Rigorous Numerics for Partial Differential Equations: the Kuramoto-Sivashinsky equation

by

Piotr Zgliczyński[1]
Jagiellonian University, Institute of Mathematics,
Reymonta 4, 30-059 Kraków, Poland
e-mail: zgliczyn@im.uj.edu.pl

and

Konstantin Mischaikow[2]

Center for Dynamical Systems and Nonlinear Studies
School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332
e-mail: mischaik@math.gatech.edu

Abstract

We present a new topological method for the study of the dynamics of dissipative PDE's. The method is based on the concept of the self-consistent apriori bounds, which permit the rigorous justification of Galerkin projections. As a result we obtain a low-dimensional system of ODE's subject to rigorously controlled small perturbation from the neglected modes. To these ODE's we apply the Conley index to obtain information about the dynamics of the PDE under consideration.

We applied the method to the Kuramoto-Sivashinsky equation

$$u_t = (u^2)_x - u_{xx} - \nu u_{xxxx},\ u(x,t) = u(x+2\pi,t),\ u(x,t) = -u(-x,t)$$

We obtained a computer assisted proof the existence of several fixed points for various values of $\nu > 0$.

AMS Subject classification numbers: 37B30, 37L65, 65M60, 35Q35

[1]Research supported in part by Polish KBN grants 2P03A 021 15, 2 P03A 011 18 and NSF–NATO grant DGE–98–04459.
[2]Research supported in part by NSF grant DMS-9805584.

$a_1 = 1.07934 \times 10^{-37}$	$a_2 = 1.25665$	$a_3 = -1.92524 \times 10^{-37}$
$a_4 = -0.559867$	$a_5 = 7.81863 \times 10^{-38}$	$a_6 = 0.0881138$
$a_7 = -1.56596 \times 10^{-38}$	$a_8 = -0.0122945$	$a_9 = 2.54974 \times 10^{-39}$
$a_{10} = 0.00143504$	$a_{11} = -3.4963 \times 10^{-40}$	$a_{12} = -0.000156065$
$a_{13} = 4.35072 \times 10^{-41}$	$a_{14} = 1.59816 \times 10^{-05}$	$a_{15} = -5.02979 \times 10^{-42}$
$a_{16} = -1.57158 \times 10^{-06}$	$a_{17} = 5.50953 \times 10^{-43}$	$a_{18} = 1.49677 \times 10^{-07}$
$a_{19} = -5.62586 \times 10^{-44}$	$a_{20} = -1.39049 \times 10^{-08}$	$a_{21} = -8.26547 \times 10^{-45}$
$a_{22} = 1.26591 \times 10^{-09}$	$a_{23} = 1.30584 \times 10^{-43}$	$a_{24} = -1.13347 \times 10^{-10}$
$a_{25} = -9.46577 \times 10^{-43}$	$a_{26} = 1.0008 \times 10^{-11}$	$a_{27} = 1.1614 \times 10^{-40}$
$a_{28} = -8.73294 \times 10^{-13}$		

Table 1: Coefficients for the function $u(x)$.

1 Introduction

Even in the setting of infinite dimensional dynamics many of the dynamical objects of interest are low dimensional, e.g. equilibria, periodic orbits, connecting orbits, horseshoes, etc. In this paper we introduce techniques which, in principle, allow for the rigorous verification of such solutions for a variety of partial differential equations. Our approach is to combine rigorous computer calculations with topological invariants to obtain accurate existence statements. To demonstrate these techniques we have chosen to study the the Kuramoto-Sivashinsky equation [13, 20]

$$u_t = -\nu u_{xxxx} - u_{xx} + 2uu_x \qquad (t,x) \in [0,\infty) \times (-\pi,\pi) \tag{1}$$

subject to periodic and odd boundary conditions

$$u(t,-\pi) = u(t,\pi) \quad \text{and} \quad u(t,-x) = -u(t,x). \tag{2}$$

The following theorem is a prototype for the results which can be obtained.

Theorem 1.1 *For $\nu = 0.1$ there exists an equilibrium $u^*(x)$ of (1) such that*

$$||u^* - u||_{L^2} < 2.71547 \times 10^{-13} \quad \text{and} \quad ||u^* - u||_{C^0} < 2.06706 \times 10^{-13}$$

where $u(x) = \sum_{k=1}^{28} -2a_k \sin(kx)$ with the a_k given in Table 1.

Having stated this theorem we now try to put the result into the context of the goals of our methods. To begin with it needs to be emphasized that the computations which lead to this result are rigorous in the sense that we have employed interval arithmetic to overcome all errors introduced by the fact that we are using floating point arithmetic in our calculations.

As will become clear in the later sections, this result is obtained by studying the full partial differential equation rather than attempting to solve a boundary value problem. While from the point of view of traditional numerical analysis this approach may appear inefficient, it is an important point. To be more

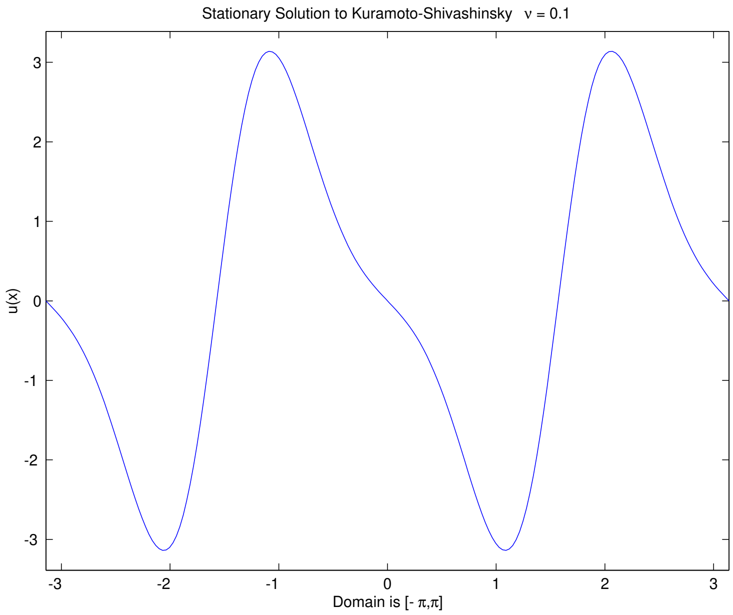

Figure 1: The equilibrium solution $u^*(x)$ for $\nu = 0.1$

precise, our method does not attempt to directly approximate any particular solution to the partial differential equation. Rather we essentially compute the Conley index of a compact region, called an *isolating neighborhood*, of phase space. The diameter of this region provides the error bounds stated in the theorem. The index guarantees the existence of the equilibrium solution.

The Conley index theory is a far reaching topological generalization of Morse theory. In particular, this index can be used to prove the existence of periodic orbits, connecting orbits, and chaotic dynamics [5, 21, 19, 3, 17]. It has been numerically observed that for various parameter values (1) contains these types of dynamical objects. In principle, combining earlier rigorous numerical methods [15, 16, 18, 2, 8, 26, 27] with the techniques described in this paper and the above mentioned index theorems will lead to rigorous proofs of the existence of periodic orbits and even chaotic dynamics. However, we do not pursue these more complicated structures in this paper for two reasons. First, finding the appropriate isolating neighborhoods is more complicated in these cases and our goal here is to emphasize the fundamental ideas associated with the methods. The second, and more important point, is that a straightforward application of the earlier numerical methods would lead to large computations - which we believe can be avoided by alternative methods (see for example [25]). This latter point is currently being investigated.

Returning to our discussion of Theorem 1.1, an obvious question concerns the stability of u^*. For this we have no definitive answer. As was indicated before our method does not directly approximate u^* and therefore we do not obtain uniqueness results or hyperbolicity results. On the other hand, being a generalization of the Morse index the Conley index does contain some information about the stability or instability of the dynamics in the isolating neighborhood. Thus, what can be asserted is the following. Assume that u^* is a hyperbolic fixed point, i.e. all eigenvalues have nonzero real part, and that u^* is the only solution which remains within either the L^2 or C^0 bounds of u for all time, then u^* has exactly two unstable eigenvalues, i.e. its unstable manifold is two dimensional. We hope to treat this problem in a subsequent paper.

It should be mentioned that even though we are doing the computations via an approximation of the full partial differential equation, we never integrate the equations. Rather, as will be made clear in Section 2 the computations are reduced to solving a set of inequalities. It is for this reason that we are able to get such sharp bounds on the equilibria. As the following theorem demonstrates we can, in fact obtain bounds on the level of the floating point accuracy.

Theorem 1.2 *For $\nu = 0.75$ there exists an equilibrium $u^*(x)$ for (1) such that $||u^* - u||_{L^2} < 8.83145 \times 10^{-16}$ and $||u^* - u||_{C^0} < 7.11323 \times 10^{-16}$ where $u(x) = \sum_{k=1}^{30} -2a_k \sin(kx)$ (the values of a_k can be found in the appendix).*

By now it is a well demonstrated principle that the asymptotic behavior of a wide variety of infinite dimensional dynamical systems is finite dimensional [10, 9, 23]. The Kuramoto-Sivashinsky equation (1) is a particularly well studied example of such a system [7]. In fact, it is known that (1) posses an inertial manifold and therefore, that there exists a family of ordinary differential equations that exactly describes the asymptotic dynamics. Unfortunately, the estimates for the dimension of these manifolds make them impractical for our purposes [12].

We mention these methods to emphasize that our approach does not directly make use of any of these results. What appears to be essential for our techniques is that the spectrum of the linear operator for the evolution equation is not clustered near the imaginary axis. This is in contrast to the inertial manifold techniques which strongly rely on gap conditions of the spectrum or cone conditions from the flow. Our approach is to use the computer to restrict our attention to that portion of phase space in which the desired dynamics (for this paper the fixed points) occur. Obviously, by restricting the phase space one can get much better estimates. This sets up a loop by which one can continuously improve the estimates until the desired bounds are reached.

Our analysis of the fixed points for (1) was motivated in part by the work of Jolly, Keverkidis, and Titi [11]. In particular, using a 12 mode traditional Galerkin approximation, they produced a bifurcation diagram for $\nu \in (0.057, \infty)$. We used their reported solutions to test our methods. In particular, as is indicated below we were able to find and prove the existence of an equilibrium point on each of their stable branches. Unfortunately, we used a fairly primitive search procedure and therefore missed a few unstable branches. Our expectation is that by combining our methods with a continuation package, one could produce a rigorous bifurcation diagram with fairly precise bounds in a computationally inexpensive manner.

Below we include some of the steady states we found.

- $\nu = 0.5$. Two stable unimodal fixed points.
- $\nu = 0.3$. Two stable unimodal fixed points.
- $\nu = 0.127$, $\nu = 0.125$. A stable and unstable bimodal fixed point. Negative branch is stable, positive one is unstable with apparently two-dimensional unstable manifold.

