AN INTERVAL METHOD FOR FINDING PERIODIC ORBITS OF INFINITE DIMENSIONAL DISCRETE DYNAMICAL SYSTEMS

ZBIGNIEW GALIAS[†][§] AND PIOTR ZGLICZYŃSKI[‡][§]

Abstract. In this work we describe methods for proving the existence of periodic orbits of infinite dimensional discrete dynamical systems and for finding all periodic orbits with a given period enclosed in a specified region. As an example we consider the Kot-Schaffer growth-dispersal model, for which we find all fixed points and period–2 orbits enclosed in the region containing the attractor observed numerically.

Key words. infinite dimensional system, periodic orbit, Krawczyk method, interval arithmetic

AMS subject classifications. 37L65, 37M99, 65G20, 65G30, 35B10

1. Introduction. The goal of this paper is to present tools for rigorous computer assisted studies of existence of periodic orbits in dissipative infinite dimensional discrete dynamical systems. By a dissipative system for the purpose of this work it is enough to think of it as of a system which is well approximated by its Galerkin projection.

We show how combining modern tools of interval analysis [1, 5, 6] and the technique for reduction of infinite dimensional dynamics to finite dimension introduced in [8] (see also [7]) one can prove the existence of periodic orbits and find all periodic orbits of low period, similarly as it was done before in finite dimension [3, 4] for the Hénon map and the Ikeda map.

As an illustration of our approach we consider the Kot-Schaffer growth-dispersal model (see [7] and references given there). It consists of a map $\Phi : L^2([-\pi,\pi]) \mapsto L^2([-\pi,\pi])$ of the form

$$\Phi(a)(y) = \frac{1}{2\pi}\int_{-\pi}^{\pi} b(x,y)\mu a(x)\left(1-\frac{a(x)}{c(x)}\right)dx, \tag{1.1}$$

where $\mu > 0$ and $b(x,y) = b(x-y)$.

As in [7] we represent elements of $L^2([-\pi,\pi]$ using the basis $e_k = e^{ikx}$. Let us denote by a_k, b_k and c_k the coefficients of Fourier expansion of functions a, b and $1/c$, respectively. The regularity properties of functions b and c are reflected in the decay of coefficients b_k and c_k. We assume that the functions a, b and c are even. From the assumption that the functions a, b and c are real valued we have $a_{-k} = a_k$, $b_{-k} = b_k$ and $c_{-k} = c_k$. In this basis the equation (1.1) becomes equivalent to the countable system of maps

$$f_k(a) = \mu b_k \left(a_k - \sum_{j+l+n=k} c_j a_l a_n\right), \qquad k \in \mathbb{Z}. \tag{1.2}$$

Using the symmetry conditions $a_{-k} = a_k$, $b_{-k} = b_k$ and $c_{-k} = c_k$ and denoting

[†]Department of Electrical Engineering, AGH University of Science and Technology, Kraków, Poland (galias@agh.edu.pl).

[‡]Department of Mathematics, Jagiellonian University, Kraków, Poland (Piotr.Zgliczynski@im.uj.edu.pl).

[§]The work partially supported by KBN, grant no. 1393/P03/2002/22.

$a = (a_0, a_1, a_2, \ldots)$ we obtain

$$f_k(a) = \mu b_k \left(a_k - \sum_{j+l+n=k} c_{|j|} a_{|l|} a_{|n|} \right), \qquad k \geq 0. \tag{1.3}$$

In [7] the system (1.3) was investigated using the concept of self-consistent bounds with the objective of proving the existence of nontrivial dynamics using Conley index tools as it was originally proposed in [8] in the PDEs context. For some particular choices of parameters μ, b_k and c_k the existence of invariant sets with nontrivial symbolic dynamics was established. The topological nature of arguments given in [7] does not allow to derive further interesting information from the dynamical point of view, for example what is the number of fixed points and periodic orbits, what are their stability properties, etc.

In this work we develop tools, which allow to prove the existence and uniqueness of fixed points and periodic orbits for the infinite dimensional maps and to count all low-period cycles for the map. Our approach is based on the Krawczyk method for proving the existence of zeros of finite dimensional maps [1, 6], which is briefly recalled in section 2.

In section 3 the infinite dimensional Krawczyk operator is defined. The main result of the paper is encapsulated in Theorem 3.5 which provides a condition for the existence of zeros of infinite dimensional maps. In sections 3.1 and 3.2 we show how to use this theorem for proving the existence of fixed points and periodic orbits, respectively. In section 3.3 we discuss the problem of finding all low period cycles enclosed in a given region of the state space.

In the last part of the paper we apply the methods introduced in section 3 to the map (1.3). For certain values of parameters we find all fixed points and period–2 orbits enclosed in a certain region of the state space and one period–4 cycle. The results and numerical data from the proofs are presented in section 4.

Following the convention in the literature on interval arithmetic by bold letters we will denote interval objects — intervals, interval vectors and interval matrices. For a given interval $\mathbf{x} = [a, b]$ by $\mathrm{mid}(\mathbf{x})$ we denote its midpoint, by $\mathrm{diam}(\mathbf{x})$ its diameter and by $|\mathbf{x}|$ its magnitude

$$\mathrm{mid}(\mathbf{x}) = (a+b)/2, \qquad \mathrm{diam}(\mathbf{x}) = b - a, \qquad |\mathbf{x}| = \sup_{x \in \mathbf{x}} |x|.$$

2. Krawczyk method in finite dimension. First, let us describe the Krawczyk method for finding zeros of n–dimensional functions. Let $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a C^1 function. We would like to solve the equation

$$f(x) = 0. \tag{2.1}$$

2.1. Basic idea of Krawczyk method. We begin by explaining the basic idea of the Krawczyk method [1, 6]. One of the methods frequently used in numerical analysis to locate zeros of f is based on the *Newton operator*

$$N(x) = x - f'(x)^{-1} f(x). \tag{2.2}$$

It is well known that if $f(x^*) = 0$ and $f'(x^*)$ is nonsingular, then x^* is an attracting fixed point for $N(x)$. It turns out that the same is true if we replace $f'(x)^{-1}$ by a fixed matrix C, which is sufficiently close to $f'(x^*)^{-1}$. The *modified Newton operator* is given by

$$\tilde{N}(x) = x - Cf(x). \tag{2.3}$$

The main question is how can we use $\tilde{N}$ to prove the existence of solution of (2.1). The answer is quite obvious. If U is compact and convex and if $\tilde{N}(U) \subset U$ then from the Brouwer's theorem it follows, that there exists $x^* \in U$ such that $\tilde{N}(x^*) = x^*$. Since C is invertible we obtain $f(x^*) = 0$. To prove the uniqueness it is enough to show that $\tilde{N}$ is a contraction on U.

Observe that it is impossible to verify directly the condition $\tilde{N}(U) \subset U$ in interval arithmetic, i.e. to show that for some interval vector $\mathbf{x}$ (a product of intervals) we have $\tilde{N}(\mathbf{x}) \subset \mathbf{x}$. The reason is that the diameter of $\mathbf{x} - Cf(\mathbf{x})$ is equal to $\mathrm{diam}(\mathbf{x}) + \mathrm{diam}(Cf(\mathbf{x}))$ and obviously it cannot be smaller than $\mathrm{diam}(\mathbf{x})$.

It turns out that the mean value form of $\tilde{N}$ can cure this deficiency. The mean value form is a common way to reduce overestimation in the evaluation of functions. Using the mean value theorem one can show that for a C^1 function f, the interval vector $\mathbf{x}$ and arbitrary $\hat{x} \in \mathbf{x}$ the condition $f(x) \in f(\hat{x}) + f'(\mathbf{x})(\mathbf{x} - \hat{x})$ holds for all $x \in \mathbf{x}$, where $f'(\mathbf{x})$ is the interval matrix containing the real matrices $f'(x)$ for $x \in \mathbf{x}$.

Applying the mean value form to $\tilde{N}$ we obtain

$$\tilde{N}(x) \in \tilde{N}(\hat{x}) + \tilde{N}'(\mathbf{x}) \cdot (\mathbf{x} - \hat{x}) = \hat{x} - Cf(\hat{x}) + (I - Cf'(\mathbf{x}))(\mathbf{x} - \hat{x}). \tag{2.4}$$

The Krawczyk operator is defined as the right hand side of the above formula

$$K(\mathbf{x}) = \hat{x} - Cf(\hat{x}) + (I - Cf'(\mathbf{x}))(\mathbf{x} - \hat{x}). \tag{2.5}$$

The discussion presented above explains the relation between the Krawczyk operator and the existence of zeros of f.

2.2. Properties of the Krawczyk operator. The following theorem summarizes main properties of the Krawczyk operator [6].

THEOREM 2.1. *Let $\mathbf{x} \subset \mathbb{R}^n$ be an interval vector. Let us choose $\hat{x} \in \mathbf{x}$ and an invertible matrix $C \in \mathbb{R}^{n\times n}$. The Krawczyk operator K is defined as in (2.5).*

1. *If $x^* \in \mathbf{x}$ and $f(x^*) = 0$, then $x^* \in K(\mathbf{x})$.*
2. *If $K(\mathbf{x}) \subset \mathrm{int}\,\mathbf{x}$, then there exists exactly one zero of f in $\mathbf{x}$.*

Second part of the above theorem allows us to prove the existence of a unique zero within a given interval vector, while the first one provides a nonexistence criterion. Usually one chooses $\hat{x}$ to be the center of $\mathbf{x}$ and C the inverse of $f'(\hat{x})$.

