Attracting fixed points for the Kuramoto-Sivashinsky equation - a computer assisted proof

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Abstract

We present a computer assisted proof of an existence of several attracting fixed points for the Kuramoto-Sivashinsky equation

$$u_t = (u^2)_x - u_{xx} - \nu u_{xxxx}, \quad u(x,t) = u(x+2\pi,t), \quad u(x,t) = -u(-x,t),$$

where $\nu > 0$. The method is general and can be applied to other dissipative PDEs.

Keywords: dissipative PDEs, fixed points, Galerkin projection, computer assisted proof

MSC numbers: 35B35, 35B45, 65G20, 65N30

1 Introduction

The goal of this paper is to extend the method of self-consistent a priori bounds developed in [ZM, Z] for a rigorous study of dynamics of dissipative PDEs. We present an approach which allows to show that a given fixed point for a PDE is asymptotically stable. We apply the method to the Kuramoto-Sivashinsky equation (KS equation) subject to periodic and odd boundary conditions

$$u_t = (u^2)_x - u_{xx} - \nu u_{xxxx}, \quad u(x,t) = u(x+2\pi,t), \quad u(x,t) = -u(-x,t). \quad (1)$$

where $\nu > 0$. While the method will be explained in details later, we would like to stress here its basic ingredients, which will also explain how this paper is dependent on [Z] and [ZM] and what is new here.

The approach starts as in [ZM]:

1. we have to find an approximate attracting fixed point x_0 for some Galerkin projection of (1). Then we construct of a trapping region around x_0 (which is an

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example of self-consistent a priori bounds defined in [ZM]) using the algorithm presented in [ZM]. From this we conclude that there exists a fixed point $x^* \in R$, but we cannot claim its asymptotic stability.
2. In paper [Z] we obtained estimates for the Lipschitz constants for the flow induced by the Navier-Stokes equations on two-dimensional torus. Here we adopt this approach to construct a norm for which the induced flow is a contraction around x^*.

In the present work for each attracting branch from a nonrigorous steady state bifurcation diagram presented in [JKT] we picked up a point on it and we proved that it is attracting.

Below we include some of the attracting steady states we had proved rigorously to exist.

- $\nu \in 0.75 + [-10^{-2}, 10^{-2}]$ two stable unimodal fixed points.
- $\nu \in 0.5 + [-10^{-4}, 10^{-4}]$ two stable unimodal fixed points.
- $\nu \in 0.3 + [-10^{-4}, 10^{-4}]$ two stable unimodal fixed points.
- $\nu \in 0.125 + [-10^{-4}, 10^{-4}]$ one stable bimodal fixed point.
- $\nu \in 0.1 + [-10^{-4}, 10^{-4}]$ one bimodal stable fixed point.
- $\nu \in 0.08 + [-10^{-6}, 10^{-6}]$ one bimodal stable fixed point. A pair of stable fixed points close to R_3t_2 (see [JKT]).
- $\nu \in 0.062+[-10^{-6}, 10^{-6}]$, two stable trimodal points and two stable points from giant branch.
- $\nu \in 0.045 + [-10^{-6}, 10^{-6}]$, $\nu \in 0.04 + [-10^{-7}, 10^{-7}]$, two stable points from giant branch.

In the above listing when we write, that for $\nu \in 0.75 + [-10^{-2}, 10^{-2}]$ we have two stable fixed points, then this means that for all ν in this interval these stable fixed points exist. In Section 4 we present an example of precise theorem about an existence of fixed point obtained using our method.

In Sections 2 and 3 we present the method in details and we prove Theorem 8, which is a main tool in our approach. In Section 4 we present an example of precise theorem, we give an outline of the algorithm and present numerical data from the proof. In Section 5 we derive various estimates for KS-equation required in the rigorous check of assumptions of Theorem 8. In Section 6 we discuss the directions in which this work can be extended further.

2 Uniform convergence of Galerkin projections on a trapping region

We adopt here the notations used in Sections 4 and 5 in [Z]. Let H be a real Hilbert space. Let $e_1, e_2, \dots$ form an orthonormal basis in H.

In the sequel we will quite often denote the elements of H by x and we hope it will not be confused with the space variable in (1)

Let $A_n : H \to \mathbb{R}$ denote a projection onto 1-dimensional subspace $\langle e_n \rangle$, i.e $x = \sum A_n(x) e_n$ for all $x \in H$. By X_n we will denote a space spanned by $\{e_1, \dots, e_n\}$. Let P_n denote the projection onto X_n, $Q_n = I - P_n$.

For $x \in \mathbb{R}^n$ or $x \in H$ we set $|x|$ to be a standard (Euclidean) norm ,$|x|_\infty = \max_i |x_i|$ and $|x|_1 = \sum_i |x_i|$.

We investigate the Galerkin projections of the following problem

$$x' = F(x) = L(x) + N(x), \tag{2}$$

where L is a linear operator and N is a nonlinear part of F. We assume that the basis $e_1, e_2, \dots$ of H is build from eigenvectors of L. We assume that the corresponding eigenvalues λ_k (i.e. $Le_k = \lambda_k e_k$) are ordered so that

$$\lambda_1 \geq \lambda_2 \geq \dots, \quad \text{and} \quad \lim_{k\to\infty} \lambda_k = -\infty.$$

Hence L can have only a finite number of positive eigenvalues.

Definition 1 *Let $W \subset H$ and $F : dom(F) \to H$, W is closed. We say that W and F satisfy conditions C1,C2,C3 if*

C1 *There exists $M \geq 0$, such that $P_n(W) \subset W$ for $n \geq M$*

C2 *Let $\hat{u_k} = \max_{x\in W} |A_k x|$. Then, $\hat{u} = \sum \hat{u_k} e_k \in H$. In particular, $|\hat{u}| < \infty$.*

C3 *The function $x \mapsto F(x)$ is continuous on W and $f = \sum_k f_k e_k$, given by $f_k = \max_{x\in W} |A_k F(x)|$ is in H. In particular, $|f| < \infty$.*

Observe that if $W_m \subset X_m$ and $\{a_k^-, a_k^+\}$ form self-consistent a priori bounds (see [ZM, Def. 2.1]) for F, then $W = W_m \oplus \Pi_{k=m+1}^\infty [a_k^-, a_k^+]$ and F satisfy conditions *C1, C2, C3*.

Definition 2 *We say that $W \subset H$ and $F = N + L$ satisfy condition* D *if for any i, j the function*

$$\frac{\partial N_i}{\partial x_j} : W \to \mathbb{R} \tag{3}$$

is continuous and the following condition hold

D *there exists $l \in \mathbb{R}$ such that for all $k = 1, 2, \dots$*

$$1/2 \sum_{i=1}^\infty \left|\frac{\partial N_k}{\partial x_i}\right|(W) + 1/2 \sum_{i=1}^\infty \left|\frac{\partial N_i}{\partial x_k}\right|(W) + \lambda_k \leq l \tag{4}$$

The main idea behind condition D is to ensure that Lipschitz constants of flows induced by Galerkin projections of (2) are uniformly bounded (see the proof of Theorem 13 in [Z] for more details).

Definition 3 *Consider an ODE*

$$x' = f(x), \tag{5}$$

where $x \in \mathbb{R}^n$. The compact set $W \subset \mathbb{R}^n$ is called a trapping region for (5) *if for any solution, $x(t)$, of (5) if $x(0) \in W$, then $x(t) \in W$ for all $t > 0$.*

The following easy lemma was used throughout this paper as a criterion for a set be a trapping region.

Lemma 1 *Assume that W is a closure of an open set, with a piecewise smooth boundary. For any $x \in \partial W$ let $\nu(x)$ denote an outward normal vector to ∂W.*

If for all $x \in \partial W$ we have $\nu(x) \cdot f(x) < 0$, then W is a trapping region for (5).

