \leq \\ \|x_1(t) - x_2(t) + h(f(x_1(t), y_c) - f(x_2(t), y_c)\| + h\|[\delta]\| + O(h^2) \end{aligned}$$

Now we apply the mean value theorem to $x + hf(x)$ and to obtain

$$m(t+h) \leq \left(\max_{\eta \in [x_1(t), x_2(t)]} \|Id + h\frac{\partial f}{\partial x}(\eta, y_c)\| \right) m(t) + h\|[\delta]\| + O(h^2)$$

and finally (since $h > 0$)

$$\begin{aligned} \frac{m(t+h) - m(t)}{h} \leq \max_{\eta \in [x_1(t), x_2(t)]} \frac{\|Id + h\frac{\partial f}{\partial x}(\eta, y_c)\| - 1}{h} m(t) + \\ \|[\delta]\| + O(|h|) \end{aligned}$$

Hence

$$D_+ m(t) \leq lm(t) + \|[\delta]\|. \tag{43}$$

From differential inequalities we obtain (compare for example the proof of Lemma 6)

$$m(t) \leq m(t_0) e^{l(t-t_0)} + \frac{\|[\delta]\|}{l} \left(e^{l(t-t_0)} - 1 \right). \tag{44}$$

∎

4 The Lohner-type algorithm for perturbations of ODEs

For a given function $y : [0, \infty) \to \mathbb{R}^m$ let $\varphi(t, x_0, y)$ denotes a solution of equation (1) with an initial conditions $x(0) = x_0$. For a given $y_0 \in \mathbb{R}^m$ let $\overline{\varphi}(t, x_0, y_0)$ be a solution of the following Cauchy problem

$$x' = f(x, y_0), \quad x(0) = x_0 \tag{45}$$

with the same initial condition $x(0) = x_0$. Observe that system (45) is a particular case of (1) with $y(t) = y_0$.

We are interested in finding rigorous bounds for $\phi(t, [x_0], [y_0])$, where $[x_0] \subset \mathbb{R}^n$ and $[y_0] \subset C([0, \infty), \mathbb{R}^m)$. The set $[y_0]$ might be defined be some dynamical process, in this case we may need to compute something for each time step or be just given by the specifying the bounds, for example $y \in [y_0]$ iff $y(t) \in [-\epsilon, \epsilon]^m$.

To this end we propose a modification of the original Lohner algorithm [Lo, Lo1]. Our presentation and notation follows a description of the C^0-Lohner algorithm presented in [Z1].

4.1 One step of the algorithm

In the description below the objects with an index k refer to the current values and those with an index $k+1$ are the values after the next time step.

We define

$$[y_k] = \{y \in C([0,\infty], \mathbb{R}^m) \mid y(t) = z(t_k + t) \quad \text{for some } z \in [y_0]\}.$$

We will also us e the following notation for $[y] \subset C([0,\infty), \mathbb{R}^s)$ we set

$$[y]([t_1,t_2]) = \{z(t) \mid z \in [y], t \in [t_1,t_2]\}.$$

One step of the Lohner algorithm is a shift along the trajectory of system (1) with following input and output data:

Input data:

- t_k is a current time
- h_k is a time step
- $[x_k] \subset \mathbb{R}^n$, such that $\varphi(t_k, [x_0], [y_0]) \subset [x_k]$
- eventually some bounds for $[y_k]$

Output data:

- $t_{k+1} = t_k + h_k$ is a new current time
- $[x_{k+1}] \subset \mathbb{R}^n$, such that $\varphi(t_{k+1}, [x_0], [y_0]) \subset [x_{k+1}]$
- eventually some bounds for $[y_0][0, t_{k+1})$.

We do not specify here a form (a representation) of sets $[x_k]$. They can be interval sets, balls, doubletons etc. (see [MZ, Z1]). This issue is very important in handling of the wrapping effect and is discussed in detail in [Lo, Lo1] (see also Section 3 in [Z1]).

One step of the algorithm consists from the following parts:

1. Generation of a priori bounds for φ and $[y_0]([t_k, t_{k+1}])$.

We find a convex and compact set $[W_2] \subset \mathbb{R}^n$ and a convex set $[W_y] \subset \mathbb{R}^m$, such that

$$\varphi([0,h_k],[x_k],[y_k]) \subset [W_2] \tag{46}$$

$$[y_k]([0,h_k]) \subset [W_y] \tag{47}$$

2. We fix $y_c \in [W_y]$.

3. Computation of an unperturbed x-projection. We apply one step of the C^0-Lohner algorithm to (45) with a time step h_k and an initial condition given by $[x_k]$ and $y_0 = y_c$. As a result we obtain $[\overline{x}_{k+1}] \subset \mathbb{R}^n$ and a convex and compact set $[W_1] \subset \mathbb{R}^n$, such that

$$\begin{aligned} \overline{\varphi}(h_k,[x_k],y_c) &\subset [\overline{x}_{k+1}] \\ \overline{\varphi}([0,h_k],[x_k],y_c) &\subset [W_1] \end{aligned}$$

