Abundance of homoclinic and heteroclinic orbits and rigorous bounds for the topological entropy for the Hénon map

Zbigniew Galias †§ and Piotr Zgliczyński ‡||

† Institute for Nonlinear Science, University of California, San Diego
9500 Gilman Dr., Dept. 0402, San Diego, CA 92093-0402, USA
e-mail: `galias@ucsd.edu`
‡ and
Jagiellonian University, Institute of Mathematics, Reymonta 4, 30-059 Kraków, Poland.
e-mail: `zgliczyn@im.uj.edu.pl`

Abstract. We show how to link topological tools with a local hyperbolic behavior to prove the existence of homoclinic and heteroclinic trajectories for a map. We apply this technique for the Hénon map h with classical parameter values ($a = 1.4, b = 0.3$). For this map we give a computer assisted proof of the existence of abundance of homoclinic and heteroclinic trajectories. We also introduce the method for computation of the lower bound of the topological entropy of a map based on the covering relations involving different iterations of a map and we prove the that topological entropy of h is larger than 0.3381.

§ To whom correspondence should be addressed. Permanent address: Department of Electrical Engineering, University of Mining and Metallurgy, al. Mickiewicza 30, 30-059 Kraków, Poland.
|| Current address: Department of Mathematics, Indiana University, Bloomington, Indiana,USA

1. Introduction

The goal of this paper is to present methods, which can be rigorously implemented in computer, to establish an existence of periodic points, symbolic dynamics, homoclinic and heteroclinic trajectories and give lower bounds for topological entropy.

To obtain an existence of periodic points and/or symbolic dynamics, it is enough to use topological tools only. The basic tool in this context is that of *covering relation*, which was introduced in [15] and is recalled in section 2. The problem of the existence of homoclinic and heteroclinic trajectories requires an approach, which combines covering relations ($\mathcal{C}^0$ tools) with a local hyperbolic behavior ($\mathcal{C}^1$ tools). We introduce such method in section 3.

As an example we consider the Hénon map [5]

$$h(x,y) = (1 + y - ax^2, bx), \tag{1}$$

where $a = 1.4$ and $b = 0.3$ are the standard parameter values. Numerical simulations of this map show the *strange attractor*, which was and still is the subject of intense research (see [14] and references given there) . Below we recall the number of relevant rigorous results about this map.

Let us start with the $\mathcal{C}^1$ results. Misiurewicz and Szewc showed analytically [7] that the stable and unstable manifolds of the fixed point intersect transversally, giving rise to the Smale's horseshoe dynamics for some unknown iterate of h. Recently, Stoffer and Palmer [12] gave a computer assisted proof of the existence of the horseshoe dynamics for 25-th iterate of h.

Concerning the $\mathcal{C}^0$ results computer assisted proofs of the existence of symbolic dynamics for h^7 and h^2 are presented by the authors in [15] and [2] respectively. It is also worth to mention in this context the result of Szymczak [13]. Using Conley index theory he showed with computer assistance that for the Hénon map there exist periodic points of all period but 3 and 5. The same result was later shown using covering relations in [2].

In this paper we link the symbolic dynamics established in [15, 2] with local hyperbolic behavior. We prove the existence of orbits homoclinic to the fixed point, orbits homoclinic to the period-2 orbit, heteroclinic orbits connecting the fixed point outside the attractor with the fixed point inside the attractor. These trajectories can be arbitrarily complex in the sense that they can realize any admissible symbolic sequence of finite length.

The last section of our paper is devoted to computation of the rigorous lower bound for the topological entropy for the Hénon map, denoted here by $\mathrm{H}(h)$. The number of cycles with small period [3] and non-rigorous numerics [10] suggest that $\mathrm{H}(h) \approx 0.465$.

It follows from the results given in [7] that $\mathrm{H}(h) > 0$. The existence of symbolic dynamics shown in [15, 12] give an explicit positive lower bound of $(\log 2)/7 > 0.099$ and $(\log 2)/25 > 0.0277$ respectively. The best lower bound to date is given in [2], namely $\mathrm{H}(h) \geq 0.5 \log[(\sqrt{5}+1)/2] > 0.2406$. In this paper we construct covering relations, which give better lower bound for $\mathrm{H}(h)$, namely 0.3381. In this computation we develop a method, which allows to estimate the topological entropy from the existence of covering relations for various iterates of the map.

Certain assertions presented in this paper are proved by means of interval arithmetic implemented on a computer. For the introduction to the interval arithmetic underlying the methods discussed here see [8]. During computations we used procedures for interval arithmetic from BIAS and PROFIL packages [6]. In this paper we use bold letters to denote intervals, interval vectors and matrices and usual math italic to denote real quantities.

2. $\mathcal{C}^0$ tools. Topological covering

In this section we describe a topological method we use to prove the existence of periodic trajectories and symbolic dynamics. The method is based on the concept of covering. We present here an intuitive description of covering relation for planar maps, for precise formulation the reader is referred to [15, 17, 16]. For covering relations in higher dimension see [18].

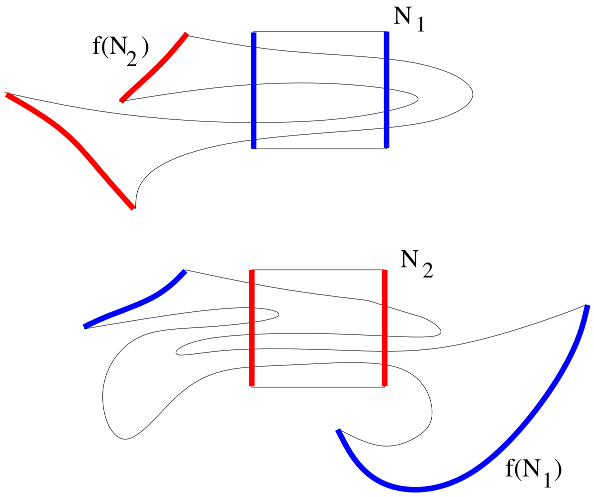

Figure 1. Two examples of possible position of the set $f(N_i)$ with respect to the set N_j, horizontal edges and their images are plotted using thick lines, N_1 f–covers N_2 but N_2 does not f-cover N_1 (images of horizontal edges of N_2 lie geometrically on the same side of N_1)

Let us consider a continuous two–dimensional map f. Let us choose p not necessarily disjoint quadrangles $N_1, N_2, \ldots, N_p$. For each N_i we choose two opposite edges and call them "horizontal". The two others are called "vertical". We say that N_i f–covers N_j and we use the notation $N_i \stackrel{f}{\Rightarrow} N_j$ if

(i) the image of N_i under f has empty intersection with the horizontal edges of N_j,

(ii) the images of vertical edges of N_i has empty intersection with N_j and they are located geometrically on the opposite sides of N_j ($f(N_i)$ can be continuously deformed to the set enclosed in the stripe defined by the horizontal edges of N_j

without any intersection with the edges of N_j and then images of vertical edges of N_i lie in this stripe on the opposite sides of N_j).