The following theorem was proved in [Z]

Theorem 2 *[Z, Thm. 13] Assume that $R \subset H$ and F satisfy conditions* C1,C2,C3,D *and R is convex. Assume that $P_n(R)$ is a trapping region for the n-dimensional Galerkin projection of (2) for all $n > M_1$. Then*

1. **Uniform convergence and existence** *For a fixed $x_0 \in R$, let $x_n : [0, \infty] \to P_n(R)$ be a solution of $x' = P_n(F(x))$, $x(0) = P_n x_0$. Then x_n converges uniformly on compact intervals to a function $x^* : [0, \infty] \to R$, which is a solution of (2) and $x^*(0) = x_0$. The convergence of x_n on compact time intervals is uniform with respect to $x_0 \in R$.*

2. **Uniqueness within** R. *There exists only one solution of the initial value problem (2), $x(0) = x_0$ for any $x_0 \in R$, such that $x(t) \in R$ for $t > 0$.*

3. **Lipschitz constant**. *Let $x : [0, \infty] \to R$ and $y : [0, \infty] \to R$ be solutions of (2), then*
$$|y(t) - x(t)| \leq e^{lt} |x(0) - y(0)|$$

4. **Semidynamical system.** *The map $\varphi : \mathbb{R}_+ \times R \to R$, where $\varphi(\cdot, x_0)$ is a unique solution of equation (2), such that $\varphi(0, x_0) = x_0$ defines a semidynamical system on R, namely*

   - *φ is continuous*
   - *$\varphi(0, x) = x$*
   - *$\varphi(t, \varphi(s, x)) = \varphi(t + s, x)$*

In the context of this paper the statement about the Lipschitz constant in Theorem 2 is of special importance. We can formulate it as follows

$$|\varphi(t, x) - \varphi(t, y)| \leq e^{lt} |x - y|, \quad t \geq 0. \tag{6}$$

where l is given in condition D and $x, y \in R$.

Assume that we have a trapping region, R, satisfying assumptions of Theorem 2. Next step is to prove that the induced semiflow is contracting. This maybe hard to achieve in the original norm, but in Section 3 we construct another norm (similar to $|\cdot|_\infty$-norm) for which we are able to show that (6) holds with $l < 0$ for the steady states for KS-equation mentioned in the Introduction.

3 Diagonalization and construction of 'contracting' norm

As was mentioned in Section 2 we would like to construct a 'contracting' norm on trapping region R ($l<0$ in equation (6)).

3.1 Block decomposition

Our construction will be based on the approximate diagonalization of the matrix $\frac{\partial F}{\partial x}$. We want this matrix to be dominated by diagonal terms. This is achieved by an approximate diagonalization in case of real eigenvalues. The case of the complex eigenvalues forces us to consider blocks on the diagonal. We formalize this as follows

Definition 4 *A decomposition of H, into a sum of subspaces is called* a block decomposition of H *if the following conditions are satisfied*

1. $H=\bigoplus_i H_i$,

2. *for every i $h_i = dim\ H_i \leq h_{\max} < \infty$,*

3. *for every i $H_i = \langle e_{i_1}, e_{i_2}, \ldots, e_{i_{h_i}}\rangle$,*

4. *If $dim\, H=\infty$, then there exists k such that for $i>k$ $h_i=1$.*

For a block decomposition of H we adopt the following notation, which makes a distinction between blocks and one dimensional subspaces spanned by $\langle e_i\rangle$. For the blocks we use $H_{(i)} = \langle e_{i_1}, \ldots e_{i_k}\rangle$, where $(i)=(i_1,\ldots i_k)$. The symbol H_i will always mean the subspace generated by e_i. For one-dimensional block (i) we adopt the following convention, the only element of (i) will be denoted by the same letter i.

For a given block decomposition of H and block (i), we set

$$\dim\ (i) = \dim\ H_{(i)}.$$

For any $x\in H$ by $x_{(i)}$ we will denote a projection of x onto $H_{(i)}$. For any a and $(i)=(i_1,\ldots,i_k)$ we will say that $(i)\leq a$ if $i_s\leq a$ for all $s=1,\ldots,k$ and we say that $(i)>a$ if $i_s>a$ for all $s=1,\ldots,k$.

On each component $H_{(i)}$ we will use a norm induced from H. By $P_{(i)}$ we will denote an orthogonal projection onto $H_{(i)}$. By $\mathrm{Lin}(H_{(i)},H_{(j)})$ we denote a set of all linear maps from $H_{(i)}$ to $H_{(j)}$ equipped with an operator norm $|A| = \max_{|v|=1, v\in H_{(i)}} |Av|$.

We have the following easy

Lemma 3 *Assume that we have a block decomposition of H and let $W\subset H$. If for any k,l the function*

$$\frac{\partial F_k}{\partial x_l} : W \to \mathbb{R}$$

is continuous, then for every (i) *and* (j) *the map*

$$\frac{\partial F_{(i)}}{\partial x_{(j)}} : W \to Lin(H_{(j)}, H_{(i)}) \approx \mathbb{R}^{dim\ (j) \times dim\ (i)}$$

is continuous.

For any square matrix $Q \in \mathbb{R}^{\dim H \times \dim H}$ (a linear map $Q : \mathrm{dom}(Q) \to H$) and for any blocks $(i), (j)$ we define a matrix $Q_{(i)(j)}$ as the matrix corresponding to an induced linear map $Q_{(i)(j)} : H_{(j)} \to H_{(i)}$ given by $Q_{(i)(j)}(x) = P_{(i)}(QP_{(j)}x)$.

3.2 A block-infinity norm for block decomposition

For a fixed block decomposition of H we define the norm (*the block-infinity norm*) by

$$|x|_{b,\infty} = \max_{(i)} |P_{(i)}x|. \tag{7}$$

We have the following easy

Lemma 4 *Assume that* $W \subset H$, W *is closed and satisfies condition* **C2**. *Then on* W *the convergence in the norm* $|\cdot|$ *is equivalent to the convergence in the norm* $|\cdot|_{b,\infty}$, *namely*

For any sequence $\{x_n\} \subset W$, $|x_n - x^*| \to 0$ *if and only if* $|x_n - x^*|_{b,\infty} \to 0$.

Now we turn to the computation of the logarithmic norm for $|\cdot|_{b,\infty}$.

For any norm $\|\cdot\|$ on $\mathbb{R}^n$ following [HNW] we introduce the notion of logarithmic norm of a matrix by

Definition 5 *Let* Q *be a square matrix, then we call*

$$\mu(Q) = \limsup_{h>0, h\to 0} \frac{\|I + hQ\| - 1}{h} \tag{8}$$

the logarithmic norm *of* Q.

Definition 5 differs slightly from Definition I.10.4 in [HNW], because to avoid a question of an existence of the limit in equation (8) we use the $\limsup$.

By $\mu(Q)$ we denote the logarithmic norm induced by the Euclidean norm, for all other norms we will use a subscript identifying it.

The following theorem was proved in [HNW]

Theorem 5 *[HNW, Th. I.10.5] The logarithmic norm is obtained by the following formulas*

$$\mu(Q) = \textit{the largest eigenvalue of} \quad 1/2(Q + Q^T) \tag{9}$$

$$\mu_\infty(Q) = \max_k \left(q_{kk} + \sum_{i, i\neq k} |q_{ki}| \right) \tag{10}$$

$$\mu_1(Q) = \max_i \left(q_{ii} + \sum_{k, k\neq i} |q_{ki}| \right) \tag{11}$$

Next lemma tells us how to compute the logarithmic norm induced by the block-infinity norm.