In order to investigate the existence of fixed points of f in $\mathbf{x}$ one applies the Krawczyk operator to the map $g = \mathrm{id} - f$

$$K(\mathbf{x}) = \hat{x} - C(\hat{x} - f(\hat{x})) + (I - C(I - f'(\mathbf{x})))(\mathbf{x} - \hat{x}).$$

3. Krawczyk method in infinite dimension. In this section we describe a rigorous method for proving the existence of solutions of $f(x) = 0$ in the infinite dimension. The method is based on the infinite dimensional Krawczyk operator.

Let us consider a Hilbert space H with the orthogonal basis $\{e_i\}_{i=0}^{\infty}$. Assume that $f\colon \mathrm{dom}\,(f) \mapsto H$, $\mathrm{dom}\,(f) \subset H$. Let $X_k = \mathrm{span}(e_0, e_1, \ldots, e_k)$. By P_k we denote the projection onto the subspace spanned by $X_k = \{e_0, e_1, \ldots, e_k\}$. For $x \in H$, by x_i we denote the i–th coordinate of x with respect to the basis $\{e_i\}$ and for a function f we define $f_i(x)$ in the same way.

Let us choose an infinite dimensional interval vector $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1, \ldots)$. We would like to investigate the existence and uniqueness of solution of $f(x) = 0$ in $\mathbf{x}$.

Definition 3.1. *The infinite dimensional interval vector $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1, \ldots)$ satisfies the* convergence conditions *for the map f if $\mathbf{x}$ is closed and*

C1. $0 \in \mathbf{x}_k$ *for all* $k \geq k_0$,

C2. $\sum_k |\mathbf{x}_k|^2 < \infty$, *where* $|\mathbf{x}_k| = \max_{x_k \in \mathbf{x}_k} |x_k|$,

C3. f *is continuous on* $\mathbf{x}$ *and* $\sum_k |f_k(\mathbf{x})|^2 < \infty$ *where* $|f_k(\mathbf{x})| = \max_{x \in \mathbf{x}} |f_k(x)|$,

F1. $\frac{\partial f_i}{\partial x_j} : \mathbf{x} \mapsto \mathbb{R}$ *is continuous for all* i, j,

F2. $\sum_k \left|\frac{\partial f_i}{\partial x_k}(\mathbf{x})\right| \operatorname{diam}(\mathbf{x}_k) < \infty$ *for every* i, *where* $\left|\frac{\partial f_i}{\partial x_k}(\mathbf{x})\right| = \max_{x \in \mathbf{x}} \left|\frac{\partial f_i}{\partial x_k}(x)\right|$.

Convergence conditions are part of the definition of self-consistent a priori bounds (see [8, 7]). For applications of self-consistent a priori bounds for proving the existence of zeros, fixed points and application of the Conley index theory see [7, 8].

Observe that for the map considered in this work it is very easy to satisfy the convergence conditions. It is enough to assume that $|\mathbf{x}_k|$ decay at polynomial or exponential rate. Observe also that these conditions have nothing to do with the dynamics, they just define a compact set on which we are looking for zeros.

Lemma 3.2. *Assume that $\mathbf{x}$ satisfies the convergence conditions for f. Then for every $i = 1, 2, \ldots$ and $x, y \in \mathbf{x}$ we have*

$$f_i(x) - f_i(y) = \sum_{j=0}^{\infty} c_{ij}(x_j - y_j), \tag{3.1}$$

where $c_{ij} \in \frac{\partial f_i}{\partial x_j}(\mathbf{x}) = \{\frac{\partial f_i}{\partial x_j}(x)\}_{x \in \mathbf{x}}$.

Proof. Let us fix i. Let us denote

$$g_j(x, y) = \int_0^1 \frac{\partial f_i}{\partial x_j}(y + t(x - y))dt. \tag{3.2}$$

We consider the map $f_i \colon X_n \mapsto \mathbb{R}$, where $n \geq 0$. For each $x, y \in X_n$ we have

$$f_i(x) - f_i(y) = \sum_{j=0}^{n} \left(\int_0^1 \frac{\partial f_i}{\partial x_j}(y + t(x - y))dt \right) \cdot (x_j - y_j) = \sum_{j=0}^{n} g_j(x, y)(x_j - y_j). \tag{3.3}$$

Now we consider the map $f_i \colon \mathbf{x} \mapsto \mathbb{R}$ Let us choose $x, y \in \mathbf{x}$. From (3.3) it follows immediately that

$$f_i(P_n x) - f_i(P_n y) = \sum_{j=0}^{n} g_j(P_n x, P_n y)(x_j - y_j). \tag{3.4}$$

We want now to pass to the limit $n \to \infty$ in the above equation. From **C3** it follows that $f_i(P_n x) \to f_i(x)$ and $f_i(P_n y) \to f_i(y)$. We will show that

$$\lim_{n \to \infty} \sum_{j=0}^{n} g_j(P_n x, P_n y)(x_j - y_j) = \sum_{j=0}^{\infty} g_j(x, y)(x_j - y_j). \tag{3.5}$$

Let us fix $\epsilon > 0$. Observe that from **F2** it follows immediately that there exists n_0 such that for $n \geq n_0$ we have

$$\left| \sum_{j=n_0+1}^{\infty} g_j(x, y)(x_j - y_j) \right| < \epsilon, \qquad \left| \sum_{j=n_0+1}^{n} g_j(P_n x, P_n y)(x_j - y_j) \right| < \epsilon. \tag{3.6}$$

Hence for $n \geq n_0$ we obtain

$$\left|\sum_{j=0}^{n} g_j(P_n x, P_n y)(x_j - y_j) - \sum_{j=0}^{\infty} g_j(x,y)(x_j - y_j)\right| \leq$$
$$2\epsilon + \left|\sum_{j=0}^{n_0} (g_j(P_n x, P_n y) - g_j(x,y))(x_j - y_j)\right|.$$

To finish the proof of (3.5) observe that from **F1** and the compactness of $\mathbf{x}$ it follows that for each j the functions $[0,1] \ni t \mapsto \frac{\partial f_i}{\partial x_j}(P_n y + t(P_n x - P_n y))$ converge uniformly on $[0,1]$ to the function $[0,1] \ni t \mapsto \frac{\partial f_i}{\partial x_j}(y + t(x-y))$.

Hence we obtain

$$f_i(x) - f_i(y) = \sum_{j=0}^{\infty} g_j(x,y)(x_j - y_j). \tag{3.7}$$

This completes the proof since $g_j(x,y) = \int_0^1 \frac{\partial f_i}{\partial x_j}(y + t(x-y))dt \in \frac{\partial f_i}{\partial x_j}(\mathbf{x})$. $\square$

In the finite dimensional Krawczyk method it is assumed that the matrix C is invertible. For the infinite dimensional version we have to make some more assumptions about $C = (c_{i,j})_{i,j\in\mathbb{N}}$.

We assume that C is *bounded*, i.e. there exists A such that $|c_{i,j}| \leq A$, for all i,j. We also assume that C is a matrix of *finite dependence*, i.e. there exists p such that $c_{i,j} = 0$, if $|i-j| > p$. Further, we assume that C is *nonsingular*, in the following sense

$$Cx = 0 \Rightarrow x = 0, \qquad \text{for } x = (x_i)_{i\in\mathbb{N}} \text{ such that } \sup_i |x_i| < \infty. \tag{3.8}$$

We define the Krawczyk operator as

$$K(\mathbf{x}) = \hat{x} - Cf(\hat{x}) + (I - Cf'(\mathbf{x}))(\mathbf{x} - \hat{x}),$$

where $\hat{x} \in \mathbf{x}$. The kth element of the vector $K(\mathbf{x})$ can be computed as

$$K_k(\mathbf{x}) = \hat{x}_k - \sum_{p=0}^{\infty} c_{k,p} f_p(\hat{x}) + \sum_{p=0}^{\infty}\left(\delta_{k,p} - \sum_{i=0}^{\infty} c_{k,i}\frac{\partial f_i}{\partial x_p}(\mathbf{x})\right)(\mathbf{x}_p - \hat{x}_p).$$

Since C is a matrix of finite dependence it follows that for each k only a finite number of $c_{k,p}$ is non-zero, and hence the first sum in the above equation is finite. From finite dependence and the condition **F2** it follows that the second sum converges.

Before proceeding let us make the following simple observation about the infinite dimensional modified Newton operator $\mathrm{id} - Cf$.

PROPOSITION 3.3. *If C is a bounded matrix of finite dependence and $\mathbf{x}$ satisfies convergence conditions* **C1,C2,C3,F1,F2** *for f, then these conditions and also Lemma 3.2 are true for* $\mathrm{id} - Cf$.

The following two theorems state what is the relation between $K(\mathbf{x})$ and zeros of f in $\mathbf{x}$.

THEOREM 3.4. *Let $\mathbf{x}$ be an infinite dimensional interval vector satisfying the convergence conditions for f and let us choose $\hat{x} \in \mathbf{x}$. Let us assume that C is a bounded, nonsingular matrix of finite dependence. If $x^* \in \mathbf{x}$ and $f(x^*) = 0$, then*

$$x^* \in K(\mathbf{x}) = \hat{x} - Cf(\hat{x}) + (I - Cf'(\mathbf{x}))(\mathbf{x} - \hat{x}). \tag{3.9}$$

Proof. From Proposition 3.3 and Lemma 3.2 it follows that

$$x^* = x^* - Cf(x^*) \in \hat{x} - Cf(\hat{x}) + (I - Cf'(\mathbf{x}))(x^* - \hat{x}) \in K(\mathbf{x}). \tag{3.10}$$

□

THEOREM 3.5. *Let $\mathbf{x}$ be an infinite dimensional interval vector satisfying the convergence conditions for f and let us choose $\hat{x} \in \mathbf{x}$. Let us assume that C is a bounded, nonsingular matrix of finite dependence. If*

$$K(\mathbf{x}) = \hat{x} - Cf(\hat{x}) + (I - Cf'(\mathbf{x}))(\mathbf{x} - \hat{x}) \subset \mathbf{x}, \tag{3.11}$$

and there exists $\alpha < 1$, such that for all k

$$\mathrm{diam}(K_k(\mathbf{x})) \le \alpha \, \mathrm{diam}(\mathbf{x}_k), \tag{3.12}$$

then there exists a unique $x^* \in \mathbf{x}$, such that $f(x^*) = 0$.

