THE HÉNON HEILES HAMILTONIAN NEAR THE CRITICAL ENERGY LEVEL - SOME RIGOROUS RESULTS

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Abstract. We continue the investigation of the Hénon-Heiles system started in [AZ] and we provide new results in four directions. We prove the existence of infinitely many solutions which are homoclinic or heteroclinic to periodic solutions. We prove the existence of infinitely many symmetric hyperbolic periodic solutions. We provide a new topological method to prove the existence of elliptic periodic solutions and we apply it to the system. We give all the results in an explicitly given energy interval containing the critical value of 1/6. All proofs are computer assisted.

1. Introduction

We investigate the Hénon–Heiles system introduced in [HH] defined by the Hamiltonian

$$H = \frac{p_x^2 + p_y^2}{2} + \frac{x^2 + y^2}{2} + x^2 y - \frac{y^3}{3}; \tag{1}$$

the corresponding Hamilton equations are:

$$\left\{ \begin{array}{rcl} \dot{p}_x & = & -x - 2xy \\ \dot{p}_y & = & -x^2 + y^2 - y \\ \dot{x} & = & p_x \\ \dot{y} & = & p_y \end{array} \right. \tag{2}$$

The system is autonomous, therefore an integral of motion is provided by the energy H. In other words, the energy levels $\{(x, y, p_x, p_y) \in \mathbb{R}^4 : H(p_x, p_y, x, y) = h\}$, denoted by $H = h$, are invariant under the flow generated by (2).

Hénon and Heiles presented the Hamiltonian (1) as a model problem for galactic potentials admitting an axis of symmetry. Later it was observed in [LF] that a wide class of three particle systems can be reduced to a Hénon–Heiles type of Hamiltonian by considering only the first three terms in the Taylor expansion (see also [BGM] for a list of problems related to this model).

The numerical simulations of the Poincaré return map of system (2) at the section $x = 0, \dot{x} > 0$ performed by Hénon and Heiles [HH] show an interesting dynamical behavior for the energy range $0 < h \leq 1/6$. When h is small the system

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appears to be integrable, with large elliptic islands dominating the picture of the dynamics. Then, as the energy increases, some small chaotic regions arise, which tend to grow at the expense of the elliptic islands. Finally, at $h = 1/6$ the chaotic regions dominate the whole picture, with apparently only two small elliptic islands left.

When $h < 0$ the dynamics of the Hénon–Heiles system is rather uninteresting, since all solutions escape to infinity in both time directions. When $h = 0$ a critical point appears at the origin, and as h becomes positive eight main families of periodic orbits Π_i , $i = 1, \ldots, 8$ arise. All these periodic orbits lie on a compact component of the manifold $H = h$ and exist in the whole energy range $0 < h < 1/6$. At $h = \frac{1}{6}$ three new symmetrically placed critical points appear, each bifurcating into a hyperbolic periodic orbit when $h > 1/6$. An extensive numerical investigation of periodic orbits, their bifurcations and stability properties for Hénon-Heiles hamiltonian can be found in [DHB].

Much of the rigorous research on the Hénon–Heiles system aimed to explain this striking behavior was done in 70's, see the survey paper by Churchill, Pecelli and Rod [CPR] and also [CKR]. From later rigorous work we would like to mention papers [GR1, GR2, MHO, Le], where the existence of horseshoe has been shown for $h > 1/6$.

In [AZ] we proved that the Hénon–Heiles system embeds a rich symbolic dynamic structure and it has an infinite number of periodic points for energies in the range $h \in \left(\frac{1}{6} - \delta, \frac{1}{6} + \delta\right)$ for some $\delta > 0$. Roughly speaking, we showed the following: there exists an invariant set S which contains three periodic orbits. On the set S we proved the existence of the same symbolic dynamics we would get, if we knew that the stable manifolds of all orbits intersected transversally the unstable manifolds of all orbits. On the other hand we did not prove the hyperbolicity of S: a basic feature of our procedure was that we did not have any assumption involving derivatives. Since the properties of the derivatives of the Poincaré map were not investigated, we were did not infer the existence of homoclinic or heteroclinic solutions, but we could only prove the existence of solutions which remain close to a periodic solution for an infinite period of (forward or backward) time. More precisely, we did prove the existence of a horseshoe dynamics only from a topological point of view, disregarding any differentiability property.

In this paper we change the approach. Although we do not prove the hyperbolicity of S, we use a technique which involves the computation of the derivatives of the Poincaré map in order to prove the existence of infinitely many homoclinic and heteroclinic solutions.

The second improvement of this paper concerns the existence of symmetric periodic solutions. Indeed, the Hénon-Heiles Hamiltonian is symmetric with respect to a rotation of $2\pi/3$, a reflection with respect to the x axis and also with respect to time. We prove the existence of solutions having exactly the same symmetry, together with solutions which are not symmetric. For this purpose we use the topological method introduced by Wilczak in [W]. This approach is similar in the spirit to the Devaney's approach [D], which is based on invariant manifolds.

In [AZ] we did not take into account any elliptic fixed point of the Poincaré map, since we were mainly interested in chaotic behavior which is related to hyperbolic points. In this paper we provide a purely topological tool which can be used to prove the existence of an elliptic fixed point. We apply this method to prove the existence of elliptic periodic points for the same Poincaré map, in the proposed in [DHB] classification scheme of periodic orbits this is a d_a-orbit.

Finally, all the result in this paper hold for all values of the energy $h \in \left[\frac{1}{6} - \delta, \frac{1}{6}\right]$, where the value of $\delta > 0$ is given explicitly.

2. $\mathcal{C}^0$ TOOLS FOR HYPERBOLIC POINTS

In this section we describe a topological method we use to prove the existence of periodic trajectories and symbolic dynamics. The method is based on the concept of covering relation by a map. We present here an intuitive description of covering relation for planar maps, for precise formulation the reader is referred to [AZ] and references given there. For covering relations in higher dimension see [GZ3].

Definitions 2.1 and 2.3 were introduced in [AZ]. Definitions 2.5 and 2.6 were introduced in [A] to deal with the new difficulties of the restricted 3-body problem and are used here to provide a much shorter computation time.

Definition 2.1. *A **triple set** (or t-set) is a triple $N = (|N|, N^l, N^r)$ of closed subsets of $\mathbb{R}^2$ satisfying the following properties:*

1: *$|N|$ is a closed parallelogram*
2: *N^l and N^r are closed half-planes*
3: *the sets $N^{le} := N^l \cap |N|$ and $N^{re} := N^r \cap |N|$ are two nonadjacent edges of $|N|$*

We call $|N|$, N^l, N^r, N^{le} and N^{re} the support, the left side, the right side, the left edge and the right edge of the t-set N respectively. One observes that $\mathbb{R}^2 \setminus (|N| \cup N^l \cup N^r)$ consists of two disjoint sets. We call N^t and N^b the closure of such sets (the top and bottom sides of the triple set). We also set $N^{te} := N^t \cap |N|$ and $N^{be} := N^b \cap |N|$ (the top and bottom edges of N).

Remark 2.2. Since all theorems we use concern topological properties of the t-sets, then one can choose any continuous deformation of the t-set defined above, obtaining the same results, see also [GZ3].

Definition 2.3. *Let $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a map and let N_1 and N_2 be two triple sets. We say that N_1 $f-$covers N_2 ($N_1 \overset{f}{\Longrightarrow} N_2$) if:*

a: $f(|N_1|) \subset \operatorname{int}(N_2^l \cup |N_2| \cup N_2^r)$
b: *either* $f(N_1^{le}) \subset \operatorname{int}(N_2^l)$ *and* $f(N_1^{re}) \subset \operatorname{int}(N_2^r)$
or $f(N_1^{le}) \subset \operatorname{int}(N_2^r)$ *and* $f(N_1^{re}) \subset \operatorname{int}(N_2^l)$

The following lemma says that we can reduce the condition (a) in the above definition to the boundary of $|N_1|$ if we know that the map f is defined on $|N_1|$ and it is injective. This lemma plays a very important role in the computer assisted verification of the covering relations, as it reduces the computations to the boundary of $|N_1|$.