Lemma 6 *Assume that we have a block decomposition of $\mathbb{R}^n$, then*

$$\mu_{b,\infty}(Q) \leq \max_{(i)} \left(\mu(Q_{(i)(i)}) + \sum_{(k)\neq(i)} |Q_{(i)(k)}| \right), \tag{12}$$

Proof: Since for any $h > 0$, (i) and $x \in \mathbb{R}^n$ we have

$$\begin{aligned} |P_{(i)}(I+hQ)x| = &\left| \left(I_{(i)(i)} + hQ_{(i)(i)}\right) x_{(i)} + h \sum_{(j),(j)\neq(i)} Q_{(i)(j)} x_{(j)} \right| \leq \\ &\left| \left(I_{(i)(i)} + hQ_{(i)(i)}\right) x_{(i)} \right| + h \sum_{(j),(j)\neq(i)} \left| Q_{(i)(j)} x_{(j)} \right| \leq \\ &\left(\left| I_{(i)(i)} + hQ_{(i)(i)} \right| + h \sum_{(j),(j)\neq(i)} \left| Q_{(i)(j)} \right| \right) |x|_{b,\infty}, \end{aligned}$$

then

$$|(I+hQ)|_{b,\infty} \leq \max_{(i)} \left(\left| I_{(i)(i)} + hQ_{(i)(i)} \right| + h \sum_{(j),(j)\neq(i)} \left| Q_{(i)(j)} \right| \right). \tag{13}$$

From the above equation and the definition of the logarithmic norm one easily obtains the assertion of the theorem. ∎

From Theorem 5 it follows that, when all blocks are one-dimensional then we have an equality in (12), but observe that this is not true in general as it is shown by the following example.

Let $n = 4$. Consider the following blocks $(e_1, e_2), (e_3), (e_4)$ and the matrix

$$Q = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{14}$$

An easy computation shows that

$$\mu_{b,\infty}(Q) = 2 < \max_{(i)} \left(\mu(Q_{(i)(i)}) + \sum_{(k)\neq(i)} |Q_{(i)(k)}| \right) = 2\sqrt{2}$$

3.3 Lipschitz constants in block-infinity norm and main theorem

The following theorem has exactly the same proof as Theorem 13 in [Z]. The only difference is that the standard norm in H is replaced by the block-infinity norm.

Theorem 7 *Assume that $R \subset H$, R is convex and F satisfy conditions* C1, C2, C3. *Assume that we have a block decomposition of H, such that condition* **Db** *holds*

Db *there exists $l \in \mathbb{R}$, such that for any (i) and $x \in R$*

$$\mu\left(\frac{\partial F_{(i)}}{\partial x_{(i)}}(x)\right) + \sum_{(k),\,(k)\neq(i)} \left|\frac{\partial F_{(i)}}{\partial x_{(k)}}(x)\right| \leq l$$

Assume that $P_n(R)$ is a trapping region for the n-dimensional Galerkin projection of (2) for all $n > M_1$. Then

1. **Uniform convergence and existence** *For a fixed $x_0 \in R$, let $x_n : [0,\infty] \to P_n(R)$ be a solution of $x' = P_n(F(x))$, $x(0) = P_n x_0$. Then x_n converges uniformly in a max-infinity norm on compact intervals to a function $x^* : [0,\infty] \to R$, which is a solution of (2) and $x^*(0) = x_0$. The convergence of x_n on compact time intervals is uniform with respect to $x_0 \in R$.*

2. **Uniqueness within** R. *There exists only one solution of the initial value problem (2), $x(0) = x_0$ for any $x_0 \in R$, such that $x(t) \in R$ for $t > 0$.*

3. **Lipschitz constant**. *Let $x : [0,\infty] \to R$ and $y : [0,\infty] \to R$ be solutions of (2), then*
$$|y(t) - x(t)|_{b,\infty} \leq e^{lt}|x(0) - y(0)|_{b,\infty}$$

4. **Semidynamical system.** *The map $\varphi : \mathbb{R}_+ \times R \to R$, where $\varphi(\cdot, x_0)$ is a unique solution of equation (2), such that $\varphi(0, x_0) = x_0$ defines a semidynamical system on R, namely*
   - *φ is continuous*
   - $\varphi(0, x) = x$
   - $\varphi(t, \varphi(s, x)) = \varphi(t + s, x)$

The following theorem is the main tool in proving an existence of attracting fixed points.

Theorem 8 *Same assumptions on R,F and a block decomposition of H as in Theorem 7. Assume that $l < 0$.*

Then there exists a fixed point for (2) $x^* \in R$, unique in R, such that for every $y \in R$

$$|\varphi(t, y) - x^*|_{b,\infty} \leq e^{lt}|y - x^*|_{b,\infty}, \quad \text{for } t \geq 0$$
$$\lim_{t\to\infty} \varphi(t, y) = x^*$$

Proof: It is enough to prove an existence of $x^* \in R$, such that $F(x^*) = 0$, because the uniqueness in R and all other assertions follow directly from the assumption $l < 0$ and the Lipschitz constant estimates given in Theorem 7.

It is easy to see that for all $n > M_1$ there exists $x_n \in P_n R$ a fixed point for the n-th Galerkin projection. Passing to the limit with n (by picking a subsequence eventually) we obtain x^* (see [ZM, Thm 2.16] for details). ∎

4 An example of a theorem and a description of an algorithm

In this section we present an example of theorem we prove using our method, give a description of an algorithm and provide some numerical data from the proof.

Theorem 9 *Let*

$$u(x) = -2\left(0.711691\sin(x) - 0.123059\sin(2x) + 0.01011\sin(3x)\right).$$

For any $\nu \in 0.75 + [-10^{-2}, 10^{-2}]$ there exists an equilibrium solution $u_\nu(x)$ to (1) such that

- *u_ν is attracting*
- *$\|u_\nu - u\|_{L^2} \leq 0.104$, $\|\partial_x u_\nu - \partial_x u\|_{L^2} \leq 0.132$, $\|u_\nu - u\|_{C^0} \leq 0.084$.*

The attracting fixed point from above theorem had already been discovered (nonrigorously) by Jolly *et al.* in [JKT]. In the terminology used there this is a unimodal fixed point.

The proof consists of two parts

1. A construction of topologically self-consistent a priori bounds (see Def. 2.11 in [ZM]) for KS-equation, i.e. $W \subset X_m$ and $\{a_k^-, a_k^+\}_{k>m}$, an isolating block $N \subset W$ with empty exit set. We define a set R given by

$$R = N \oplus \Pi_{k=m+1}^{\infty}[a_k^-, a_k^+]. \tag{15}$$

From the construction it follows that $P_n(R)$ a trapping region for n-th Galerkin projection of KS equation for $n \geq m$.

From results in [ZM] it follows that there exists $u_\nu \in R$ such that $F(u_\nu) = 0$.

2. Computation of l. Now if $l < 0$ then from Theorem 8 it follows that u_ν is attracting.

The method of construction of a the tail in self-consistent bounds i.e. the numbers $a_k^\pm$ for $k > m$ is described in Section 3.3 in [ZM], but the construction of an isolating block N was not presented there, therefore we present here an outline of this algorithm.

First we introduce some notation and fix some terminology. We recall condition *C4a* from [ZM]

C4a Let $u \in W \oplus \prod_{k=m+1}^{\infty}[a_k^-, a_k^+]$. Then, for $k > m$

$$A_k u = a_k^+ \quad \Rightarrow \quad A_k F_\nu(u)) < 0 \tag{16}$$

$$A_k u = a_k^- \quad \Rightarrow \quad A_k F_\nu(u)) > 0. \tag{17}$$

Definition 6 *In the context of a block decomposition of a finite dimensional space, $H = \bigoplus H_{(i)}$, we consider a system of differential inclusions, which is a product of inclusions for each block of the following form*

$$x'_k \in \lambda_k x_k + (b_k, B_k) \tag{18}$$

or 2-dim block $(k) = (k_1, k_2)$

$$\begin{aligned} x'_{k_1} &\in \alpha_{(k)} x_{k_1} - \beta_{(k)} x_{k_2} + (b_{k_1}, B_{k_1}) \\ x'_{k_1} &\in -\beta_{(k)} x_{k_1} + \alpha_{(k)} x_{k_2} + (b_{k_2}, B_{k_2}). \end{aligned} \tag{19}$$

Let $N = \bigoplus N_{(i)}$, where $N_{(i)} \subset H_{(i)}$ and

$$\begin{aligned} N_{(i)} &= [n^-_{(i)}, n^+_{(i)}], \qquad \textit{if } \dim(i) = 1 \\ N_{(i)} &= \overline{B}(0, n_{(i)}), \qquad \textit{if } \dim(i) = 2. \end{aligned}$$

We will say that we have an isolation for (k)-block on N *if the following holds*

$$\lambda_k n^+_k + (b_k, B_k) < 0, \qquad \lambda_k n^-_k + (b_k, B_k) > 0, \qquad \textit{if } \dim(k) = 1, \tag{20}$$

if $\dim(k) = 2$ *we require that*

$$\lambda_{(k)} n_{(k)} + \sqrt{(b_{k_1}, B_{k_1})^2 + (b_{k_2}, B_{k_2})^2} < 0. \tag{21}$$

The following easy lemma explains why we care for an isolation.