Proof. From Proposition 3.3 and Lemma 3.2 it follows that for $x \in \mathbf{x}$

$$x_i - Cf(x)_i = \hat{x}_i - Cf(\hat{x})_i + \sum_j d_{ij}(x_j - \hat{x}_j) \in \mathbf{x}_i, \tag{3.13}$$

where $d_{ij} \in \frac{\partial(\mathrm{id}-Cf)_i}{\partial x_j}(\mathbf{x})$.

In order to prove the existence of a zero of f in $\mathbf{x}$ we consider Galerkin projections of $\mathrm{id} - Cf$. Let us choose $k > k_0$ and let $x \in P_k(\mathbf{x})$. Applying P_k to (3.13) we obtain

$$P_k(\mathrm{id} - Cf)(x) \in P_k K(\mathbf{x}) \subset P_k(\mathbf{x}), \tag{3.14}$$

and hence $P_k(\mathrm{id} - Cf)(P_k(\mathbf{x})) \subset P_k(\mathbf{x})$. From Brouwer theorem it follows that there exists a fixed point $x_k \in \mathbf{x} \cap X_k$ for the map $x \mapsto x - P_k Cf(x)$. Since by **C2** the set $\mathbf{x}$ is compact then (possibly choosing a convergent subsequence) we can assume that $x_k \to x^* \in \mathbf{x}$. We will show that $f(x^*) = 0$.

Let us fix n. For all k we have $P_k x_k = x_k = P_k(\mathrm{id} - Cf)(x_k)$. Since $P_n P_k = P_n$ for $k \ge n$ we obtain

$$P_n x_k = P_n(\mathrm{id} - Cf)(x_k), \qquad \text{for } k \ge n. \tag{3.15}$$

Now we pass to the limit $k \to \infty$ on both sides of equation (3.15). We have $P_n x_k \to P_n x^*$ and from the continuity (Proposition 3.3 and condition **C3**) of $\mathrm{id} - Cf$ on $\mathbf{x}$ we obtain $P_n(\mathrm{id} - Cf)(x_k) \to P_n(\mathrm{id} - Cf)(x^*)$. Hence for every n

$$P_n x^* = P_n(\mathrm{id} - Cf)(x^*). \tag{3.16}$$

From the above equation it follows that

$$\begin{aligned} &x^* = (\mathrm{id} - Cf)(x^*) = x^* - Cf(x^*), \\ &Cf(x^*) = 0. \end{aligned}$$

From the non-singularity of C (condition (3.8)) it follows that $f(x^*) = 0$.

Now we prove the uniqueness of solutions. Let $x, y \in \mathbf{x}$ be two solutions of $f(x) = 0$. From Lemma 3.2 it follows that

$$x - y = x - Cf(x) - y + Cf(x) \in (I - Cf'(\mathbf{x})) \cdot (x - y). \tag{3.17}$$

Let us denote $\mathbf{A} = I - Cf'(\mathbf{x})$ and $d_k = \mathrm{diam}(\mathbf{x}_k)$.
From the assumptions it follows that for all $k = 0, 1, 2, \ldots$

$$\mathrm{diam}(K_k(\mathbf{x})) \leq \alpha\, \mathrm{diam}(\mathbf{x}_k), \tag{3.18}$$

$$\mathrm{diam}\left(\sum_{i=0}^{\infty} \mathbf{A}_{ki}(\mathbf{x}_i - \hat{x}_i)\right) \leq \alpha\, \mathrm{diam}(\mathbf{x}_k). \tag{3.19}$$

Using the following two properties of the diam operator (see [6])

$$\mathrm{diam}(\mathbf{a} + \mathbf{b}) = \mathrm{diam}(\mathbf{a}) + \mathrm{diam}(\mathbf{b}), \qquad \mathrm{diam}(\mathbf{ab}) \geq |\mathbf{a}|\, \mathrm{diam}(\mathbf{b}), \tag{3.20}$$

we obtain

$$\sum_{i=0}^{\infty} \mathrm{diam}\left(\mathbf{A}_{ki}(\mathbf{x}_i - \hat{x}_i)\right) \leq \alpha\, \mathrm{diam}(\mathbf{x}_k), \tag{3.21}$$

$$\sum_{i=0}^{\infty} |\mathbf{A}_{ki}|\, \mathrm{diam}(\mathbf{x}_i - \hat{x}_i) \leq \alpha\, \mathrm{diam}(\mathbf{x}_k), \tag{3.22}$$

$$\sum_{i=0}^{\infty} |\mathbf{A}_{ki}| d_i \leq \alpha d_k. \tag{3.23}$$

Let us define $e_k = |x_k - y_k|$ for $k \geq 0$. Let us choose $\beta \in (1, \alpha^{-1})$, for example $\beta = (1 + \alpha^{-1})/2$. Let m be such that $\frac{e_m}{d_m} \cdot \beta \geq \sup_{i \geq 0} \frac{e_i}{d_i}$. From (3.17)

$$e_m = |x_m - y_m| \leq \sum_{i=0}^{\infty} |\mathbf{A}_{mi}| e_i \leq \frac{e_m \beta}{d_m} \sum_{i=0}^{\infty} |\mathbf{A}_{mi}| d_i \leq \frac{e_m \beta}{d_m} \cdot \alpha d_m = \alpha\beta e_m. \tag{3.24}$$

Since $\alpha\beta < 1$ it follows that $e_m = 0$ and $e_i = 0$ for all i. Hence $x = y$. □

3.1. Proving the existence of fixed points. Let us consider the map $f = (f_0, f_1, f_2, \ldots)$. In order to prove the existence of a unique fixed point in $\mathbf{a} = (\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \ldots)$ we use the infinite dimensional Krawczyk operator for the map $\mathrm{id} - f$

$$K(\mathbf{a}) = \hat{a} - C(\hat{a} - f(\hat{a})) + (I - C(I - f'(\mathbf{a})))(\mathbf{a} - \hat{a}).$$

In order to be able to perform rigorous computations we have to choose a method to represent the infinite data in a computer. We assume that there is an exponential representation of the tail of $\mathbf{a}$, i.e. $\mathbf{a}_k = As^{-k}[-1, 1]$ for $k \geq M$, where $A > 0$ and $s > 1$. The infinite dimensional interval vector $\mathbf{a}$ is then stored in a computer as a finite dimensional vector $(\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_{M-1})$ and two real numbers A and s representing the tail ($k \geq M$). It is also possible to use other representations of the tail (for example polynomial: $\mathbf{a}_k = Ak^{-s}$).

First let us describe how we choose $\hat{a}$ and C. Let m be a positive integer number with $m \leq M$. We set $\hat{a}$ and C for the variables $a_0, a_1, \ldots, a_{m-1}$ as in the finite dimensional Krawczyk method applied for the mth projection of the infinite dimensional system. If m is sufficiently high then the other variables are not very important.

The point $\hat{a}$ is chosen as $\hat{a}_k = \mathrm{mid}(\mathbf{a}_k)$ for $k < M$ and $\hat{a}_k = 0$ for $k \geq M$.

Let $C^{(m)}$ be the inverse of $I-(P_m f)'(\hat{a}^{(m)})$, where P_m is the projection onto the first m variables, and $\hat{a}^{(m)} = (\hat{a}_0, \hat{a}_1, \ldots, \hat{a}_{m-1})$. The matrix C is defined as

$$C = \left(\begin{array}{cccc|cccc} c^{(m)}_{0,0} & c^{(m)}_{0,1} & \cdots & c^{(m)}_{0,m-1} & & & & \\ c^{(m)}_{1,0} & c^{(m)}_{1,1} & \cdots & c^{(m)}_{1,m-1} & & 0 & & \\ \vdots & & \ddots & & & & & \\ c^{(m)}_{m-1,0} & c^{(m)}_{m-1,1} & \cdots & c^{(m)}_{m-1,m-1} & & & & \\ \hline & & & & 1 & & & 0 \\ & & & & & 1 & & \\ & 0 & & & & & 1 & \\ & & & & 0 & & & \ddots \end{array}\right) \tag{3.25}$$

Observe that C is a bounded, nonsingular matrix of finite dependence.