4. Computation of the influence of the perturbation. Using formulas from Lemmas 7 or 8 we find a set $[\Delta] \subset \mathbb{R}^n$, such that

$$\varphi(t_{k+1}, [x_0], [y_0]) \subset \overline{\varphi}(h_k, [x_k], y_c) + [\Delta]. \tag{48}$$

Hence

$$\varphi(t_{k+1}, [x_0], [y_0]) \subset [x_{k+1}] = [\overline{x}_{k+1}] + [\Delta] \tag{49}$$

5. Eventually we do some computation to obtain $[y_{k+1}]$

4.2 Part 1 - comments

In the context of an nonautonomous ODE with small and uniformly bounded $[\delta]$ we can set $[W_y] = \mathbb{R}$. To obtain $[W_2]$ any rough enclosure procedure devised for ODEs should work.

In the context of a dissipative PDE the whole story is more complicated and we refer the interested reader to [Z2].

4.3 Part 4 - details

In Lemmas 7 or 8 we have presented two ways to compute $[\Delta] = [\Delta](h)$ for $0 \leq h \leq h_k$.

An approach based on component-wise estimates (compare Lemma 7)

1. We set

$$\begin{aligned}
[\delta] &= [\{f(x, y_c) - f(x, y) \mid x \in [W_1], y \in [W_y]\}]_I \\
C_i &= \text{right}(|[\delta_i]|), \quad i = 1, \dots, n \\
J_{ij} &= \begin{cases} \text{right}\left(\frac{\partial f_i}{\partial x_i}([W_2], [W_y])\right) & \text{if } i = j, \\ \text{right}\left(\left|\frac{\partial f_i}{\partial x_j}([W_2], [W_y])\right|\right). & \text{if } i \neq j. \end{cases}
\end{aligned}$$

2. $D = \int_0^h e^{J(h-s)} C \, ds$

3. $[\Delta_i] = [-D_i, D_i]$, for $i = 1, \dots, n$

It remains to explain how we compute $\int_0^t e^{A(t-s)} C \, ds$. First observe that

$$\int_0^t e^{A(t-s)} C \, ds = t \left(\sum_{n=0}^{\infty} \frac{(At)^n}{(n+1)!} \right) \cdot C. \tag{50}$$

We fix any norm $\|\cdot\|$, for example L^∞-norm, i.e. $\|x\|_\infty = \max_i |x_i|$ (*we should rather chose a norm for which* $\|\tilde{A}\|$ *is the smallest one*). Let us set

$$\tilde{A} = At, \qquad A_n = \frac{\tilde{A}^n}{(n+1)!}.$$

In this notation

$$\sum_{n=0}^{\infty} \frac{(At)^n}{(n+1)!} = \sum_{n=0}^{\infty} A_n$$

$$A_0 = \mathrm{Id}, \qquad A_{n+1} = A_n \cdot \frac{\tilde{A}}{n+2}$$

For the remainder term we will use the following estimate

$$\|A_{N+k}\| \leq \|A_N\| \cdot \left\| \frac{\tilde{A}}{N+2} \right\|^k$$

Hence if $\left\| \frac{\tilde{A}}{N+2} \right\| < 1$, then

$$\left\| \sum_{n>N} A_n \right\| \leq \|A_N\| \cdot \left\| \frac{\tilde{A}}{N+2} \right\| \cdot \left(1 - \left\| \frac{\tilde{A}}{N+2} \right\|\right)^{-1} \tag{51}$$

An approach based on logarithmic norms:(compare Lemma 8) We fix any norm $\| \cdot \|$, preferably the L^∞-norm, i.e. $\|x\|_\infty = \max_i |x_i|$

1. $[\delta] = [\{f(x, y_c) - f(x, y) \mid x \in [W_1], y \in [W_y]\}]_I$.
2. $C = \|[\delta]\|$
3. $l = \mathrm{right}\left(\mu(\frac{\partial f}{\partial x}([W_2], y_c))\right)$
4. If $l \neq 0$, then $D = \frac{C(e^{lh}-1)}{l}$.
   If $l = 0$, then $D = Ch$
5. $[\Delta] = [-D, D]^n$

4.4 Rearrangement

The rearrangement is an essential ingredient in the Lohner algorithm, designed to reduce the wrapping effect [Lo, Lo1, Mo]. We will not discuss this issue here, but we will only include necessary formulas (see [Z1] for more details and the motivation).