Lemma 10 *Let a block decomposition of H, set N and differential inclusions be as in Def. 6. If we have an isolation for all blocks on N, then the set N is a trapping region (in particular it is an isolating block).*

We perform our computations in an interval arithmetic [Mo]. We use the following notations and conventions:

By arabic letters we denote both single valued objects like vectors, real numbers, matrices and sets of these objects. Sometimes we will use square brackets, for example $[r]$, to denote sets. Usually this will be some set constructed in an algorithm. In situations when we want to stress, that we have a set in a formula involving both single-valued objects and sets together we will rather use square bracket, hence we prefer to write $[S]$ instead of S to represent a set. From this point of view $[S]$ and S are different symbols in the alphabet used to name variables and formally speaking there is no relation between the set represented by $[S]$ and the object represented by S. Sometimes both variables $[S]$ and S are used simultaneously, then usually $S \in [S]$ in this situation, but this is always stated explicitly.

For a set $[S]$ by $[S]_I$ we denote an interval hull of $[S]$, i.e. the smallest product of intervals containing $[S]$. The symbol $\mathrm{hull}(x_1, \ldots, x_k)$ will denote an interval hull of intervals $x_1, \ldots, x_k$. For any interval set $[S] = [S]_I$ by $\mathrm{m}([S])$ we will denote a center point of $[S]_I$. For any interval $[a, b]$ we define a diameter by $\mathrm{diam}([a, b]) = b - a$. For an interval vector or an interval matrix $[S] = [S]_I$ by $\mathrm{diam}([S])$ we will denote the maximum of diameters of its components. For an interval $[x^-, x^+]$ we set $right([x^-, x^+]) = x^+$ and $left([x^-, x^+]) = x^-$.

4.1 A detailed outline of algorithm

Input data:

- m, M - dimensions describing self-consistent bounds
- $[\nu] = \nu_0 + [-\delta\nu, \delta\nu]$ - a range of parameters
- $x_0 \in \mathbb{R}^m$, such that $P_m F_{\nu_0}(x_0) \approx 0$ - this is our candidate for a fixed point
- Δ - a parameter defining an initial size of N

1. *An approximate diagonalization of $dP_mF_{\nu_0}(x_0)$, a generation of new coordinates in $X_m = \mathbb{R}^m$ and a block decomposition of H.*

From an approximate diagonalization of $dP_mF_{\nu_0}(x_0)$ we obtain new coordinates, which will be called *the block coordinates*. The coordinates a_k will be refered to as *the standard coordinates*. The block coordinates are obtained from standard coordinates through an affine transformation $T : \mathbb{R}^m \to \mathbb{R}^m$

$$T(x) = T_l(x - x_0), \tag{22}$$

where $T_l \in \mathbb{R}^{m \times m}$.

We define a block decomposition of $H = \bigoplus_{(i)} H_i$, such for $(i) > m$ all blocks are given by $H_{(i)} = \langle e_i \rangle$, for $(i) < m$ each block $H_{(i)}$ is an eigenspace of $dP_mF_{\nu_0}(x_0)$. Complex eigenvalues give rise to two-dimensional and real ones to one-dimensional blocks. Eigenvectors from 1-dim blocks are normalized to a unit length, in a 2-dim block the length of a longer vector from the pair is normalized to one.

From now on we change the norm in H so that the blocks $H_{(i)}$ become orthogonal and for two-dimensional blocks the real and the imaginary part of a complex eigenvector are orthogonal.

2. Preparation for the main loop

- *an initialization of variables $Iso[k]$, $1 \leq k \leq m$.* We set $Iso[k] = 0$. It will later become true (i.e. equal to 1) if we will have an isolation for the block containing k-th variable).
- *an initialization of our initial guess for $N = \bigoplus_{(i) \leq m} N_{(i)}$*, where $N_{(i)} \subset H_{(i)}$. We set

$$\begin{aligned} N_{(i)} &= [n^-_{(i)}, n^+_{(i)}] = [-\Delta, \Delta], \qquad \text{if } \dim(i) = 1 \\ N_{(i)} &= \overline{B}(0, n_{(i)}) = \overline{B}(0, \Delta), \qquad \text{if } \dim(i) = 2 \end{aligned}$$

3. Main loop

- *an initialization of a local variable $iso\_change = 0$.*
  The variable *iso_change* tells us if there is any new isolation for $1 \leq k \leq m$ or if our set N has changed giving us the chance that repeating a loop once again will result in a better tail, which may produce new isolations.

- *a computation of W.* $W = [T^{-1}(N)]_I$. It is enough to define W as $W = T^{-1}(N)$, i.e. the set N in standard coordinates. But if we evaluate this formula in interval arithmetics, we obtain the set $[T^{-1}N]_I$ which is larger than $T^{-1}(N)$ due to round-off errors and the wrapping effect [Mo].

- *a generation of self-consistent tail.* Using formulas derived in Section 3 in [ZM], for current values of W, m, M, we find $\{a_k^-, a_k^+\}$, such that conditions *C1,C2,C3* and *C4a* are satisfied on $W \oplus \Pi_{k=m+1}^{\infty}[a_k^-, a_k^+]$.

  In principle the procedure of generation $\{a_k^\pm\}$ may fail, in this case we interrupt the algorithm and return *fail*.

- *a computation of an influence of the tail $V = \Pi_{k=m+1}^{\infty}[a_k^-, a_k^+]$ onto the m-dimensional Galerkin projection.* Using formulas from Section 3 in [ZM] we find an interval vector $[\epsilon] \subset \mathbb{R}^n$, such that

  $$P_m(F_\nu(x)) - P_m(F_\nu(P_m x)) \subset [\epsilon], \qquad \text{for } x \in W \oplus V \tag{23}$$

  Our goal for the next step is to construct an isolating block N for an equation (in fact a differential inclusion)

  $$x' \in P_m(F_\nu(x)) + [\epsilon], \qquad \text{where } x \in W \tag{24}$$

- *a transformation of equation (24) to the block coordinates and 'an interval diagonalization'.*

  We transform equation (24) to the block coordinates, as a result we obtain for $\nu \in \nu_0 + [-\delta\nu, \delta\nu]$, for one-dimensional blocks (i)

  $$x_i' \in \lambda_i(\nu)x_i + f_i(x) + [\tilde{\epsilon}_i] \tag{25}$$

  and for 2-dimensional blocks $(i) = (i_1, i_2)$

  $$\begin{aligned} x_{i_1}' &\in \alpha_{(i)}(\nu)x_{i_1} + \beta_{(i)}(\nu)x_{i_2} + f_{i_1}(x) + [\tilde{\epsilon}_{i_1}] \\ x_{i_2}' &\in -\beta_{(i)}(\nu)x_{i_1} + \alpha_{(i)}(\nu)x_{i_2} + f_{i_2}(x) + [\tilde{\epsilon}_{i_2}]. \end{aligned} \tag{26}$$

  Observe that, since we are using the block coordinates, which diagonalize dP_mFP_m then for a small N the values $f_i(x)$ for $x \in N$ will be usually very small.