With the definitions of $\hat{a}$ and C given above the formula for $K_k(\mathbf{a})$ reads

$$\begin{aligned} K_k(\mathbf{a}) =& \hat{a}_k - \sum_{p=0}^{m-1} c_{k,p}(\hat{a}_p - f_p(\hat{a})) + \sum_{p=0}^{m-1}\left(\delta_{k,p} - c_{k,p} + \sum_{i=0}^{m-1} c_{k,i}\frac{\partial f_i}{\partial a_p}(\mathbf{a})\right)(\mathbf{a}_p - \hat{a}_p) \\ &+ \sum_{p=m}^{\infty}\left(\sum_{i=0}^{m-1} c_{k,i}\frac{\partial f_i}{\partial a_p}(\mathbf{a})\right)(\mathbf{a}_p - \hat{a}_p), \qquad \text{for } k < m \end{aligned} \tag{3.26}$$

$$K_k(\mathbf{a}) = f_k(\hat{a}) + \sum_{p=0}^{\infty}\frac{\partial f_k}{\partial a_p}(\mathbf{a})(\mathbf{a}_p - \hat{a}_p), \qquad \text{for } k \geq m. \tag{3.27}$$

3.1.1. Checking the assumptions of the existence theorem. In order to prove the existence of a fixed point within $\mathbf{a}$ we have to prove that the assumptions of the Theorem 3.5 hold. Below we describe how this can be done for the map (1.3) using a computer. Since the number of inequalities we have to check is infinite and also each inequality contains infinite sums it is clear that we cannot use a computer directly to check these assumptions. We have to derive formulas which allow us to evaluate the enclosure for $K_k(\mathbf{a})$ in a finite number of arithmetic operations and to reduce the number of conditions which has to be checked to a finite number.

There are two integer numbers which are important in the computer assisted proof. m is the number of modes of Galerkin projection based on which we choose the matrix C. $M > m$ is the number of variables which are represented as ordinary intervals. The tail is composed of variables a_k with $k \geq M$. We also introduce the third number denoted by N. N is the number of elements in infinite sums for which the computations are done in interval arithmetic, other elements are replaced by a single interval containing the enclosure for the sum. We assume that $N > M$ and $N \geq m - 1 + \max\{j : c_j \neq 0\}$.

Let us define some quantities, which appear in the subsequent formulas.

$$A_{\mathrm{m}} = \max_{0 \leq k < M}\{A, |a_k^-|s^k, |a_k^+|s^k\}, \tag{3.28}$$

$$B_1 = |c_0| + \sum_{j=1}^{\infty}|c_j|(s^{-j} + s^j), \quad B_2 = \sum_{j=1}^{\infty}|c_j|j(s^{-j} + s^j), \tag{3.29}$$

$$t_1 = \frac{s^2+1}{s^2-1}, \quad t_2 = \frac{2}{s^2-1}. \tag{3.30}$$

In the appendix we derive formulas for finding enclosures for $f_k(\mathbf{a})$, derivates $\frac{\partial f_k}{\partial x_j}(\mathbf{a})$, and enclosures of infinite sums. Other lemmas are necessary to prove the enclosures $K_k(\mathbf{a}) \subset \mathbf{a}_k$ for large k all at once.

Now, we describe how to find enclosures for $K_k(\mathbf{a})$. We consider three cases depending on k.

Case 1: $k < m$. We split the last component of (3.26) into two parts. The first part (sum over $p = 0, \ldots, N-1$) is left unmodified. The enclosure for the second part (sum for $p \geq N$) is obtained from lemma A.3

$$\begin{aligned}K_k(\mathbf{a}) \subset &\hat{a}_k - \sum_{p=0}^{m-1} c_{k,p}(\hat{a}_p - f_p(\hat{a})) + \sum_{p=0}^{m-1}\left(\delta_{k,p} - c_{k,p} + \sum_{i=0}^{m-1} c_{k,i}\frac{\partial f_i}{\partial a_p}(\mathbf{a})\right)(\mathbf{a}_p - \hat{a}_p)\\ &+ \sum_{p=m}^{N-1}\left(\sum_{i=0}^{m-1} c_{k,i}\frac{\partial f_i}{\partial a_p}(\mathbf{a})\right)(\mathbf{a}_p - \hat{a}_p) + \frac{4\mu b A^2 B_1}{s^{2N-2}(s^2-1)}\sum_{i=0}^{m-1}|c_{ki}|\frac{s^i}{2^i}[-1,1].\end{aligned}$$

The above formula is evaluated in interval arithmetic. Enclosures for $f_p(\hat{a})$ and derivatives $\frac{\partial f_i}{\partial x_p}(\mathbf{a})$ are found using (A.1), (A.11)) (see appendix). Observe that since only a finite number of elements of $\hat{a}$ is nonzero, the expression $f_p(\hat{a})$ can be computed in a finite number of arithmetic operations.

Case 2: $k \geq m$, $k < M$. Splitting the infinite sum in (3.26) into two parts and using lemma A.3 we obtain the following enclosure

$$K_k(\mathbf{a}) \subset f_k(\hat{a}) + \sum_{p=0}^{N-1}\frac{\partial f_k}{\partial a_p}(\mathbf{a})(\mathbf{a}_p - \hat{a}_p) + \frac{4\mu b A_{\mathrm{m}}^2 B_1 s^k}{s^{2N-2}(s^2-1)2^k}[-1,1]. \tag{3.31}$$

To find an enclosure of $K_k(\mathbf{a})$ we evaluate the first two terms in interval arithmetic and add the interval containing the enclosure for the infinite sum containing derivatives for $p \geq N$.

Case 3: $k \geq M$. From Lemmas A.1, A.4 A.4 we obtain the following enclosure:

$$|K_k(\mathbf{a})| \leq \left|f_k(\hat{a}) + \frac{\partial f_k}{\partial a_0}(\mathbf{a})(\mathbf{a}_0 - \hat{a}_0) + \sum_{p=1}^{\infty}\frac{\partial f_k}{\partial a_p}(\mathbf{a})(\mathbf{a}_p - \hat{a}_p)\right| \tag{3.32}$$

$$\leq \frac{\mu b}{2^k s^k}(3A_{\mathrm{m}} + A_{\mathrm{m}}^2(t_1 B_1 + 4t_2 B_1 + 4B_1 + 5B_2) + 5kA_{\mathrm{m}}^2 B_1), \tag{3.33}$$

where A_{m}, B_1, B_2, t_1, t_2 and defined in (3.28), (3.29) and (3.30).

In order to prove that $K_k(\mathbf{a}) \subset \mathbf{a}_k$ for all $k \geq M$ we use the following lemma

LEMMA 3.6. *Let us assume that $\mathbf{a}_k = [-1,1]As^{-k}$ for $k \geq M$. If*

$$\frac{\mu b}{2^M}(3A_{\mathrm{m}} + A_{\mathrm{m}}^2(t_1 B_1 + 4t_2 B_1 + 4B_1 + 5B_2) + 5MA_{\mathrm{m}}^2 B_1) < A \tag{3.34}$$

then $|K_k(\mathbf{a})| < As^{-k}$ for all $k \geq M$.

Proof. See proof of Lemma A.1. □

3.2. Periodic orbits. In this section we describe a method for proving the existence of a unique periodic orbit within a sequence of infinite dimensional interval vectors $(\mathbf{a}^0, \mathbf{a}^1, \ldots, \mathbf{a}^{r-1})$, i.e. that there exists an orbit a^0, a^1, a^{r-1}, such that $a^i \in \mathbf{a}^i$ and $f(a^i) = a^{(i+1) \bmod r}$. In order to prove the existence of periodic orbits with period

r one can use the Krawczyk operator for the map $\mathrm{id} - f^r$. A better choice, however, is to use a different map.

Let x be a vector with coordinates of all points a^j for $j = 0, 1, \dots, r-1$ arranged in the following way: $x = (a_0^0, a_0^1, \dots, a_0^{r-1}, \dots, a_i^0, a_i^1, \dots, a_i^{r-1}, \dots)$. In this representation $x_k = x_{k_1+k_2 r} = a_{k_2}^{k_1}$, where $k = k_1 + k_2 r$, $k_1 = k \bmod r$, $k_2 \geq 0$.

Let us define the map g:

$$g\left(\begin{array}{c} a_0^0 \\ a_0^1 \\ \dots \\ a_0^{r-1} \\ \hline a_1^0 \\ a_1^1 \\ \dots \\ a_1^{r-1} \\ \hline \dots \end{array}\right) = \left(\begin{array}{c} a_0^0 - f_0(a^{r-1}) \\ a_0^1 - f_0(a^0) \\ \dots \\ a_0^{r-1} - f_0(a^{r-2}) \\ \hline a_1^0 - f_1(a^{r-1}) \\ a_1^1 - f_1(a^0) \\ \dots \\ a_1^{r-1} - f_1(a^{r-2}) \\ \hline \dots \end{array}\right). \tag{3.35}$$

Using the decomposition $k = k_1 + k_2 r$, where $k_1 = k \bmod r$ and $k_2 \geq 0$ the formula for $g_k(x)$ reads

$$g_k(x) = g_{k_1+k_2 r}(x) = a_{k_2}^{k_1} - f_{k_2}(a^{(k_1-1) \bmod r}). \tag{3.36}$$

It is clear that zeros of g correspond to period–r orbits of f. Our aim is to find conditions, which can be checked with a computer under which the existence and uniqueness of a zero of g, and hence a periodic orbit of f is ensured.