  Our primitive search procedure did not find a solution on the bi-tri branch.
- $\nu = 0.1$ An bimodal stable and unstable (2 unstable directions) and two unstable trimodal fixed points (both with 1-dimensional unstable manifold)

  We did not find an unstable branch connecting bi-tri branch with quadrimodal branch.
- $\nu = 0.08$ A bimodal stable (neg. branch) and unstable (2 unstable directions) fixed points. A pair of stable fixed points close to $R_3 t_2$ (see [11]). A pair of unstable trimodal fixed points (1 unstable direction).

  We did not find a branch connecting bi-tri and quadimodal branches.
- $\nu = 0.0666..$, $\nu = 0.063$ Two unstable bimodal points, two stable trimodal points and two stable solutions apparently belonging to the *giant* branch.

  We are lacking two unstable branches which are present in [11].
- $\nu = 0.062$, Two stable trimodal points and two stable points from giant branch
- $\nu = 0.045$, Two stable points from giant branch and pairs of unstable tri- and quadrimodal fixed points
- $\nu = 0.04$, Two stable giant fixed points. Two stable quadrimodal fixed points. Two unstable trimodal points.
- $\nu = 0.029$ Two unstable quadrimodal points.

Since the submission of this paper, these techniques have been extended in two directions. In [28] the method of self-consistent apriori bounds is applied to yield a new ODE-type proof of the uniqueness and existence for the Navier-Stokes equations on the plane with periodic boundary conditions. In [1] these ideas are applied to an infinite dimensional map for which rigorous and accurate computations of periodic orbits, connecting orbits, and chaotic symbolic dynamics is obtained.

2 The Method

Our method begins with the reduction of the full dynamical system to a lower dimensional system which can be studied numerically. In particular, we begin with a nonlinear evolution equation in a Hilbert space H (L^2 in our treatment of Kuramoto-Sivashinsky) of the form

$$\frac{du}{dt} = F(u) \tag{3}$$

where the domain of F is dense in H. Furthermore, we assume that $\{\phi_i \mid i = 0, 1, \ldots\}$ forms a complete orthogonal basis for H.

In the case of the Kuramoto-Sivashinsky equation $F(u) = Lu + B(u, u)$, where L is the linear part and B is the nonlinear part, the functions $\{\varphi_i\}$ are chosen to be eigenvalues of L.

Fix $m \in \mathbb{N}$ and let

$$P = P_m : H \to X_m = X$$

be the orthogonal projection from H onto the finite dimensional subspace spanned by $\{\phi_1, \phi_2, \ldots, \phi_m\}$. Let

$$Q = Q_m := I - P : H \to Y = Y_m$$

be the complementary orthogonal projection. Finally, let

$$A_k : H \to \mathbb{R}$$

be the orthogonal projection onto the subspace generated by ϕ_k.

Given $u \in H$, let $Pu = p$ and $Qu = q$. Equation (3) can be rewritten as

$$\frac{dp}{dt} = PF(p, q) \tag{4}$$

$$\frac{dq}{dt} = QF(p, q) \tag{5}$$

The strategy adopted is fairly simple: study the dynamics of the low dimensional Galerkin projection (4) to draw conclusions about the dynamics of (3). Before turning to the precise conditions, consider the following heuristic description of our approach.

Let $W \subset X = X_m$. For $j > m$, let $W_j \subset X_j$ such that $P(P_j^{-1}(W_j)) = W$, (i.e. $W_j = W \oplus (I - P)W_j$). Similarly, let $V \subset Y$ and set $V_j = Q_j(V)$. Furthermore, given $q_j \in V_j$ assume that $\lim_{j\to\infty} ||q_j|| = 0$. Our only knowledge concerning the higher order modes or "tails" of the solutions is that they project into V. This implies that our knowledge of the vector field is reduced to the following differential inclusion

$$\frac{dp}{dt} \in PF(p, V)$$

where $p \in W$. Numerical calculations on this equation are used to find topological invariants (the Conley index, the fixed point index) which guarantee the existence of specific dynamics, e.g. fixed points, periodic orbits, symbolic dynamics, positive entropy, etc. It is simultaneously argued that the topological invariant is the same for any Galerkin system of the form

$$\frac{dp_j}{dt} \in PF(p_j, V_j)$$

where $p_j \in W_j$. Thus, the same dynamical object exists for each sufficiently high Galerkin approximation. Finally, it is shown that the limit of these objects leads to the desired dynamics for the full system (3).

2.1 Self-consistent Bounds

As one might expect the orthonormal basis $\{\phi_i\}$ and the sets W and V must be chosen with care. The first issue that needs to be dealt with is analytic in nature - solutions to the ordinary differential equations must approximate solutions of the partial differential equation. This leads to the following definition.

Definition 2.1 Let $m, M \in \mathbb{N}$ with $m \leq M$. A compact set $W \subset X_m$ and a sequence of pairs $\{a_k^\pm \in \mathbb{R} \mid a_k^- < a_k^+,\ k \in \mathbb{N}\}$ form *self-consistent apriori bounds* for (3) if the following conditions are satisfied:

C1 For $k > M$, $a_k^- < 0 < a_k^+$.

C2 Let $\hat{a}_k := \max |a_k^\pm|$ and set $\hat{u} = \sum_{k=0}^\infty \hat{a}_k \phi_k$. Then, $\hat{u} \in H$ and in particular, $\hat{u}$ is bounded in the norm on H ($||\hat{u}|| < \infty$).

C3 The function $u \mapsto F(u)$ is continuous on

$$W \oplus \prod_{k=m+1}^\infty [a_k^-, a_k^+] \subset H.$$

Moreover, if we define $f_k = \max_{u \in W \oplus \prod_{k=m+1}^\infty [a_k^-, a_k^+]} |A_k F(u)|$ and set $\hat{f} = \sum f_k \phi_k$, then $\hat{f} \in H$. In particular, $||\hat{f}|| < \infty$.

In practice $W \subset \prod_{k=1}^m [a_k^-, a_k^+]$. Given self-consistent apriori bounds W and $\{a_k^\pm\}$, let

$$V := \prod_{k=m+1}^\infty [a_k^-, a_k^+] \subset Y_m.$$

Our goal is to numerically solve (4) on W and draw conclusions about the dynamics of (3) on the set $W \oplus V \subset H$. To do this we will make use of the following results, the first two of them are obvious.

Lemma 2.2 *Given self-consistent apriori bounds W and $\{a_k^\pm\}$, $W \oplus V$ is a compact subset of H.*

Lemma 2.3 *Given self-consistent apriori bounds W and $\{a_k^\pm\}$, $W \oplus V$, then*

$$\lim_{n\to\infty} Q_n(F(u)) = 0, \quad \textit{uniformly for } u \in W \oplus V$$

Proposition 2.4 *Let W and $\{a_k^\pm\}$ be self-consistent bounds for (3). A function $a : [0,T] \to W \oplus V$ given by*

$$a(t) := \sum_{k=0}^{\infty} a_k(t)\phi_k$$

is a solution to (3), if and only if, for each $k \in \mathbb{N}$ and all $t \in [0,T]$

$$\frac{da_k}{dt} = A_k F(a). \tag{6}$$

Proof. $(\Rightarrow)$ This direction follows directly from the projection of (3) onto each of the basis elements.

$(\Leftarrow)$ Assume that (6) is satisfied for each $k \in \mathbb{N}$ and all $t \in [0,T]$. Let

$$a(t) := \sum_{k=0}^{\infty} a_k(t)\phi_k \in H$$

First observe that from **C3** it follows immediately that $\sum_{k=1}^{\infty} \frac{da_k}{dt}\phi_k = F(a) \in H$.

It needs to be shown that

$$\frac{da}{dt} = \lim_{h\to 0} \frac{a(t+h) - a(t)}{h} = \sum_{k=0}^{\infty} \frac{da_k}{dt}\phi_k.$$

This is equivalent to showing that

$$\lim_{h\to 0} \left| \frac{1}{h} \sum_{k=1}^{\infty} (a_k(t+h) - a_k(t))\phi_k - \sum_{k=1}^{\infty} \frac{da_k}{dt}\phi_k \right| = 0$$

for all $t \in [0,T]$.

Fix $h > 0$, then for any $n \in \mathbb{N}$

$$\begin{aligned} \left| \frac{1}{h} \sum_{k=1}^{\infty} (a_k(t+h) - a_k(t))\phi_k - \sum_{k=1}^{\infty} \frac{da_k}{dt}\phi_k \right| &\leq \\ \left| \frac{1}{h} \sum_{k=1}^{n} (a_k(t+h) - a_k(t))\phi_k - \sum_{k=1}^{n} \frac{da_k}{dt}\phi_k \right| & \\ + \left| \frac{1}{h} \sum_{k=n+1}^{\infty} (a_k(t+h) - a_k(t))\phi_k \right| + \left| \sum_{k=n+1}^{\infty} \frac{da_k}{dt}\phi_k \right| & \end{aligned}$$

We will estimate the three terms on the right hand side separately. From lemma 2.3 it follows for a given $\epsilon > 0$ there exists n_0 such that $n > n_0$ implies

$$\left| \sum_{k=n+1}^{\infty} \frac{da_k}{dt} \phi_k \right| = |Q_n(F(a))| < \epsilon/3.$$

From now on fix $n > n_0$. Condition *C3* and the mean value theorem implies

$$\begin{aligned} \left| \sum_{k=n+1}^{\infty} \frac{1}{h}(a_k(t+h) - a_k(t))\phi_k \right| &= \left| \sum_{k=n+1}^{\infty} \frac{da_k}{dt}(t + \theta_k h)\phi_k \right| \\ = \left| \sum_{k=n+1}^{\infty} (A_k F(a(t + \theta_k h))\phi_k \right| &\leq \left| \sum_{k=n+1}^{\infty} f_k \phi_k \right| < \epsilon/3. \end{aligned}$$

Finally, for h sufficiently small,

$$\left| \frac{1}{h} \sum_{k=1}^{n} (a_k(t+h) - a_k(t))\phi_k - \sum_{k=1}^{n} \frac{da_k}{dt} \phi_k \right| < \epsilon/3$$

and hence the desired limit is obtained. ∎

2.2 Conley Index

Proposition 2.4 indicates that given self-consistent apriori bounds W and $\{a_k^{\pm}\}$, finite time solutions to (6) are solutions to the full partial differential equation. Thus, the goal of this paper is to find solutions to (6). Of course, numerically one can only study (4) restricted to W and then argue that the resulting numerical solution is an approximation to a solution to (3). Hence, rather than attempting to approximate specific trajectories in W directly, the objective is to compute a Conley index for (4) and then show that this index information is sufficient to guarantee a solution for (3).

In order to describe this index the following definitions are needed. Let $\varphi : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ be a continuous flow generated by a differential equation $\dot{z} = f(z)$.