Lemma 2.4. *Let $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a map and let N_1 and N_2 be two triple sets. Assume that f is an injective map on $|N_1|$, then $N_1 \overset{f}{\Longrightarrow} N_2$ if and only if*

a′: $f(\partial|N_1|) \subset \operatorname{int}(N_2^l \cup |N_2| \cup N_2^r)$
b: *either* $f(N_1^{le}) \subset \operatorname{int}(N_2^l)$ *and* $f(N_1^{re}) \subset \operatorname{int}(N_2^r)$
or $f(N_1^{le}) \subset \operatorname{int}(N_2^r)$ *and* $f(N_1^{re}) \subset \operatorname{int}(N_2^l)$

As we pointed out above, in some circumstances it is easier to compute the inverse flow, then the direct flow. To this purpose we give the following definitions:

Definition 2.5. *Let N_1 and N_2 be two triple sets. We say that N_1 $f-$backcovers N_2 ($N_1 \overset{f}{\Longleftarrow} N_2$) whenever:*

- **a:** $f : \Omega_1 \subset \mathbb{R}^2 \to \Omega_2 \subset \mathbb{R}^2$ *is a homeomorphism*
- **b:** $|N_2| \subset \Omega_2$
- **c:** $f^{-1}(\partial|N_2|) \subset \mathrm{int}(N_1^t \cup |N_1| \cup N_1^b)$
- **d:** *either* $f^{-1}(N_2^{te}) \subset \mathrm{int}(N_1^t)$ *and* $f^{-1}(N_2^{be}) \subset \mathrm{int}(N_1^b)$ *or* $f^{-1}(N_2^{te}) \subset \mathrm{int}(N_1^b)$ *and* $f^{-1}(N_2^{be}) \subset \mathrm{int}(N_1^t)$

Definition 2.6. *Let N_1 and N_2 be two triple sets. We say that N_1 generically $f-$covers N_2 ($N_1 \overset{f}{\Longleftrightarrow} N_2$) if N_1 $f-$covers N_2 or there exists $n \geq 1$ t-sets M_i, $i = 1, \dots, n$ and $n+1$ maps g_j, $j = 0, \dots, n$ such that $f = g_j \circ \cdots \circ g_1 \circ g_0$, $N_1 \overset{g_0}{\Longrightarrow} M_1$, $M_i \overset{g_i}{\Longrightarrow} M_{i+1}$ for all $i = 1, \dots, n-1$ and $M_n \overset{g_n}{\Longrightarrow} N_2$.*

Remark 2.7. *In* [A] *it was proved that backcovering implies generic covering. Note that, although covering, backcovering and generic covering occur often simultaneously and they are indeed a very similar phenomenon, they are not equivalent. In fact it may even happen that a map f is not defined on the whole support of N_1, and still N_1 $f-$backcovers N_2 or N_1 generically $f-$covers N_2. On the other hand, the result of a backcovering or of a generic covering relation is very similar to the result of a covering relation as far as the results we are interested in are concerned.*

Remark 2.8. *The generic covering relation is not symmetric, despite that the symbol we use for it is symmetric, but the dynamics works always from the left side to the right side.*

The following theorem was proved in [A] and follows from Theorem 4 in [Z3] and the definition of generic covering (see [GZ3] for the higher dimensional version). Together with Theorem 2.14 it is the main topological tool we use:

Theorem 2.9. *Given t-sets $M_i \subset \mathbb{R}^2$ and n continuous maps $f_i : \Omega_{1i} \subset \mathbb{R}^2 \to \Omega_{2i} \subset \mathbb{R}^2$ such that*

$$M_0 \overset{f_0}{\Longleftrightarrow} M_1 \overset{f_1}{\Longleftrightarrow} M_2 \overset{f_2}{\Longleftrightarrow} M_2 \dots \overset{f_{n-1}}{\Longleftrightarrow} M_0 = M_n,$$

then there exists $x \in \mathrm{int}|M_0|$, such that $f_k \circ \cdots \circ f_1 \circ f_0(x) \in \mathrm{int}|M_{k+1}|$, for $k = 0, \dots, n-1$ and $x = f_{n-1} \circ \cdots \circ f_1 \circ f_0(x)$.

Assume that we have n t-sets N_i, $i = 1, \dots, n$, with some covering relations. Let $N = \bigcup_i |N_i|$; the following definitions are standard.

Definition 2.10. *Let f be injective. The invariant set of N is defined by* $\mathrm{Inv}(N, f) := \{x \in N : f^i(x) \in N \text{ for all } i \in \mathbb{Z}\}$.

Definition 2.11. *The transition matrix $T(j, i)$, $i, j = 1, \dots, n$, is defined as follows:*

$$T(j,i) = \begin{cases} 1 & \text{if } N_i \overset{f}{\Longleftrightarrow} N_j \\ 0 & \text{otherwise} \end{cases}$$

Let Σ_n be the set of bi-infinite sequences of n symbols.

Definition 2.12. *A sequence $\{x_k\} \in \Sigma_n$ is said to be* admissible *(for T) if $T(x_{k+1}, x_k) = 1$ for all k. We denote by $\Sigma_T \subset \Sigma_n$ the set of all admissible sequences.*

Definition 2.13. *Assume $|N_i| \cap |N_j| = \emptyset$, for $i \neq j$. The projection $\pi : \mathrm{Inv}(N, f) \to \Sigma_T$ is defined by setting $\pi(x)_i = j$ where j satisfies $f^i(x) \in |N_j|$ for all $i \in \mathbb{Z}$.*

The set Σ_T inherits the topology from Σ_n; the shift map $\sigma : \Sigma_T \to \Sigma_T$ is continuous. We prove a semiconjugacy between σ and f, i.e. we prove that $\sigma \circ \pi = \pi \circ f|_{\mathrm{Inv}(N,f)}$. In particular this implies that there exists a symbolic dynamics structure on $\mathrm{Inv}(N, f)$.

The following theorem was proved in [Z3] (see Theorems 5 and 6) for the case $n = 2$. The following is a natural extension to a general number of sets and the proof is exactly the same.

Theorem 2.14. *The projection π is onto, and if $\{x_n\} \in \Sigma_T$ is a periodic sequence, then $\pi^{-1}(\{x_n\})$ contains a periodic point.*

3. $\mathcal{C}^1$ tools. Perturbations around a hyperbolic fixed point

In this section we give a brief account of the method introduced in [GZ2] (see also [WZ]) which we use in Section 5 to prove the existence of homo-heteroclinic orbits.

Consider a two–dimensional function $f(x) = (f_1(x), f_2(x))^T$, where $x = (x_1, x_2)^T$. Assume that, in the neighborhood of 0, the action of the function f is well approximated by multiplication by the diagonal matrix

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{3}$$

with $|\lambda_2| < 1 < |\lambda_1|$.

More precisely, assume that there exists a set $U \ni 0$ such that for $x \in U$ we have $f_1(x) \in \lambda_1 x_1 + \varepsilon_1 x_2$ and $f_2(x) \in \varepsilon_2 x_1 + \lambda_2 x_2$, where λ_1, λ_2, ε_1 and ε_2 are intervals. This can be written formally as

$$f(x) \in Ax \text{ for } x \in U, \tag{4}$$

where A is the interval matrix defined by

$$A = \begin{pmatrix} \lambda_1 & \varepsilon_1 \\ \varepsilon_2 & \lambda_2 \end{pmatrix}.$$

Let

$$\begin{array}{ll} \varepsilon_1' = \sup\{|\varepsilon| : \varepsilon \in \varepsilon_1\}, & \varepsilon_2' = \sup\{|\varepsilon| : \varepsilon \in \varepsilon_2\}, \\ \lambda_1' = \inf\{|\lambda_1| : \lambda_1 \in \lambda_1\}, & \lambda_2' = \sup\{|\lambda_2| : \lambda_2 \in \lambda_2\}. \end{array} \tag{5}$$

For the nondiagonal elements of the matrix A we have the following estimates: $\varepsilon_i \subset [-\varepsilon_i', \varepsilon_i']$. λ_1' and λ_2' are the worst case values if we consider the system as a perturbation of a linear hyperbolic system with diagonal Jacobian matrix. In what follows we assume that the perturbation of the system does not destroy the hyperbolicity, i.e. $\lambda_1' > 1$ and $\lambda_2' < 1$.

Define the rectangle N_{α_1,α_2} by

$$N_{\alpha_1,\alpha_2} = [-\alpha_1, \alpha_1] \times [-\alpha_2, \alpha_2].$$

The first result of this section is summarized in the following theorem (see [GZ2] for the proof).