  For $1 \leq i \leq m$ we compute an interval (b_i, B_i) such that

  $$f_i(W) + \tilde{\epsilon}_i \subset (b_i, B_i). \tag{27}$$

  Instead of equations (25) and (26) we will consider now equations

  $$x_i' \in \lambda_i(\nu)x_i + (b_i, B_i) \tag{28}$$

  and

  $$\begin{aligned} x_{i_1}' &\in \alpha_{(i)}x_{i_1} + \beta_{(i)}x_{i_2} + (b_{i_1}, B_{i_1}) \\ x_{i_2}' &\in -\beta_{(i)}x_{i_1} + \alpha_{(i)}x_{i_2} + (b_{i_2}, B_{i_2}), \end{aligned} \tag{29}$$

respectively.

Let us stress that in our computations we have uniform bounds for $\lambda_{(i)}(\nu)$, $\alpha_{(i)}(\nu)$ and $\beta_{(i)}(\nu)$. Namely we have intervals $[\lambda_{(i)}]$, $[\alpha_{(i)}]$ and $[\beta_{(i)}]$ such that for all $\nu \in [\nu_0 - \delta\nu, \nu_0 + \delta\nu]$ holds

$$\lambda_{(i)}(\nu) \in [\lambda_{(i)}], \quad \alpha_{(i)}(\nu) \in [\alpha_{(i)}], \quad \beta_{(i)}(\nu) \in [\beta_{(i)}]$$

- *an isolation for* $1 \leq i \leq m$.

  First observe that since we are looking for an attracting fixed point, then $\lambda_i < 0$ and $\alpha_{(i)} < 0$ (provided x_0 is a good approximation).

  For each block we try to find an isolation as follows: for 1-dim block (i) we set

$$d_i^+ = \text{right}\left(-\frac{B_i}{[\lambda_i]}\right), \qquad d_i^- = \text{left}\left(-\frac{b_i}{[\lambda_i]}\right) \tag{30}$$

  Now if

$$[d_i^-, d_i^+] \subset [n_i^-, n_i^+] \tag{31}$$

  then an easy computation shows that we have an isolation for the (i)-th block.

  For 2-dimensional blocks $(i) = (i_1, i_2)$ we set

$$d_{(i)} = \text{right}\left(-\frac{B}{[\alpha_{(i)}]}\right), \tag{32}$$

  where $B = \text{right}\left(\sqrt{(b_{i_1}, B_{i_1})^2 + (b_{i_2}, B_{i_2})^2}\right)$. It is easy to see that we have an isolation for the (i)-th block on N if

$$n_{(i)} \geq d_{(i)}. \tag{33}$$

  If (31) holds for 1-dim block (i), then we set

$$\begin{aligned} \mathit{iso\_change} &= 1 \\ n_i^+ &= d_i^+, \quad n_i^- = d_i^- \\ \mathit{Iso}[i] &= 1 \end{aligned}$$

  If (33) holds for 2-dim block $(i) = (i_1, i_2)$, then we set

$$\begin{aligned} \mathit{iso\_change} &= 1 \\ n_{(i)} &= d_{(i)} \\ \mathit{Iso}[i_1] &= 1, \qquad \mathit{Iso}[i_2] = 1 \end{aligned}$$

  Observe that with these updates we achieve the following: if $\mathit{iso\_change} = 1$, then the current set N is a proper subset of the set N at the beginning of the loop. This guarantees that W and then also $[\epsilon]$ will be smaller in the next iterate. This implies also that, if in one loop we have an isolation for a block (i), then we will have an isolation for this block for all following iterations of the loop and it creates a possibility for obtaining an isolation for other blocks in the next iterations.

- *a verification of an isolation for all blocks.* We check if for all $1 \leq k \leq m$ $Iso[k] = 1$. If this is a case then we leave the loop, because N is the desired trapping region.

  Otherwise we check if $iso\_change = 0$. If this is the case then the algorithm failed. If $iso\_change = 1$ we repeat the loop again.

4. Computation of l. From previous steps we have a block decomposition of H, a change of coordinates T, a trapping region N and a tail $[a_k^-, a_k^+]_{k>m}$.

We compute l using formulas from Section 5 on the set $W = [T^{-1}(N)]_I \oplus \Pi_{k>m}[a_k^-, a_k^+]$.

5. Output. We set $x_c = \mathrm{m}(N)$ (a center point of N). Let $x_* = T^{-1}x_c$. x_* is a center of a trapping region N expressed in the standard coordinates. We estimate $|y - x_*|$ in various norms for all $y \in T^{-1}(N) \oplus \Pi_{k>m}[a_k^-, a_k^+]$ (for example L_2, H^1 etc). If $l < 0$ then we can conclude that close to x_* there exists an attracting fixed point, otherwise we can claim only an existence of a fixed point.

End of an algorithm

4.2 Numerical data from the proof of Theorem 9

We have chosen $m = 3$ and $M = 10$. In fact to complete successfully full algorithm, without checking that $l < 0$, i.e. to prove just an existence of fixed point it is enough to take $m = 2$ (see the proof of Theorem 4.1 in [ZM]), but in this case we were unable to verify that $l < 0$.

Other starting parameters for the algorithm were given by

$$\begin{aligned} x_0 &= (0.712361,\ -0.12324,\ 0.0101787) \\ \Delta &= 0.03125 \end{aligned}$$

x_0 was found by simply integrating forward a 3-dim Galerkin projection of (1). We tried first $\Delta = 10^{-5}$ and then with $\Delta = 5\Delta$ (we multiplied the current value of Δ by 5) until we were able to successfully complete the algorithm.

From the approximate diagonalization we list approximate eigenvalues, several most significant digits of interval matrixes T_l and T_l^{-1}. Diameters of entries in T_l and T_l^{-1} were smaller than 10^{-15}.

$$\lambda_1 \approx -51.46617, \qquad \lambda_2 \approx -7.7545, \qquad \lambda_3 \approx -0.52575.$$

We see that in our block decomposition we have only 1-dimensional blocks.

$$T_l = \begin{bmatrix} -0.0090621 & 0.09949 & 1.0087 \\ -0.39312 & -1.1041 & -0.069407 \\ -1.1408 & -0.21674 & -0.0066056 \end{bmatrix},$$

$$T_l^{-1} = \begin{bmatrix} 0.0065846 & 0.18519 & -0.94042 \\ -0.065071 & -0.97779 & 0.33745 \\ 0.99786 & 0.098106 & -0.041732 \end{bmatrix},$$

We obtained an isolation for $1 \leq i \leq 3$ after 4 iterates of the main loop for the set N is given by

$$N = [-3.176878e-04, 2.272739e-04] \times [-3.618637e-03, 3.756375e-03] \\ \times [-2.566221e-02, 2.711549e-02]$$

For l we have

$$l < -0.05716.$$

Below we list some other data obtained in the algorithm. The set $W = [T^{-1}(N)]_I$ is given by

$$\begin{aligned} W &= [a_1^-, a_1^+] \times [a_2^-, a_2^+] \times [a_3^-, a_3^+] \\ [a_1^-, a_1^+] &= 0.711691 + [-0.0255012, 0.0255012] \\ [a_2^-, a_2^+] &= -0.123059 + [-0.0125283, 0.0125283] \\ [a_3^-, a_3^+] &= 0.01011 + [-0.00173494, 0.00173494] \end{aligned}$$

In Table 1 we list $a_k^\pm$ for $k > 4$.

k	$[a_k^-, a_k^+]$
4	$-6.77647 \cdot 10^{-4} + [-1.48185, 1.48185] \cdot 10^{-4}$
5	$3.95994 \cdot 10^{-5} + [-1.10021, 1.10021] \cdot 10^{-5}$
6	$-2.14113 \cdot 10^{-6} + [-7.10907, 7.10907] \cdot 10^{-7}$
7	$1.09725 \cdot 10^{-7} + [-4.21541, 4.21541] \cdot 10^{-8}$
8	$-5.41181 \cdot 10^{-9} + [-2.35893, 2.35893] \cdot 10^{-9}$
9	$2.60507 \cdot 10^{-10} + [-2.10033, 2.10033] \cdot 10^{-10}$
10	$-1.22454 \cdot 10^{-11} + [-3.57734, 3.57734] \cdot 10^{-10}$
> 10	$[-1, 1] \cdot 4176.07/k^{10}$

