# Rigorous numerics for dissipative PDEs

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## Outline of this talk

1. Model problem - Kuramoto-Sivashinsky Eq. and our results about the dynamics

2. Algorithm for rigorous integration of dissipative PDEs

3. Some data from the proofs

4. Conclusions, future work

# A Model Problem - Kuramoto-Sivashinsky PDE

Consider the Kuramoto-Sivashinsky (KS ) eq.

$$u_t = -\nu u_{xxxx} - u_{xx} + 2uu_x, \qquad \nu > 0$$

where $(t, x) \in [0, \infty) \times \mathbf{R}$ subject to periodic and odd boundary conditions

$$\begin{aligned} u(t, 0) &= u(t, 2\pi) \\ u(t, -x) &= -u(t, x) \end{aligned}$$

For various values of $\nu$ a variety of dynamics,

fixed points,
periodic orbits,
heteroclinic orbits,
chaotic dynamics,

have been observed numerically.

Goal: A rigorous means of proving these numerical results.

# A Model Problem - Kuramoto-Sivashinsky PDE, Fourier expansion

Fourier expansion is: $u(t,x) = \sum_{k=-\infty}^{\infty} b_k(t) e^{ikx}$

Substituting in KS and applying boundary conditions gives:

$$\dot{a}_k = k^2(1-\nu k^2)a_k - k \sum_{n=1}^{k-1} a_n a_{k-n} + 2k \sum_{n=1}^{\infty} a_n a_{n+k}$$

where $b_k = i a_k$ and $k = 1, 2, 3, \ldots$.

Linearization: $\dot{a}_k = k^2(1-\nu k^2)a_k$

- $k$-th mode is unstable for $k < \frac{1}{\sqrt{\nu}}$
- $k$-th mode is stable for $k > \frac{1}{\sqrt{\nu}}$
- the modes with $k >> \frac{1}{\sqrt{\nu}}$ should be irrelevant for the dynamics

## A Model Problem - Kuramoto-Sivashinsky PDE, known results

Known results:

- the existence of global attractor, the functions from attractor are analytic - Fourier series converge at geometric rate (Foias, Temam)

- the existence of finite dimensional inertial manifold (Foias, Nicolaenko, Sell, Temam, Rossa, Jolly) ( not of much use in rigorous numerics)

No analytical results dynamics more complicated than fixed points bifurcating from zero solution

## Our rigorous results for Kuramoto-Sivashinsky PDE

- the existence of multiple periodic orbits for various parameter values $\nu \approx 0.1215$, $0.1212$, $0.125$, $0.032$, $0.02991$, both stable and unstable orbits

- the existence of multiple fixed points for various values o f $\nu$ and their bifurcations (joint with K. Mischaikow)

- the existence of attractive fixed points for various values of $\nu$

# Periodic point for KS-equation

## $\mu = 0.127$

Symmetric attracting orbit

Theorem: Let $u_0(x) = \sum_{k=1}^{10} -2a_k \sin(kx)$, where $a_k$ are given in table below. There exists a function $u^*(t,x)$ , the classical solution of KS for $\nu = 0.127$, such that

$$\|u_0 - u^*(0,\cdot)\|_{L_2} < 8.1 \cdot 10^{-4},$$
$$\|u_0 - u^*(0,\cdot)\|_{C^0} < 6.5 \cdot 10^{-4}$$

such that $u^*$ is periodic with respect to $t$.

| | |
|---|---|
| $a_1 = 2.012088e-01$ | $a_2 = 1.289978$ |
| $a_3 = 2.012152e-01$ | $a_4 = -3.778654e-01$ |
| $a_5 = -4.231056e-02$ | $a_6 = 4.316137e-02$ |
| $a_7 = 6.940373e-03$ | $a_8 = -4.156441e-03$ |
| $a_9 = -7.945097e-04$ | $a_{10} = 3.315994e-04$ |