The Jacobian of g has the following block structure:

$$g'(x) = I - \left(\begin{array}{c|c|c} M_{0,0} & M_{0,1} & \dots \\ \hline M_{1,0} & M_{1,1} & \dots \\ \hline \vdots & \vdots & \ddots \end{array}\right), \tag{3.37}$$

where each block is an $r \times r$ matrix with r nonzero entries

$$M_{i,j} = \begin{pmatrix} & & & \frac{\partial f_i}{\partial a_j}(a^{r-1}) \\ \frac{\partial f_i}{\partial a_j}(a^0) & & & \\ & \ddots & & \\ & & \frac{\partial f_i}{\partial a_j}(a^{r-2}) & \end{pmatrix}. \tag{3.38}$$

Let us decompose the numbers k and l in the following way: $k = k_1 + k_2 r$, $l = l_1 + l_2 r$, where $k_1 = k \bmod r$, $l_1 = l \bmod r$ and $k_2, l_2 \geq 0$. Then we have:

$$\frac{\partial g_k}{\partial x_l} = \frac{\partial g_{k_1+k_2 r}}{\partial x_{l_1+l_2 r}} = \delta_{k_1+k_2 r, l_1+l_2 r} - \delta_{k_1, (l_1+1) \bmod r} \frac{\partial f_{k_2}}{\partial a_{l_2}}(a^{(k_1-1) \bmod r}). \tag{3.39}$$

To prove the existence of periodic orbit in $(\mathbf{a}^0, \mathbf{a}^1, \dots, \mathbf{a}^{r-1})$ we evaluate the Krawczyk operator for the map g

$$K(\mathbf{x}) = \hat{x} - Cg(\hat{x}) + (I - Cg'(\mathbf{x}))(\mathbf{x} - \hat{x}), \tag{3.40}$$

where C is the preconditioning matrix. In the computer assisted proof we use the matrix C of the following form

$$C = \begin{pmatrix} c_{0,0} & c_{0,1} & \dots & c_{0,mr-1} & 0 & 0 & \dots \\ c_{0,0} & c_{0,1} & \dots & c_{0,mr-1} & 0 & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \\ c_{mr-1,0} & c_{mr-1,1} & \dots & c_{mr-1,mr-1} & 0 & 0 & \dots \\ 0 & 0 & \dots & 0 & 1 & 0 & \dots \\ 0 & 0 & \dots & 0 & 0 & 1 & \dots \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{3.41}$$

where the submatrix $(c_{i,j})_{i,j=0}^{mr-1}$ is obtained from the finite dimensional version of the Krawczyk method (we apply Krawczyk method to period r orbit of m–dimensional map). For $i \geq mr$ or $j \geq mr$ we set $c_{i,j} = 1$ if $i = j$ and $c_{i,j} = 0$ otherwise.

The point $\hat{x}$ is chosen as: $\hat{x}_k = \operatorname{mid}(\mathbf{x}_k)$ for $k < Mr$ and $\hat{x}_k = 0$ for $k \geq Mr$.

Let us derive the formula for $K_k(\mathbf{x})$

$$\begin{aligned} K_k(\mathbf{x}) =& \hat{x}_k - \sum_{j=0}^{\infty} c_{k,j} g_j(\hat{x}) + \sum_{j=0}^{\infty} \left(\delta_{k,j} - \sum_{i=0}^{\infty} c_{k,i} \frac{\partial g_i}{\partial x_j}(\mathbf{x}) \right) (\mathbf{x}_j - \hat{x}_j) \\ =& \hat{a}_{k_2}^{k_1} + \sum_{j=0}^{r-1} \sum_{i=0}^{\infty} c_{k,j+ir} (f_i(\hat{a}^{(j-1) \bmod r}) - \hat{a}_i^j) \\ &+ \sum_{j=0}^{r-1} \sum_{i=0}^{\infty} \left(\delta_{k,j+ir} + \sum_{l=0}^{\infty} c_{k,(j+1) \bmod r + lr} \frac{\partial f_l}{\partial a_i}(\mathbf{a}^j) - c_{k,j+ir} \right) (\mathbf{a}_i^j - \hat{a}_i^j). \end{aligned}$$

In order to prove the existence of a periodic orbit we have to check that $K_k(\mathbf{x}) \subset \mathbf{x}_k$ and $\operatorname{diam}(K_k(\mathbf{x})) \leq \alpha \operatorname{diam}(\mathbf{x}_k)$. To this end we have to find enclosures of $K_k(\mathbf{x})$. We consider two cases depending on k.

Case 1: Enclosure for $K_k(\mathbf{x})$, $k < mr$.

$$\begin{aligned} K_k(\mathbf{x}) =& \hat{a}_{k_2}^{k_1} + \sum_{j=0}^{r-1} \sum_{i=0}^{m-1} c_{k,j+ir} (f_i(\hat{a}^{(j-1) \bmod r}) - \hat{a}_i^j) \\ &+ \sum_{j=0}^{r-1} \sum_{i=0}^{m-1} \left(\sum_{l=0}^{m-1} c_{k,(j+1) \bmod r + lr} \frac{\partial f_l}{\partial a_i}(\mathbf{a}^j) - c_{k,j+ir} + \delta_{k,j+ir} \right) (\mathbf{a}_i^j - \hat{a}_i^j) \\ &+ \sum_{j=0}^{r-1} \sum_{i=m}^{N-1} \left(\sum_{l=0}^{m-1} c_{k,(j+1) \bmod r + lr} \frac{\partial f_l}{\partial a_i}(\mathbf{a}^j) \right) (\mathbf{a}_i^j - \hat{a}_i^j) \\ &+ \sum_{j=0}^{r-1} \sum_{i=N}^{\infty} \left(\sum_{l=0}^{m-1} c_{k,(j+1) \bmod r + lr} \frac{\partial f_l}{\partial a_i}(\mathbf{a}^j) \right) \mathbf{a}_i^j. \end{aligned}$$

$K_k(\mathbf{x})$ is computed in interval arithmetic using the above formula. The enclosure for the last term is found using Lemma A.3 applied r times for $\mathbf{a}^0, \mathbf{a}^1, \dots, \mathbf{a}^{r-1}$.

Case 2: Enclosure for $K_k(\mathbf{x})$, $k \geq mr$. It can by shown that for $k \geq mr$

$$K_k(\mathbf{x}) = f_{k_2}(\hat{a}^l) + \sum_{i=0}^{\infty} \frac{\partial f_{k_2}}{\partial a_i}(\mathbf{a}^l)(\mathbf{a}_i^l - \hat{a}_i^l),$$

where $l = (k_1 - 1) \bmod r$.

Observe that the formula is basically the same as for the fixed point. Hence, we can proceed in a similar manner. For $mr \leq k < Mr$ the infinite sum is split into two parts:

$$K_k(\mathbf{x}) = f_{k_2}(\hat{a}^l) + \sum_{i=0}^{N-1} \frac{\partial f_{k_2}}{\partial a_i}(\mathbf{a}^l)(\mathbf{a}_i^l - \hat{a}_i^l) + \sum_{i=N}^{\infty} \frac{\partial f_{k_2}}{\partial a_i}(\mathbf{a}^l)(\mathbf{a}_i^l - \hat{a}_i^l) \tag{3.42}$$

The enclosure for the last term is found from Lemma A.3.

Lemma 3.6 is used r times for $\mathbf{a}^0, \mathbf{a}^1, \ldots, \mathbf{a}^{r-1}$ to prove that the condition $K_k(\mathbf{x}) \subset \mathbf{x}_k$ holds for all $k \geq Mr$.

3.3. Finding all fixed points and periodic orbits. If we want to find all fixed points enclosed in a certain compact set we apply a combination of the Krawczyk operator and the generalized bisection algorithm. We evaluate the Krawczyk operator on the vector $\mathbf{a}$. If the intersection $K(\mathbf{a}) \cap \mathbf{a}$ is empty then there is no fixed points of f in $\mathbf{a}$. If the assumptions of the theorem 3.5 hold then there exists exactly one fixed point in $\mathbf{a}$. If none of these two conditions is satisfied we divide the interval vector $\mathbf{a}$ into smaller parts are repeat the computations for each part independently. In this way we can search the whole region and find all fixed points in this region.

The main problem in the construction of an efficient algorithm is the way how we divide the infinite dimensional interval vector. Remember that the infinite dimensional interval vector is represented as a finite dimensional vector $\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_{M-1}$ and two real numbers s and A. The components of the vector for $k \geq M$ are $\mathbf{a}_k = [-As^{-k}, As^{-k}]$.

In the implementation of the bisection algorithm used in this work, the divisions are only done in the first m dimensions. First we make a division of $\mathbf{a}_0$. In the next division level we divide $\mathbf{a}_1$ and so on. After dividing $\mathbf{a}_{m-1}$ we divide $\mathbf{a}_0$.

Since we make no divisions in the variables a_k for $k \geq m$ we need a method to reduce the size of $\mathbf{a}_k$ for $k \geq m$. In order to achieve that we find the bound for $K_k(\mathbf{a})$ and if $K_k(\mathbf{a}) \subset \mathbf{a}_k$ then we reduce $\mathbf{a}_k$ to be $K_k(\mathbf{a})$. It is obvious that if $K_k(\mathbf{a}) \subset \mathbf{a}_k$ then there is no fixed points in $\mathbf{a}$ lying outside $K_k(\mathbf{a})$.

In order to find all periodic orbits in a certain region of the state space we apply generalized bisection algorithm and the Krawczyk method for the map g defined by equation (3.35).

4. Results for the Kot-Schaffer growth-dispersal model. In this section we study the existence of fixed points and periodic orbits for the map (1.3) with the following parameter values:

$$\mu = 3.5, \quad b_k = 2^{-k}, \quad c_0 = 0.8, \quad c_1 = -0.2, \quad c_k = 0 \text{ for } k > 1. \tag{4.1}$$

This is one of the sets of parameter values considered in [7].