Definition 2.5 A compact set $N \subset \mathbb{R}^n$ is an *isolating neighborhood* if

$$\mathrm{Inv}\,(N, \varphi) := \{z \in N \mid \varphi(\mathbb{R}, z) \subset N\} \subset \mathrm{int}\, N.$$

If, in addition, for any $z \in \partial N$, there exists $t_z > 0$ such that

$$\varphi((0, t_z), z) \cap N = \emptyset \quad \text{or} \quad \varphi((-t_z, 0), z) \cap N = \emptyset, \tag{7}$$

then N is an *isolating block*. Given an isolating neighborhood N, the associated maximal invariant set $\mathrm{Inv}\,(N, \varphi)$ is an *isolated invariant set*.

The easiest way to verify the existence of an isolating block is through local sections.

Definition 2.6 $\Xi \subset \mathbb{R}^n$ is a *local section* for φ if for some $\epsilon > 0$

$$\varphi : (-\epsilon, \epsilon) \times \Xi \to \varphi((-\epsilon, \epsilon), \Xi) \tag{8}$$

is a homeomorphism and $\varphi((-\epsilon, \epsilon), \Xi)$ is an open subset of $\mathbb{R}^n$.

A special form of local section is a hypersurface which is transverse to the flow. More formally, let $\Xi \subset \mathbb{R}^n$ be an $n-1$ dimensional manifold with normal vector $\mu(z)$ at $z \in \Xi$. Ξ is a local section if for each $z \in \Xi$,

$$\mu(z) \cdot f(z) \neq 0. \tag{9}$$

It is straightforward to check that N is an isolating block if ∂N can be written as the union of the closure of local sections with the property that (7) is satisfied at every point in the intersection of the closure of the sections.

In this paper the focus is both on proving the existence of equilibria and providing tight bounds on the location of the equilibria. To do this requires have good isolating blocks. With this in mind consider the linear ordinary differential equation

$$\dot{z} = Bz, \quad z \in \mathbb{R}^n. \tag{10}$$

Assume that the origin is a hyperbolic fixed point. Without loss of generality it can be assumed that B is in Jordan normal form. Generically, to each real eigenvalue there is associated a 1-dimensional eigenspace and to each pair of complex conjugate eigenvalues there is an associated 2-dimensional eigenspace. Thus, $\mathbb{R}^n$ can be decomposed into the product of eigenspaces, i.e.

$$\mathbb{R}^n = V_1 \times V_2 \times \cdots \times V_k$$

where V_i is either $\mathbb{R}$ or $\mathbb{R}^2$. In what follows we will use the following notation, $z_i \in V_i$, $i = 1, \ldots, k$, and if $V_i \cong \mathbb{R}^2$, then $z_i = (x_i, y_i)$.

Our interest is not on the dynamics of (10) on the entire phase space, but rather on a prescribed compact subset. Since our goal is to understand the equilibria of (10) consider a neighborhood of the origin,

$$N = I_1 \times I_2 \times \cdots \times I_k$$

where

$$I_i := \begin{cases} [b_i^-, b_i^+], \ b_i^- < 0 < b_i^+ & \text{if } V_i \cong \mathbb{R}, \\ \{(x_i, y_i) \in V_i \mid \sqrt{x_i^2 + y_i^2} \leq b_i, \ b_i > 0\} & \text{if } V_i \cong \mathbb{R}^2. \end{cases}$$

The following result is obvious, but to make a point crucial to the results of this paper we will provide the proof.

Lemma 2.7 *The compact set N is an isolating block for (10).*

Proof. Since B is in Jordan normal form the system decouples according to the decomposition $\mathbb{R}^n = V_1 \times V_2 \times \cdots \times V_k$.

If $V_i = \mathbb{R}$, then (10) reduces to $\dot{z}_i = \lambda_i z_i$. Since B is hyperbolic $\lambda_i \neq 0$, and hence at $z_i = b_i^{\pm}$ (9) becomes

$$\lambda_i b_i^{\pm} \neq 0.$$

If $V_i = \mathbb{R}^2$, then (10) reduces to

$$\begin{aligned} \dot{x}_i &= \alpha_i x_i + \beta_i y_i \\ \dot{y}_i &= -\beta_i x_i + \alpha_i y_i \end{aligned}$$

where by hyperbolicity $\alpha_i \neq 0$. So again for $\sqrt{x_i^2 + y_i^2} = b_i$ (9) becomes

$$(x_i, y_i) \cdot (\alpha_i x_i + \beta_i y_i, -\beta_i x_i + \alpha_i y_i)^t = \alpha_i b_i^2 \neq 0.$$

∎

To see why this trivial argument is of importance, consider the more interesting example of

$$\dot{z} = Bz + f(z) + E(z) \tag{11}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is $o(||z||^2)$ at 0 and E represents a known bounded error. In our situation E arises from numerical errors and approximations. More precisely, we assume that there are known constants c_i such that

$$\sup_{z \in N} ||E_i(z)|| \leq c_i.$$

Observe that a sufficient condition for N to be an isolating block for (11) is the following: for each i such that $V_i = \mathbb{R}$,

$$\lambda_i b_i^{\pm} + f_i(z) + E_i(z) \tag{12}$$

has the same sign as $\lambda_i b_i^{\pm}$ over the set $\{z \in N \mid z_i = b_i^{\pm}\}$, and for each i such that $V_i = \mathbb{R}^2$,

$$(x_i, y_i) \cdot (\alpha x_i + \beta y_i + f_{i_1}(x) + E_{i_1}(x), -\beta x_i + \alpha y_i + f_{i_2}(x) + E_{i_2}(x))^t \tag{13}$$

has the same sign as α_i over the set $\{z \in N \mid \sqrt{x_i^2 + y_i^2} = b_i\}$.

For the linear case the eigenvalues of B are assumed to be known and the $b_i^{\pm}$ can be chosen arbitrarily. Therefore, one can also interpret (12) and (13) as providing a set of inequalities that if simultaneously solved for $b_i^{\pm}$ provide an isolating block even in the context of numerical errors and approximations. In particular, finding isolating blocks need not involve numerically solving the ordinary differential equation.

In itself the knowledge that N is an isolating block does not imply anything about $\mathrm{Inv}\,(N, \varphi)$. To gain information concerning the isolated invariant set we will make use of the Conley index. For our purposes we need only a very small portion of the index theory and so we give a minimal operational definition (see [5, 21, 19, 3, 17] for further information).

Definition 2.8 Let N be an isolating block and let $\partial N = L^+ \cup L^-$ where $L^\pm$ are closed sets. Furthermore, assume that $z \in L^-$ implies that

$$\varphi((0,\epsilon),z) \cap N = \emptyset$$

for a sufficiently small $\epsilon > 0$. Similarly, assume that if $z \in \cap L^+$, then

$$\varphi((-\epsilon,0),z) \cap N = \emptyset$$

for a sufficiently small $\epsilon > 0$. The *Conley index* of $S = \mathrm{Inv}\,(N,\varphi)$ is

$$CH_*(S) := H_*(N,L).$$

No knowledge of relative homology groups is required for the applications described in this paper. The following theorem gives a formula for the index of a hyperbolic fixed point.

Proposition 2.9 *Let q be the number of eigenvalues of B with positive real part. Assume that for all $i = 1,\ldots,k$ either the condition associated with (12) or the condition associated with (13) are satisfied . Then,*

$$CH_j(\mathrm{Inv}\,(N,\varphi)) \cong \begin{cases} \mathbb{Z} & \text{if } j = q \\ 0 & \text{otherwise.} \end{cases}$$

The following theorem due to McCord [14, Corollary 5.9] (see also [22, Thm. 3]), indicates that if the Conley index is a that of Proposition 2.9, then there exists a fixed point in N.

Theorem 2.10 *If the Conley index has the form*

$$CH_j(\mathrm{Inv}\,(N,\varphi)) \cong \begin{cases} \mathbb{Z} & \text{if } j = q \\ 0 & \text{otherwise,} \end{cases}$$

for some q, then N contains a fixed point.

To indicate how these index ideas will be used in this paper let us return to the system (11). Observe that the only assumption on the error term E was that it is bounded, therefore, it is no longer apriori true that the origin is a fixed point or that there even exists a fixed point to (11). On the other hand the sets N and L remain unchanged. Therefore, the Conley index implies the existence of the fixed point.

2.3 A Singular Perturbation Result

As was indicated in the previous section, it is possible to find an isolating block for a finite dimensional ordinary differential equation about a fixed point by solving an appropriate set of inequalities. However, to do this requires a good estimate of the location of the fixed point, knowledge of the eigenvalues, the ability to evaluate the nonlinear terms, and estimates of associated errors. Therefore, the dimension to which one can hope to apply this procedure is obviously limited. In this section we will describe a singular perturbation result which allows one to "lift" the index computations of the previous sections to arbitrary dimensions.

The definition of self-consistent bounds related individual solutions of the infinite family of ordinary differential equations to solutions in partial differential equation. We now need to extend this definition in order to know that the index computations we perform for the finite dimensional approximation have implications for the partial differential equation.

Definition 2.11 Let $m, M \in \mathbb{N}$ with $m \leq M$. A pair of compact sets $N \subset W \subset X_m$ and a sequence of pairs $\{a_k^{\pm} \in \mathbb{R} \mid a_k^- < a_k^+,\ k \in \mathbb{N}\}$ are *topologically self-consistent* if W and $\{a_k^{\pm}\}$ are self-consistent apriori bounds and the following conditions are satisfied.

C4 Let $u \in W \oplus \prod_{k=m+1}^{\infty}[a_k^-, a_k^+]$. Then, for $k > m$

$$A_k u = a_k^{\pm} \quad \Rightarrow \quad A_k F(u) \neq 0. \tag{14}$$

C5 N is an isolating block for (4) for all $q \in \prod_{k>m}[a_k^-, a_k^+]$.

For Kuramoto-Sivashinsky we will make use of the following stricter form of **C4**.

C4a Let $u \in W \oplus \prod_{k=m+1}^{\infty}[a_k^-, a_k^+]$. Then, for $k > m$

$$A_k u = a_k^+ \quad \Rightarrow \quad A_k F(u)) < 0 \tag{15}$$

$$A_k u = a_k^- \quad \Rightarrow \quad A_k F(u)) > 0. \tag{16}$$

Using the line of reasoning that was described in the analysis of (11), condition **C5** can be replaced by the following assumption.

C5a Let N be an isolating block for (4). Let $\nu^{\pm}(p)$ be the outward normal at $p \in L^{\pm}$. If $u \in W \oplus \prod_{k=m+1}^{\infty}[a_k^-, a_k^+]$ such that $Pu \in L^{\pm}$, then

$$PF(u) \cdot \nu^+(Pu) < 0 \qquad PF(u) \cdot \nu^-(Pu) > 0.$$

We shall now discuss two singular perturbation results. The first allows one to lift isolating blocks.