Table 1: Estimates for the intervals $[a_k^-, a_k^+]$ representing self-consistent a priori bounds in the proof of Theorem 9

For $[\epsilon] = ([\epsilon_1], [\epsilon_2], [\epsilon_3])$ we obtained the following numbers

$$\begin{aligned} [\epsilon_1] &= -1.42733 \cdot 10^{-5} + [-5.37438, 5.37438] \cdot 10^{-6} \\ [\epsilon_2] &= 0.000342671 + [-0.000107623, 0.000107623] \\ [\epsilon_3] &= -0.00294653 + [-0.00074762, 0.00074762]. \end{aligned}$$

After passing to the block coordinates and an interval diagonalization we obtained on $N \oplus \Pi_{k>m}[a_k^-, a_k^+]$

$$\begin{aligned} x_1' &\in [-0.01608764, 0.01150572] + [-52.28307, -50.65041]x_1 \\ x_2' &\in [-0.02740801, 0.02844858] + [-7.932687, -7.574724]x_2 \\ x_3' &\in [-0.01291743, 0.01364734] + [-0.5486830, -0.5033749]x_3 \end{aligned}$$

In Table 2 we list the computed upper bounds for l_i on $N \oplus \Pi_{k>m}[a_k^-, a_k^+]$. It is easy to see that $l = l_3 < -0.05716$.

k	l_k
1	-44.76
2	-6.06
3	-0.05716
4	-160.6
5	-425.8
6	-914.7
7	-1726.9
8	-2979.6
9	-4807.8
10	-7364.5
> 10	-10820.7

Table 2: Estimates from above on l_i from the computation of l in the proof of Theorem 9

Observe that the value of $l = -0.05716$ is close to zero. This is essentially due to the fact that we wanted to extend the as much as possible the parameter interval. If we just choose $\nu = 0.75$ with the same x_0, m, M then we obtain $l = -0.348938$. By increasing m and M we can further decrease l up to λ_3. For example for $m = 15$ and $M = 45$ we obtained $l = -0.52581$.

5 Details for KS-equation

The goal of this section is to derive formulas from which, for a given block decomposition and trapping region, the constants $l_{(i)}$ for the KS equation may be computed, thus implementing step **4** of the algorithm of Section 4.1.

5.1 KS-equation in Fourier representation

For KS-equation in one space dimension with periodic and odd boundary conditions we have the following infinite ladder of equations for the Fourier coefficients (see [ZM])

$$\begin{aligned} \dot{a}_k &= F_k(a) = \lambda_k a_k + N_k(a) && (34)\\ \lambda_k &= k^2(1 - \nu k^2) && (35)\\ N_k(a) &= -k \sum_{n=1}^{k-1} a_n a_{k-n} + 2k \sum_{n=1}^{\infty} a_n a_{n+k} && (36) \end{aligned}$$

Hence we obtain

$$\frac{\partial N_i}{\partial a_j} = 2ia_{i+j}, \quad \text{for } i = j$$

$$
\begin{aligned}
\frac{\partial N_i}{\partial a_j} &= -2ia_{i-j} + 2ia_{i+j}, \quad \text{for } j < i \\
\frac{\partial N_i}{\partial a_j} &= 2ia_{j-i} + 2ia_{i+j}, \quad \text{for } j > i \\
\frac{\partial F_i}{\partial a_j} &= i^2(1-\nu i^2)\delta_{ij} + \frac{\partial N_i}{\partial a_j}
\end{aligned}
$$

5.2 Coordinate change and block-decomposition

The following lemma does not require any proof.

Lemma 11 *Let $A : H \to H$ be a linear coordinate change of the form*

$$
\begin{aligned}
A : X_m \oplus Y_m &\to X_m \oplus Y_m \\
A(x \oplus y) &= Ax \oplus y.
\end{aligned}
$$

Let $\tilde{F} = A \circ F \circ A^{-1}$ ($\tilde{F}$ is F expressed in new coordinates).

$$
\begin{aligned}
\frac{\partial \tilde{F}_i}{\partial x_j} &= \sum_{k,l=1}^{m} A_{ik} \frac{\partial F_k}{\partial x_l} A^{-1}_{lj} && \text{for } i \leq m \text{ and } j \leq m \\
\frac{\partial \tilde{F}_i}{\partial x_j} &= \sum_{k \leq m} A_{ik} \frac{\partial F_k}{\partial x_j} && \text{for } i \leq m \text{ and } j > m \\
\frac{\partial \tilde{F}_i}{\partial x_j} &= \sum_{l \leq m} \frac{\partial F_i}{\partial x_l} A^{-1}_{lj} && \text{for } i > m \text{ and } j \leq m \\
\frac{\partial \tilde{F}_i}{\partial x_j} &= \frac{\partial F_i}{\partial x_j} && \text{for } i > m \text{ and } j > m
\end{aligned}
$$

Consider now the KS equation and assume that $W = N \oplus \Pi_{k=m+1}^{\infty}[a_k^-, a_k^+]$ is a trapping region, representing self-consistent bounds for a fixed point. Let the numbers $m < M$ be as in conditions **C1,C2,C3** and we assume that $a_k^{\pm} = \pm \frac{C}{k^s}$ for $k > M$ (as in [ZM]).

Let $A \in \mathbb{R}^{m \times m}$ be a coordinate change around an approximate fixed point in X_m for the m-dimensional Galerkin projection of (34). This matrix induces a coordinate change in H. For our purpose it is optimal to choose A so that the m-dimensional Galerkin projection of F is very close to a diagonal matrix (or to a block diagonal matrix when complex eigenvalues are present).

From now on we will use these new coordinates on H. We also change the norm so that the new coordinates become orthogonal. We define the splitting of $P_m H$ into blocks which are either 2-dimensional (complex eigenvalue) or one-dimensional (real eigenvalue). For the KS equation there was no need to consider more complicated situations. For $(i) > m$ all blocks are 1-dimensional (these coordinates are not affected by our coordinate change).

We would like to derive the formula for

$$
l_{(i)} := \sup_{x \in W} \mu\left(\frac{\partial \tilde{F}_{(i)}}{\partial x_{(i)}}(x)\right) + \sum_{(j),(j)\neq(i)} \sup_{x \in W} \left|\frac{\partial \tilde{F}_{(i)}}{\partial x_{(j)}}(x)\right| \tag{37}
$$

We define $S(l)$ by

$$
S(l) = \sum_{k \geq l} \sup_{a \in W} |a_k|.
$$

We estimate $S(l)$ from above using the following

Lemma 12 *Assume that* $|a_k(W)| \leq \frac{C}{k^s}$ *for* $k > M$, $s > 1$, *then*

$$S(l) < \sum_{k=l}^{M} |a_k(W)| + \frac{C}{(s-1)M^{s-1}}, \quad \textit{for } l \leq M$$

$$S(l) < \frac{C}{(s-1)(l-1)^{s-1}}, \quad \textit{for } l > M.$$

Proof: Observe that

$$\sum_{k=l}^{\infty} \frac{1}{k^s} < \int_{l-1}^{\infty} \frac{dx}{x^s} = \frac{1}{(s-1)(l-1)^{s-1}} \tag{38}$$

∎

We set

$$\overline{S}(l) = \sum_{k=l}^{M} |a_k(W)| + \frac{C}{(s-1)M^{s-1}}, \quad \text{for } l \leq M \tag{39}$$

$$\overline{S}(l) = \frac{C}{(s-1)(l-1)^{s-1}}, \quad \text{for } l > M. \tag{40}$$

5.3 Formulas for 1-dimensional blocks

Observe that if $\dim(i) = 1$, then $\frac{\partial \tilde{F}_{(i)}}{\partial x_{(i)}} \in \mathbb{R}$, hence

$$\mu\left(\frac{\partial \tilde{F}_{(i)}}{\partial x_{(i)}}\right) = \frac{\partial \tilde{F}_{(i)}}{\partial x_{(i)}} \tag{41}$$