Proof uses Brouwer Thm. and rigorous integration of KS-PDE

## Periodic point for KS-equation

## $\mu = 0.1215$

non-symmetric attracting orbit past period doubling

Theorem: Let $u_0(x) = \sum_{k=1}^{13} -2a_k \sin(kx)$, where $a_k$ are given in table below. There exists a function $u^*(t,x)$ , the classical solution of KS for $\nu = 0.1215$, such that

$$\|u_0 - u^*(0,\cdot)\|_{L_2} < 9.9 \cdot 10^{-5},$$
$$\|u_0 - u^*(0,\cdot)\|_{C^0} < 6.2 \cdot 10^{-5}$$

such that $u^*$ is periodic with respect to $t$.

| | |
|---|---|
| $a_1 = 2.559310e-01$ | $a_2 = 1.096696$ |
| $a_3 = 2.559302e-01$ | $a_4 = -3.079615e-01$ |
| $a_5 = -4.780276e-02$ | $a_6 = 3.002052e-02$ |
| $a_7 = 7.352633e-03$ | $a_8 = -2.530197e-03$ |
| $a_9 = -7.561938e-04$ | $a_{10} = 1.624861e-04$ |
| $a_{11} = 6.833008e-05$ | $a_{12} = -8.789182e-06$ |
| $a_{13} = -5.429523e-06$ | |

Proof uses Brouwer Thm. and rigorous integration of KS-PDE

# Periodic point for KS-equation

# $\mu = 0.1215$

Symmetric unstable orbit, past period doubling

Theorem: Let $u_0(x) = \sum_{k=1}^{11} -2a_k \sin(kx)$, where $a_k$ are given in table below. There exists a function $u^*(t,x)$ , the classical solution of KS for $\nu = 0.1215$, such that

$$\|u_0 - u^*(0,\cdot)\|_{L_2} < 1.27 \cdot 10^{-3},$$
$$\|u_0 - u^*(0,\cdot)\|_{C^0} < 8.26 \cdot 10^{-4}$$

such that $u^*$ is periodic with respect to $t$.

| | |
|---|---|
| $a_1 = 2.450027e-01$ | $a_2 = 1.041500e+00$ |
| $a_3 = 2.449985e-01$ | $a_4 = -2.760754e-01$ |
| $a_5 = -4.371320e-02$ | $a_6 = 2.531380e-02$ |
| $a_7 = 6.345919e-03$ | $a_8 = -1.996779e-03$ |
| $a_9 = -6.177148e-04$ | $a_{10} = 1.184863e-04$ |
| $a_{11} = 5.269771e-05$ | |

Proof uses Miranda Thm. and rigorous integration of KS-PDE, the orbit is apparently unstable

# Periodic point for KS-equation

## $\mu = 0.032$

symmetric attracting orbit, close to chaotic region

Theorem: Let $u_0(x) = \sum_{k=1}^{23} -2a_k \sin(kx)$, where $a_k$ are given in table below. There exists a function $u^*(t,x)$ , the classical solution of KS for $\nu = 0.032$, such that

$$\|u_0 - u^*(0,\cdot)\|_{L_2} < 8.9 \cdot 10^{-4},$$
$$\|u_0 - u^*(0,\cdot)\|_{C^0} < 9.5 \cdot 10^{-4}$$

such that $u^*$ is periodic with respect to $t$.

| | |
|---|---|
| $a_1 = 3.506682e-01$ | $a_2 = 2.522889e-02$ |
| $a_3 = 3.506665e-01$ | $a_4 = -2.276745e+00$ |
| $a_5 = -1.115325e+00$ | $a_6 = -3.693057e-01$ |
| $a_7 = 4.603873e-01$ | $a_8 = -4.604564e-01$ |
| $a_9 = -3.115024e-01$ | $a_{10} = -1.449674e-01$ |
| $a_{11} = 5.104894e-02$ | $a_{12} = -2.165916e-02$ |
| $a_{13} = -3.413293e-02$ | $a_{14} = -2.613508e-02$ |
| $a_{15} = 1.307623e-03$ | $a_{16} = 8.752424e-05$ |
| $a_{17} = -2.115586e-03$ | $a_{18} = -2.891477e-03$ |
| $a_{19} = -5.007345e-04$ | $a_{20} = 3.374289e-05$ |
| $a_{21} = -4.423567e-05$ | $a_{22} = -2.280484e-04$ |
| $a_{23} = -9.029570e-05$ | |