Theorem 3.1. *Let U, f and A are as above. Assume that*

$$\varepsilon_1'\varepsilon_2' < (1 - \lambda_2')(\lambda_1' - 1). \tag{6}$$

Then for each pair (α_1, α_2) such that the following conditions are satisfied

$$\frac{\varepsilon_1'}{\lambda_1' - 1} < \frac{\alpha_1}{\alpha_2} < \frac{1 - \lambda_2'}{\epsilon_2'}, \tag{7}$$

$$N_{\alpha_1,\alpha_2} \subset U, \tag{8}$$

the following statements hold

1. *if $f^k(x) \in N_{\alpha_1,\alpha_2}$ and $k \geq 0$, then $\lim_{k\to\infty} f^k(x) = 0$*
2. *if $y_k \in N_{\alpha_1,\alpha_2}$ and $f(y_{k-1}) = y_k$ for $k \leq 0$, then $\lim_{k\to-\infty} y_k = 0$.*

From assumption (6) it follows easily that there exists (α_1, α_2) such that the conditions (7,8) are satisfied. The wider the interval $[\varepsilon_2'/(1-\lambda_2'), (\lambda_1'-1)/\varepsilon_1']$ is the more freedom we have in choosing the ratio α_1/α_2. In this context the value of $(\lambda_1'-1)(1-\lambda_2')$ can be understood as the measure of the hyperbolicity of the considered system: the larger this value is, the more "hyperbolic" the system is and we can accept larger errors coming from non–diagonal elements.

If f is invertible, then the second statement of Theorem 3.1 becomes:

2. if $f^k(x) \in N_{\alpha_1,\alpha_2}$ for $k \leq 0$, then $\lim_{k\to-\infty} f^k(x) = 0$.

Now we consider a general case of an arbitrary hyperbolic fixed point of a 2-dimensional map. We reduce this case to the previous one by introducing a coordinate change transforming the Jacobian matrix of the map at the fixed point to the diagonal form (3).

Definition 3.2. *Let x_* be a fixed point for the map f. We say that f is* hyperbolic *on $N \ni x_*$, if there exists a coordinate system around x_* such that, in this coordinate system $x_* = 0$, there exists a set U in which the condition (4) holds, and $N = N_{\alpha_1,\alpha_2} \subset U$, where α_1, α_2 are such that all assumptions of the Theorem 3.1 are satisfied.*

The theorem below, which follows directly from the Theorems 2.9 and 3.1, shows how we can combine $\mathcal{C}^0$ and $\mathcal{C}^1$ tools to prove the existence of asymptotic orbits with prescribed itinerary (see [GZ2]).

Theorem 3.3. *Assume that g is hyperbolic on N_m and f hyperbolic on N_0. Let $x_g \in N_m$ be a fixed point for g and $x_f \in N_0$ be a fixed point for f. If*

$$N_0 \overset{f}{\Longleftrightarrow} N_0 \overset{f_0}{\Longleftrightarrow} N_1 \overset{f_1}{\Longleftrightarrow} N_2 \overset{f_2}{\Longleftrightarrow} \dots \overset{f_{m-1}}{\Longleftrightarrow} N_m \overset{g}{\Longleftrightarrow} N_m \tag{9}$$

then there exists a sequence $(x_k)_{k=-\infty}^0$, $f(x_k) = x_{k+1}$ for $k < 0$ such that

$$\begin{gathered} x_k \in N_0, \quad k \leq 0, \\ f_{i-1} \circ f_{i-2} \circ \cdots \circ f_0(x_0) \in N_i \text{ for } i = 1, \dots, m, \\ g^n \circ f_{m-1} \circ \cdots \circ f_0(x_0) \in N_m \text{ for } n > 0, \\ \lim_{k\to-\infty} x_k = x_f, \\ \lim_{k\to\infty} g^k \circ f_{m-1} \circ \cdots \circ f_0(x_0) = x_g. \end{gathered}$$

The above theorem can be used without any modifications for proving the existence of trajectories converging to periodic orbits. All we need to do is to consider higher iterates of maps f and g.

4. The Poincaré map and the symmetries

The potential function $V(x, y) = \frac{x^2+y^2}{2} + x^2y - \frac{y^3}{3}$ in the Hénon-Heiles Hamiltonian possesses the symmetry group of the equilateral triangle. It contains the following isometries: rotations by an angle $\pm 2\pi/3$, denoted by $R_\pm$ and three reflections with respect to lines $L = \{y = 0\}$, $R_\pm L$, denoted respectively by I_L, $I_\pm$.

It is easy to see that any of the linear symmetries M of $V(x, y)$ induces a a linear symmetry of Hénon-Heiles system, $((x, y), (p_x, p_y)) \mapsto (M(x, y), M(p_x, p_y))$. In the sequel we will use the same notation for the symmetries of V and of the full system.

In order to study the system at some fixed energy h we consider the Poincaré return map $P : D(P) \subset \mathbb{R}^2 \to \mathbb{R}^2$ defined in the following way. Given (y, p_y) there exists a unique positive value of $p_x = p_x(y, p_y) = \sqrt{2h - y^2 - p_y^2 + \frac{2}{3}y^3}$ such that $H(0, p_x, y, p_y) = h$. $P(y, p_y)$ is defined as the projection on the (y, p_y) plane of the first intersection of the hyperplane $x = 0$ of the trajectory of the system starting at $(0, p_x, y, p_y)$. In other words, P is the the projection on the (y, p_y) plane of the return map on the section $x = 0$, $p_x > 0$. We call Θ_1 this Poincaré section. For $0 < h \leq 1/6$ we define $D_1 \subset \Theta_1$ as an open set of points inside the curve $2h = p_y^2 + y^2 - \frac{2}{3}y^3$. If $h < 1/6$ we set $\Xi_1 = D_1$ and for $h = 1/6$ $\Xi_1 = D_1 \setminus \left\{\left(0, \pm\frac{1}{2\sqrt{3}}\right)\right\}$. It was shown in [AZ, Lemma 2 and Remark 1] that the return map is well defined on Ξ_1. Let Ξ_2 and Ξ_3 be the symmetric images of Ξ_1 with respect to R_- and R_+. Let $P_i : \Xi_i \to \Xi_i$ be the Poincaré maps and let $\tilde{P}_i : \Xi_i \to \Xi_{i+1}$ be the map defined by the flow (for simplicity let $\Xi_4 \equiv \Xi_1$). In above notation $P = P_1$.

Definition 4.1. *Let $S_{ij} : \Theta_i \to \Theta_j$ be defined by $S_{ij}(x, y, p_x, p_y) = R_-(x, y, p_x, p_y)$ if $(i, j) = (1, 2)$, $(i, j) = (2, 3)$ or $(i, j) = (3, 1)$, $S_{ij}(x, y, p_x, p_y) = R_+(x, y, p_x, p_y)$ if $(i, j) = (2, 1)$, $(i, j) = (3, 2)$ or $(i, j) = (1, 3)$ and $S_{ii} = identity$.*

5. Homoclinic and heteroclinic solutions

In this section we prove that the Hénon-Heiles system admits infinitely many solutions homoclinic or heteroclinic to periodic orbits.

The following lemma was proved with computer assistance in [AZ] (the definition of the t-sets are given in Section 4.2 in [AZ] except for the set M_4, see Remark 5.3). The coverings are intended with respect to the Poincaré map P_1, unless otherwise stated.

Lemma 5.1. *Let $h = \frac{1}{6}$. The following covering relations hold:*

$$M_0 \Longrightarrow M_0 \Longrightarrow M_1 \Longrightarrow M_2 \Longrightarrow M_3 \Longrightarrow M_4 \Longrightarrow \begin{matrix} M_0 \\ M_1 \end{matrix}, \tag{10}$$

$$N_0 \Longrightarrow N_0 \Longrightarrow N_1 \Longrightarrow N_2 \Longrightarrow S_{12}(N_3), \tag{11}$$

$$N_3 \Longrightarrow \begin{matrix} N_0 \\ N_1 \end{matrix}, \tag{12}$$

$$M_0 \Longrightarrow N_0 \text{ and } N_0 \overset{id}{\Longrightarrow} M_0. \tag{13}$$

We extend the previous result as follows:

Lemma 5.2. *The covering relations listed above hold for all* $h \in \left[\frac{1}{6} - \frac{1}{1000}, \frac{1}{6}\right]$.