Lemma 13 *Assume* $(i) \leq m$ *and dim* $(i) = 1$.

$$\sup_{x\in W} \mu\left(\frac{\partial \tilde{F}_{(i)}}{\partial x_{(i)}}(x)\right) + \sum_{(j)\neq(i)} \sup_{x\in W} \left|\frac{\partial \tilde{F}_{(i)}}{\partial x_{(j)}}(x)\right| \leq$$

$$\bar{l}_i := \sup_{x\in W} \frac{\partial \tilde{F}_i}{\partial x_i}(x) + \sum_{j\neq i, j\leq m} \sup_{x\in W} \left|\frac{\partial \tilde{F}_i}{\partial x_j}(x)\right|$$

$$+2\sum_{k\leq m} k|A_{ik}|(\overline{S}(m-k+1) + \overline{S}(m+k+1)).$$

Proof: Observe that when $(j) = (j_1, j_2)$ is a 2-dimensional block, then we need to compute the norm of $\left[\frac{\partial \tilde{F}_i}{\partial x_{j_1}}(x), \frac{\partial \tilde{F}_i}{\partial x_{j_2}}(x)\right]$. It is easy to see that this norm is less than or equal to

$$\left|\frac{\partial \tilde{F}_i}{\partial x_{j_1}}(x)\right| + \left|\frac{\partial \tilde{F}_i}{\partial x_{j_2}}(x)\right|.$$

This means that ignoring the block structure is safe (we have an inequality in the correct direction), hence we obtain

$$\sum_{(j)\neq(i)} \left| \frac{\partial \tilde{F}_{(i)}}{\partial x_{(j)}}(W) \right| \leq \sum_{j\neq i} \left| \frac{\partial \tilde{F}_i}{\partial x_j}(W) \right| \tag{42}$$

To finish the proof it is enough to show that

$$\sum_{j>m} \left| \frac{\partial \tilde{F}_i}{\partial x_j}(W) \right| \leq 2\sum_{k\leq m} k|A_{ik}|(S(m-k+1)+S(m+k+1)). \tag{43}$$

Observe that from Lemma 11 it follows that

$$\begin{aligned}\sum_{j>m} \left| \frac{\partial \tilde{F}_i}{\partial x_j}(W) \right| &\leq \sum_{j>m}\sum_{k\leq m} |A_{ik}|2k(|a_{j-k}(W)|+|a_{j+k}(W)|) = \\ &\sum_{k\leq m} |A_{ik}|2k \left(\sum_{j>m} |a_{j-k}(W)|+|a_{j+k}(W)| \right) = \\ &\sum_{k\leq m} |A_{ik}|2k \left(S(m-k+1)+S(m+k+1)\right)\end{aligned}$$

∎

Observe that from our assumptions about the decomposition of H it follows that all blocks (i), such that $(i)>m$, are one-dimensional.

Lemma 14 *For $m<i\leq M$ we have*

$$\sup_{x\in W} \mu\left(\frac{\partial \tilde{F}_{(i)}}{\partial x_{(i)}}(x) \right) + \sum_{(j),(j)\neq(i)} \sup_{x\in W} \left| \frac{\partial \tilde{F}_{(i)}}{\partial x_{(j)}}(x) \right| \leq$$

$$\bar{l}_i = i^2(1-\nu i^2) + \sum_{j\leq M} \sup_{x\in W} \left| \frac{\partial \tilde{N}_i}{\partial x_j}(x) \right| + 2i(\overline{S}(M+1-i)+\overline{S}(i+M+1))$$

Proof: Just as in the proof of Lemma 13 we can ignore the block structure here. It is easy to see that

$$\sup_{x\in W} \frac{\partial \tilde{F}_i}{\partial x_i}(x) + \sum_{j\neq i} \sup_{x\in W} \left| \frac{\partial \tilde{F}_i}{\partial x_j}(x) \right| \leq i^2(1-\nu i^2) + \sum_{j=1}^{\infty} \sup_{x\in W} \left| \frac{\partial \tilde{N}_i}{\partial x_j}(x) \right| \tag{44}$$

Therefore to finish the proof it is enough to show that

$$\sum_{j=M+1}^{\infty} \sup_{x\in W} \left| \frac{\partial \tilde{N}_i}{\partial x_j}(x) \right| < 2i(S(M+1-i)+S(i+M+1)) \tag{45}$$

We proceed as follows

$$\sum_{j=M+1}^{\infty} \sup_{x\in W} \left| \frac{\partial \tilde{N}_i}{\partial x_j}(x) \right| = \sum_{j=M+1}^{\infty} \sup_{x\in W} \left| \frac{\partial N_i}{\partial x_j}(x) \right| \leq$$

$$\sum_{j=M+1}^{\infty} 2i(|a_{j-i}(W)| + |a_{i+j}(W)|) \leq 2i(S(M+1-i) + S(M+1+i)$$

∎

Lemma 15 *For $(i) > m$ we have*

$$\sup_{x\in W} \mu\left(\frac{\partial \tilde{F}_{(i)}}{\partial x_{(i)}}(x)\right) + \sum_{(j)\neq(i)} \sup_{x\in W} \left|\frac{\partial \tilde{F}_{(i)}}{\partial x_{(j)}}(x)\right| \leq \bar{l}_i := i^2(1-\nu i^2)$$
$$+2im(\overline{S}(i-m) + \overline{S}(i+1)) \max_{k,l=1,\ldots,m} |A^{-1}_{kl}|$$
$$+2i(\overline{S}(i+m+1) + 2\overline{S}(1))$$

Proof: Just as in the proof of Lemma 13 we can ignore the block structure here. It is easy to see that

$$\sup_{x\in W} \frac{\partial \tilde{F}_i}{\partial x_i}(x) + \sum_{j\neq i} \sup_{x\in W} \left|\frac{\partial \tilde{F}_i}{\partial x_j}(x)\right| \leq i^2(1-\nu i^2) + \sum_{j=1}^{\infty} \sup_{x\in W} \left|\frac{\partial \tilde{N}_i}{\partial x_j}(x)\right| \tag{46}$$

Therefore to finish the proof it is enough to show that

$$\sum_{j=1}^{m} \sup_{x\in W} \left|\frac{\partial \tilde{N}_i}{\partial x_j}(x)\right| \leq 2im(S(i-m) + S(i+1)) \max_{k,l=1,\ldots,m} |A^{-1}_{kl}|, \tag{47}$$

$$\sum_{j=m+1}^{\infty} \sup_{x\in W} \left|\frac{\partial \tilde{N}_i}{\partial x_j}(x)\right| \leq 2i(S(i+m+1) + 2S(1)). \tag{48}$$

To prove (47) observe that

$$\sum_{j=1}^{m} \sup_{x\in W} \left|\frac{\partial \tilde{N}_i}{\partial x_j}(x)\right| = \sum_{j=1}^{m} \sup_{x\in W} \left|\sum_{l=1}^{m} \frac{\partial N_i}{\partial x_l}(x) A^{-1}_{lj}\right| \leq$$
$$\sum_{j=1}^{m}\sum_{l=1}^{m} 2i(|a_{i-l}(W)| + |a_{i+l}(W)|)|A^{-1}_{lj}| \leq$$
$$2i\sum_{j=1}^{m}(S(i-m) + S(i+1)) \max_{k,l=1,\ldots,m} |A^{-1}_{kl}| =$$
$$2im(S(i-m) + S(i+1)) \max_{k,l=1,\ldots,m} |A^{-1}_{kl}|.$$

To prove (48) we proceed as follows

$$\sum_{j=m+1}^{\infty} \sup_{x\in W} \left|\frac{\partial \tilde{N}_i}{\partial x_j}(x)\right| = \sum_{j=m+1}^{\infty} \sup_{x\in W} \left|\frac{\partial N_i}{\partial x_j}(x)\right| \leq$$

$$\sum_{m<j<i} (2i(|a_{i-j}(W)| + |a_{i+j}(W)|))$$

$$+2i|a_{2i}(W)| + \sum_{j>i} 2i(|a_{j-i}(W)| + |a_{i+j}(W)|) \leq$$

$$2i \left(\sum_{j>m} |a_{i+j}(W)| + \sum_{m<j<i} |a_{i-j}(W)| + \sum_{j>i} |a_{j-i}(W)| \right) <$$

$$2i \left(S(i+m+1) + 2S(1) \right)$$

∎

The following lemma shows how to handle the case of large i.