The intuitive meaning of covering relation is clear, when one looks at the example shown in Fig. 1. We draw two possible layouts of sets $f(N_i)$ with respect to N_j. N_1 f–covers N_2 because $f(N_1)$ has empty intersection with horizontal edges of N_2 and vertical edges of N_1 are mapped by f to different sides of N_2. However N_2 does not f-cover N_1 because the vertical edges of N_2 lie on the same side of N_1.

The most important property of the concept of covering is summarized in the following theorem

Theorem 1. *Let $f_i : N_i \to \mathbb{R}^2$ for $i = 0, 1, \dots, m-1$ are continuous maps. Let us assume that*

$$N_0 \overset{f_0}{\Rightarrow} N_1 \overset{f_1}{\Rightarrow} N_2 \overset{f_2}{\Rightarrow} \dots \overset{f_{m-1}}{\Rightarrow} N_m. \tag{2}$$

Then there exists a point $x_* \in N_0$ such that

$$f_{i-1} \circ f_{i-2} \circ \dots \circ f_0(x_*) \in N_i \quad \textit{for} \quad i = 1, \dots, m. \tag{3}$$

If additionally the sequence forms a loop ($N_m = N_0$) then x_* can be chosen such that

$$f_{m-1} \circ f_{m-2} \circ \dots \circ f_0(x_*) = x_*. \tag{4}$$

In the sequel we say that the point x^* (and the sequence $(x_0, \dots, x_m)$, where $x_0 = x_*$ and $f(x_k) = x_{k+1}$ for $k = 0, \dots, m-1$) existence of which is asserted in Theorem 1 realizes the sequence of coverings (2).

Proof. Assume first that $N_m = N_0$. The existence of x^* satisfying (3) and (4) follows immediately from Theorem 4 in [17].

Consider now the general case. We extend the chain of covering relations (2) to form a loop by adding the covering relation

$$N_m \overset{A}{\Rightarrow} N_0,$$

where A is a suitably chosen affine map. Now we have the closed loop of covering relations. From the first part of the proof applied to this chain we obtain the existence of x^* satisfying (3). □

In applications of the above theorem the functions f_i can be various iterates of a map or various Poincare maps for a differential equation. Now let us consider a case when the first and/or the last covering in a sequence is a self–covering. Then we can produce a sequence of coverings of arbitrary length. As a consequence of Theorem 1 we obtain the following result concerning the existence of infinite (or bi–infinite) sequences of points realizing the sequence of coverings.

Theorem 2. **1.** *If*

$$N_0 \overset{f_0}{\Rightarrow} N_1 \overset{f_1}{\Rightarrow} N_2 \overset{f_2}{\Rightarrow} \dots \overset{f_{m-1}}{\Rightarrow} N_m \overset{g}{\Rightarrow} N_m \tag{5}$$

then there exists $x_0 \in N_0$ *such that*

$$\begin{aligned} &f_{i-1} \circ f_{i-2} \circ \ldots \circ f_0(x_0) \in N_i \quad && for \quad i = 1, \ldots, m, \\ &g^k \circ f_{m-1} \circ \ldots \circ f_0(x_0) \in N_m \quad && for \quad k > 0. \ldots \end{aligned}$$

2. *If*

$$N_0 \overset{\mathrm{f}}{\Rightarrow} N_0 \overset{\mathrm{f_0}}{\Rightarrow} N_1 \overset{\mathrm{f_1}}{\Rightarrow} N_2 \overset{\mathrm{f_2}}{\Rightarrow} \ldots \overset{\mathrm{f_{m-1}}}{\Rightarrow} N_m \tag{6}$$

then there exists a sequence (x_k) *for* $k \leq 0$, $f(x_k) = x_{k+1}$ *for* $k < 0$ *such that*

$$\begin{aligned} &f_{i-1} \circ f_{i-2} \circ \ldots \circ f_0(x_0) \in N_i \quad && for \quad i = 1, \ldots, m, \\ &x_k \in N_0 && for \quad k \leq 0. \end{aligned}$$

3. *If*

$$N_0 \overset{\mathrm{f}}{\Rightarrow} N_0 \overset{\mathrm{f_0}}{\Rightarrow} N_1 \overset{\mathrm{f_1}}{\Rightarrow} N_2 \overset{\mathrm{f_2}}{\Rightarrow} \ldots \overset{\mathrm{f_{m-1}}}{\Rightarrow} N_m \overset{\mathrm{g}}{\Rightarrow} N_m \tag{7}$$

then there exists a sequence $(x_k)_{k=-\infty}^{0}$, $f(x_k) = x_{k+1}$ *for* $k < 0$ *such that*

$$\begin{aligned} &f_{i-1} \circ f_{i-2} \circ \ldots \circ f_0(x_0) \in N_i \quad && for \quad i = 1, \ldots, m, \\ &g^n \circ f_{m-1} \circ \ldots \circ f_0(x_0) \in N_m && for \quad n > 0, \\ &x_k \in N_0 && for \quad k \leq 0. \end{aligned}$$

We would like to use above theorem to obtain orbits which are asymptotic to given fixed points or periodic orbits. Consider for example case 1. Since $N_m \overset{\mathrm{g}}{\Rightarrow} N_m$, then from Theorem 1 it follows that there exists $x_* \in N_m$, which is a fixed point of g, i.e. $g(x_*) = x_*$. Let x_0 by a point obtained from theorem 2. We have

$$g^k \circ f_{m-1} \circ \ldots \circ f_0(x_0) \in N_m \quad \text{for} \quad k \geq 0, \tag{8}$$

but we cannot claim anything about asymptotic behavior of $g^k \circ f_{m-1} \circ \ldots \circ f_0(x_0)$. In the next section we discuss conditions which guarantee that the realizing sequence of points converges to x_*.

$$\lim_{k \to \infty} g^k \circ f_{m-1} \circ \ldots \circ f_0(x_0) = x_*. \tag{9}$$

3. $\mathcal{C}^1$ tools. Perturbations around hyperbolic fixed point

In this section we consider a two–dimensional function $f(x) = (f_1(x), f_2(x))^T$, where $x = (x_1, x_2)^T$. We assume that in the neighborhood of 0 the action of the function f is well approximated by multiplication by the diagonal matrix

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{10}$$

with $|\lambda_2| < 1 < |\lambda_1|$.