Remark 5.3. *In order to have all covering relations to hold, we had to slightly enlarge the set* M_4*: more precisely we changed the lines of parameters* $(-.827, .068)$ *and* $(-.827, .03)$ *with the lines of parameters* $(-.827, .08)$ *and* $(-.827, .02)$.

Proof. The proof is by computer assistance. In order to obtain an explicit range for h we set its value as an interval variable and proceeded as in [AZ]. □

Let $y_0 = -.249776$. Then $(y_0, 0)$ is the approximate position of a fixed point. More precisely, let $\delta = .000011$:

Lemma 5.4. *For all* $h \in \left[\frac{1}{6} - 10^{-5}, \frac{1}{6}\right]$ *the Poincaré map* P_1 *admits a unique fixed point* $(\tilde{y}, 0)$ *in the interval set* $I = [y_0 - \delta, y_0 + \delta] \times \{0\}$.

Proof. Let P_{11} and P_{12} be the components of the Poincaré map. We check by computer assistance that $P_{12}(y_0 - \delta, 0) < 0$, $P_{12}(y_0 + \delta, 0) > 0$, $P_{11}(y_0, 0) \in [y_0 - \delta, y_0 + \delta]$ and $\frac{\partial P_{12}(y)}{\partial y} > 0$ for all $y \in [y_0 - \delta, y_0 + \delta]$. This implies that there exists a unique point $\tilde{y}$ such that $P_{11}(\tilde{y}, 0) = \tilde{y}$ and $P_{12}(\tilde{y}, 0) = 0$; the result follows by Lemma 6.11. □

Lemma 5.5. *Let* $U = [y_0 - .0001, y_0 + .0001] \times [-.0002, .0002]$ *and let*

$$M = \begin{pmatrix} [3.49553, 3.89926] & [-1.94232, -1.70826] \\ [-7.3115, -6.57506] & [3.48093, 3.9135] \end{pmatrix}.$$

Then for all $(y, y_p) \in U$ *and for all* $h \in \left[\frac{1}{6} - 10^{-5}, \frac{1}{6}\right]$ *we have* $dP(x) \in M$.

Proof. By computer assistance. □

A simple computation yields the following:

Lemma 5.6. *Consider the linear coordinate change given by the matrix*

$$Q = \begin{pmatrix} .454 & -.454 \\ -.89 & -.89 \end{pmatrix}; \tag{14}$$

in the new coordinates the matrix M *reads*

$$A = \begin{pmatrix} \lambda_1 & \varepsilon_1 \\ \varepsilon_2 & \lambda_2 \end{pmatrix} = \begin{pmatrix} [6.83963, 7.67504] & [-0.399616, 0.435792] \\ [-0.435972, 0.399436] & [-0.28043, 0.554978] \end{pmatrix}.$$

Let ε_1',ε_2',λ_1' *and* λ_2' *be as in (5). Then* $\frac{\varepsilon_1}{\lambda_1 - 1} < 0.0747$ *and* $\frac{1-\lambda_2}{\varepsilon_2} > 1.02$.

Let $\alpha_1 = .00005$ and $\alpha_2 = .00013$; the pair (α_1, α_2) satisfies (7). Let $N_{\alpha_1,\alpha_2} = [-\alpha_1, \alpha_1] \times [-\alpha_2, \alpha_2]$. For all $y \in [y_0 - .000011, y_0 + .000011]$ we define a t-set R_y as follows: $|R_y| = \{z = (y, 0) + Qx | x \in N_{\alpha_1,\alpha_2}\}$ where Q is given in (14). The right and left sides are defined as for the sets $N_4 - N_7$ (see Definition 2.1). A direct check shows that $R_y \subset U$ for all $y \in [y_0 - \delta, y_0 + \delta]$ (see Figure 1).

Lemma 5.7. *There exists t-sets* N_4, N_5, N_6 *and* N_7 *such that the following covering relations hold:*

$$N_0 \Longleftrightarrow N_4 \Longleftrightarrow N_5 \Longleftrightarrow N_6 \Longleftrightarrow N_7 \Longleftrightarrow R_{\tilde{y}} \Longleftrightarrow R_{\tilde{y}}$$

for all $h \in \left[\frac{1}{6} - 10^{-5}, \frac{1}{6}\right]$. *Furthermore all the inverse covering relations also hold. All coverings are* P *coverings, except for* $N_7 \Longleftrightarrow R_{\tilde{y}}$*, which is by the identity map.*

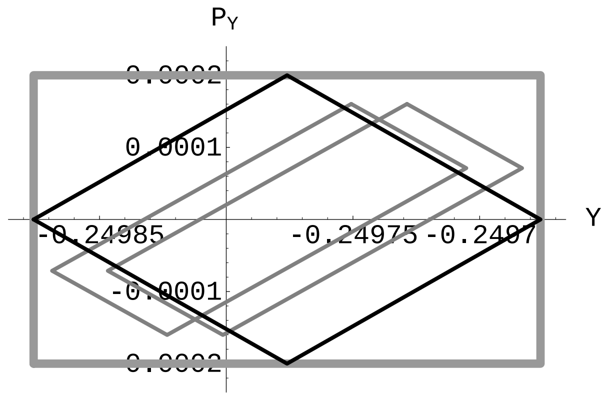

FIGURE 1. The sets U (large grey rectangle), N_7 (black parallelogram), $R_{y-\delta}$ and $R_{y+\delta}$ (grey parallelograms)

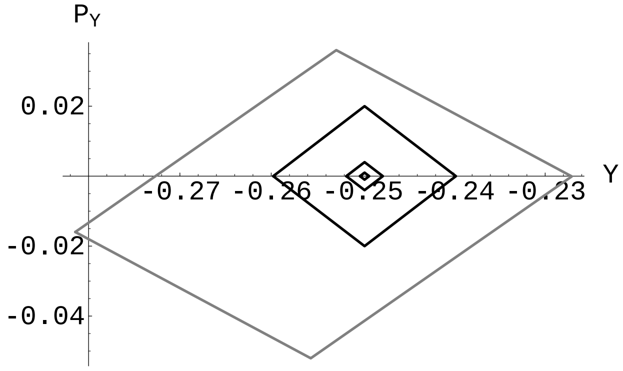

FIGURE 2. The sets N_0 (grey) and $N_4 - N_6$ (black). The set N_7 is too small to be displayed.

Proof. The definition of the t-sets is in Section 8, see also Figures 1 and 2. The proof of the coverings $N_0 \Longrightarrow N_4$, $N_4 \Longrightarrow N_0$, $N_4 \Longrightarrow N_5$, $N_5 \Longrightarrow N_6$, $N_6 \Longrightarrow N_7$ and $R_y \Longrightarrow N_7$ for all $y \in [y_0 - \delta, y_0 + \delta]$ (and therefore for $y = \tilde{y}$) is by computer assistance. The coverings $N_5 \Longleftarrow N_4$, $N_6 \Longleftarrow N_5$ and $N_7 \Longleftarrow N_6$ follow by the fact that the sets N_4, N_5, N_6 and N_7 are symmetric with respect to S. The proof of $N_7 \overset{id}{\Longleftrightarrow} R_{\tilde{y}}$ for all $y \in [y_0 - \delta, y_0 + \delta]$ is by inspection. $R_{\tilde{y}} \Longleftrightarrow R_{\tilde{y}}$ follows from $R_{\tilde{y}} \Longrightarrow N_7$ and $N_7 \overset{id}{\Longleftrightarrow} R_{\tilde{y}}$. □

Note that the range of the energy allowed in Lemmas 5.4-5.7 is one order of magnitude smaller than that in Lemma 5.2; this is not a failure of the numerical procedure, but it is due to the fact that the value of y for the fixed point does change when the energy changes, therefore in order to bound it in an interval of size $2\delta = .000022$ it is only possible to have a small variation of the energy. The same goes for Lemma 5.7, since the sets N_7 and R_y are much smaller than the sets N_0 and N_4.