Let us start with non-rigorous computations. We consider Galerkin projection of (1.3) with 21 modes. Starting from random initial conditions we have computed the trajectory composed of 10000 points and found the ranges of variables. The results

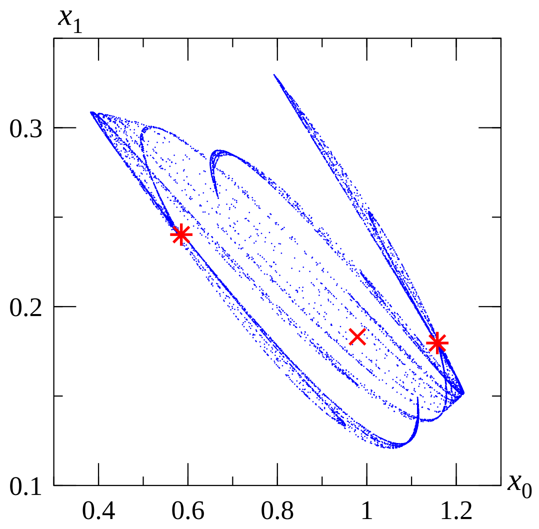

FIG. 4.1. *Trajectory of the Galerkin projection with 21 modes, position of the fixed point* ($\times$) *and the period–2 orbit* ($\ast$) *embedded within the attractor*

are shown in Fig. 4.1 and Table. 4.1. One can clearly see that the size of the observed attractor decreases very fast with the mode number. This indicates that the influence of higher modes on the dynamics of first modes is small.

TABLE 4.1
Variables' ranges

k	range	k	range
0	$[0.404, 1.262]$	11	$[3.010, 17.58] \cdot 10^{-13}$
1	$[0.143, 0.313]$	12	$[4.036, 10.34] \cdot 10^{-14}$
2	$[-0.00093, 0.0490]$	13	$[1.073, 2.684] \cdot 10^{-15}$
3	$[0.00163, 0.00849]$	14	$[1.309, 10.783] \cdot 10^{-17}$
4	$[-0.000169, 0.000646]$	15	$[1.982, 6.862] \cdot 10^{-18}$
5	$[2.393, 5.267] \cdot 10^{-5}$	16	$[-9.717, 10.369] \cdot 10^{-20}$
6	$[8.354, 41.75] \cdot 10^{-7}$	17	$[2.216, 6.007] \cdot 10^{-21}$
7	$[1.176, 4.679] \cdot 10^{-7}$	18	$[-0.182, 92.83] \cdot 10^{-24}$
8	$[-9.29, 12.793] \cdot 10^{-9}$	19	$[1.43, 2.63] \cdot 10^{-24}$
9	$[5.131, 9.779] \cdot 10^{-10}$	20	$[3.60, 7.77] \cdot 10^{-26}$
10	$[1.481, 4.365] \cdot 10^{-11}$		

4.1. Rigorous analysis of Galerkin projections. Here we apply the finite dimensional Krawczyk method for the Galerkin projection with m modes. Using Krawczyk method and bisection algorithm we have found all fixed points in the interval vector $\mathbf{a} = (\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_{m-1})$, where $\mathbf{a}_k = [-2, 2] \cdot 2^{-k}$ for $k \geq 0$. Computation results are collected in Table 4.2. For each m we report the number of fixed points found and the number of evaluations of Krawczyk operator. For $m > 1$ the number of fixed points remains constant and is equal to 4. For each Galerkin projection the origin is a fixed point. For each nontrivial fixed point we give the interval vector containing its first coordinate. One can see that the position of the fixed point converges.

Similar properties can be observed for other coordinates.

TABLE 4.2

Fixed points for Galerkin projection with m modes, p is the number of fixed points, c is the number of evaluations of the Krawczyk operator

m	p	c	intervals containing first coordinates of the fixed points		
1	2	25	$0.892857142857714^{33}_{25}$		
2	4	271	$0.171687825826289^{3}_{0}$	$0.513202527925668^{49}_{39}$	$1.011538217676599^{9}_{7}$
3	4	1949	$0.253007602568858^{30}_{26}$	$0.584167078589910^{94}_{80}$	$1.016847126042066^{8}_{6}$
4	4	10111	$0.265171408097915^{7}_{2}$	$0.617340723962466^{75}_{59}$	$1.017008600770327^{7}_{5}$
5	4	39187	$0.265586887333343^{301}_{295}$	$0.622927007989566^{82}_{65}$	$1.017012164806444^{4}_{2}$
6	4	113409	$0.265586768759000^{22}_{16}$	$0.623695679912000^{44}_{27}$	$1.017012223932998^{8}_{6}$
7	4	281091	$0.265586935874199^{7}_{0}$	$0.623757964371711^{49}_{30}$	$1.017012224691266^{2}_{0}$
8	4	657753	$0.265586946359222^{23}_{16}$	$0.623761685696651^{81}_{62}$	$1.017012224698914^{4}_{2}$
9	4	1453077	$0.265586946492822^{43}_{36}$	$0.623761851442855^{78}_{57}$	$1.017012224698976^{6}_{4}$
10	4	3024889	$0.265586946493310^{45}_{37}$	$0.623761857301011^{94}_{74}$	$1.017012224698976^{6}_{4}$

TABLE 4.3

Period–2 points for Galerkin projections with m modes, the number p of periodic points found, the number c of evaluations of the Krawczyk operator

m	p	c	intervals containing first coordinates of the period–2 points		
1	3	91	$0.535714285714428^{66}_{48}$	$1.071428571428557^{2}_{1}$	$0.892857142857714^{42}_{15}$
2	3	1539	$0.619404035207474^{117}_{085}$	$1.195966439229699^{6}_{4}$	1.011538217676^{6}_{595}
3	3	32961	$0.612765170336844^{4}_{05}$	$1.200466183162622^{9}_{6}$	$1.016847126042066^{9}_{4}$
4	3	258789	$0.612410015978585^{802}_{768}$	$1.200674696455^{701}_{699}$	$1.017008600770322^{9}_{4}$
5	3	1610369	$0.612404301036060^{124}_{088}$	$1.200680995964911^{5}_{2}$	$1.017012164806444^{5}_{0}$

In Table 4.3 we show the results on the existence of period–2 points for Galerkin projections with low number of modes. This time the search was restricted to the region $\mathbf{a} = (\mathbf{a}_0, \mathbf{a}_1, \ldots)$, where $\mathbf{a}_0 = [0.3, 1.3]$, $\mathbf{a}_1 = [0.1, 0.35]$, and $\mathbf{a}_k = [-0.5, 0.5] \cdot 2^{-k}$ for $k \geq 2$. This region contains the attractor observed numerically (compare Table 4.1). For each $m = 1, 2, \ldots, 5$ we have found three fixed points of the second iterate of the map.

4.2. Fixed points for the infinite dimensional map. As a first example of application of the infinite dimensional Krawczyk method we prove the existence of a fixed point for the infinite dimensional system (1.3).

THEOREM 4.1. *There exists a single fixed point in* $\mathbf{a} = (\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \ldots)$*, where* $\mathbf{a}_0 = [1.0165, 1.0175]$*,* $\mathbf{a}_1 = [0.194, 0.1947]$*,* $\mathbf{a}_2 = [0.0302, 0.0308]$*,* $\mathbf{a}_3 = [0.0035, 0.0042]$*,* $\mathbf{a}_k = [-0.1, 0.1] \cdot 2^{-k}$ *for* $k \geq 4$.

In the computed assisted proof we have used $M = 13$, $m = 4$, and $N = 20$. The results of the computer evaluation of $K(\mathbf{a})$ are presented in Table 4.4. One can easily check that $K_k(\mathbf{a}) \subset \operatorname{int} \mathbf{a}_k$ and that $A' < A$ for $k \geq M$. The existence of a single fixed point in $\mathbf{a}$ follows. We have iterated the Krawczyk operator and obtained the sharp bound for the position of the fixed point (last column of Table 4.4).

Now we report the results on application of the method for finding all fixed points in a certain region. Let us consider the interval vector $\mathbf{a} = (\mathbf{a}_0, \mathbf{a}_1, \ldots)$, where $\mathbf{a}_k = [-2, 2] \cdot 2^{-k}$ for $k \geq 0$. Using infinite dimensional Krawczyk operator and the bisection method we have shown that there are four fixed points in $\mathbf{a}$. The origin

TABLE 4.4
Details of the computer assisted proof of the existence of the fixed point

k	$\mathbf{a}_k$	$K_k(\mathbf{a})$	sharp bound
0	$[1.0165, 1.0175]$	$[1.01660, 1.01742]$	1.01701222^{58}_{36}
1	$[0.194, 0.1947]$	$[0.19406, 0.19460]$	0.19433733^{73}_{61}
2	$[0.0302, 0.0308]$	$[0.030278, 0.030741]$	0.030518985^{55}_{08}
3	$[0.0035, 0.0042]$	$[0.0035517, 0.00418302]$	0.003884168^{22}_{01}
4	$0.00625[-1, 1]$	$[-0.0047240, 0.0056177]$	0.0004066^{902}_{896}
5	$0.003125[-1, 1]$	$[-0.0018224, 0.0018881]$	$3.5968^{74}_{59} \cdot 10^{-5}$
6	$0.0015625[-1, 1]$	$[-0.00051562, 0.00051946]$	$2.753^{16}_{09} \cdot 10^{-6}$
7	$0.00078125[-1, 1]$	$[-0.00013436, 0.00013453]$	$1.85^{82}_{75} \cdot 10^{-7}$
8	$0.000390625[-1, 1]$	$3.53435 \cdot 10^{-5}[-1, 1]$	$1.1^{28}_{13} \cdot 10^{-8}$
9	$0.0001953125[-1, 1]$	$9.35659 \cdot 10^{-6}[-1, 1]$	$[3.36, 8.86] \cdot 10^{-10}$
10	$9.765625 \cdot 10^{-5}[-1, 1]$	$2.46934 \cdot 10^{-6}[-1, 1]$	$[-0.988, 1.049] \cdot 10^{-9}$
11	$4.8828125 \cdot 10^{-5}[-1, 1]$	$6.49886 \cdot 10^{-7}[-1, 1]$	$[-3.233, 3.237] \cdot 10^{-9}$
12	$2.44140625 \cdot 10^{-5}[-1, 1]$	$1.70616 \cdot 10^{-7}[-1, 1]$	$[-8.621, 8.623] \cdot 10^{-9}$
M	$0.1 \cdot 2^{-k}[-1, 1]$	$0.089158 \cdot 2^{-k}[-1, 1]$	$0.08908 \cdot 2^{-k}[-1, 1]$

$(0, 0, \ldots)$ is a fixed point of the system. Interval vectors containing other fixed points are given in Table 4.5. The number of evaluations of Krawczyk operator was 51995 and the computation time was 259.36 s. During the search the infinite dimensional Krawczyk operator with $m = 4$ and $M = 13$ was used.