More precisely we assume that there exists a set $U \ni 0$ such that for $x \in U$ we have $f_1(x) \in \boldsymbol{\lambda}_1 x_1 + \boldsymbol{\varepsilon}_1 x_2$ and $f_2(x) \in \boldsymbol{\varepsilon}_2 x_1 + \boldsymbol{\lambda}_2 x_2$, where $\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_2$, $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$ are intervals. This means that $f(x) \in \boldsymbol{A}x$ for $x \in U$, where $\boldsymbol{A}$ is the interval matrix defined by

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{\lambda}_1 & \boldsymbol{\varepsilon}_1 \\ \boldsymbol{\varepsilon}_2 & \boldsymbol{\lambda}_2 \end{pmatrix}. \tag{11}$$

Let

$$\varepsilon_i' = \sup\{|\varepsilon| : \varepsilon \in \boldsymbol{\varepsilon}_i\}, \quad i = 1, 2. \tag{12}$$

Hence for the nondiagonal elements of the matrix $\boldsymbol{A}$ we have the following estimates: $\boldsymbol{\varepsilon}_i \subset [-\varepsilon_i', \varepsilon_i']$.

Let

$$\lambda_1' = \inf\{|\lambda_1| : \lambda_1 \in \boldsymbol{\lambda}_1\}, \tag{13}$$
$$\lambda_2' = \sup\{|\lambda_2| : \lambda_2 \in \boldsymbol{\lambda}_2\}. \tag{14}$$

λ_1' and λ_2' are the worst case values if we consider the system as a perturbation of a hyperbolic system. In what follows we assume that the perturbation of the system does not destroy hyperbolicity: $\lambda_1' > 1$ and $\lambda_2' < 1$.

Let us define the rectangle N_{α_1,α_2} by

$$N_{\alpha_1,\alpha_2} = [-\alpha_1, \alpha_1] \times [-\alpha_2, \alpha_2].$$

The main result of this section is summarized in the following theorem.

Theorem 3. *Let U, f and $\boldsymbol{A}$ are as above. Assume that*

$$\varepsilon_1'\varepsilon_2' < (1 - \lambda_2')(\lambda_1' - 1). \tag{15}$$

Then for each pair (α_1, α_2) such that the following conditions are satisfied

$$\frac{\varepsilon_1'}{\lambda_1' - 1} < \frac{\alpha_1}{\alpha_2} < \frac{1 - \lambda_2'}{\epsilon_2'}, \tag{16}$$
$$N_{\alpha_1,\alpha_2} \subset U, \tag{17}$$

holds

1. *if $f^k(x) \in N_{\alpha_1,\alpha_2}$ for $k \geq 0$, then $\lim_{k\to\infty} f^k(x) = 0$*

2. *if $y_k \in N_{\alpha_1,\alpha_2}$ and $f(y_{k-1}) = y_k$ for $k \leq 0$, then $\lim_{k\to-\infty} y_k = 0$*

From the assumption (15) it follows easily that there exists (α_1, α_2) such that conditions (16,17) are satisfied. The wider the interval $[\varepsilon_2'/(1 - \lambda_2'), (\lambda_1' - 1)/\varepsilon_1']$ is the more freedom in choosing the ratio α_1/α_2 we have. In this context the value of $(\lambda_1' - 1)(1 - \lambda_2')$ can be understood as the measure of the hyperbolicity of our system. The larger this value is the more "hyperbolic" the system is and we can accept larger errors coming from non–diagonal elements.

If f is invertible then assertion 2 of Theorem 3 can be stated as follows:
if $f^k(x) \in N_{\alpha_1,\alpha_2}$ for $k \leq 0$, then $\lim_{k\to-\infty} f_k(x) = 0$.

Before we give the proof of the above theorem we need the following two lemmas.

Lemma 1. *Assume that $N_{\alpha_1,\alpha_2} \subset U$. Let $r = \frac{\alpha_1}{\alpha_2}$. Let $\mu < 1$ be such that*

$$\varepsilon_2' r + \lambda_2' < \mu, \tag{18}$$
$$\frac{r}{\lambda_1' r - \varepsilon_1'} < \mu. \tag{19}$$

If $x, f(x), f^2(x) \in N_{\alpha_1,\alpha_2}$, then $f(x) \in N_{\mu\alpha_1,\mu\alpha_2}$.

Proof. Since $f_2(x) \in \boldsymbol{\varepsilon}_2 x_1 + \boldsymbol{\lambda}_2 x_2$, $|x_1| \le \alpha_2$ and $|x_2| \le \alpha_1$ we have

$$|f_2(x)| \le \varepsilon_2' \alpha_1 + \lambda_2' \alpha_2 < \mu\alpha_2.$$

It is easy to see the last inequality is equivalent to condition (18). Hence we showed that

$$|f_2(x)| < \mu\alpha_2. \tag{20}$$

It remains to show that $|f_1(x)| < \mu\alpha_1$. We start by showing that

$$\text{if } |x_1| > \mu\alpha_1 \text{ and } |x_2| < \mu\alpha_2, \text{ then } |f_1(x_1,x_2)| > \alpha_1. \tag{21}$$

Since $f_1(x) \in \boldsymbol{\lambda}_1 x_1 + \boldsymbol{\varepsilon}_1 x_2$ we have

$$\begin{aligned} &|f_1(x_1,x_2)| > \lambda_1' \mu\alpha_1 - \varepsilon_1' \mu\alpha_2 = \mu(\lambda_1'\alpha_1 - \varepsilon_1'\alpha_2) \\ &\ge \frac{\alpha_1}{\lambda_1'\alpha_1 - \varepsilon_1'\alpha_2}(\lambda_1'\alpha_1 - \varepsilon_1'\alpha_2) = \alpha_1. \end{aligned}$$