Let T be a transition matrix induced by the chains of covering relations from Lemma 5.1 and their symmetric images by S_{ij} as in [AZ]. The above lemmas and Theorem 3.3 yield the existence of infinitely many orbits homoclinic or heteroclinic to periodic orbits. More precisely:

Theorem 5.8. *For all $h \in \left[\frac{1}{6} - 10^{-5}, \frac{1}{6}\right]$ and for all sequences $\{x_k\} \in \Sigma_T$ which are asymptotically constant, there exists an orbit $\{y_k\}$ of the Poincaré map which satisfies $\pi(y_k) = x_k$ for all k, $y_k \to y_1$ as $k \to -\infty$ and $y_k \to y_2$ as $k \to +\infty$, where $y_{1,2} \in \{(\tilde{y}, 0), S_{12}((\tilde{y}, 0)), S_{21}((\tilde{y}, 0))\}$.*

Proof. Let $\tilde{T}$ be a transition matrix induced by the chains of covering relations from Lemmas 5.1 and 5.7 and their symmetric images by S_{ij}. By Lemma 5.7 there exists an admissible sequence $\{\tilde{x}_k\} \in \Sigma_{\tilde{T}}$ such that $\tilde{x}_k = R_{\tilde{y}}$, $\tilde{x}_k = S_{12}(R_{\tilde{y}})$ or $\tilde{x}_k = S_{21}(R_{\tilde{y}})$ for large k, where $(\tilde{y}, 0)$ is a fixed point. The set $R_{\tilde{y}}$ correspond to the set N_{α_1,α_2} in Definition 3.2 and the definition applies due to Lemma 5.6, therefore $(\tilde{y}, 0)$ is hyperbolic. The conclusion follows by Theorem 3.3. □

Corollary 5.9. *For all $h \in \left[\frac{1}{6} - 10^{-5}, \frac{1}{6}\right]$ the system admits infinitely many solutions homoclinic or heteroclinic to periodic solutions.*

6. Symmetric periodic points

6.1. **General definitions.** The concept of symmetries in dynamical systems has been extensively covered in the literature, see [La, La1] and references given there. Here we follow [La1] with a few unessential modifications.

Definition 6.1. *Let Ω be a manifold. Let $\varphi : \mathbb{R} \times \Omega \to \Omega$ be a dynamical system induced by*

$$\frac{d}{dt}x = F(x). \tag{15}$$

A diffeomorphism $M : \Omega \to \Omega$ is called a symmetry *of φ (or of F) if for all $x \in \Omega$*

$$\frac{d}{dt}M\varphi(t,x)_{|t=0} = F \circ M(x). \tag{16}$$

Observe that condition (16) is equivalent with the following condition

$$dM(x)F(x) = F(M(x)), \qquad \forall x \in \Omega,$$

where $dM(x)$ is the derivative of M at x.

It is easy to see that a symmetry transforms trajectories into trajectories, namely we have

$$M(\varphi(t,x)) = \varphi(t, M(x)) \text{ for all } t \text{ and } x$$

For the discrete time case we have the following

Definition 6.2. *An invertible transformation $M : \Omega \longrightarrow \Omega$ is called a* symmetry *of the discrete dynamical system induced by $f : \Omega \longrightarrow \Omega$ if*

$$M \circ f = f \circ M.$$

As in continuous case, it is easy to see that a symmetry of a discrete dynamical system transforms trajectories into trajectories.

Alternatively, a dynamical system may admit transformations $S : \Omega \to \Omega$ that map trajectories into other trajectories reversing the time direction. Such transformations are called *reversing symmetries.* More precisely, we have the following definitions [La1]:

Definition 6.3. *Let φ be a continuous dynamical system induced by (15).*

A diffeomorphism $S : \Omega \to \Omega$ is called a reversing symmetry *of φ (or of F) if for all $x \in \Omega$*

$$\frac{d}{dt} S\varphi(t,x)_{|t=0} = -F \circ S(x). \tag{17}$$

It is easy to see that condition (17) is equivalent to the following

$$dS(x)F(x) = -F(S(x)) \text{ for all } x \in \Omega.$$

Moreover, we have

$$S(\varphi(t,x)) = \varphi(-t, S(x)).$$

Definition 6.4. *An invertible transformation $S : \Omega \longrightarrow \Omega$ is called a reversing symmetry for the discrete dynamical system induced by $f : \Omega \longrightarrow \Omega$ if*

$$S \circ f = f^{-1} \circ S,$$

or equivalently $S^{-1} \circ f \circ S \circ f = Id|_\Omega$.

Observe that

- The composition of two (reversing) symmetries is a symmetry.
- The composition of a reversing symmetry and a symmetry is a reversing symmetry.
- The inverse of a (reversing) symmetry is a (reversing) symmetry.

From the above observations it follows that E, the set of symmetries and reversing symmetries, forms a group called a *reversing symmetry group*.

Definition 6.5. *Let $\varphi : \mathbb{T} \times \Omega \to \Omega$ (where $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$) be a dynamical system. For each point $x_0 \in \Omega$, we define its orbit $\Gamma(x_0)$ by*

$$\Gamma(x_0) = \{x \mid x = \varphi(t, x_0) \textit{ for some } t \in \mathbb{T}\}$$

To describe the symmetry properties of orbits, following [GSS, La1] we introduce the notion of isotropy subgroup.

Definition 6.6. *Consider a dynamical system on a phase space Ω with the reversing symmetry group E. The isotropy subgroup Σ_{x_0} of a point $x_0 \in \Omega$ is defined as*

$$\Sigma_{x_0} = \{U \in E \mid U(x_0) = x_0\}.$$

Similarly, the isotropy subgroup of the orbit $\Gamma(x_0)$ is defined as

$$\Sigma_{\Gamma(x_0)} = \{U \in E \mid U(\Gamma(x_0)) = \Gamma(x_0)\}.$$

If $U \in \Sigma_{x_0}$ ($U \in \Sigma_{\Gamma(x_0)}$) then we say that the point x_0 (the orbit $\Gamma(x_0)$) is U-symmetric. If U is clear from the context, then we will often drop it and speak of a symmetric point (orbit).

6.2. **Symmetries and reversing symmetries in the Hénon-Heiles system.** Let us recall from Section 4 that the Hénon-Heiles system possesses the symmetries $R_\pm$, I_L and $I_\pm$. It is also invariant with respect to the time reversal. This means that the map T, given by $T(x, y, p_x, p_y) = (x, y, -p_x, -p_y)$, is a reversing symmetry of this system.

Let E' be the subgroup of the reversing symmetry group for the Hénon-Heiles system generated by the reversing symmetry T and the above mentioned symmetries.

Definition 6.7. *Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be the map defined by $S(x, y) = (x, -y)$.*

The following result, which is a consequence of the composition of time reversal and I_L symmetries for the Hénon-Heiles system, was proved in [AZ, Lemma 1].

Lemma 6.8. *$P(y_1, p_1) = (y_2, p_2)$ if and only if $PS(y_2, p_2) = S(y_1, p_1)$.*

Remark 6.9. *Lemma 6.8 can be rephrased by saying that S is a reversing symmetry for P.*

We have the following easy

Lemma 6.10. *Let $x_0 \in \Xi_1$ be such that its P-orbit is S-symmetric; then the continuous orbit for the Hénon-Heiles system of this point is $T \circ I_L$-symmetric. In particular its projection onto the (x, y)-plane is symmetric with respect to the reflection $(x, y) \mapsto (-x, y)$.*

A standard method for studying the symmetric periodic orbits for a reversing symmetry is the Fixed Set Iteration (FSI) method [LQ, La1] (also known as De-Vogelaere method [DV]). This method consist of finding the intersections of the iterates of the fixed sets of the reversing symmetries S and $P \circ S$ (see [La1, Prop. 1.2.2] for more details). For our purposes and context we will formulate it in form of two lemmas

Lemma 6.11. *If $P(y_1, 0) = (y_2, 0)$, then $P(y_2, 0) = (y_1, 0)$ and therefore $(y_1, 0)$ is either a fixed point (if $y_1 = y_2$) or a period 2 point (if $y_1 \neq y_2$) for the Poincaré map.*

Proof. It is a direct consequence of the reversing symmetry S. $\square$

Lemma 6.12. *If $P^k(y_1, 0) = (y_2, 0)$, $y_1 \neq y_2$, then $P^{2k}(y_1, 0) = (y_1, 0)$, i.e. $(y_1, 0)$ is periodic of period $2k$ for the Poincaré map.*

The method in [CPR] used to obtain $T \circ I_L$-symmetric orbits Π_i for $i = 4, 5, 6$ for Hénon-Heiles system can be also seen as a variant of FSI method, as the authors of that paper are looking for solutions of $P_{1/2}(y, 0) = (y, 0)$, where $P_{1/2}$ is a return map to Θ_1 with negative p_x.