## The method of self-consistent bounds

$H$ - Hilbert space,
$e_1, e_2, \dots$ - an orthogonal basis in $H$
The corresponding projections are

$$\begin{aligned} p_m = P_m a &:= (a_1, a_2, \dots, a_m) \\ q_m = Q_m a &:= (a_{m+1}, a_{m+2}, \dots) \end{aligned}$$

The problem:

$$\dot{a} = F(a) \tag{1}$$

$F$ is not continuous, with dense domain in $H$.

$F_k \circ P_n$ is a $C^1$-function for $n, k \in \mathbf{N}$

Later $F(a) = L(a) + N(a)$, $L$ - linear, $N$- non-linear
$e_1, e_2, \dots$ - eigenvectors of $L$ - very helpful

The method:

Def. Fix $m, M$ ( $m \leq M$). *A compact set $W \subset P_m(H)$ and a sequence of pairs $\{a_k^{\pm} \in \mathbf{R} \mid a_k^- < a_k^+,\ k \in \mathbf{Z}^+\}$ are self-consistent a-priori bounds for $F$ if:*

**C1** *For $k > M$, $a_k^- < 0 < a_k^+$.*

**C2** *Let $\hat{a}_k := \max |a_k^{\pm}|$ and set $\hat{u} = \sum_{k=0}^{\infty} \hat{a}_k e_k$. Then, $\hat{u} \in H$, $(\{\hat{a}_k\} \in l_2)$*

**C3** *The function $u \mapsto F(u)$ is continuous on*

$$W \oplus \prod_{k=m+1}^{\infty} [a_k^-, a_k^+] \subset H.$$

*Moreover, if we define $\widehat{f}_k = \max_{u \in W \oplus \prod_{k=m+1}^{\infty} [a_k^-, a_k^+]} |F_k(u)|$ and set $\widehat{f} = \sum \widehat{f}_k e_k$, then $\widehat{f} \in H$. $(\{\widehat{f}_k\} \in l_2)$*

Notation: $T = \prod_{k=m+1}^{\infty} [a_k^-, a_k^+]$ - Tail

## ISOLATION for $n > m$

For $a \in W \oplus T$ and $k > m$ holds

$$\begin{aligned} a_k = a_k^+ \quad &\Rightarrow \quad \dot{a}_k < 0 \\ a_k = a_k^- \quad &\Rightarrow \quad \dot{a}_k > 0 \end{aligned}$$

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C1,C2,C3 - give convergence

C4 - gives a priori bounds

C1,C2,C3,C4 - easy to satisfy (later)

## Finite dimensional rigorous computations in $m$ first variables

Basic Differential Inclusion:

$$\dot{p} \ \in \ P_m F(p) + \Gamma_m, \quad p \in \mathbf{R}^m, \tag{2}$$

where $\Gamma_m = \{P_m F(p+q) - P_m F(p) \mid q \in T\}$

We say a multivalued map $p_I : [0,h] \to H$ is *upper attainable set (uas) map* for (2) if the following is true

- any $C^1$ function satisfying (2) and defined on the maximum interval of existence is defined on $[0,h]$
- if a $C^1$-function $p : [0,h] \to X_m$ satisfies (2), then $p(t) \in p_I(t)$ for $t \in [0,h]$

Theorem: Assume $W \oplus T$ are self-consistent bounds for $F$. If $p_I : [0, t_1] \to X_m = P_m(H)$ is uas map for (2), such that $p_I([0, t_1]) \subset W$.