Hence (21) holds. Now let us assume that $|f_1(x)| > \mu\alpha_1$. From (20) and (21) it follows that $|f_1(f(x))| > \alpha_1$. This contradicts the assumption that $f^2(x) \notin N_{\alpha_1,\alpha_2}$. In this way we showed that

$$|f_1(x)| \le \mu\alpha_1. \tag{22}$$

This observation and inequality (20) imply the assertion of the lemma. □

Lemma 2. *The same assumptions as in lemma 1. Let n be a positive integer.*
If $f^k(x) \in N_{\alpha_1,\alpha_2}$ for $k = 0, 1, \ldots, 2n$, then $f^n(x) \in N_{\mu^n\alpha_1,\mu^n\alpha_2}$.

Proof. Observe that if assumptions of lemma 1 hold for the pair (α_1, α_2) then they hold also for any pair of the form $(s\alpha_1, s\alpha_2)$, where $0 < s < 1$. The case $n = 1$ is covered by lemma 1. We show an argument for $n = 2$, the general case goes along the same lines.

Assume that $x, f(x), f^2(x), f^3(x), f^4(x) \in N_{\alpha_1,\alpha_2}$. By applying lemma 1 to the triplets $\{x, f(x), f^2(x)\}$, $\{f(x), f^2(x), f^3(x)\}$ and $\{f^2(x), f^3(x), f^4(x)\}$ we conclude that

$$f(x), f^2(x), f^3(x) \in N_{\mu\alpha_1,\mu\alpha_2}.$$

After applying lemma 1 to the above triple we obtain

$$f^2(x) \in N_{\mu^2\alpha_1,\mu^2\alpha_2}.$$

□

Now we are ready to prove the main theorem.

Proof of Theorem 3. Observe that from (16) it follows easily that

$$\varepsilon_2' r + \lambda_2' < 1, \quad \frac{r}{\lambda_1' r - \varepsilon_1'} < 1$$

Hence there exists $\mu < 1$ such that the assumptions of lemmas 1 and 2 are satisfied. Assume now that $f^k(x) \in N_{\alpha_1,\alpha_2}$ for $k > 0$. From lemma 2 it follows that

$$f^k(x) \in N_{\mu^k\alpha_1,\mu^k\alpha_2} \tag{23}$$

Hence $f^k(x) \to 0$ for $k \to \infty$.

The proof of second assertion is analogous. □

Definition 1. *Let p be a fixed point for the map f. We say that f is* hyperbolic *on N, if there exists a coordinate system around p such that, in this coordinate system $N = N_{\alpha_1,\alpha_2}$, where α_1, α_2 are such that all assumptions of theorem 3 are satisfied.*

The theorem below, which follows directly from theorems 2 and 3, shows how we can combine $\mathcal{C}^0$ and $\mathcal{C}^1$ tools to obtain asymptotic orbits with prescribed itinerary.

Theorem 4. *Assume that g is hyperbolic on N_m and f hyperbolic on N_0. Let $x_g \in N_m$ be a fixed point for g and $x_f \in N_0$ be a fixed point for f.*

1. *If*

$$N_0 \overset{f_0}{\Rightarrow} N_1 \overset{f_1}{\Rightarrow} N_2 \overset{f_2}{\Rightarrow} \ldots \overset{f_{m-1}}{\Rightarrow} N_m \overset{g}{\Rightarrow} N_m \tag{24}$$

then there exists $x_0 \in N_0$ such that

$$\begin{aligned} &f_{i-1} \circ f_{i-2} \circ \ldots \circ f_0(x_0) \in N_i \qquad &&\textit{for} \quad i = 1, \ldots, m, \\ &g^k \circ f_{m-1} \circ \ldots \circ f_0(x_0) \in N_m &&\textit{for} \quad k > 0, \\ &\lim_{k\to\infty} g^k \circ f_{m-1} \circ \ldots \circ f_0(x_0) = x_g. \end{aligned}$$

2. *If*

$$N_0 \overset{f}{\Rightarrow} N_0 \overset{f_0}{\Rightarrow} N_1 \overset{f_1}{\Rightarrow} N_2 \overset{f_2}{\Rightarrow} \ldots \overset{f_{m-1}}{\Rightarrow} N_m \tag{25}$$

then there exists a sequence $\{x_k\}$ for $k \leq 0$, $f(x_k) = x_{k-1}$ such that

$$\begin{aligned} &f_{i-1} \circ f_{i-2} \circ \ldots \circ f_0(x_0) \in N_i \qquad &&\textit{for} \quad i = 1, \ldots, m, \\ &x_k \in N_0 &&\textit{for} \quad k \leq 0, \\ &\lim_{k\to-\infty} x_k = x_f. \end{aligned}$$

3. *If*

$$N_0 \overset{f}{\Rightarrow} N_0 \overset{f_0}{\Rightarrow} N_1 \overset{f_1}{\Rightarrow} N_2 \overset{f_2}{\Rightarrow} \ldots \overset{f_{m-1}}{\Rightarrow} N_m \overset{g}{\Rightarrow} N_m \tag{26}$$

then there exists a sequence $\{x_k\}$ for $k \leq 0$, $f(x_k) = x_{k-1}$ such that

$$\begin{aligned} &f_{i-1} \circ f_{i-2} \circ \ldots \circ f_0(x_0) \in N_i \qquad &&\textit{for} \quad i = 1, \ldots, m, \\ &g^n \circ f_{m-1} \circ \ldots \circ f_0(x_0) \in N_m &&\textit{for} \quad n > 0, \\ &\lim_{k\to\infty} g^k \circ f_{m-1} \circ \ldots \circ f_0(x_0) = x_g, \\ &x_k \in N_0, \quad k \leq 0, \\ &\lim_{k\to-\infty} x_k = x_f. \end{aligned}$$

The above theorem can be used without any modifications for proving the existence of trajectories converging to periodic orbits. All we need to do is to consider higher iterates of maps f and g in (24), (25) or (26).