Theorem 2.12 *Let $m, M \in \mathbb{N}$ with $m \leq M$. Assume $N \subset W \subset X_m$ and the sequence of pairs $\{a_k^{\pm} \in \mathbb{R} \mid a_k^- < a_k^+,\ k \in \mathbb{N}\}$ are topologically self-consistent. Fix an integer $r > m$. Then for any $q = \sum_{k=r+1}^{\infty} q_k \phi_k$, such that $q \in \Pi_{k=r+1}^{\infty}[a_k^-, a_k^+]$ and $q_k = 0$ for $k > M$ the set*

$$\tilde{N} := N \times [a_{m+1}^-, a_{m+1}^+] \times [a_{m+2}^-, a_{m+2}^+] \times \cdots \times [a_r^-, a_r^+]$$

is an isolating block for the system of equations

$$\dot{x}_k = A_k F(\sum_{i=1}^{k} x_i \phi_i + q) \qquad k = 1, \ldots, r \tag{17}$$

where $x \in \mathbf{R}^r$.

Proof. Let $u = (w, v) \in W \oplus \prod_{k=m+1}^{r}[a_k^-, a_k^+]$. From **C1** it follows that $u + q \in W \oplus \Pi_{k=m+1}^{\infty}[a_k^-, a_k^+]$. If $u \in \partial\tilde{N}$, then either w is in ∂N or v is in $\partial \prod_{k=m+1}^{r}[a_k^-, a_k^+]$. In the first case **C5** forces the vector field to be transverse at the boundary. In the second case transversality follows from **C4**. ∎

Remark 2.13 *For $r > M$ equations (17) are the Galerkin projection of $\dot{u} = F(u)$.*

The direction of the vector field influences the index computation. With this in mind define

$$\operatorname{dir}(k) := \begin{cases} -1 & \text{if } A_k u = a_k^+ \ \Rightarrow\ A_k F(u)) < 0 \text{ and} \\ & \quad\ A_k u = a_k^- \ \Rightarrow\ A_k F(u)) > 0 \\ 0 & \text{if } A_k u = a_k^+ \ \Rightarrow\ A_k F(u)) < 0 \text{ and} \\ & \quad\ A_k u = a_k^- \ \Rightarrow\ A_k F(u)) < 0 \\ 0 & \text{if } A_k u = a_k^+ \ \Rightarrow\ A_k F(u)) > 0 \text{ and} \\ & \quad\ A_k u = a_k^- \ \Rightarrow\ A_k F(u)) > 0 \\ 1 & \text{if } A_k u = a_k^+ \ \Rightarrow\ A_k F(u)) > 0 \text{ and} \\ & \quad\ A_k u = a_k^- \ \Rightarrow\ A_k F(u)) < 0 \end{cases}$$

Theorem 2.14 *Let $m, M \in \mathbb{N}$ with $m \leq M$. Assume $N \subset W \subset X_m$ and the sequence of pairs $\{a_k^\pm \in \mathbb{R} \mid a_k^- < a_k^+,\ k \in \mathbb{N}\}$ are topologically self-consistent. Fix an integer $r > m$. Let $q = \sum_{k=r+1}^{\infty} q_k \phi_k$, such that $q \in \Pi_{k=r+1}^{\infty}[a_k^-, a_k^+]$ and $q_k = 0$ for $k > M$. Let*

$$\tilde{N} := N \times [a_{m+1}^-, a_{m+1}^+] \times [a_{m+2}^-, a_{m+2}^+] \times \cdots \times [a_r^-, a_r^+]$$

Consider the dynamical system induced by (17).

If for some $j \in \{m+1, \ldots, r\}$, $\operatorname{dir}(j) = 0$, then

$$CH_*(\operatorname{Inv}(\tilde{N})) = 0.$$

Assume that for all $j \in \{m+1, \ldots, r\}$, $\operatorname{dir}(j) \neq 0$, and let d be the number of $j \in \{m+1, \ldots, r\}$ such that $\operatorname{dir}(j) = 1$, then

$$CH_{s+d}(\operatorname{Inv}(\tilde{N})) \cong CH_s(\operatorname{Inv}(N)). \tag{18}$$

Proof. By Theorem 2.12, $\tilde{N}$ is an isolating block.

We will present the proof of the second part of the theorem, only.

Assume that for all $j \in \{m+1, \ldots, r\}$, $\operatorname{dir}(j) \neq 0$. Let $\mathcal{J} := \{j \mid m < j \leq r,\ \operatorname{dir}(j) = 1\}$. Set

$$\begin{aligned} \tilde{L}^- := \ & \left(L^- \times \prod_{k=m+1}^{r}[a_k^-, a_k^+]\right) \cup \\ & \bigcup_{j \in \mathcal{J}} \left(N \times \prod_{k=m+1}^{j-1}[a_k^-, a_k^+] \times \{a_j^\pm\} \times \prod_{k=j+1}^{r}[a_k^-, a_k^+]\right) \end{aligned}$$

and

$$\tilde{L}^+ := \left(L^+ \times \prod_{k=m+1}^{r} [a_k^-, a_k^+]\right) \cup$$
$$\bigcup_{j \notin \mathcal{J}} \left(N \times \prod_{k=m+1}^{j-1} [a_k^-, a_k^+] \times \{a_j^{\pm}\} \times \prod_{k=j+1}^{r} [a_k^-, a_k^+]\right)$$

Let $\varphi : \mathbb{R} \times \mathbb{R}^r \to \mathbb{R}^r$, be any flow generated by

$$\dot{a}_k = A_k F(u+q) \quad k = 1, \ldots, r$$

where $a_k = A_k u$ and $u \in W \oplus \prod_{k=m+1}^{r} [a_k^-, a_k^+]$.

Clearly, if $P_r u \in L^-$, then $\varphi((0,\epsilon), P_r u) \notin \tilde{N}$ for small $\epsilon > 0$. Similarly, if $P_r u \in L^+$, then $\varphi((-\epsilon, 0), P_r u) \notin \tilde{N}$ for small $\epsilon > 0$.

Let $u = (w, v) \in W \oplus \prod_{k=m+1}^{r} [a_k^-, a_k^+]$. If $u \in \partial\tilde{N}$, then either w is in ∂N or v is in $\partial \prod_{k=m+1}^{r} [a_k^-, a_k^+]$. Therefore, $\partial N = \tilde{L}^+ \cup \tilde{L}^-$.

Thus, the Conley index of Inv $(\tilde{N})$ is given by

$$CH_*(\text{Inv}\,(\tilde{N}) \cong H_*(\tilde{N}, \tilde{L}^-).$$

A simple argument using the Mayer-Vietoris sequence gives the desired homology groups. ∎

Observe that **C4a** implies that $\text{dir}(k) = -1$ for all $k > m$. Therefore, one has the following result.

Corollary 2.15 *Assume $N \subset W \subset X_m$ and the sequence of pairs $\{a_k^{\pm} \in \mathbb{R} \mid a_k^- < a_k^+,\ k \in \mathbb{N}\}$ are topologically self-consistent and satisfy* **C4a**. *Fix an integer $r > m$ and let*

$$\tilde{N} := N \times [a_{m+1}^-, a_{m+1}^+] \times [a_{m+2}^-, a_{m+2}^+] \times \cdots \times [a_r^-, a_r^+].$$

Then

$$CH_*(\text{Inv}\,(\tilde{N})) \cong CH_*(\text{Inv}\,(N)).$$

The following theorem is used for all the results described in the Introduction.

Theorem 2.16 *Assume $N \subset W \subset X_m$ and the sequence of pairs $\{a_k^{\pm} \in \mathbb{R} \mid a_k^- < a_k^+,\ k \in \mathbb{N}\}$ are topologically self-consistent and satisfy* **C4a**. *Assume*

$$CH_j(\text{Inv}\,(N, \varphi)) \cong \begin{cases} \mathbb{Z} & \text{if } j = l, \\ 0 & \text{otherwise,} \end{cases}$$

for some l, then there exists

$$u^* \in N \times \prod_{k=m+1}^{\infty} [a_k^-, a_k^+],$$

a fixed point for the partial differential equation (3).

Proof. Combining Theorems 2.14 and 2.10, immediately gives that for each $r > M$ there exists a fixed point

$$z_r \in N \times \prod_{k=m+1}^{r} [a_k^-, a_k^+]$$

for the Galerkin projection onto the first r coordinates.

Since $N \times \prod_{k=m+1}^{\infty} [a_k^-, a_k^+]$ is compact the collection $\{z_r \mid r = m+1, m+2, \ldots\}$ contains a limit point u^*. From the continuity of $P_n \circ F$ on $W \oplus \prod_{k=m+1}^{\infty} [a_k^-, a_k^+]$ it follows that $P_n \circ F(u^*) = 0$ for each $n \in \mathbb{N}$. By Proposition 2.4 u^* is a fixed point for (3). ∎

2.4 Remarks on Related Work

We are aware of at least two other results that are closely related to the methods described in this Section. The first is work of L. Cesari [4] from the early 60's which in spirit is very similar to ours. His method can be characterized as follows [24]. Let B be a Banach space. Let X be a finite dimensional subspace of B and let $P : B \to X$ be a projection. Let $\tilde{N} \subset B$ be closed with the property that $P\tilde{N} = N \subset X$ is compact and for every $x \in N$, $P^{-1}(x) \cap \tilde{N}$ is closed. Consider a continuous map $f : \tilde{N} \to B$. The goal is to find fixed points for f by studying the behavior of the projection of the map onto X.

It is obvious that u^* is a fixed point of f, if and only if $Pu^* = Pfu^*$, and $u^* = Pu^* + (I - P)f(u^*)$.

Cesari's method applies if and only if the following three conditions are satisfied:

(i) For each $x \in N$,

$$P + (I - P)f : P^{-1}(x) \cap \tilde{N} \to P^{-1}(x) \cap \tilde{N}$$

is a contraction.

(ii) Given condition (i), for each $x \in N$, there exists a unique $u(x) \in \tilde{N}$ such that $Pu(x) + (I - P)f(u(x)) = u(x)$. The function $u : N \to \tilde{N}$ is continuous.

(iii) There are no fixed points of $Pfu : N \to N$ on the boundary of N.

Relating this back to the context of this paper, observe that a fixed point for the partial differential equation is a fixed point for any nonzero constant time map of the corresponding semi-flow. (iii) is closely related to the condition **C5**. As stated (ii) is not well defined unless (i) holds. **C4** is the analogous assumption to (i) and differs in two significant ways. A necessary condition to have a contraction, is for the stronger assumption of **C4a** to hold. However, **C4a** is not sufficient. An important point is that we do not make any assumptions on the direction of the vector field within $\tilde{N}$. Thus, condition **C4a** is in principle easier to verify than (i). On the other hand, this makes it clear that we cannot guarantee uniqueness of the fixed point given our assumptions.

The other work is due to C. Conley and P. Fife [6] and is closely related to Theorem 2.14. Formulas of the form (18) are classical in the context of product systems (see [5]). In [6] one finds a similar formula, but in that context at the parameter value for which one computes the index in the lower dimensional system, there is no higher dimensional dynamics defined. However, the higher dimensional system is defined for an arbitrarily small perturbation. The key idea is that in the proper context the lower dimensional system is normally hyperbolic. In this paper we circumvent this type of assumption using isolating blocks, **C5**, and imposing **C4**.