Then for any $q_0 \in T$, the problem $u' = F(u)$ (and all its Galerkin projections $u' = P_n F(u)$, $n > M$) has a solution $u(t) = (p(t), q(t))$ for $t \in [0, t_1]$, such that

$$p(t) \in p_I(t), \qquad q(t) \in T, \qquad \text{for } t \in [0, t_1]$$

## Why it is a easy to find a good tail = self-consistent bounds

$$u_t = Lu + N(u, Du, \dots, D^r u)$$

$x \in \mathbf{T}^n$ (periodic boundary conditions),
$L$ - linear, diagonal, $N$ - polynomial

Fourier expansion $u(t) = \sum_{k \in \mathbf{Z}^n} a_k(t) e^{ik \cdot x}$

Lemma. Let $s > s_0$. If $|a_k| \leq C/|k^s|$, $|a_0| \leq C$, then there exists $D = D(C, s)$

$$|N_k| \leq \frac{D}{|k|^{s-r}}, \qquad |N_0| \leq D$$

Isolation. Assume $L(a)_k = -|k|^p a_k$, $p > r$.

Assume $|a_k| \leq \frac{C}{|k^s|}$, $|a_{k_0}| = \frac{C}{|k_0|^s}$, then

$$\frac{d|a_{k_0}|}{dt} \leq -|k_0|^p |a_{k_0}| + |N_{k_0}(a)| \leq$$

$$-C|k_0|^{p-s} + D|k_0|^{r-s}$$

$$\frac{d|a_{k_0}|}{dt} < 0, \qquad |k_0| > M$$

# Rigorous integration for dissipative PDEs

$(x, y) \in X_m \oplus Y_m \subset H$ - Hilbert space,
dim$X_m = m < \infty$, dim$Y_m \leq \infty$
$P_m : H \to H$, projection onto $X_m$, $Q_m = I - P_m$
$F$ - our PDE in some basis on $H$

$$x' = PF(x, y) \tag{3}$$
$$y' = QF(x, y) \tag{4}$$

Idea: Replace (3 - 4) by

$$x'(t) \in P_m F(x(t), Tail(t)) \tag{5}$$
$$y(t) \in Tail(t), \tag{6}$$

where $Tail(t)$ has *finite representation* and can be computed *in finite number of operations*. $Tail_k(t)$ should decay fast enough.

We want also that: $x(t) \oplus P_n Q_m Tail(t)$, *for* $n > M$ *, is a rigorous estimate to* $n$*-dimensional Galerkin projection of* $F$*, for the initial condition* $x(0) \oplus P_n Q_m Tail(0)$

## Integration of dissipative PDEs - II

$$x'(t) \in P_m F(x(t), Tail(t)) \tag{7}$$
$$y(t) \in Tail(t), \tag{8}$$

$x' = P_m F(x)$ - Galerkin projection, induces $\varphi_m$

**One time step:**
initial condition $Z \oplus Tail(0) \subset X_m \oplus Y_m$, $h > 0$
**1** • find $W \oplus T[0,h]$ (rough enclosure)

$$\begin{aligned} P_m F(x,y) - P_m F(x,0) &\subset \Gamma, \qquad x \in Z \\ \varphi_{m,\Gamma}([0,h], Z) &\subset W \\ Tail([0,h]) &\subset T[0,h]. \end{aligned}$$

**2** • instead of (7) consider $x' \in P_m F(x,0) + \Gamma$ - use algorithm for differential inclusions, to obtain $x(h)$ for $(x,y) \in Z \oplus Tail(0)$.