4. Hyperbolic quadrangles for Hénon map

In this section we find hyperbolic neighborhoods for three periodic points of the Hénon map [5]

$$h(x_1, x_2) = (1 + x_2 - ax_1^2, bx_1), \tag{27}$$

where $a = 1.4$ and $b = 0.3$ are the standard parameter values. There are two fixed points and one period–2 orbit for this map. One of the fixed points (point u) lies in the trapping region containing the attractor observed numerically while the other (point v) lies outside the trapping region. We choose one point from the period–2 orbit and we call it w. Rigorous bounds for coordinates of these points are given in the second column of Table 1.

For each of these orbits we locate the quadrangles N_5, N_6, N_7 such that $u \in N_5$, $v \in N_6$, $w \in N_7$, h is hyperbolic on N_5 and N_6, h^2 is hyperbolic on N_7.

We do this in the following way. First using the interval Newton method a very good rigorous approximation $\boldsymbol{x} = (\boldsymbol{x}_1, \boldsymbol{x}_2)$ of the position of the periodic point is found [2, 3] (in case of period–1 and period–2 orbits the approximate position could also be found from an analytical formula). Using this approximation we compute Jacobian matrix at $(\boldsymbol{x}_1, \boldsymbol{x}_2)$, its eigenvalue and eigenvectors. Using the eigenvectors we construct the change of coordinates matrix $\boldsymbol{E}$ and compute its inverse $\boldsymbol{E}^{-1}$. Then we construct the interval vector $\boldsymbol{U} = (\boldsymbol{x}_1 + [-\delta, \delta], \boldsymbol{x}_2 + [-\delta, \delta])$, where δ is a parameter defining the size of $\boldsymbol{U}$. We compute the interval evaluation of the Jacobian matrix over $\boldsymbol{U}$ ($\boldsymbol{J} = f'(\boldsymbol{U})$) and find $\boldsymbol{A} = \boldsymbol{E}^{-1}\boldsymbol{J}\boldsymbol{E}$. By construction the interval matrix $\boldsymbol{A}$ is close to diagonal and has the form (11). Using formula (16) we find the interval $\boldsymbol{r}$ containing the allowed values of the ratio α_2/α_1. Then we choose the values α_1 and α_2 in such a way that their ratio belongs $\boldsymbol{r}$. Finally we check whether the set $R = \{z = x + Ey \colon x \in \boldsymbol{x}_*, E \in \boldsymbol{E}, y \in N_{\alpha_1,\alpha_2}\}$ is contained in $\boldsymbol{U}$. In practice we check the condition $\boldsymbol{x} + \boldsymbol{E}N_{\alpha_1,\alpha_2} \subset \boldsymbol{U}$ using interval arithmetic. Observe that since we do not know the exact position of the periodic point the set R is not exactly a quadrangle. The results of this procedure for the three periodic points considered are summarized in Table 1.

Table 1. Hyperbolic quadrangles for periodic points u, v and w. $\boldsymbol{x}_{1,2}$ are rigorous bounds for the position of the periodic point, δ represents the size of $\boldsymbol{U}$ ($\boldsymbol{U} = (\boldsymbol{x}_1 + [-\delta, \delta], \boldsymbol{x}_2 + [-\delta, \delta])$), $\boldsymbol{r}$ is the interval containing the allowed values of α_2/α_1 for the given δ, $\alpha_{1,2}$ are the actually chosen values

	$\boldsymbol{x}_1, \boldsymbol{x}_2$	δ	$\boldsymbol{r}$	α_1	α_2
u_1	$0.631354477089504^{98402}_{42891}$	0.18	$[0.11, 2.1]$	0.08	0.16
u_2	$0.189406343126851^{53406}_{31202}$				
v_1	$-1.13135447708950^{43179}_{52061}$	0.5	$[0.151, 2.24]$	0.3	0.3
v_2	$-0.339406343126851^{25096}_{52851}$				
w_1	$-0.47580005117505^{591583}_{652645}$	0.04	$[0.21, 1.7]$	0.02	0.03
w_2	$0.29274001535251^{703565}_{67581}$				

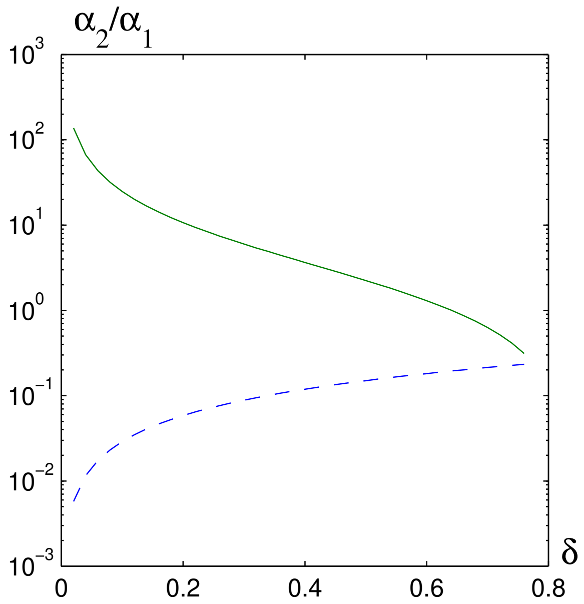

Figure 2. Allowed values of the ratio α_2/α_1 versus the size of the neighborhood $\boldsymbol{U}$ for the point u.

There is a trade–off for the choice of the size of $\boldsymbol{U}$. We would like to choose $\boldsymbol{U}$ as large as possible so that large rectangles would fit into it. On the other hand if we choose larger $\boldsymbol{U}$ then the elements of $\boldsymbol{A}$ have large diameters and we may obtain very narrow or even empty interval for the allowed values of α_2/α_1.

In Fig. 2 we show how the interval $\boldsymbol{r}$ changes with δ for the case of the fixed point inside the attractor. It is clear that for very small $\boldsymbol{U}$ the error terms ε_i' in matrix $\boldsymbol{A}$ are very small and hence the interval $\boldsymbol{r}$ containing the allowed values for the ratio α_2/α_1 is very wide. When we increase the size of $\boldsymbol{U}$ the interval $\boldsymbol{r}$ becomes thinner and for a certain value of δ there are no allowed values of α_2/α_1.

The rectangle $\boldsymbol{U}$ and the quadrangle R for all the three periodic orbits considered are plotted in Fig. 3. In all cases the condition $R \subset \boldsymbol{U}$ is fulfilled.