TABLE 4.5
Positions of all fixed points found with the infinite dimensional Krawczyk method

k	interval vectors containing fixed points		
0	0.265586946^{57}_{41}	0.6237618^{61}_{54}	1.01701222^{58}_{36}
1	-0.233319240^{05}_{14}	0.43585749^{48}_{20}	0.19433733^{73}_{61}
2	-0.101969705^{03}_{12}	-0.08528861^{61}_{85}	0.030518985^{56}_{12}
3	-0.0193410159^{20}_{58}	0.04306229^{34}_{07}	0.003884168^{19}_{06}
4	-0.0017412981^{54}_{96}	-0.0101167^{59}_{67}	0.0004066^{901}_{897}
5	$5.6571^{82}_{64} \cdot 10^{-6}$	0.0023097^{58}_{24}	$3.5968^{70}_{63} \cdot 10^{-5}$
6	$2.15251^{59}_{42} \cdot 10^{-5}$	-0.00041368^{48}_{68}	$2.7531^{34}_{16} \cdot 10^{-6}$
7	$2.831^{11}_{07} \cdot 10^{-6}$	$6.613^{52}_{28} \cdot 10^{-5}$	$1.857^{98}_{68} \cdot 10^{-7}$
8	$1.86^{53}_{41} \cdot 10^{-7}$	$-9.22^{41}_{87} \cdot 10^{-6}$	$1.1^{26}_{14} \cdot 10^{-8}$
9	$[3.897, 4.451] \cdot 10^{-9}$	$1.1^{64}_{54} \cdot 10^{-6}$	$[3.47, 8.74] \cdot 10^{-10}$
10	$[-1.186, 0.317] \cdot 10^{-9}$	$[-1.389, -1.246] \cdot 10^{-7}$	$[-0.977, 1.038] \cdot 10^{-9}$
11	$[-1.569, 1.463] \cdot 10^{-9}$	$[0.285, 2.454] \cdot 10^{-8}$	$[-3.223, 3.226] \cdot 10^{-9}$
12	$[-2.326, 2.320] \cdot 10^{-9}$	$[-1.408, 1.146] \cdot 10^{-8}$	$[-8.611, 8.611] \cdot 10^{-9}$
M	$0.019400 \cdot 2^{-k}[-1, 1]$	$0.06576 \cdot 2^{-k}[-1, 1]$	$0.08908 \cdot 2^{-k}[-1, 1]$

If we restrict the search to the region $\mathbf{a} = (\mathbf{a}_0, \mathbf{a}_1, \ldots)$, where $\mathbf{a}_0 = [0.3, 1.3]$, $\mathbf{a}_1 = [0.1, 0.35]$, and $\mathbf{a}_k = [-0.5, 0.5] \cdot 2^{-k}$ for $k \geq 2$ the computation time can be significantly reduced. This region contains the numerically observed attractor (compare Table 4.1). For this search space the Krawczyk operator is evaluated only 249 times and the computation time is 0.97 s. The search procedure finds one fixed point in this region.

4.3. Period–2 orbits. Analysis of Galerkin projections indicates that there exists one period–2 orbit for the system.

As a first step of analysis of period–2 points for the infinite dimensional system we have proved the existence of one period–2 orbit. Namely, we have shown that

THEOREM 4.2. *There exists a single period two orbit in* $\mathbf{a}^0, \mathbf{a}^1$*, where* $\mathbf{a}_0^0 = [0.6044, 0.6204]$, $\mathbf{a}_1^0 = [0.2375, 0.2435]$, $\mathbf{a}_2^0 = [0.0417, 0.0437]$, $\mathbf{a}_3^0 = [0.0026, 0.0046]$, $\mathbf{a}_0^1 = [1.1956, 1.2056]$, $\mathbf{a}_1^1 = [0.1895, 0.1935]$, $\mathbf{a}_2^1 = [0.0180, 0.0192]$, $\mathbf{a}_3^1 = [0.0022, 0.0036]$, $\mathbf{a}_k^0 = \mathbf{a}_k^1 = [-As^{-k}, As^{-k}]$ *for* $k \geq 4$*, where* $A = 0.1$ *and* $s = 2$*.*

In the proof we have used parameters $m = 5$, $M = 14$, $N = 30$ of the numerical procedure. Iterating the Krawczyk operator we have found the interval vectors with diameter below 10^{-7} to which the period–2 orbit belongs.

We now describe the results on the search of period–2 points in the region containing the chaotic attractor observed numerically. We have found all period–2 points belonging to the interval vector $\mathbf{a} = (\mathbf{a}_0, \mathbf{a}_1, \ldots)$, where $\mathbf{a}_0 = [0.3, 1.3]$, $\mathbf{a}_1 = [0.1, 0.35]$, and $\mathbf{a}_k = [-0.5, 0.5]2^{-k}$ for $k \geq 2$. During the computations we have used $m = 4$ and $M = 13$. The number of evaluations of Krawczyk operator was 918043 and the computation time was 14226.45 s.

Three fixed points of f^2 were found. The interval vectors containing these points are collected in Table 4.6. First two points constitute the unique period–2 orbit belonging to the region where the chaotic attractor is observed in simulations. The last point is a fixed point of the system (compare Table 4.5). The fixed point and the period–2 orbit found are shown in Fig. 4.1.

TABLE 4.6
Positions of period–2 orbits found with the infinite dimensional Krawczyk method.

k	Interval vectors containing fixed points of f^2		
0	0.6124043_{08}^{22}	1.2006811_{04}^{12}	1.01701222_{09}^{85}
1	0.2405484_{04}^{11}	0.19150592_{26}^{69}	0.1943373_{33}^{41}
2	0.04278031_{6}^{8}	0.018621_{89}^{91}	0.03051898_{43}^{63}
3	0.00371012_{05}^{12}	0.002934520_{34}^{8}	0.00388416_{79}^{83}
4	0.00028687_{18}^{29}	0.00049308_{51}^{6}	0.0004066_{89}^{91}
5	$2.857_{58}^{61} \cdot 10^{-5}$	$4.8383_{69}^{87} \cdot 10^{-5}$	$3.5968_{59}^{74} \cdot 10^{-5}$
6	$2.4611_{26}^{65} \cdot 10^{-6}$	$3.164_{36}^{41} \cdot 10^{-6}$	$2.7531_{11}^{39} \cdot 10^{-6}$
7	$1.6_{09}^{11} \cdot 10^{-7}$	$1.78_{87}^{92} \cdot 10^{-7}$	$1.857_{68}^{99} \cdot 10^{-7}$
8	$[9.52, 9.73] \cdot 10^{-9}$	$[1.001, 1.014] \cdot 10^{-8}$	$[1.114, 1.127] \cdot 10^{-8}$
9	$[2.98, 8.92] \cdot 10^{-10}$	$[1.88, 8.69] \cdot 10^{-10}$	$[3.47, 8.74] \cdot 10^{-10}$
10	$[-8.72, 9.41] \cdot 10^{-10}$	$[-1.87, 1.92] \cdot 10^{-9}$	$[-9.76, 10.38] \cdot 10^{-10}$
11	$[-4.29, 4.29] \cdot 10^{-9}$	$[-5.54, 5.55] \cdot 10^{-9}$	$[-3.23, 3.23] \cdot 10^{-9}$
12	$[-1.37, 1.37] \cdot 10^{-8}$	$[-1.14, 1.14] \cdot 10^{-8}$	$[-8.62, 8.62] \cdot 10^{-9}$
M	$A = 0.1236$	$A = 0.1236$	$A = 0.08908$

4.4. Period–4 orbit. As a last example we have proved the existence of a period-4 orbit. We have shown that

THEOREM 4.3. *There exists a single period four orbit in* $\mathbf{a}^0, \mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3$*, where* $\mathbf{a}_0^0 = [1.099, 1.131]$, $\mathbf{a}_1^0 = [0.2107, 0.2347]$, $\mathbf{a}_2^0 = [0.00532, 0.01932]$, $\mathbf{a}_3^0 = [0.001, 0.006]$, $\mathbf{a}_0^1 = [0.8208, 0.8708]$, $\mathbf{a}_1^1 = [0.1701, 0.1981]$, $\mathbf{a}_2^1 = [0.03897, 0.05297]$, $\mathbf{a}_3^1 = [0.001, 0.006]$, $\mathbf{a}_0^2 = [1.1997, 1.2317]$, $\mathbf{a}_1^2 = [0.1631, 0.1911]$, $\mathbf{a}_2^2 = [0.0157, 0.0297]$, $\mathbf{a}_3^2 = [0.001, 0.006]$, $\mathbf{a}_0^3 = [0.5285, 0.5765]$, $\mathbf{a}_1^3 = [0.2518, 0.2798]$, $\mathbf{a}_2^3 = [0.03114, 0.04514]$, $\mathbf{a}_3^3 = [0.001, 0.006]$, $\mathbf{a}_k^0 = \mathbf{a}_k^1 = \mathbf{a}_k^2 = \mathbf{a}_k^3 = [-0.05, 0.05] \cdot 2^{-k}$ *for* $k \geq 4$*.*

In the computer assisted proof we have used parameters $M = 15$, $m = 5$ and $N = 20$.