3 Estimates for Kuramoto-Sivashinsky equation

As the Hilbert space H for the Kuramoto-Sivashinsky equation (1) we choose the subspace of $L^2(-\pi,\pi)$ consisting of 2π-periodic and odd functions.

Since $u(t,x)$ is odd its Fourier expansion takes the form

$$u(t,x)=\sum_{k=-\infty}^{k=\infty} b_k(t)\exp(ikx) \tag{19}$$

Since $u(t,x)$ is real, $b_k=\bar{b}_{-k}$. Substituting (19) into (1) gives the following equations

$$\dot{b}_k=(k^2-\nu k^4)b_k+\mathrm{i}k\sum_{m=-\infty}^{m=\infty} b_m b_{k-m} \tag{20}$$

Since we are interested in solutions with odd symmetry it follows that b_k are pure imaginary. Let

$$a_k:=\sqrt{-1}\,b_k.$$

Then, $a_k=-a_{-k}$ and $a_0=0$ which results in the following infinite system of ordinary differential equations

$$\dot{a}_k=k^2(1-\nu k^2)a_k-k\sum_{n=1}^{k-1} a_n a_{k-n}+2k\sum_{n=1}^{\infty} a_n a_{n+k} \quad k=1,2,3,\ldots \tag{21}$$

and

$$u(t,x)=\sum_{k=1}^{\infty}-2a_k\sin(kx). \tag{22}$$

We will use these equations to draw rigorous conclusions about Kuramoto-Sivashinsky by finding self-consistent apriori bounds for (1) that satisfy the stronger condition **C4a** and then applying Theorem 2.16. To do this, however, we will need to understand the errors contributed by ignoring the higher modes and the errors introduced by the use of floating point arithmetic.

Let $m, M \in \mathbb{N}$ be fixed with $m \leq M$. Let $W \subset \mathbb{R}^m$ and $\{a_k^\pm \in \mathbb{R} \mid k \in \mathbb{N}\}$ satisfy conditions **C1** - **C3** with the added constraints that

$$W = \prod_{k=1}^{m} [a_k^-, a_k^+]$$

and

$$a_k^\pm = \pm \frac{C_s}{k^s}, \quad k > M \tag{23}$$

for some constant $C_s > 0$ and integer $s > 1$.

Though technically incorrect, it is perhaps useful for the reader to think of the numerical approximation of the dynamics being computed with respect to the finite dimensional system

$$\dot{a}_k = k^2(1 - \nu k^2)a_k - k \sum_{n=1}^{k-1} a_n a_{k-n} + 2k \sum_{n=1}^{M-k} a_n a_{n+k} \quad k = 1, \ldots, m \tag{24}$$

where $a_k = (a_k^- + a_k^+)/2$ for $k = m+1, \ldots M$. In doing so it becomes clear that there are essentially three levels of approximation that need to be dealt with. The first involves the terms in the infinite tail $\{a_k \mid k > M\}$. These are completely absent from (24) and therefore must be absorbed as a fixed error term (think of the term $E(x)$ in (11)). The power decay rule (23) will be used to determine this quantity. The second, involves the terms $\{a_k \mid m < k \leq M\}$. In principle, one could set $m = M$, however our strategy is to try to obtain better estimates for these terms than can be expected by the general decay of (23). However, these terms act as constants and hence can be viewed as parameters for the system (24). Finally, the terms $\{a_k \mid k = 1, \ldots, m\}$ are the actual variables for the dynamical system being studied. It should also be kept in mind that we need to lift the index information, and therefore need to be able to verify **C4a** for all k.

Of course, our goal is that of rigorous computations. Therefore each of the above mentioned a_i is actually an *interval*. The intervals associated with $\{a_k \mid k = 1, \ldots, m\}$ are essentially determined by the floating point approximations. For $k > m$, the intervals are

$$a_k = [a_k^-, a_k^+].$$

We will let

$$|a_k| := \max\{|a_k^-|, |a_k^+|\}.$$

To compute the above mentioned errors we return to (21) and observe that in addition to the linear part there is a finite sum of terms

$$FS(k) = \sum_{n=1}^{k-1} a_n a_{k-n} \tag{25}$$

and an infinite sum of terms

$$IS(k) = \sum_{n=1}^{\infty} a_n a_{n+k}. \tag{26}$$

Obviously bounds on these terms are necessary.

3.1 $1 \le k \le M$

Since $FS(k)$ is a finite sum and we have already chosen the interval values for a_n, we can explicitly compute $FS(k)$. Perhaps it is worth noting that to evaluate $FS(k)$ only involves the intervals $\{a_n \mid n = 1, \ldots, M-1\}$ which are chosen in such a way that we expect them to be reasonably good approximations of the actual terms.

Lemma 3.1 *Assume* $1 \le k \le M$. *Then,*

$$\begin{aligned} IS(k) \quad \subset \quad & \sum_{n=1}^{M-k} a_n a_{k+n} + C_s \sum_{n=M-k+1}^{M} \frac{|a_n|}{(k+n)^s}[-1,1] + \\ & \frac{C_s^2}{(k+M+1)^s (s-1) M^{s-1}}[-1,1] \end{aligned}$$

Proof. By definition,

$$IS(k) = \sum_{n=1}^{M-k} a_n a_{k+n} + \sum_{n=M-k+1}^{M} a_n a_{k+n} + \sum_{n=M+1}^{\infty} a_n a_{k+n}.$$

With regard to the second sum

$$\sum_{n=M-k+1}^{M} a_n a_{k+n} \quad \subset \quad \sum_{n=M-k+1}^{M} |a_n| \frac{C_s}{(k+n)^s}[-1,1].$$

Finally, the third sum produces

$$\begin{aligned} \sum_{n=M+1}^{\infty} a_n a_{k+n} \quad & \subset \quad \sum_{n=M+1}^{\infty} \frac{C_s}{n^s} \frac{C_s}{(n+k)^s}[-1,1] \\ & \subset \quad \frac{C_s^2}{(k+M+1)^s}[-1,1] \sum_{n=M+1}^{\infty} \frac{1}{n^s} \\ & \subset \quad \frac{C_s^2}{(k+M+1)^s (s-1) M^{s-1}}[-1,1]. \end{aligned}$$

In above derivation we used the following estimate

$$\sum_{n=M+1}^{\infty} \frac{1}{n^s} < \int_M^{\infty} \frac{dx}{n^s} = \frac{1}{(s-1)M^{s-1}}$$

∎

Remark 3.2 This estimate and some of those that follow can be improved by noting that

$$\sum_{n=M+1}^{\infty} \frac{1}{n^s(n+k)^s} < \int_M^{\infty} \frac{dx}{x^s(x+k)^s}.$$

Of course, the right hand side has an explicit rational expression, but it is rather complicated for large s and so was not utilized here.

A simple extension of Lemma 3.1 leads to the following corollary.

Corollary 3.3 *Let $1 \leq k \leq m$. Then,*

$$\begin{aligned} \sum_{n=m-k+1}^{\infty} a_n a_{n+k} \quad \subset \quad & \sum_{n=m-k+1}^{M-k} a_n a_{n+k} + C_s \sum_{n=M-k+1}^{M} \frac{|a_n|}{(k+n)^s}[-1,1] + \\ & \frac{C_s^2}{(k+M+1)^s(s-1)M^{s-1}}[-1,1] \end{aligned}$$

Observe that collorary 3.3 estimates the error in the vector field due to the Galerkin projection, namely

$$A_k F(p+q) - A_k F(p) = 2k \sum_{n=m-k+1}^{\infty} a_n a_{n+k} \tag{27}$$

3.2 $k > M$

Throughout this section it is assumed that $k > M$. Let

$$e(k) := \begin{cases} 1 & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

Lemma 3.4 *Let $M < k \leq 2M$. Then,*

$$FS(k) \subset 2 \sum_{n=k-M}^{\lfloor k/2 \rfloor} a_n a_{k-n} + e(k) a_{k/2}^2 + 2C_s \sum_{n=1}^{k-M-1} \frac{|a_n|}{(k-n)^s}[-1,1].$$

Proof. Expanding (25) gives

$$\begin{aligned} FS(k) &= \sum_{n=1}^{k-M-1} a_n a_{k-n} + \sum_{n=k-M}^{M} a_n a_{k-n} + \sum_{n=M+1}^{k-1} a_n a_{k-n} \\ &\subset 2C_s \sum_{n=1}^{k-M-1} \frac{|a_n|}{(k-n)^s}[-1,1] + \sum_{n=k-M}^{M} a_n a_{k-n} \\ &\subset \sum_{n=k-M}^{\lfloor k/2 \rfloor} a_n a_{k-n} + e(k) a_{k/2}^2 + 2C_s \sum_{n=1}^{k-M-1} \frac{|a_n|}{(k-n)^s}[-1,1]. \end{aligned}$$

∎

Lemma 3.5 *Let $k > 2M$. Then,*

$$FS(k) \subset \frac{C_s}{k^{s-1}}\left(\frac{2^{s+1}}{2M+1}\sum_{n=1}^{M}|a_n| + \frac{C_s 4^s}{(2M+1)^{s+1}} + \frac{C_s 2^s}{(s-1)M^s}\right)[-1,1].$$

Proof. From (25) it follows that

$$\begin{aligned}
FS(k) &= \sum_{n=1}^{k-1} a_n a_{k-n} \\
&= 2\sum_{n=1}^{\lfloor k/2 \rfloor} a_n a_{k-n} + e(k)a_{k/2}^2 \\
&= 2\sum_{n=1}^{M} a_n a_{k-n} + 2\sum_{n=M+1}^{\lfloor k/2 \rfloor} a_n a_{k-n} + e(k)a_{k/2}^2.
\end{aligned}$$

Each of these terms will be estimated separately. The first one results in:

$$\begin{aligned}
\sum_{n=1}^{M} a_n a_{k-n} &\subset \sum_{n=1}^{M} \frac{|a_n| C_s}{(k-M)^s}[-1,1] \\
&\subset \frac{C_s}{k^s(1-M/k)^s}[-1,1]\sum_{n=1}^{M}|a_n| \\
&\subset \frac{2^s C_s}{k^s}[-1,1]\sum_{n=1}^{M}|a_n| \\
&\subset \frac{2^s C_s}{k^{s-1}(2M+1)}[-1,1]\sum_{n=1}^{M}|a_n|.
\end{aligned}$$

The second term leads to

$$\begin{aligned}
\sum_{n=M+1}^{\lfloor k/2 \rfloor} a_n a_{k-n} &\subset C_s^2 \sum_{n=M+1}^{\lfloor k/2 \rfloor} \frac{1}{n^s(k-n)^s}[-1,1] \\
&= \frac{C_s^2}{k^s}\sum_{n=M+1}^{\lfloor k/2 \rfloor} \frac{1}{n^s(1-n/k)^s}[-1,1] \\
&\subset \frac{C_s^2 2^s}{k^s}\sum_{n=M+1}^{\lfloor k/2 \rfloor} \frac{1}{n^s}[-1,1] \\
&\subset \frac{C_s^2 2^s}{k^s}\int_M^\infty \frac{dx}{x}[-1,1] \\
&= \frac{C_s^2 2^s}{k^s(s-1)M^{s-1}}[-1,1] \\
&\subset \frac{C_s^2 2^{s-1}}{k^s(s-1)M^s}[-1,1].
\end{aligned}$$

Finally, the third term gives rise to

$$e(k)a_{k/2}^2 \subset \frac{C_s^2 2^{2s}}{k^{2s}}[-1,1] \subset \frac{1}{k^{s-1}} \frac{C_s^2 4^s}{(2M+1)^{s+1}}[-1,1].$$

∎

Turning now to the infinite sum we can obtain the following estimate.