**3** • compute $Tail(h)$.

## Representation used for KS equation

We look for solutions in

$$W \oplus T = W \oplus \Pi_{k=m+1}^{k \leq M} \oplus \Pi_{k>M} \left[\frac{-C}{k^s}, \frac{C}{k^s}\right] \quad (9)$$

where $W \subset X_m$.

$$N_k(W \oplus T) \subset [N_k^-, N_k^+], \qquad k = m+1, \dots, M$$
$$N_k(W \oplus T) \subset \left[\frac{-D(W \oplus T)}{k^{s-2}}, \frac{D(W \oplus T)}{k^{s-2}}\right], \qquad k > M$$

We solve (estimate rigorously) the solutions of the following system of differential inclusions

$$\begin{aligned} x' &\in P_m F(x) + \Gamma, & x \in W \subset X_m \\ x_k' &\in \lambda_k x_k + [N_k^-, N_k^+], & k = m+1, \dots, \end{aligned}$$

$x_k$ for $k > M$ are given by a single formula.

## Rigorous integration for ODEs and differential inclusions - basic principles

$x \in \mathbf{R}^n$, $f : \mathbf{R}^n \to \mathbf{R}^n$ - $C^1$.,

$$x' = f(x), \quad x(0) = x_0 \qquad \text{(ODE)}$$

induces $\varphi(t, x_0) \in \mathbf{R}^n$

**One time step:**
initial condition: $X_0 \subset \mathbf{R}^n$, $h > 0$ is a time step
**1•** find $W \subset \mathbf{R}^n$ (rough enclosure), such that $\varphi([0, h], X_0) \subset W$

**2•** apply the Taylor method to (ODE), evaluate the error term on $W$ to obtain $X_1 \subset \mathbf{R}^n$, such that

$$\varphi(h, X_0) \subset X_1$$

# Rigorous integration for ODEs - comments

- all computations are performed in **interval arithmetic**

- one should be very careful in the way how step **2** is executed, straightforward interval evaluation leads to *the wrapping effect*.

- we use *the Lohner algorithm*

# Rigorous integration of differential inclusion

Differential inclusion : $\Gamma \subset \mathbf{R}^n$

$$x' \in f(x) + \Gamma, \quad x(0) = x_0$$

induces $\varphi_\Gamma(t, x_0) \subset \mathbf{R}^n$

**One time step:**
initial condition: $X_0 \subset \mathbf{R}^n$, $h > 0$ is a time step
**1•** compute $X_1$, such that $\varphi(h, X_0) \subset X_1$

**2•** find $W_2 \subset \mathbf{R}^n$ (rough enclosure), such that

$$\varphi_\Gamma([0, h], X_0) \subset W_2, \quad x \in X_0$$

**3•** use Gronwall type lemma to find $\Delta \subset \mathbf{R}^n$,

$$\varphi_\Gamma(h, x) - \varphi(h, x) \in \Delta$$

This step requires $\frac{\partial f}{\partial x}(W_2)$

**4•**

$$\varphi_\Gamma(h, X_0) \subset X_1 + \Delta$$

# Differential inclusions - Fundamental Lemma

For a fixed $y_c \in \mathbf{R}^{n_2}$ we compare the solutions of two ODEs

$$\begin{aligned} x_1' &= f(x_1, y_c), \\ x_2' &= f(x_2, y_c) + (f(x_2, y(t)) - f(x_2, y_c)) \\ & \quad x_1(t_0) = x_2(t_0) = x_0 \end{aligned}$$

where $y(t)$ is given (but unknown) function.

Lemma: Let:

$[W_y] \subset \mathbf{R}^{n_2}$, convex, $y([t_0, t_0 + h]) \subset [W_y]$.

$[W_1] \subset [W_2] \subset \mathbf{R}^{n_1}$ - convex and compact.

$x_1([t_0, t_0 + h]) \subset [W_1]$, $x_2([t_0, t_0 + h]) \subset [W_2]$ for any continuous function $y : [t_0, t_0 + h] \to [W_y]$.