5. Symbolic dynamics for the Hénon map

We assume that the reader is familiar with the notions of symbolic dynamics, subshift of finite type, transition matrices as presented in any standard textbook on dynamical systems (see for example [11]).

First we recall the results on the existence of symbolic dynamics for h^2 [2] and h^7 [15]. The existence of symbolic dynamics was proved for the quadrangles

$$\begin{aligned}
N_1 &= ((-0.75,0.29),(-0.8,0.39),(-0.31,0.34),(-0.22,0.24)),\\
N_2 &= ((0,0.19),(0.08,0.29),(0.42,0.2),(0.34,0.1)),\\
N_3 &= ((0.46,0.01),(0.595,0.28),(0.691,0.28),(0.556,0.01)),\\
N_4 &= ((0.588,0.01),(0.723,0.28),(0.755,0.28),(0.62,0.01)),
\end{aligned} \tag{28}$$

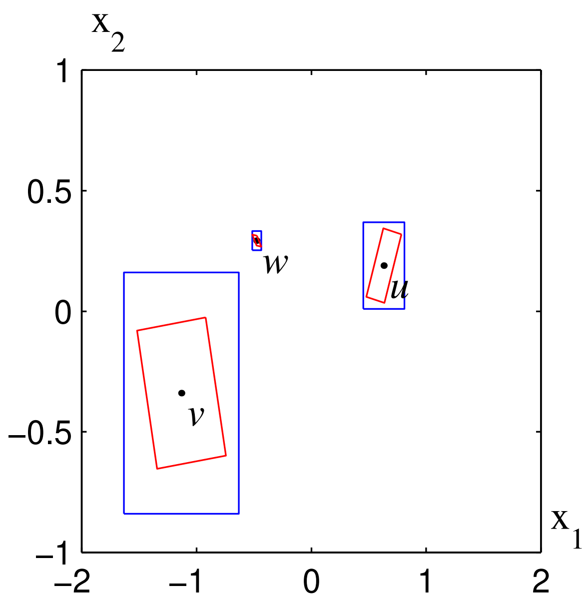

Figure 3. The rectangle $\boldsymbol{U}$ and the quadrangle $R \subset \boldsymbol{U}$ for the fixed point u, the fixed point v and the period–2 point w.

shown in Fig. 4.

In [2, 15] it was shown that

$$N_1 \overset{h^2}{\Longrightarrow} N_1, \quad N_1 \overset{h^2}{\Longrightarrow} N_2, \quad N_2 \overset{h^2}{\Longrightarrow} N_1 \tag{29}$$

$$N_3 \overset{h^7}{\Longrightarrow} N_3, \quad N_3 \overset{h^7}{\Longrightarrow} N_4, \quad N_4 \overset{h^7}{\Longrightarrow} N_3, \quad N_4 \overset{h^7}{\Longrightarrow} N_4. \tag{30}$$

From (29,30) and Theorem 1 it follows immediately that there is a subshift of finite type on sets N_1, N_2 described by the following transition matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

embedded in h^2 and for h^7 there is the full shift on two symbols embedded.

Let us recall from section 4 that N_5, N_6, N_7 are sets on which h is hyperbolic. Now we establish covering connections between the sets N_i, which we later use to prove the existence of many homoclinic and heteroclinic orbits.

In order to prove that N_i h^n–covers N_j we divide the edges of N_i into a finite number of rectangles (two–dimensional intervals) and using interval arithmetics we compute their images under h^n. Then we check whether the images of rectangles have empty intersection with horizontal edges of N_j and if the images of vertical edges of N_i lie on the opposite sides of N_j. In our rigorous checks each edge was covered by up to 30 rectangles.

First we have found self–covering relations for the sets N_5, N_6, N_7:

$$N_5 \overset{h}{\Longrightarrow} N_5, \quad N_6 \overset{h}{\Longrightarrow} N_6, \quad N_7 \overset{h^2}{\Longrightarrow} N_7. \tag{31}$$

The computation results for the set N_7 are shown in Fig. 5, where images of vertical and horizontal edges of N_7 under h^2 are plotted.

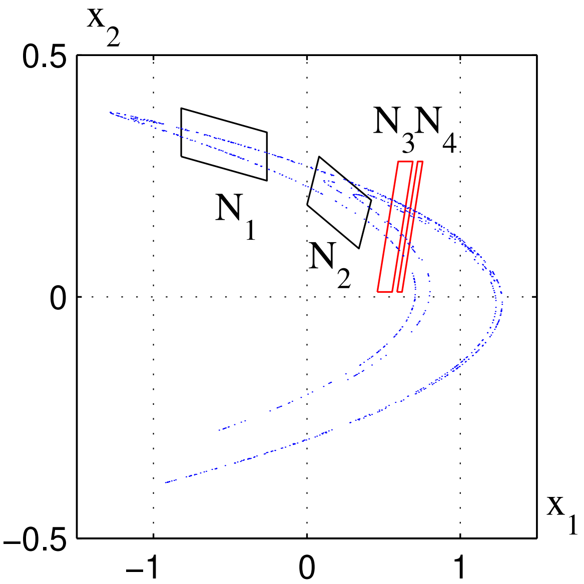

Figure 4. The sets N_1 and N_2 for the proof of the existence of symbolic dynamics for h^2 from [2] and the sets N_3 and N_4 for the proof of the existence of symbolic dynamics for h^7 from [15].