Appendix. Rigorous estimates.

In the appendix we derive several rigorous estimates involving the map f defined by equation (1.3). These estimates are used to carry out computer assisted proofs of existence of periodic orbits. In order to simplify the formulas let us define the value of c_0 to be equal to $c_0/2$.

A.1. Estimates for f_k. The following formula for f_k which is useful for computational purposes can be derived from the symmetry conditions $a_k = a_{-k}$, $b_k = b_{-k}$, $c_k = c_{-k}$

$$f_k(a) = \mu b_k \left(a_k - \sum_{j=0}^{\infty} c_j (S_{|k-j|}(a) + S_{k+j}(a)) \right), \tag{A.1}$$

where

$$S_k(a) = \sum_{l+n=k} a_l a_n = \sum_{0 \le n \le k} a_n a_{k-n} + 2 \sum_{n \ge 1} a_n a_{k+n}. \tag{A.2}$$

In the following we assume that for the infinite dimensional interval vector $\mathbf{a}$ we have the exponential representation of the tail. Assume that $a_k \in \mathbf{a}_k = [a_k^-, a_k^+]$ for $k < M$ and $\mathbf{a}_k = As^{-k}[-1,1]$ for $k \ge M$. We can find an enclosure for $S_k(\mathbf{a})$ using the formula

$$S_k(\mathbf{a}) \subset \sum_{0 \le n \le k} \mathbf{a}_n \mathbf{a}_{k-n} + 2 \sum_{1 \le n < M} \mathbf{a}_n \mathbf{a}_{n+k} + \frac{2A^2}{s^{2M+k-2}(s^2-1)}[-1,1], \tag{A.3}$$

In order to obtain a rigorous estimate for $f_k(a)$ we use (A.1) and (A.3). If $c_j = 0$ for $j > j_0$ then the above formula allows to find the enclosure of $f_k(a)$ in a finite number of interval computations. In the following we assume that $c_j = 0$ for $j > 2$ so indeed the sum in (A.1) is finite.

Let us assume that $s > 1$, $A > 0$, $a_k \in \mathbf{a}_k$, where $\mathbf{a}_k = [a_k^-, a_k^+]$ for $k < M$ and $\mathbf{a}_k = As^{-k}[-1,1]$ for $k \ge M$. Let A_{m} be defined as in (3.28). It is clear that $|a_k| \le A_{\mathrm{m}} s^{-k}$ for all $k \ge 0$. With the definition of A_{m} we can easily obtain the following estimate for $S_k(a)$:

$$|S_k(a)| \le \frac{A_{\mathrm{m}}^2}{s^k} \left(k + \frac{s^2+1}{s^2-1} \right). \tag{A.4}$$

LEMMA A.1. *Let us assume that $b_k = b2^{-k}$ for all $k \ge 0$. Let $s > 1$, $A > 0$, $a_k \in \mathbf{a}_k$, where $\mathbf{a}_k = [a_k^-, a_k^+]$ for $k < M$ and $\mathbf{a}_k = As^{-k}[-1,1]$ for $k \ge M$. Let A_{m}, B_1, B_2, t_1 be defined as in (3.28), (3.29), and (3.30). Let us assume that*

$$D = A + A_{\mathrm{m}}^2 (t_1 B_1 + B_2) \ge 0 \tag{A.5}$$

$$E = A_{\mathrm{m}}^2 B_1 \ge 0, \tag{A.6}$$

$$A' = \mu b (D + EM) 2^{-M} \le A. \tag{A.7}$$

Then $|f_k(a)| \le As^{-k}$ for all $k \ge M$. Further, we can take A' and the assertions will also hold, i.e.

$$\left(|a_k| \le A' s^{-k} \quad \forall k \ge M \right) \Rightarrow \left(|f_k(a)| \le A' s^{-k} \quad \forall k \ge M \right). \tag{A.8}$$

Proof. It can be shown that

$$|f_k(a)| = \left|\mu b_k\left(a_k - \sum_{j=0}^{\infty} c_j(S_{|k-j|}(a) + S_{k+j}(a))\right)\right| \le \frac{\mu b}{2^k s^k}\,(D+kE)\,.$$

Now we show that for all $k \ge M$

$$\mu b(D+Ek)2^{-k} \le A. \tag{A.9}$$

From the assumptions it follows that the above equation holds for M. It is sufficient to show that if it holds for $k \ge M$, then it also holds for $k+1$. It is clear that $kE + D \ge E$. Hence $2(D+kE) \ge D + E(k+1)$. It follows that

$$\mu b(D+E(k+1))2^{-(k+1)} \le \mu b(D+Ek)2^{-k} \le A.$$

Finally

$$|f_k(a)| \le \frac{\mu b}{2^k s^k}(D+kE) \le \frac{A}{s^k} \quad \text{for all } k \ge M.$$

It is clear that A' is also good. Since $A' \le A$ the new values of $A_{\mathrm{m}}(A')$, $D(A')$ and $E(A')$ are smaller than those for the original value of A. Thus $\mu b(D(A') + E(A')k)2^{-k} \le A'$ and we can continue in the same way as previously. □

A.2. Derivatives.

Lemma A.2. *$k \ge 0$, $p \ge 0$*

$$\frac{\partial S_k(a)}{\partial a_p} = \begin{cases} 2a_k & \text{for } p = 0,\\ 2(a_{k+p} + a_{|k-p|}) & \text{for } p > 0,\end{cases} \tag{A.10}$$

$$\frac{\partial f_k(a)}{\partial a_p} = \begin{cases} \mu b_k\left(\delta_{0,k} - \sum_{j=0}^{\infty} 2c_j(a_{|k-j|} + a_{k+j})\right), & p = 0,\\ \mu b_k\left(\delta_{p,k} - \sum_{j=0}^{\infty} 2c_j(a_{|k-j|+p} + a_{||k-j|-p|} + a_{k+j+p} + a_{|k+j-p|})\right), & p > 0.\end{cases} \tag{A.11}$$

A.3. Enclosures for infinite sums. Now we present lemmas necessary to find enclosures for infinite sums and to prove conditions $K_k(\mathbf{x}) \subset \operatorname{int}(\mathbf{x}_k)$ for $k \ge M$ in one step.

Lemma A.3. *$N \ge m - 1 + \max\{j\colon c_j \ne 0\}$, $N \ge m$, $|a_k| \le As^{-k}$ for all $k \ge 0$,*

$$\left|\sum_{p=N}^{\infty} \frac{\partial f_i}{\partial a_p}(\mathbf{a})\mathbf{a}_p\right| \le \frac{4\mu b A^2 B_1 s^i}{s^{2N-2}(s^2-1)2^i} \quad \text{for } 0 \le i < m, \tag{A.12}$$

$$\left|\sum_{p=N}^{\infty}\left(\sum_{i=0}^{m-1} c_{ki}\frac{\partial f_i}{\partial a_p}(\mathbf{a})\right)\mathbf{a}_p\right| \le \frac{4\mu b A^2 B_1}{s^{2N-2}(s^2-1)}\sum_{i=0}^{m-1}|c_{ki}|\frac{s^i}{2^i}. \tag{A.13}$$

LEMMA A.4. *$k > 1$, $|a_k| \le As^{-k}$, $\hat{a} \in \mathbf{a}$,*

$$\left|\frac{\partial f_k}{\partial a_0}(\mathbf{a})(\mathbf{a}_0 - \hat{a}_0)\right| \le \frac{4\mu b A^2 B_1}{2^k s^k}, \tag{A.14}$$

$$\left|\sum_{p=1}^{\infty} \frac{\partial f_k}{\partial a_p}(\mathbf{a})(\mathbf{a}_p - \hat{a}_p)\right| \le \frac{2\mu b A}{2^k s^k}\left(1 + 2At_2B_1 + 2AB_2 + 2kAB_1\right). \tag{A.15}$$

REFERENCES

[1] G. ALEFELD, *Inclusion Methods for Systems of Nonlinear Equations – The Interval Newton Method and Modifications*, in Topics in Validated Computations, Proceedings of the IMACS-GAMM International Workshop on Validated Computation, J. Herzberger, ed., Elsevier, 1994, pp. 7–26.

[2] L. CESARI, *Functional analysis and Galerkin's method*, Mich. Math. Jour., 11 (1964), pp. 383–414.

[3] Z. GALIAS, *Interval methods for rigorous investigations of periodic orbits*, Int. J. Bifurcation and Chaos, 11 (2001), pp. 2427–2450.

[4] ———, *Rigorous investigations of Ikeda map by means of interval arithmetic*, Nonlinearity, 15 (2002), pp. 1759–1779.

[5] R. MOORE, *Interval Analysis*, Prentice Hall, Englewood Cliffs, NJ, 1966.

[6] A. NEUMAIER, *Interval methods for systems of equations*, Cambridge University Press, 1990.

[7] O. J. S. DAY AND K. MISCHAIKOW, *A rigorous numerical method for the global analysis of an infinite dimensional discrete dynamical system.* to appear in SIAM Journal on Applied Dynamical Systems.

[8] P. ZGLICZYŃSKI AND K. MISCHAIKOW, *Rigorous numerics for partial differential equations: the Kuramoto-Sivashinsky equation*, Foundations of Computational Mathematics, 1 (2001), pp. 255–288.