Then the following inequality holds for $t \in [t_0, t_0 + h]$ and for $i = 1, \ldots, n_1$

$$|x_{1,i}(t) - x_{2,i}(t)| \leq \left( \int_{t_0}^{t} e^{J(t-s)} C \, ds \right)_i, \qquad (10)$$

where

$$\begin{aligned}
[\delta] &= \{ f(x, y_c) - f(x, y) \mid x \in [W_1], y \in [W_y] \}, \\
C_i &\geq \sup |[\delta_i]|, \qquad i = 1, \ldots, n_1 \\
J_{ij} &\geq \sup \frac{\partial f_i}{\partial x_j}([W_2], [W_y]) \text{if } i = j, \\
J_{ij} &\geq \sup \left| \frac{\partial f_i}{\partial x_j}([W_2], [W_y]) \right| \text{if } i \neq j.
\end{aligned}$$

# Tail evolution

Our problem $a'_k = \lambda_k a_k + N_k(a)$,
$\lambda_k \to -\infty$, for $|k| \to \infty$

$W$, $T([0,h])$ - the rough enclosure for $Z \oplus T(0)$ for $t \in [0,h]$

For $k > m$ we have

$$N_k^{\pm} = N_k^{\pm}(W, T([0,h]))$$
$$\lambda_k a_k + N_k^- < \frac{da_k}{dt} < \lambda_k a_k + N_k^+,$$

hence

$$b_k^{\pm} = \frac{N_k^{\pm}}{-\lambda_k}, \qquad (11)$$
$$T(h)_k^{\pm} = \left(T(0)_k^{\pm} - b_k^{\pm}\right) e^{\lambda_k h} + b_k^{\pm} \qquad (12)$$

It remains to put $T(h)$ for $k > M$ in the form

$$T(h)_k^{\pm} = \frac{\pm C(T(h))}{k^{s(T(h))}}$$

For $k > M$ we have

$$
\begin{aligned}
0 < b_k^+ &\leq \frac{C(b)}{k^{s(b)}}\\
T(0)_k^+ &= \frac{C(T(0))}{k^{s(T(0))}}\\
T(h)_k^+ \leq T(0)_k^{\pm} e^{\lambda_k h} &+ b_k^{\pm}.
\end{aligned}
$$

$$T(h)_k^+ \leq \frac{C(T(0))}{k^{s(T(0))}} e^{\lambda_k h} + \frac{C(b)}{k^{s(b)}}.$$

Let

$$E = e^{h\lambda_{M+1}} (M+1)^{s(b)-s(T(0))}.$$

then (modulo some conditions on $M$, $h$)

$$e^{\lambda_k h} \leq \frac{E}{k^{s(b)-s(T(0))}}, \quad k > M$$

and finally we can set

$$T_k^{\pm}(h) = \pm \frac{C(T(0))E + C(b)}{k^{s(b)}}.$$

## About the computations

- gnu C++
- interval arithmetic - from CAPD package devoloped in Krakow, Poland
- we use the Lohner algorithm to integrate differential inclusions

# Some computation data

On 3GHz machine, Linux, gnu C++

- $\nu = 0.127$, $m = 10$, $M = 3 * m$, $h = 1e - 3$, $order = 4$, $T/2 \approx 1.12$, computation time around 10 sec

- $\nu = 0.1215$, $m = 13$, $M = 3 * m$, $h = 4e - 4$, $order = 6$, $T \approx 3.07$, computation time around 240 sec

- $\nu = 0.032$, $m = 23$, $M = 3 * m$, $h = 1.5e - 4$, $order = 5$, $T/2 \approx 0.41$, computation time around 300 sec

## Conclusions

- rigorous numerics for dissipative PDEs is possible

- global existence and uniqueness theorems are not required, interesting solutions are constructed

- could be applied to (I hope): Ginzburg-Landau, Navier-Stokes in 2D and 3D

# Future work

- prove chaos (symbolic dynamics) for KS $\nu \approx 0.029$ or $\nu \approx 0.1212$

- Construct an *rigorous* $C^1$-*algorithm* for dissipative PDE.

This will make possible to **rigorously** apply a lot of dynamical system theory to dissipative PDEs.