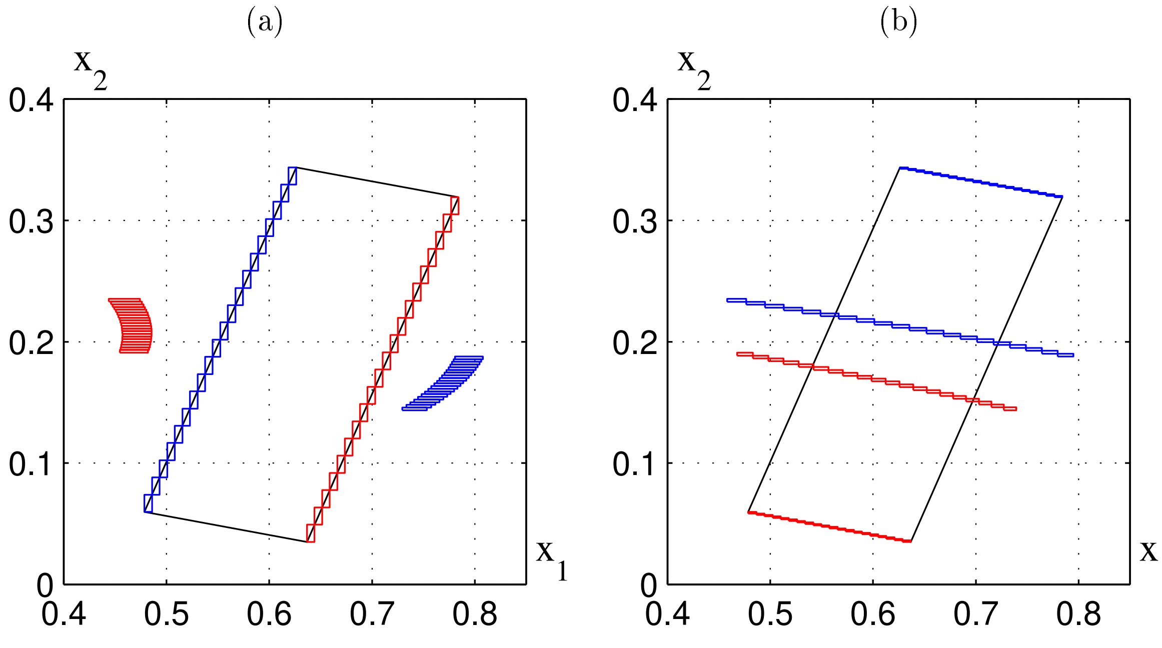

Figure 5. $N_7 \overset{h^2}{\Rightarrow} N_7$, images of vertical (a) and horizontal (b) edges of N_7

We have also proved the existence of cross coverings between the sets $N_1, \ldots, N_7$:

$$N_1 \overset{h^4}{\Rightarrow} N_3, \quad N_1 \overset{h^4}{\Rightarrow} N_4, \quad N_2 \overset{h^4}{\Rightarrow} N_3, \quad N_2 \overset{h^4}{\Rightarrow} N_4, \quad N_3 \overset{h^6}{\Rightarrow} N_1, \tag{32}$$

$$N_6 \overset{h^3}{\Rightarrow} N_5, \tag{33}$$

$$N_5 \overset{h}{\Rightarrow} N_3, \quad N_5 \overset{h}{\Rightarrow} N_4, \quad N_3 \overset{h^4}{\Rightarrow} N_5, \tag{34}$$

$$N_7 \overset{h^6}{\Rightarrow} N_1, \quad N_1 \overset{h^2}{\Rightarrow} N_7. \tag{35}$$

In Fig. 6 we show the results obtained in the proof of the last set of coverings. The first set of coverings (32) contains the covering relations in both ways between the sets N_1, N_2 and N_3, N_4. It means that we can construct very complex covering sequences, where the finite length covering sequences corresponding to symbolic dynamics on N_1, N_2 and N_3, N_4 are mixed. The covering (33) will be used to prove the existence of heteroclinic orbit from v to u. Coverings (34) and (35) will be used to prove the existence of homoclinic orbits for u and w.

The covering relations between the sets N_i are summarized in the following table:

	N_1	N_2	N_3	N_4	N_5	N_6	N_7
N_1	2	2	4	4			2
N_2	2		4	4			
N_3	6		7	7	4		
N_4			7	7			
N_5			1	1	1		
N_6					3	1	
N_7	6						2

(36)

Integer entry n in the row labelled by N_i and a column labelled by N_j means that N_i h^n–covers N_j ($N_i \overset{h^n}{\Rightarrow} N_j$).

6. The existence of homoclinic and heteroclinic trajectories

Now we are ready prove the existence of many different homoclinic and heteroclinic trajectories. First we prove the existence of **infinitely many trajectories homoclinic to the fixed point inside the attractor**. From the table of covering relations described by (36) it follows that we can construct the following sequence of covering relations:

$$N_5 \overset{h}{\Rightarrow} N_5 \overset{h}{\Rightarrow} N_3 \overset{h^7}{\Rightarrow} \cdots \overset{h^7}{\Rightarrow} N_3 \overset{h^4}{\Rightarrow} N_5 \overset{h}{\Rightarrow} N_5$$

where the middle dots can be replaced by any finite sequence admissible for the symbolic dynamics for h^7, i.e., $N_{i_1} \overset{h^7}{\Rightarrow} \dots \overset{h^7}{\Rightarrow} N_{i_m}$, where $i_1, \dots, i_m \in \{3, 4\}$. From the case 3 in Theorem 4 for each sequence in the middle we obtain a different trajectory $(y^k)_{k=-\infty}^{k=\infty}$, which is homoclinic to a fixed point in $u \in N_5$.

In a similar manner we prove that there exist an **infinite number of trajectories homoclinic to the period–2 orbit**. In order to prove this fact we use the following sequence of coverings

$$N_7 \overset{h^2}{\Rightarrow} N_7 \overset{h^6}{\Rightarrow} N_1 \overset{h^2}{\Rightarrow} \cdots \overset{h^2}{\Rightarrow} N_1 \overset{h^2}{\Rightarrow} N_7 \overset{h^2}{\Rightarrow} N_7,$$

where the middle dots can be replaced by any finite sequence admissible for the symbolic dynamics for h^2.