6.3. **Reversing symmetry and covering relations - Wilczak's method.** Lemma 6.12 shows that to obtain a symmetric periodic orbit of period k, we have to examine at least the $k/2$-iterate of P, which is an apparent obstacle for obtaining infinite number of S-symmetric periodic orbits with unbounded periods in a finite computation. Fortunately, it turns out that, by linking covering relations with reversing symmetries using the method developed recently by Wilczak [W], we are able to prove the existence of infinitely many symmetric periodic orbits. We provide here a short description of the method; for details see [W].

Definition 6.13. *Let N be a triple set and $\gamma : [a, b] \longrightarrow \mathbb{R}^2$ be a continuous curve. We call γ a* horizontal curve *in N if the following conditions hold:*

(1) $\gamma((a, b)) \subset \mathrm{int}(|N|)$
(2) *either* $\gamma(a) \in N^{le}$ *and* $\gamma(b) \in N^{re}$
or $\gamma(a) \in N^{re}$ *and* $\gamma(b) \in N^{le}$

In [W] the following lemmas were proved. We point out that they also hold (the proof being by inspection) when the generic covering is considered.

Lemma 6.14. *Let $M_0, M_1, \dots, M_n$ be triple sets, let $f_i : M_i \longrightarrow \mathbb{R}^2$ be continuous maps such that*

$$M_0 \overset{f_0}{\Longrightarrow} M_1 \overset{f_1}{\Longrightarrow} M_2 \cdots \overset{f_{n-2}}{\Longrightarrow} M_{n-1} \overset{f_{n-1}}{\Longrightarrow} M_n.$$

If $\gamma : [a,b] \longrightarrow \mathbb{R}^2$ is a horizontal curve in M_0, then there exists real numbers t_*, t^* such that the composition $(f_{n-1} \circ f_{n-2} \circ \cdots \circ f_0) \circ (\gamma|_{[t_*,t^*]})$ is well defined. Moreover

$$(f_{m-1} \circ f_{m-2} \circ \cdots \circ f_0 \circ \gamma)|_{[t_*,t^*]} \subset \mathrm{int}|M_m| \quad m = 1, \dots, n-1 \tag{18}$$

$$(f_{n-1} \circ f_{n-2} \circ \cdots \circ f_0 \circ \gamma)|_{[t_*,t^*]} \text{ is a horizontal curve in } M_n \tag{19}$$

The following theorem summarizes Wilczak's method

Theorem 6.15. *Let S be a reversing symmetry for the dynamical system induced by the map P. Assume that*

$$M_0 \overset{P}{\Longrightarrow} M_1 \overset{P}{\Longrightarrow} M_2 \cdots \overset{P}{\Longrightarrow} M_{n-1} \overset{P}{\Longrightarrow} M_n,$$

and

1. *the fixed point set for S in M_0 is a horizontal curve in M_0*
2. *every horizontal curve in M_n intersects the fixed point set for S*

Then there exists $x \in M_0$, such that

$$S(x) = x,\ P^n(x) = x$$
$$\text{the orbit of } x \text{ is } S\text{-symmetric}$$
$$P^i(x) \in M_i \text{ for } i = 1, \dots, n,$$
$$P^{n+i}(x) \in S(M_{n-i}) \text{ for } i = 1, \dots, n.$$

Definition 6.16. *Let M be a t-set. We define its symmetric image with respect to the x-axis $\tilde{M}$ as follows: $|\tilde{M}| = S(|M|)$, $\tilde{M}^l = S(M^t)$, $\tilde{M}^r = S(M^b)$, and the remaining definitions follow as in Definition 2.1.*

Remark 6.17. *Conditions 1. and 2. of Theorem 6.15 are satisfied if M_0 and M_n are symmetric.*

Theorem 6.18. *For all $h \in [\frac{1}{6} - 10^{-5}, \frac{1}{6}]$ the Hénon-Heiles system has infinitely many periodic solutions symmetric with respect to $T \circ I_L$. In particular the projections of all these orbits onto the (x,y)-plane is symmetric with respect to the reflection $(x,y) \mapsto (-x,y)$.*

Proof. By Lemmas 5.7 we have

$$N_4 \Longleftrightarrow N_0 \text{ and } N_0 \Longleftrightarrow N_4.$$

By 5.1 we have infinitely many chains of coverings linking N_0 with itself, therefore the results follows by Theorem 6.15. Assumptions (1) and (2) are satisfied because N_4 is symmetric with respect to the action of S. The result now follows from Lemma 6.10 and Theorem 6.15. □

By applying $R_\pm$ to the symmetric periodic solutions obtained in above theorem, we obtain the rotated families of $T \circ I_\pm$-symmetric periodic solutions. Figure 3 displays an example of an orbit symmetric with respect to $T \circ I_L$.

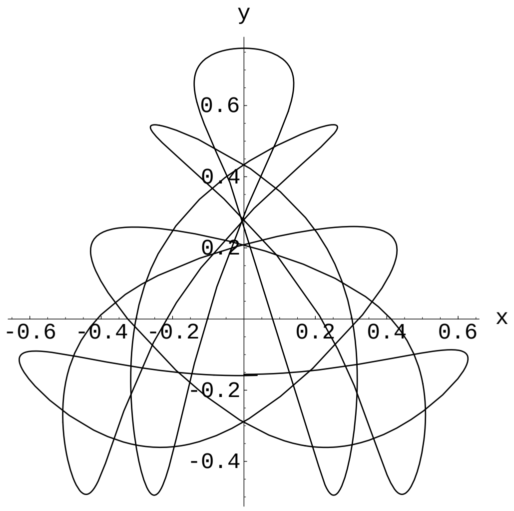

FIGURE 3. An example of an orbit symmetric with respect to $T \circ I_L$.

6.4. **Comparison to the Devaney's method.** An approach similar in the spirit to the Wilczak method was introduced by Devaney in [D] in 1984. The point of departure of Devaney's method is a symmetric fixed (periodic) point, say p. Assume we have transversal intersection of an unstable (stable) manifold of p with the fixed point set for the symmetry. From this Devaney infers the existence of an infinite number of symmetric periodic points with unbounded periods and under some additional geometric condition one also obtains a symmetric horseshoe. Hence the Devaney method is very much in the spirit of the classical dynamical system theory and when applied yields definitive results : the hyperbolic cantor-like invariant set, with the dynamics equivalent to that of the Bernouli shift; with the full symbolic characterization of symmetric and non-symmetric orbits.

Obviously Wilczak's method does not provide such definitive answers, but its assumptions are by far much easier to verify rigorously.

6.5. **Periodic orbits symmetric with respect to $\{R_\pm, I_L, I_\pm\}$.** We recall that the Hamiltonian is also symmetric with respect to a $\frac{2\pi}{3}$ rotation.

To prove the existence of solutions symmetric with respect to $\{R_\pm,\ I_L,\ I_\pm\}$ we combine ideas from [CPR] which are based on symmetry with covering relations as in Wilczak's method. See Figure 4 for an example of an orbit symmetric with respect to $\{R_\pm,\ I_L,\ I_\pm\}$.