Lemma 3.6 *Let $k > M$. Then,*

$$IS(k) \subset \frac{C_s}{k^{s-1}(M+1)} \left(\frac{C_s}{(M+1)^{s-1}(s-1)} + \sum_{n=1}^{M} |a_n| \right) [-1,1].$$

Proof. From (26) it follows that

$$IS(k) = \sum_{n=1}^{M} a_n a_{k+n} + \sum_{n=M+1}^{\infty} a_n a_{k+n}.$$

As in the previous case, each term is treated separately.

$$\begin{aligned}
\sum_{n=1}^{M} a_n a_{k+n} &\subset C_s \sum_{n=1}^{M} \frac{|a_n|}{(k+n)^s}[-1,1] \\
&= \frac{C_s}{k^s} \sum_{n=1}^{M} \frac{|a_n|}{(1+n/k)^s}[-1,1] \\
&\subset \frac{C_s}{k^{s-1}(M+1)} \sum_{n=1}^{M} |a_n|[-1,1].
\end{aligned}$$

The remaining term leads to

$$\begin{aligned}
\sum_{n=M+1}^{\infty} a_n a_{k+n} &= C_s^2 \sum_{n=M+1}^{\infty} \frac{1}{m^s(k+m)^s}[-1,1] \\
&\subset \frac{C_s^2}{(M+1)^s} \int_M^{\infty} \frac{dx}{(k+x)^s}[-1,1] \\
&= \frac{C_s^2}{(M+1)^s(s-1)(k+M)^{s-1}}[-1,1] \\
&\subset \frac{C_s^2}{(M+1)^s(s-1)k^{s-1}}[-1,1].
\end{aligned}$$

∎

3.3 Refining the Self-Consistent Bounds

The proof of our results obviously depends on having good self-consistent bounds and the precision of the final result is determined directly by these bounds. For this reason it is important to have a process by which these bounds can be improved. With this in mind consider an initial sequence of bounds $\{a_k^{\pm}\}$ which defines the sets

$$W = \prod_{k=1}^{m} [a_k^-, a_k^+] \quad \text{and} \quad V = \prod_{k=m+1}^{\infty} [a_k^-, a_k^+].$$

We will also assume that

$$1 < \nu m^2.$$

This condition means that the Fourier modes for $k \geq m$ are linearly stable.

We shall describe the refinement procedure under the assumption of **C4a**. In particular, we need that our sequence $\{a_k^{\pm}\}$ satisfy $\mathrm{dir}(k) = -1$ for all $k > m$. We also assume that since we can numerically solve (4), that the estimates for W are reasonably good.

We will inductively adjust $a_k\pm$, for $k = m+1, \ldots, M$, beginning with $a_{m+1}^{\pm}$, as follows. Let $a \in W \oplus V$ such that $a_k = a_k^+$. To satisfy **C4a** requires that $\dot{a}_k < 0$, i.e.

$$k^2(1 - \nu k^2)a_k^+ - kFS(k) + 2kIS(k) < 0.$$

This is equivalent to requiring

$$a_k^+ > \frac{2IS(k) - FS(k)}{k^3(\nu - k^{-2})}. \tag{28}$$

Of course, our goal is to make a_k^+ as small as possible within the constraints imposed by the approximations. Since we are iteratively improving our bounds, it is reasonable to assume that a worst case equality is the best guess at this stage in the procedure. Note, we are not claiming a proof at this point, we just are seeking good bounds which later will be verified to be self-consistent bounds. So using Lemma 3.1, define $f_k^{\pm}$ to be bounds for $2IS(k) - FS(k)$,

$$\begin{aligned}[f_k^-, f_k^+] := 2\sum_{n=1}^{M-k} a_n a_{k+n} + 2C_s \sum_{n=M-k+1}^{M} \frac{|a_n|}{(k+n)^s}[-1,1] + \\ \frac{2C_s^2}{(k+M+1)^s(s-1)M^{s-1}}[-1,1] - \sum_{n=1}^{k-1} a_n a_{k-n}.\end{aligned}$$

The new value of a_k^+ is given by

$$a_k^+ := \frac{f_k^+}{k^3(\nu - k^{-2})}.$$

A similar argument suggests setting

$$a_k^- := \frac{f_k^-}{k^3(\nu - k^{-2})}.$$

This approach works up to $k = M$. Recall that for $k > M$, we set $a_k^\pm = \pm C_s/k^s$. Here our goal is to improve the power of convergence, i.e. we want to increase s. Again, since we are trying to satisfy **C4a** the basic inequality which needs to be satisfied is (28). The estimates for $FS(k)$ and $IS(k)$ for $k > M$ obviously are crucial here. However, we had two sets of estimates one for $M < k \leq 2M$ and the other for $k > 2M$. Thus, we need to choose the worst of both estimates. This is done as follows.

Given an interval $I \subset \mathbb{R}$ let

$$|I| := \sup_{x \in I} |x|.$$

With the estimate from Lemma 3.4 in mind define

$$D_1(k) := \left| 2 \sum_{n=k-M}^{\lfloor k/2 \rfloor} a_n a_{k-n} + e(k) a_{k/2}^2 \right| + 2C_s \sum_{n=1}^{k-M-1} \frac{|a_n|}{(k-n)^s}.$$

Combining this with the estimate on $IS(k)$ given by Lemma 3.6 and multiplying by k^{s-1} leads to the following definition

$$D_1 := \frac{2C_s}{(M+1)} \left(\frac{2C_s}{(M+1)^{s-1}(s-1)} + \sum_{n=1}^{M} |a_n| \right) + \max_{M < k \leq 2M} k^{s-1} D_1(k).$$

Turning now to the bounds for $k > 2M$, Lemmas 3.5, 3.6, and again multiplying by k^{s-1} suggests setting

$$\begin{aligned} D_2 \quad := \quad & \frac{2C_s}{(M+1)} \left(\frac{2C_s}{(M+1)^{s-1}(s-1)} + \sum_{n=1}^{M} |a_n| \right) + \\ & \frac{C_s}{k^{s-1}} \left(\frac{2^{s+1}}{2M+1} \sum_{n=1}^{M} |a_n| + \frac{C_s 4^s}{(2M+1)^{s+1}} + \frac{C_s 2^s}{(s-1)M^s}. \right) \end{aligned}$$

From Lemmas 3.4, 3.5, and 3.6 we obtain the following result.

Corollary 3.7 *For $k > M$ and $D_s := \max\{D_1, D_2\}$*

$$|-FS(k) + 2IS(k)| < \frac{D_s}{k^{s-1}}.$$

We will use this corollary to update the decay rate for the tail terms. Again, we want (28) (which gave us **C4a**) to hold for all $k > M$. It is sufficient that

$$a_k^+ > \frac{|-FS(k) + 2IS(k)|}{k^3(\nu - (M+1)^{-2})}$$

and therefore it is sufficient that

$$a_k^+ > \frac{D_s}{k^{s-1}} \cdot \frac{1}{k^3} \cdot \frac{1}{\nu - (M+1)^{-2}} = \frac{1}{k^{s+2}} \cdot \frac{D_s}{\nu - (M+1)^{-2}}.$$

There is a similar inequality for a_k^-.

Setting this to an equality we can define

$$a_k^{\pm} := \pm \frac{C_{s+2}}{k^{s+2}} \quad \text{where} \quad C_{s+2} := \frac{D_s}{\nu - (M+1)^{-2}} \tag{29}$$

for $k > M$.

4 A Typical Proof

This section describes the proof of the following result.

Theorem 4.1 *Let*

$$u(x) = -2\left(\frac{1}{\sqrt{2}} \sin x - \frac{1}{8} \sin 2x\right).$$

For $\nu = 0.75$ there exists an equilibrium solution $u^*(x)$ to (1) such that

$$||u^* - u||_{L^2} < 0.052 \quad \text{and} \quad ||u^* - u||_{C^0} < 0.05$$

The reader should observe that this is a weaker version of Theorem 1.2. However, we present its proof since it contains all the essential features, but with a very low dimensional approximation. The rest of the results described in the introduction were proved in a similar manner.

The first step is to choose m the dimension of the Galerkin approximation and M the level to which we make specific choices for the $\{a_k^{\pm}\}$. For $m = 2$, (24) reduces to the system of equations

$$\begin{aligned} \dot{a}_1 &= \frac{1}{4} a_1 + 2a_1 a_2 \\ \dot{a}_2 &= -8a_2 - 2a_1^2. \end{aligned} \tag{30}$$

A simple algebraic computation shows that this system has exactly three fixed points: $(0,0)$-unstable, with one-dimensional unstable manifold and two attracting fixed points $u_{\pm} = (\pm\frac{1}{\sqrt{2}}, -\frac{1}{8})$. Theorem 4.1 is obtained by studying the dynamics of (30) in a neighborhood of $(\frac{1}{\sqrt{2}}, -\frac{1}{8})$.

The next step is to obtain self-consistent apriori bounds for (21). It is unrealistic to expect that goods bounds can be obtained immediately. Thus, we make a reasonable guess for bounds and then try to improve them. Let

$$W = [\frac{1}{\sqrt{2}} - 0.1, \frac{1}{\sqrt{2}} + 0.1] \times [-\frac{1}{8} - 0.1, -\frac{1}{8} + 0.1].$$

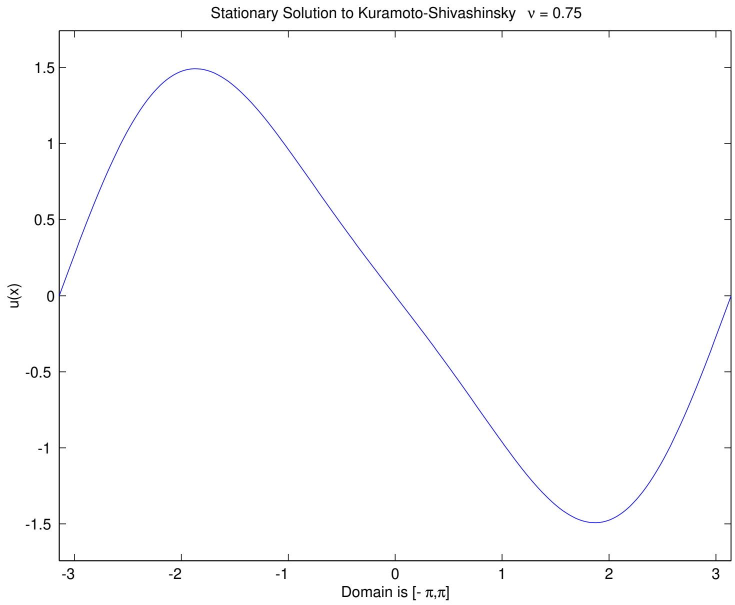

Figure 2: The computed function $u(x)$ from Theorem 4.1

The initial estimates for $[a_k^-, a_k^+]$ are given in Table 2. The formula used to derive these initial estimates will be presented in Section 5. The reason for delaying the presentation is to emphasize the fact that the initial estimates are only estimates. Obviously, choosing good estimates allows for faster convergence and choosing terrible estimates will probably result in a failure of convergence.