Lemma 16 *If for some $n > m$, $\overline{l}_n < 0$, then*

$$0 > \overline{l}_i > \overline{l}_j \qquad \textit{for} \qquad i<j, \quad i \geq n \tag{49}$$

Proof: From Lemma 15 it follows that

$$\overline{l}_i = i \left((i - \nu i^3) + 2m(\overline{S}(i-m) + \overline{S}(i+1))a + 2(\overline{S}(i+m+1) + 2\overline{S}(1)) \right)$$

where $a = \max_{k,l=1,\dots,m} |A^{-1}_{kl}|$.

Hence

$$\overline{l}_i = i \left((i - \nu i^3) + f(i) \right), \tag{50}$$

where $f(i)$ is a positive decreasing function of i. Since $\overline{l}_n < 0$, then also $(n - \nu n^3) < 0$ and it is easy to see that the function $i \mapsto (i - \nu i^3)$ is decreasing and negative for $i \geq n$. ∎

5.4 Formulas for 'complex' blocks

The purpose of this subsection is to derive a formula for $l_{(i)}$ in case of a 2-dimensional block form the diagonalization corresponding to a complex eigenvalue. Main results are summarized in Lemmas 18 and 19.

Lemma 17 *Let $Q \in \mathbb{R}^{2\times 2}$, then (in the euclidean norm)*

$$|Q| \leq \sqrt{Q_{11}^2 + Q_{12}^2} + \sqrt{Q_{21}^2 + Q_{22}^2}$$

Proof: Let $v = (v_1, v_2)$, then

$$|Qv| \leq |Q_{11}v_1 + Q_{12}v_2| + |Q_{21}v_1 + Q_{22}v_2| \leq \sqrt{Q_{11}^2 + Q_{12}^2}|v| + \sqrt{Q_{21}^2 + Q_{22}^2}|v|$$

∎

Lemma 18 *If* $(i) = (i_1, i_2)$, *then*

$$\sup_{x\in W} \mu\left(\frac{\partial \tilde{F}_{(i)}}{\partial x_{(i)}}\right) \leq \max_{k=1,2}\left(\sup_{x\in W} \frac{\partial \tilde{F}_{i_k}}{\partial x_{i_k}}(x)\right) + \sup_{x\in W} 1/2 \left|\frac{\partial \tilde{F}_{i_1}}{\partial x_{i_2}}(x) + \frac{\partial \tilde{F}_{i_2}}{\partial x_{i_1}}(x)\right|$$

Proof: The proof is an immediate consequence of Theorem 5 and the Gershgorin Theorem (see [QSS, Property 5.2])). ∎

Observe that form the diagonalization of a block corresponding to a complex eigenvalue we obtain

$$\frac{\partial \tilde{F}_{(i)}}{\partial x_{(i)}} \approx \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix},$$

hence $\sup_{x\in W} 1/2 \left|\frac{\partial \tilde{F}_{i_1}}{\partial x_{i_2}}(x) + \frac{\partial \tilde{F}_{i_2}}{\partial x_{i_1}}(x)\right|$ is usually very small.

The following lemma takes care of nondiagonal terms

Lemma 19 *If* $(i) = (i_1, i_2)$, $(i) \leq m$. *Then*

$$\sum_{(j),(j)\neq(i)} \sup_{x\in W} \left|\frac{\partial \tilde{F}_{(i)}}{\partial x_{(j)}}(x)\right| \leq \sum_{j\leq M, j\neq i_1,i_2} \sup_{x\in W} \sqrt{\left(\frac{\partial \tilde{F}_{i_1}}{\partial x_j}\right)^2 + \left(\frac{\partial \tilde{F}_{i_2}}{\partial x_j}\right)^2}$$
$$+ \sum_{l=1,2}\sum_{k\leq m} 2|A_{i_l,k}|k(\overline{S}(M+1-k) + \overline{S}(M+1+k))$$

Proof: If dim $(j) = 1$, then

$$\frac{\partial \tilde{F}_{(i)}}{\partial x_{(j)}}(x) = \left[\frac{\partial \tilde{F}_{i_1}}{\partial x_j}(x), \frac{\partial \tilde{F}_{i_2}}{\partial x_j}(x)\right]. \tag{51}$$

Therefore we obtain

$$\left|\frac{\partial \tilde{F}_{(i)}}{\partial x_{(j)}}(x)\right| = \sqrt{\left(\frac{\partial \tilde{F}_{i_1}}{\partial x_j}(x)\right)^2 + \left(\frac{\partial \tilde{F}_{i_2}}{\partial x_j}(x)\right)^2}. \tag{52}$$

From Lemma 17 it follows that we can ignore the block structure for all blocks different from (i) and use above formula for all coordinates.

Therefore we have

$$\sum_{(j)\neq(i)} \sup_{x\in W} \left|\frac{\partial \tilde{F}_{(i)}}{\partial x_{(j)}}(x)\right| \leq \sum_{j\neq i_1,i_2} \sup_{x\in W} \sqrt{\left(\frac{\partial \tilde{F}_{i_1}}{\partial x_j}\right)^2 + \left(\frac{\partial \tilde{F}_{i_2}}{\partial x_j}\right)^2}. \tag{53}$$

To finish the proof it is enough to show that

$$\sum_{j>M} \sup_{x\in W} \sqrt{\left(\frac{\partial \tilde{F}_{i_1}}{\partial x_j}\right)^2 + \left(\frac{\partial \tilde{F}_{i_2}}{\partial x_j}\right)^2} \leq$$
$$\sum_{l=1,2}\sum_{k\leq m} 2|A_{i_l,k}|k(S(M+1-k) + S(M+1+k)).$$

To make notation less cumbersome we will drop $\sup_{x\in W}$ from the computations below

$$\sum_{j>M}\sqrt{\left(\frac{\partial\tilde{F}_{i_1}}{\partial x_j}\right)^2+\left(\frac{\partial\tilde{F}_{i_2}}{\partial x_j}\right)^2}\leq\sum_{j>M}\left|\frac{\partial\tilde{F}_{i_1}}{\partial x_j}\right|+\sum_{j>M}\left|\frac{\partial\tilde{F}_{i_2}}{\partial x_j}\right| \tag{54}$$

We have for $l=1,2$ (observe that $i_l\leq m$)

$$\sum_{j>M}\left|\frac{\partial\tilde{F}_{i_l}}{\partial x_j}\right|\leq\sum_{j>M}\sum_{k\leq m}|A_{i_l,k}|\left|\frac{\partial F_k}{\partial x_j}\right|=$$

$$\sum_{k\leq m}|A_{i_l,k}|\sum_{j>M}\left|\frac{\partial F_k}{\partial x_j}\right|\leq\sum_{k\leq m}2k|A_{i_l,k}|\sum_{j>M}\left(|a_{j-k}|+|a_{j+k}|\right)\leq$$

$$\sum_{k\leq m}2k|A_{i_l,k}|\left(S(M+1-k)+S(M+1+k)\right).$$

This finishes the proof. ∎

6 Conclusions and future work

We have shown that we can prove rigorously an existence of branches of attracting fixed points for the KS-equation with odd periodic boundary conditions. Below we indicate some further possible developments and applications of the method

- a rigorous steady state bifurcation diagram for the KS-equation.
- applications other dissipative PDEs: for example Navier-Stokes equations with periodic boundary conditions on the plane
- an automatization of generation of formulas for tail (a content of Section 5 in this paper and Section 3 in [ZM]) for KS and other equations.

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