Theorem 6.19. *For all $h \in [\frac{1}{6} - 10^{-5}, \frac{1}{6}]$ the Hénon-Heiles system has infinitely many periodic solutions, such that their isotropy group contains $\{R_\pm, I_L, I_\pm\}$, i.e. they have as symmetry group of equilateral triangle.*

Proof. By Lemmas 5.1 and 5.7 we have

$$N_4 \Longleftrightarrow N_0 \Longrightarrow N_1 \Longrightarrow N_2 \Longrightarrow S_{12}(N_3) \Longrightarrow S_{12}(N_0) \Longrightarrow S_{12}(N_4). \tag{20}$$

Observe that, by Lemma 5.1, in the above chain we can replace N_0 with infinitely many different chains which start and end with N_0. As a result we obtain an infinite

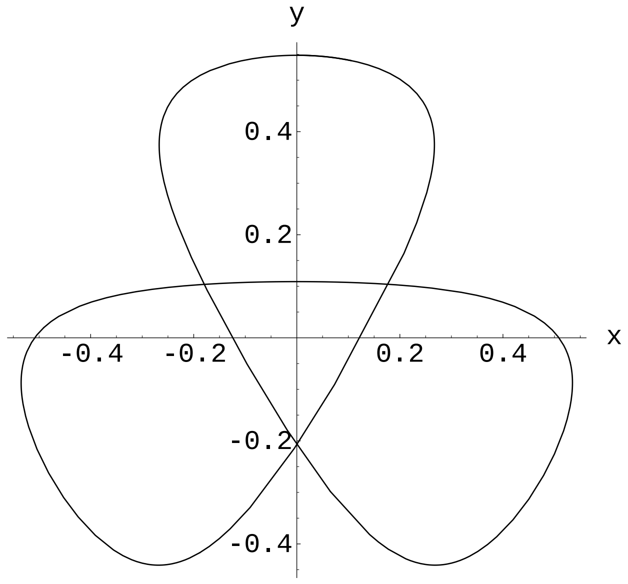

FIGURE 4. An example of an orbit symmetric with respect to $\{R_{\pm}$, I_L, $I_{\pm}\}$.

number of chains of the form

$$N_4 \Longleftrightarrow N_0 \Longleftrightarrow \dots \Longleftrightarrow N_0 \Longleftrightarrow \dots \Longleftrightarrow S_{12}(N_4) \tag{21}$$

Let us remind the reader that the set $S_{12}(N_4)$ is on $\Theta_2 = R_-\Theta_1$. Coordinates transported by R_- from a neighborhood of Θ_1 to a neighborhood of Θ_2 rise a coordinate system in which the line L_- can be written as $y = 0$. Moreover in these coordinates the definitions of N_4 and $S_{12}(N_4)$ are identical and in both cases $p_x > 0$ on N_4 and $S_{12}(N_4)$ (in the new coordinates). This means that if we look at the (x, y) projection of the trajectory realizing the chain in (21), then at the start and at end the trajectory intersects L and L_-, respectively, in counterclockwise direction.

Let C be a map obtained as the composition of all the maps in the chain of coverings in (21). Let $q_0 \in \Theta_1$ be a point realizing the chain in (21) for $0 \leq t \leq t_1$. Then the fact that the projection of the trajectory of q_0 onto the (x, y)-plane is at $t = 0$ perpendicular to L and at $t = t_1$ perpendicular to L_- can be expressed as follows:

$$q_0 = (0, y, p_x(x = 0, y, p_y = 0), 0), \qquad C(y, 0) = (y_-, 0), \tag{22}$$

where on Θ_2 we use the section coordinates transported by R_- from Θ_1. An intersection of the line $p_y = 0$ with N_4 is a horizontal line in N_4. Observe also that every horizontal line in $S_{12}(N_4)$ has a common point with the line $p_y = 0$ (we are using here transported coordinates). Hence by Lemma 6.14 there exists a solution of the equation $C(y, 0) = (y_-, 0)$. Let us fix one such solution $y = y^*$ and $y_- = y^*_-$.

Let $\pi_{(x,y)}$ and $\pi_{(p_x,p_y)}$ denote the projections onto the (x, y)-plane and the (p_x, p_y)-plane, respectively. We proved that there exist a point $q^*_0 = (0, y^*, p^*_x(0, y^*, 0), 0)$

and $t^* > 0$, such that

$$\pi_{(x,y)}\varphi(t^*, q_0^*) \in L_-, \qquad \pi_{(p_x,p_y)}\varphi(t^*, q_0^*) : \perp : L_-$$

and for $t \in [0, t^*]$ $\varphi(t, q_0^*)$ realizes the dynamics described by (21). Let $q_1^* = \varphi(t^*, q_0^*)$.

Now we apply the composition of time reversal and reflection with respect to L_- to the trajectory of q_0^* for $t \in [0, t^*]$ to obtain

$$\varphi(t, q_1^*) = T \circ I_- \varphi(t^* - t, q_0^*)$$

Observe that

$$\begin{aligned} q_2^* = \varphi(t^*, q_1^*) &= \varphi(2t^*, q_0^*) = R_+ q_0^* \in L_+ \\ \pi_{(p_x,p_y)}\varphi(t^*, q_1^*) &\perp L_+. \end{aligned}$$

Now we can reflect the trajectory joining q_1^* and q_2^* to obtain (we apply again the composition of time reversal and reflection with respect to L_+)

$$\begin{aligned} q_3^* = \varphi(t^*, q_2^*) &= \varphi(3t^*, q_0^*) \in L \\ \pi_{p_x,p_y}\varphi(t^*, q_2^*) &\perp L. \end{aligned}$$

Now we have two cases, if $q_3^* = q_0^*$, then we have already found a symmetric periodic orbit. If this is not the case, then by application of the composition of time reversal and the reflection with respect to L we can join q_3^* with q_0^* to obtain an periodic orbit of period $6t^*$. □

From the proof it is clear that we constructed infinitely many symmetric periodic solutions, which pass through the set N_4 (hence in a neighborhood of Π_4 from [CPR]). Moreover, counting the covering relations in (20) it turns out that the shortest period (for the Poincaré map P) is at least 15. This should be compared with the period 1 orbit found in [CPR] (denoted there by Π_7). From any symmetric orbit from Theorem 6.19 just as in [CPR] we obtain another one by applying a time reversal symmetry. As a result we obtain an orbit with the same $\pi_{(x,y)}$-projection, traversed in opposite direction.

We point out that the systems also admits infinitely many periodic solutions which are not symmetric, indeed it suffices to consider an admissible sequence $\{x_k\} \in \Sigma_T$ which is not symmetric and by Theorem 2.14 we obtain a non-symmetric solution.

7. $\mathcal{C}^0$ TOOLS FOR ELLIPTIC POINTS

Assume that numerical computations show that the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ has an elliptic fixed point close to $(0, 0)$ and points around the origin are approximately rotated by an angle α. We state a theorem that leads to a rigorous numerical proof of the existence of the fixed point.

Consider polar coordinates in the plane $x = \rho \cos \vartheta$ and $y = \rho \sin \vartheta$. Let $\vartheta : \mathbb{R}^2 \setminus (0, 0) \to \mathbb{R}$ and $\rho : \mathbb{R}^2 \to \mathbb{R}$ be the corresponding maps.

Theorem 7.1. *Assume that the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ is continuous and invertible and there exist $0 < \delta < R$, $0 < \varepsilon < \alpha$ such that for all x satisfying $\rho(x) = R$ we have:*

i) $\rho(f(x)) \in (R - \delta, R + \delta)$
ii) $\vartheta(f(x)) - \vartheta(x) \in (\alpha - \varepsilon, \alpha + \varepsilon)$

Then the map f has a fixed point x, satisfying $\rho(x) < R$.

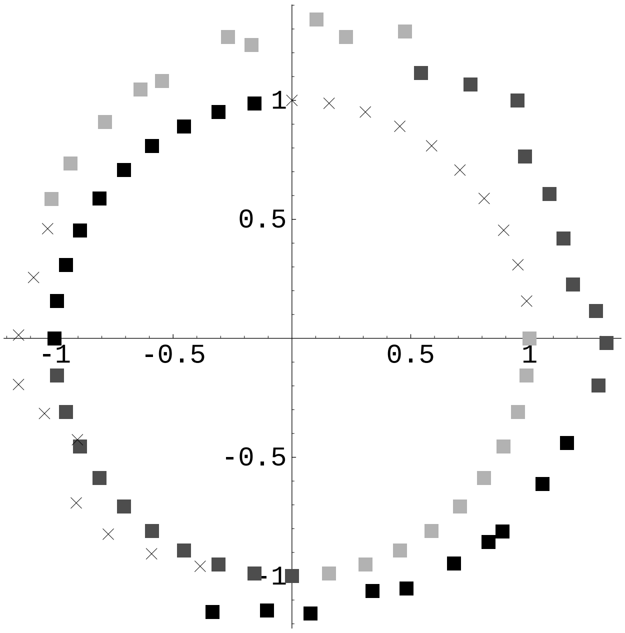

FIGURE 5. An example of a map f satisfying the assumption of Theorem 7.1

Proof. Let $R_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation in the plane around the origin by an angle α. Let $D = \{x \in \mathbb{R}^2 : \rho(x) \le R + \delta\}$ and let $F : D \to D$ be defined by

$$F(x) = \frac{\rho(x) - R}{\delta} R_\alpha(x) + \frac{(R+\delta) - \rho(x)}{\delta} f\left(\frac{xR}{\rho(x)}\right) \text{ if } R \le \rho(x) \le R + \delta$$
$$F(x) = f(x) \text{ if } \rho(x) < R$$