Beginning with the data in Table 2, the refinement procedure described in Section 3.3 is used to update $a_k^\pm$ for $k > 2$. After three iterations one obtains the estimates given in Table 3. It can now be checked that W and $\{a_k^\pm\}$ from Table 3 form self-consistent bounds.

What should be clear at this point is that the uncertainty contributed by the terms not in the Galerkin projection are extremely small. Obviously, at this point most of the uncertainty is due to the size of W.

Having controlled the errors from the Galerkin truncation, the next step is to obtain an isolating block N which is topologically self consistent with W and $\{a_k\}$. In determing N we use the vector field (30). Of course, the correct equations are given by (21):

$$\begin{aligned}\dot{a}_1 &= \frac{1}{4}a_1 + 2\sum_{n=1}^{\infty} a_n a_{n+1} \\ \dot{a}_2 &= -8a_k - 2a_1^2 + 4\sum_{n=1}^{\infty} a_n a_{n+2}.\end{aligned}$$

k	initial $[a_k^-, a_k^+]$
3	[-0.157542, 0.157542]
4	[-0.0463226, 0.0463226]
5	[-0.0183725, 0.0183725]
6	[-0.00871023, 0.00871023]
7	[-0.00465407, 0.00465407]
8	[-0.00271036, 0.00271036]
9	[-0.00168454, 0.00168454]
10	[-0.00110173, 0.00110173]
> 10	$[-1, 1] \cdot 10.9915/k^4$

Table 2: Initial estimates for the intervals $[a_k^-, a_k^+]$.

k	final $[a_k^-, a_k^+]$
3	$[0,\ 0.021055]$
4	$[-0.00192301,\ 0]$
5	$[-1.8253 \times 10^{-07},\ 0.000141734]$
6	$[-9.85549 \times 10^{-06},\ 8.64999 \times 10^{-09}]$
7	$[-6.55526 \times 10^{-10},\ 6.42034 \times 10^{-07}]$
8	$[-4.03088 \times 10^{-08},\ 9.30992 \times 10^{-11}]$
9	$[-3.51558 \times 10^{-10},\ 2.79203 \times 10^{-09}]$
10	$[-1.11597 \times 10^{-09},\ 9.71368 \times 10^{-10}]$
> 10	$[-1, 1] \cdot 10285.3/k^{10}$

Table 3: Estimates for the intervals $[a_k^-, a_k^+]$ representing self-consistent apriori bounds

Using Corollary 3.3 one obtains the following bounds on the errors, ϵ_i, $i = 1, 2$,

$$\epsilon_1 = [-0.00955626, 7.06005 \cdot 10^{-10}] \quad \text{and} \quad \epsilon_2 = [-1.544 \cdot 10^{-8}, 0.0697171].$$

Thus, the equations for which the isolating neighborhood should be found are

$$\begin{aligned} \dot{a}_1 &= \frac{1}{4} a_1 + 2 a_1 a_2 + \epsilon_1 \\ \dot{a}_2 &= -8 a_2 - 2 a_1^2 + \epsilon_2 . \end{aligned} \tag{31}$$

The construction of the isolating block around $(1/\sqrt{2}, -1/8)$ is easier if one works in coordinates determined by the eigenfunctions of the linearized equations at the fixed point. The eigenvalues are

$$\lambda_1 = -4 + 2\sqrt{3}, \quad \lambda_2 = -4 - 2\sqrt{3}$$

with corresponding unit eigenvectors

$$v_1 = \frac{1}{\sqrt{15 - 8\sqrt{3}}} (1, \frac{\lambda_1}{\sqrt{2}})^t, \quad v_2 = \frac{1}{\sqrt{15 + 8\sqrt{3}}} (1, \frac{\lambda_2}{\sqrt{2}})^t .$$

Let T be the affine change of variables from $(x,y)^t$ in this new basis to the original variables $(a_1, a_2)^t$. Then, on the set $T^{-1}(W)$ (31) becomes

$$\begin{aligned} \dot{x} &= (-4+2\sqrt{3})x + f_1(x,y) + \tilde{\epsilon}_1 \\ \dot{y} &= (-4-2\sqrt{3})y + f_2(x,y) + \tilde{\epsilon}_2 \end{aligned} \tag{32}$$

where f_1 and f_2 are polynomials containing only terms of degree two and $\tilde{\epsilon}_1$, $\tilde{\epsilon}_2$ are obtained from ϵ_1, ϵ_2 by the transformation T. In particular,

$$\tilde{\epsilon}_1 = [-0.0110098, 0.0152184], \quad \tilde{\epsilon}_2 = [-0.0764461, 0.00397075]$$

Set $\tilde{W} = [-0.0748016, 0.0748016]^2$, then $T(\tilde{W}) \subset W$. Thus, an isolating block $\tilde{N} \subset \tilde{W}$ (which satisfies the error constraints) will give rise to an isolating block $N = T(\tilde{N}) \subset W$ such that N, W and $\{a_k\}$ are topologically consistent. Since for each k, $\mathrm{dir}(k) = -1$,

$$CH_j(\mathrm{Inv}\,(N)) \cong \begin{cases} \mathbb{Z} & \text{if } j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, by Theorem 2.16, there exists the desired fixed point in

$$u^* \in N \times \prod_{k=2}^{\infty} [a_k^-, a_k^+].$$

Thus, all that remains is to construct $\tilde{N}$. In the (x,y) coordinates the error constraints become

$$\begin{aligned} f_1(x,y) + \tilde{\epsilon}_1 &\subset (b_x, B_x) := (-0.0334071, 0.0376157) \quad \text{for } (x,y) \in \tilde{W} \\ f_2(x,y) + \tilde{\epsilon}_2 &\subset (b_y, B_y) := (-0.0764461, 0.00397075) \quad \text{for } (x,y) \in \tilde{W} \end{aligned}$$

This implies that bounds on the derivative are given by

$$\lambda_1(x + \frac{b_x}{\lambda_1}) < \dot{x} < \lambda_1(x + \frac{B_x}{\lambda_1}) \quad \text{and} \quad \lambda_2(y + \frac{b_y}{\lambda_2}) < \dot{y} < \lambda_2(y + \frac{B_y}{\lambda_2}).$$

Observe that $\lambda_i < 0$, hence the box

$$\begin{aligned} \tilde{N} &= \left[-\frac{b_x}{\lambda_1}, -\frac{B_x}{\lambda_1}\right] \times \left[-\frac{b_y}{\lambda_2}, -\frac{B_y}{\lambda_2}\right] \\ &= [-0.0623385, 0.0701918] \times [-0.0132425, 0.00353264] \subset \tilde{W} \end{aligned}$$

is an isolating block.

5 Obtaining the Initial Estimates

Before beginning this section we want to once again emphasize that the proofs of the theorems in this paper are in principle independent of this section. On the other hand, good initial guesses greatly improve the speed of convergence. The estimates described in what follows apparently provide excellent initial values for the self-consistent bounds.

We will follow the arguments from [11] to produce estimates for errors in the Galerkin projection. Kuramoto-Sivashinsky can be written in the form

$$u_t + \nu A u - A^{1/2} u + 2B(u,u) = 0$$

where

$$A = \frac{\partial^4}{\partial x^4}, \quad A^{1/2} = -\frac{\partial^2}{\partial x^2}, \quad B(u,v) = u\frac{\partial v}{\partial x} \ .$$

While $A^{1/4} \neq \frac{\partial}{\partial x}$, it is still the case that

$$|A^{1/4}u|_2 = |\frac{\partial u}{\partial x}|_2.$$

To simplify the notation, let

$$|u| = |u|_2, \quad \|u\| = |A^{1/4}u|_2.$$

Since we are interested in bounded invariant sets we can without loss of generality assume the following apriori bounds for the invariant set under consideration:

$$|u(t)| \leq \rho_0, \quad \|u(t)\| \leq \rho_1, \quad \text{for } t > T(u(0)) \tag{33}$$

We will make use of the following inequality [11, Lemma 1.4]

$$|(B(u,v),w)| \leq \sqrt{2}|u|^{1/2}\|u\|^{1/2}\|v\||w| \tag{34}$$

The eigenvalues of A are $\lambda_n = n^4$, $n \in \mathbb{N}$ and the corresponding complete family of orthonormal eigenfunctions are $\{\frac{1}{\sqrt{\pi}}\sin(nx)\}$. Let $P = P_m$ be an orthogonal projection on first m eigenfunctions and set $Q = I - P$.