Evaluations 2 and 3. In this representation

$$[x_k] = x_k + [B_k][\tilde{r}_k]. \tag{52}$$

In the context of our algorithm in part 3 we obtain

$$[\overline{x}_{k+1}] = \overline{x}_{k+1} + [B_{k+1}][\overline{r}_{k+1}]. \tag{53}$$

Now we have to take into account equation (49). We set

$$\begin{aligned} x_{k+1} &= \mathrm{m}(\overline{x}_{k+1} + [\Delta]) && (54)\\ [\tilde{r}_{k+1}] &= [\overline{r}_{k+1}] + [B_{k+1}^{-1}]\left(\overline{x}_{k+1} + [\Delta] - x_{k+1}\right). && (55) \end{aligned}$$

Evaluation 4. In this representation

$$[x_k] = x_k + C_k[r_0] + [B_k][\tilde{r}_k]. \tag{56}$$

In the context of our algorithm in part 3 we obtain

$$[\overline{x}_{k+1}] = \overline{x}_{k+1} + C_{k+1}[r_0] + [B_{k+1}][\overline{r}_{k+1}]. \tag{57}$$

Equation (49) is taken into account exactly in the same way as in previous evaluations, i.e., we use (54) and (55).

5 Rigorous estimates between time steps

In order to compute the Poincaré map for differential inclusion we also need an estimate for time $t \in [t_k, t_k + h_k]$.

Input parameters:

- h_k is a time step
- $[x_k] \subset \mathbb{R}^n$, such that $\varphi(t_k, [x_0], [y_0]) \subset [x_k]$
- $[x_{k+1}] \subset \mathbb{R}^n$, such that $\varphi(t_k + h_k, [x_0], [y_0]) \subset [x_{k+1}]$
- convex and compact set $[W_2] \subset \mathbb{R}^n$ and convex set $[W_y] \subset \mathbb{R}^m$, such that

$$\begin{aligned} \varphi([t_k, t_k + h_k], [x_0], [y_0]) &\subset [W_2] && (58)\\ [y_0]([t_k, t_{k+1}]) &\subset [W_y]. && (59) \end{aligned}$$

- $y_c \in [W_y]$
- $[\overline{x}_{k+1}] \subset \mathbb{R}^n$, such that $\overline{\varphi}(h_k, [x_k], y_c) \subset [\overline{x}_{k+1}]$
- $[W_1] \subset \mathbb{R}^n$ compact and convex, such that $\overline{\varphi}([0, h_k], [x_k], y_c) \subset [W_1]$

Output:
We compute $[E_k] \subset \mathbb{R}^n$ such that

$$\varphi(t_k + [0, h_k], [x_0], [y_0]) \subset [E_k],$$

Algorithm:

- We compute $[\overline{E}_k] \subset \mathbb{R}^n$, such that

$$\overline{\varphi}([0,h_k],[x_k],y_c) \subset [\overline{E}_k] \tag{60}$$

  using a procedure for an ODE described in [Z1]. This procedure requires as input data: h_k, $[x_k]$, $[\overline{x}_{k+1}]$ and $[W_1]$.

- we compute a set $[\Delta] \subset \mathbb{R}^n$, such that

$$\varphi(t_k+h,[x_0],[y_0]) \subset \overline{\varphi}(h,[x_k],y_c) + [\Delta], \qquad \text{for } 0 \leq h \leq h_k. \tag{61}$$

  Observe that this requires y_c, $[W_1]$, $[W_2]$ and $[W_y]$.

- finally we obtain

$$\varphi(t_k+[0,h_k],[x_0],[y_0])_i \subset [E_k]_i = [\overline{E}_k]_i + [\Delta]_i. \tag{62}$$

Slightly better algorithm:

- if $0 \notin f_i([W_2],[W_y])_i$, then the i-th coordinate is strictly monotone on $[W_2] \times [W_y]$, hence we set

$$[E_k]_i = \text{hull}([x_k]_i,[x_{k+1}]_i)$$

- if $0 \in f_i([W_2],[W_y])$, then we compute $[\overline{E}_k] \subset \mathbb{R}^n$, such that

$$\overline{\varphi}([0,h_k],[x_k],y_c) \subset [\overline{E}_k] \tag{63}$$

  using a procedure for an ODE described in [Z1]. This procedure requires as input data: h_k, $[x_k]$, $[\overline{x}_{k+1}]$and $[W_1]$.

  We have

$$\varphi(t_k+[0,h_k],[x_0],[y_0])_i \subset [E_k]_i = [\overline{E}_k]_i + [\Delta]_i. \tag{64}$$

A drawback of this approach:
if we have to perform several time steps during which the computed enclosure for the trajectory has a nonempty intersection with the section, then Δ is added twice.

5.1 Computation of the Poincaré map

If as in [Z1] we assume that the section is given by $\alpha(x) = 0$ then an algorithm discussed in Section 5 in [Z1] also applies in the present context.

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