The map F is continuous, hence it has a fixed point x by Brower's theorem. Furthermore $\rho(x) < R$ (because if $\rho(x) \ge R$, then $\vartheta(F(x)) > \vartheta(x)$), therefore $F(x) = f(x) = x$. □

We apply this technique a perturbed Hénon-Heiles Hamiltonian, for which the symmetry argument cannot be used. More precisely, we considered the Hamiltonian

$$H = \frac{p_x^2 + p_y^2}{2} + \frac{x^2 + y^2}{2} + x^2 y - \frac{y^3}{3} - \varepsilon x \tag{23}$$

and the corresponding Hamilton equations:

$$\begin{cases} \dot{p}_x &= -x - 2xy - \varepsilon \\ \dot{p}_y &= -x^2 + y^2 - y \\ \dot{x} &= p_x \\ \dot{y} &= p_y. \end{cases} \tag{24}$$

By applying the method of Lemma 5.4 it is easy to check that for $\varepsilon = 0$ the Poincaré map corresponding to system (24) has a period 2 point close to $(y_1, 0)$ when $h \in \left[\frac{1}{6} - 10^{-4}, \frac{1}{6}\right]$ and $y_1 = .109$. Such point is elliptic (this is apparently the d_a periodic in the classification scheme from [DHB]); more precisely

Lemma 7.2. *For all $h \in \left[\frac{1}{6} - 10^{-4}, \frac{1}{6}\right]$ the Poincaré map P (of the unperturbed Hénon-Heiles system) admits a unique period 2 point $(\tilde{y}, 0)$ in the interval set $I = [y_0 - \delta, y_0 + \delta] \times \{0\}$. For all $x \in I$ the map P_1 satisfies $P_1(x) \in J = ([0.546687, 0.54947], [-0.00173209, 0.00232603])$ and the derivative $dP_1(x) \in M_1$, where*

$$M_1 = \begin{pmatrix} [-0.226766, 0.0659593] & [0.177345, 0.533977] \\ [-2.90765, -2.65279] & [-0.304815, 0.03848] \end{pmatrix}$$

and therefore its eigenvalues are complex conjugate and dP_1 is a rotation matrix. Furthermore, for all $x \in J$ the derivative $dP_1(x) \in M_2$, where

$$M_2 = \begin{pmatrix} [-0.178367, 0.018058] & [0.241765, 0.472579] \\ [-2.88471, -2.6612] & [-0.270883, 0.0190267] \end{pmatrix}$$

and therefore its eigenvalues are complex conjugate and dP_1 is a rotation matrix.

Proof. The proof of the existence of a fixed point is similar to the proof of Lemma 5.4: as for that case, we check by computer assistance that $P_{12}(y_0 - \delta, 0) < 0$, $P_{12}(y_0 + \delta, 0) > 0$ and $\frac{\partial P_{12}(y)}{\partial y} > 0$ for all $y \in [y_0 - \delta, y_0 + \delta]$. The difference here is that $P_{11}(y, 0) > y_0 + \delta$ for all $y \in [y_0 - \delta, y_0 + \delta]$, therefore $(\tilde{y}, 0)$ cannot be a fixed point for P_1, but it is a period 2 point.

The rigorous estimates on the derivatives are also by computer assistance. □

When $\varepsilon \neq 0$ the symmetry of the system is broken, therefore the method of the above lemma does not apply. Furthermore, since the fixed point of P_1^2 is elliptic, i.e. the map P_1^2 is locally a rotation, then the covering method of the previous sections do not apply either. By the method introduced above we prove that the elliptic fixed point persists under the perturbation.

Theorem 7.3. *For all $\varepsilon \in [0, .001]$ and for all $h \in \left[\frac{1}{6} - 10^{-4}, \frac{1}{6}\right]$ the perturbed Hénon-Heiles system (24) has an elliptic fixed point in the circle centered in $(.109, -2\varepsilon)$ of radius $.02$.*

Proof. We partition the ε-interval $[0, .001]$ in 40 equal parts $\varepsilon_i = [.000025(i - 1), .000025i]$, $i = 1, \ldots, 40$. For each such interval we consider a coordinate system such that $(\hat{x}, \hat{y}) = M \cdot (x, y) -$ $(.109, -.00006i)$, where

$$M = \begin{pmatrix} \cos(.15) & -.4\sin(.15) \\ \sin(.15) & .4\cos(.15) \end{pmatrix}.$$

In these coordinates we can apply Theorem 7.1 with $f = P_1^2$, $R = .01$ and $\alpha = \pi$. By computer assistance we prove that the assumption of the theorem hold with $\delta = .3$ and $\varepsilon = \pi/2$, hence the thesis. □

8. Computational details

8.1. **Description of the t-sets.** The procedure used to build the t-sets $N_0 - N_3$ and $M_0 - M_4$ and their definition described in [AZ]. The t-sets $N_4 - N_7$ are described in the following definition; they have been defined in order to provide a chain of coverings between N_0 and R_y in Lemma 5.7. In order to achieve this task in the easiest way we chose to build them symmetric with respect to the action of S and centered at the same point.

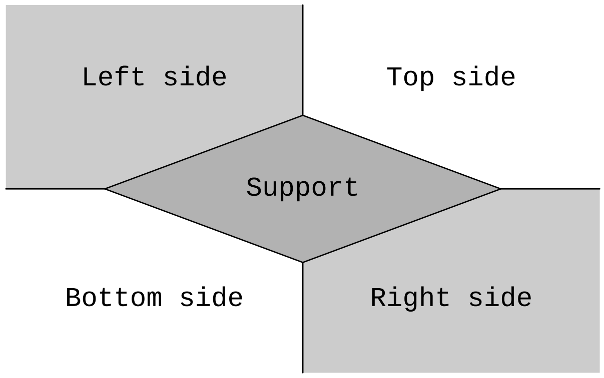

FIGURE 6. The model of the t-sets $N_4 - N_7$.

Definition 8.1. *The supports of the triple sets N_4, N_5, N_6 and N_7 are defined as follows: they are parallelograms with diagonals orthogonal to the axes, centered in $(y_0, 0)$. The length of the vertical diagonal is double the length of the horizontal diagonal, with the latter given as follows: $N_4 : .01$, $N_5 : .002$, $N_6 : .0005$, $N_7 : .0001$. The boundaries of the left and right sides of each t-set are the lines crossing two opposite vertices of the support (see Figure 6).*

8.2. **Computation techniques.** We outline here the algorithm used in the computer assisted proofs. By Lemma 2.4 the proof consists in checking that the images through the Poincaré map of the edges of the t-sets lie in some assigned regions of the plane as described by the covering relations. We do not know the exact images of such edges, since no analytical solution of the equation is available, therefore all we can do is to estimate the trajectories and compute the intersections with the Poincaré plane with rigorous error bounds. In order to compute the image of a side, we partition it in segments with small enough length and we check that every such segment is mapped in the correct region. To compute the image of a small segment, we enclose it in an interval set (a rectangle) and we compute its trajectory with a Lohner algorithm using interval arithmetics. The estimates on the derivatives of the Poincaré map in the Lemmas are done in a similar way: in fact, since the initial data for the computation of the derivatives are single points instead of intervals, it turns out that it is possible to simplify the procedure by using a simple Taylor algorithm. The wrapping effect is kept under control by using a rather large time step and a high order, in fact we used order 18. A detailed exposition of the method can be found in [Z4].

To perform the proofs the authors implemented a version of the whole algorithms in Mathematica 4.0 under the Linux O.S. In order to perform all the proofs a few hours of CPU time were required on a machine equipped with an AMD Athlon K7 1700XP processor. The reader who desires to reproduce the computer assisted proofs in this paper without writing a program can use the Mathematica notebook which can be found in [AZ2]. The notebook is provided with comments and instructions and all information concerning the computer assisted proofs. By using the commands in the notebook it is possible to make an independent check of all computations required in the proofs.

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