Using the decomposition

$$p = Pu, \quad q = Qu$$

and the same abuse of notation the equation for q is

$$\dot{q} = -\nu A q + A^{1/2} q - 2QB(u,u). \tag{35}$$

Theorem 5.1 *Under the assumptions stated above if m is large enough such that $\lambda_{m+1} > \frac{1}{\nu^2}$, then*

$$\limsup_{t\to\infty} |q(t)| \leq \frac{2\sqrt{2}\rho_0^{1/2}\rho_1^{3/2}}{\lambda_{m+1}(\nu - \lambda_{m+1}^{-\frac{1}{2}})}.$$

Proof: Beginning with (35) and taking a scalar product with q gives

$$(\frac{dq}{dt}|q) = -\nu(Aq|q) + (A^{1/2}q|q) - 2(B(u,u)|q).$$

Therefore,

$$\frac{1}{2}\frac{d}{dt}|q|^2 \leq -\nu(Aq|q) + (A^{1/2}q|q) + 2|(B(u,u)|q)| \tag{36}$$

Observe that

$$(Aq|q) = (A^{1/2}q|A^{1/2}q) = |A^{1/2}q|^2 \tag{37}$$

and

$$\begin{aligned}
(A^{1/2}q|q) &= \sum_{n=m+1}^{\infty} \lambda_n^{1/2}|q_n|^2 \\
&= \lambda_{m+1}^{-1/2}\sum_{n=m+1}^{\infty} \lambda_{m+1}^{1/2}\lambda_n^{1/2}|q_n|^2 \\
&\leq \lambda_{m+1}^{-1/2}\sum_{n=m+1}^{\infty} \lambda_n|q_n|^2 \\
&= \lambda_{m+1}^{-1/2}|A^{1/2}q|^2
\end{aligned} \tag{38}$$

Thus, (36) (37) and (38) imply that

$$\frac{d}{dt}|q|^2 \leq -2\nu|A^{1/2}q|^2 + 2\lambda_{m+1}^{-1/2}|A^{1/2}q|^2 + 4|(B(u,u)|q)|. \tag{39}$$

From (34) it follows that

$$|(B(u,u),q)| \leq \sqrt{2}|u|^{1/2}\|u\|^{3/2}|q|.$$

While, (33) implies that for $t > T(u)$

$$|(B(u,u),q)| \leq \sqrt{2}\rho_0^{1/2}\rho_1^{3/2}|q|$$

Therefore, (39) becomes

$$\frac{d}{dt}|q|^2 \leq -2(\nu - \lambda_{m+1}^{-1/2})|A^{1/2}q|^2 + 4\sqrt{2}\rho_0^{1/2}\rho_1^{3/2}|q|$$

Observe that $|A^{1/2}q|^2 \geq \lambda_{m+1}|q|^2$ and by assumption $\nu - \lambda_{m+1}^{-1/2} > 0$. Thus,

$$\begin{aligned}
\frac{d}{dt}|q|^2 &\leq -2(\nu - \lambda_{m+1}^{-1/2})\lambda_{m+1}|q|^2 + 4\sqrt{2}\rho_0^{1/2}\rho_1^{3/2}|q| \\
&= (4\sqrt{2}\rho_0^{1/2}\rho_1^{3/2} - 2(\nu - \lambda_{m+1}^{-1/2})\lambda_{m+1}|q|)|q|
\end{aligned}$$

Thus, for

$$|q| > \frac{4\sqrt{2}\rho_0^{1/2}\rho_1^{3/2}}{2(\nu - \lambda_{m+1}^{-1/2})\lambda_{m+1}}$$

$\frac{d}{dt}|q|^2 < 0$ and hence,

$$\limsup_{t->\infty} |q(t)| \leq \frac{4\sqrt{2}\rho_0^{1/2}\rho_1^{3/2}}{2(\nu - \lambda_{m+1}^{-1/2})\lambda_{m+1}}.$$

∎

Remark 5.2 Because an orthonormal collection eigenvectors were used for the calculations in this section, the coefficients q_k and a_k differ by a scaling, i.e.

$$q_k = -2\sqrt{\pi} a_n.$$

Theorem 5.1 can be used as follows. For a fixed ν, numerical experiments can suggest values for ρ_0 and ρ_1. For $m < k \leq M$ one can use the formula

$$a_k^{\pm} := \min \left\{ \pm\rho_0, \pm\frac{\rho_1}{k}, \pm\frac{4\sqrt{2\pi\rho_0\rho_1^3}}{k^4(\nu - k^-2)} \right\}.$$

For $k > M$, one defines

$$C_4 := \frac{4\sqrt{2\pi\rho_0\rho_1^3}}{\nu - (M+1)^{-2}}.$$

6 Appendix

We present here some of the numerical details of the proof of Theorem 1.2. The computation was performed on a PC with a Pentium III 450 MHz processor. The algorithm was implemented in *C++* and was compiled the Borland C++ compiler. We used double precision real numbers (64 bits, 52 bits representing the mantissa). For numbers of order 1 the floating point accuracy is $2^{-52} \approx 2.2 \times 10^{-16}$. All checks were performed using interval arithmetic. Table 4 contains the coefficients a_k of the function u from Theorem 1.2. In this proof $m = 30$ and $M = 90$. All bounds $a_k^{\pm}$ for $m < k \leq 90$ were smaller than 5×10^{-32} and for $k > M$ we had $a_k^{\pm} = \pm\frac{1.58296\times10^{12}}{k^{20}}$.

The errors in the vector field due to the Galerkin projection, the bounds for $|A_k F(p+q) - A_k F(p)|$ for $k \leq m$, where $p \in W$ and q is in the tail, ranged from 1.03×10^{-53} for $k = 1$ to 4×10^{-39} for $k = 30$.

Let us remark that from inspection of table 4 it is quite obvious that we can set $a_k = 0$ for $k \geq 16$ in Theorem 1.2 without affecting much the estimates of the distance between u^* and u in the L_2 norm and the C^0 norm.

References

[1] M. Allili, S. Day, O. Junge and K. Mischaikow, A Rigorous Numerical Method for the Global Analysis of an Infinite Dimensional Discrete Dynamical System, *in preparation*

$a_1 = 0.712424$	$a_2 = -0.12324$	$a_3 = 0.0101235$
$a_4 = -0.000671738$	$a_5 = 3.88662 \times 10^{-5}$	$a_6 = -2.07233 \times 10^{-6}$
$a_7 = 1.04399 \times 10^{-7}$	$a_8 = -5.04615 \times 10^{-9}$	$a_9 = 2.36319 \times 10^{-10}$
$a_{10} = -1.0795 \times 10^{-11}$	$a_{11} = 4.83273 \times 10^{-13}$	$a_{12} = -2.12787 \times 10^{-14}$
$a_{13} = 9.23919 \times 10^{-16}$	$a_{14} = -3.96426 \times 10^{-17}$	$a_{15} = 1.68362 \times 10^{-18}$
$a_{16} = -7.08681 \times 10^{-20}$	$a_{17} = 2.95976 \times 10^{-21}$	$a_{18} = -1.22758 \times 10^{-22}$
$a_{19} = 5.06006 \times 10^{-24}$	$a_{20} = -2.0742 \times 10^{-25}$	$a_{21} = 8.46 \times 10^{-27}$
$a_{22} = -3.43493 \times 10^{-28}$	$a_{23} = 1.38889 \times 10^{-29}$	$a_{24} = -5.59466 \times 10^{-31}$
$a_{25} = 2.24579 \times 10^{-32}$	$a_{26} = -8.98615 \times 10^{-34}$	$a_{27} = -3.58503 \times 10^{-35}$
$a_{28} = -1.42633 \times 10^{-36}$	$a_{29} = -5.66024 \times 10^{-38}$	$a_{30} = -2.24088 \times 10^{-39}$

Table 4: Coefficients for the function $u(x)$ from theorem 1.2. We included only seven most significant digits.

[2] G. Arioli and P. Zgliczyński, Symbolic dynamics for the Hénon–Heiles hamiltonian on the critical energy level, *J. Diff. Eq*, accepted

[3] L. Arnold, et. al., *Dynamical Systems*, Lect. Notes Math. 1609, R. Johnson, ed., 1995.

[4] L. Cesari, Functional analysis and Galerkin's method, *Mich. Math. Jour.* **11** (1964) 383-414.

[5] C. Conley, *Isolated Invariant Sets and the Morse Index.* CBMS Lecture Notes **38** A.M.S. Providence, R.I. 1978.

[6] C. Conley and P. Fife, Critical manifolds, travelling waves and an example from population genetics, *J. Math. Bio.* **14** (1982) 159-176.

[7] C. Foias, B. Nicolaenko, G. Sell, R. Temam, Inertial manifolds for the Kuramoto–Sivashinsky equation and an estimate of their lowest dimension. *J. Math. Pures Appl.* **67**, (1988), 197–226

[8] Z. Galias and P. Zgliczyński, Computer assisted proof of chaos in the Lorenz system, *Physica D*, **115**, 1998,165–188

[9] J. K. Hale, *Asymptotic Behavior of Dissipative Systems*, Math. Surveys and Monographs **25**, AMS 1988

[10] J. K. Hale, L. T. Magalhães, and W. M. Oliva, *An Introduction to Infinite Dimensional Dynamical Systems - Geometric Theory*, Appl. Math. Sci. **47**, Springer-Verlag 1984.

[11] M. Jolly, I. Kevrekidis, E. Titi, Approximate inertial manifolds for the Kuramoto–Sivashinsky equation: analysis and computations, *Physica D* **44**, 38-60, (1990)

[12] M. Jolly, R. Rosa, R. Temam, Evaluating the dimension of an inertial manifold for the Kuramoto–Sivashinsky Equation, preprint

[13] Y. Kuramoto, T. Tsuzuki, Persistent propagation of concentration waves in dissipative media far from thermal equilibrium, *Prog. Theor. Phys.* **55**,(1976), 365

[14] C. McCord, Mappings and homological properties in the Conley index theory, *Ergod. Th. & Dyn. Sys.* **8*** (1988) 175-198.

[15] K. Mischaikow and M. Mrozek, Isolating neighborhoods and Chaos, *Jap. J. Ind. & Appl. Math.*, **12**, 1995, 205-236..

[16] K. Mischaikow and M. Mrozek, Chaos in Lorenz equations: a computer assisted proof, *Bull. Amer. Math. Soc. (N.S.)*, **33**(1995), 66-72.

[17] K. Mischaikow and M. Mrozek, Conley Index Theory, to appear in Handbook of Dynamical Systems, Vol. 3.

[18] K. Mischaikow, M. Mrozek and A. Szymczak, Chaos in Lorenz equations: a computer assisted proof, Part III: Classical parameter values, *JDE* **169** (2001) 17-56.

[19] D. Salamon, Connected simple systems and the Conley index of isolated invariant sets. *Trans. A. M. S.* **291** (1985) 1 - 41.

[20] G.I. Sivashinsky, Nonlinear analysis of hydrodynamical instability in laminar flames – 1. Derivation of basic equations, *Acta Astron.* **4**, (1977), 1177

[21] J. Smoller, *Shock Waves and Reaction Diffusion Equations*, Springer Verlag, New York, 1980.

[22] R. Srzednicki, Periodic and Constant Solutions Via Topological Principle of Ważewski. *Universitatis Iagellonicae, Acta Mathematica*, **XXVI**, 183–190, (1987)

[23] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, Springer-Verlag 1988.

[24] S. Williams, *On the Cesari Fixed Point Method in a Banach Space*, Ph.D. Thesis, California Institute of Technology, 1967.

[25] K. Wójcik and P. Zgliczyński, Isolating segments, fixed point index and symbolic dynamics, *J. Diff. Eq*, 161, 245–288, (2000)

[26] P. Zgliczyński, Computer assisted proof of chaos in the Hénon map and in the Rössler equations, *Nonlinearity*,**10**, 1997, No. 1, 243–252

[27] P. Zgliczyński, Multidimensional perturbations of one-dimensional maps and stability of Sharkovskii ordering, *Int. J. of Bifurcation and Chaos*, **9**, No. 9, 1999, 1867–1876

[28] P. Zgliczyński, Trapping regions and an ODE-type proof of an existence and uniqueness for Navier-Stokes equations with periodic boundary conditions on the plane, *http://arXiv.org/abs/math/0103053*