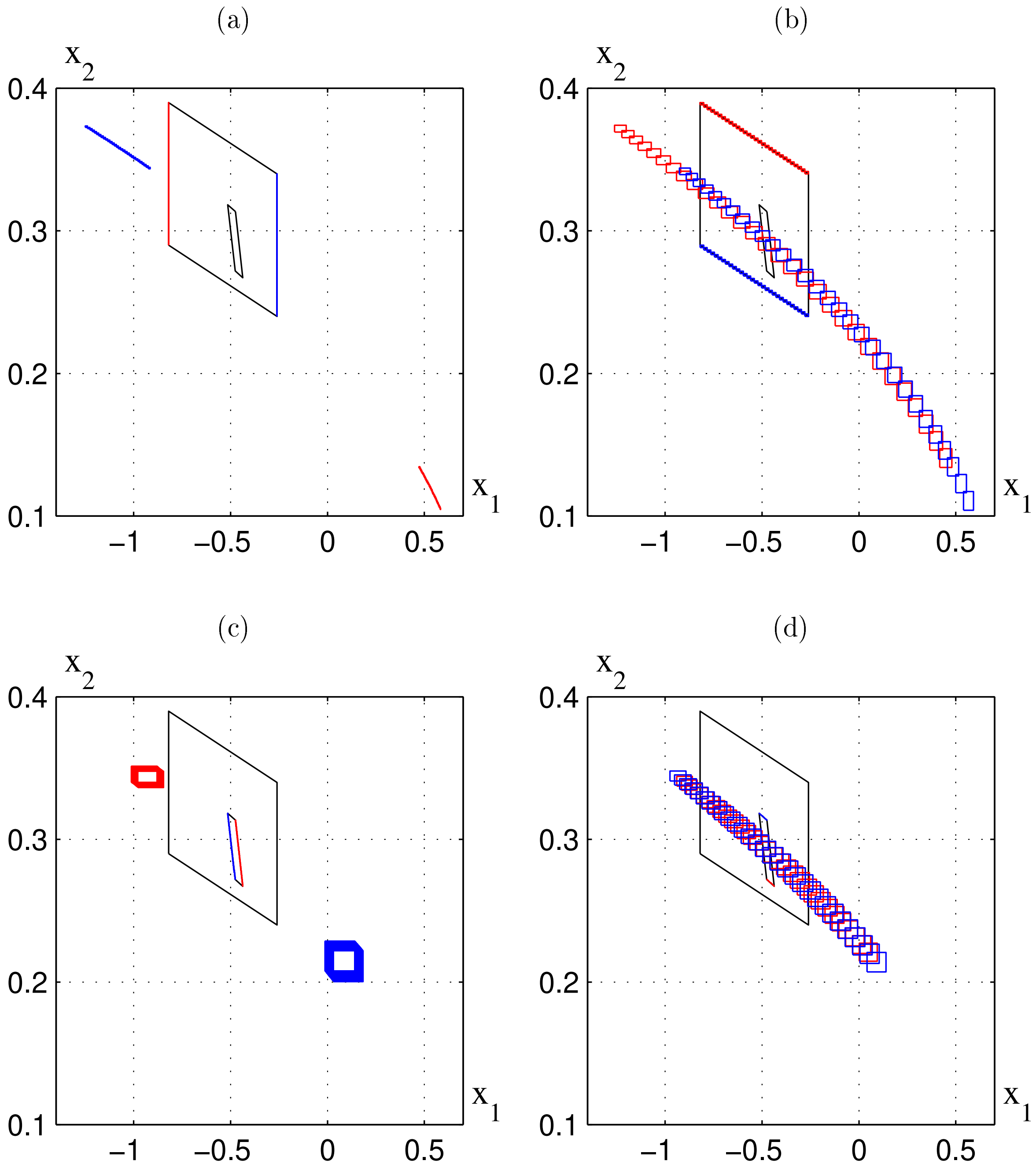

Figure 6. Covering relations between N_1 and N_7: $N_1 \overset{h^2}{\Rightarrow} N_7$ (images under h^2 of vertical (a) and horizontal (b) edges of N_1), $N_7 \overset{h^6}{\Rightarrow} N_1$ (images under h^6 of vertical (c) and horizontal (d) edges of N_7).

In order to prove the existence of a **heteroclinic orbit connecting the fixed point outside the attractor with the fixed point inside the attractor** we consider the sequence of coverings:

$$N_6 \overset{\mathrm{h}}{\Rightarrow} N_6 \overset{\mathrm{h}^3}{\Rightarrow} N_5 \overset{\mathrm{h}}{\Rightarrow} N_5. \tag{37}$$

To show the existence of **infinitely many heteroclinic trajectories connecting the fixed point outside the attractor with the fixed point inside the attractor** we add in the middle of the above sequence a finite length sequence corresponding to the existence of symbolic dynamic for h^7:

$$N_6 \overset{\mathrm{h}}{\Rightarrow} N_6 \overset{\mathrm{h}^3}{\Rightarrow} N_5 \overset{\mathrm{h}}{\Rightarrow} N_3 \overset{\mathrm{h}^7}{\Rightarrow} \ldots \overset{\mathrm{h}^7}{\Rightarrow} N_3 \overset{\mathrm{h}^4}{\Rightarrow} N_5 \overset{\mathrm{h}}{\Rightarrow} N_5$$

If the sequence in the middle is composed of only N_3 then the trajectory obtained may be equal to the one obtained before for the sequence (37). We show that if the sequences in the middle contain N_4 we obtain a trajectory different from the one produced by the sequence (37). This is not obvious because the set N_5 has non–empty intersection with both of the sets N_3 and N_4. This is however true because $h^2(N_4) \cup N_5 = \emptyset$. Now in order to produce different heteroclinic trajectories we limit ourselves to symbolic sequences ending with N_4 only. They obviously give rise to different trajectories.

Because there exist connections in both ways between the sets N_1, N_2 and N_3 we are also able to prove the existence of many other homoclinic and heteroclinic trajectories. For example one can easily construct a sequence of coverings such that the corresponding orbit is a heteroclinic trajectory connecting the fixed point inside the attractor and the period–2 orbit. In the meantime the trajectory may be very complicated. It may for example come as close as we wish to the fixed point or period–2 orbit and this may happen several times. It may also realize any symbolic sequence admissible by one of the symbolic dynamics.

7. Coverings and topological entropy

In this section we describe the method for computation of lower bound of topological entropy of the map based on coverings involving not necessarily the same iteration of f. The method is then applied to estimate the topological entropy for the Hénon map.

Let us first recall the definition of topological entropy. Topological entropy $\mathrm{H}(f)$ of a map f characterizes "mixing" of points by f. One of the equivalent definitions of topological entropy is based on the notion of (n, ε)–separated sets (see [1]).

Definition 2. *A set $E \subset X$ is called* (n, ε)–separated *if for every two different points $x, y \in E$, there exists $0 \leq j < n$ such that the distance between $f^j(x)$ and $f^j(y)$ is greater than ε. Let us define the number $s_n(\varepsilon)$ as the cardinality of a maximum (n, ε)–separated set:*

$$s_n(\varepsilon) = \max\{\operatorname{card} E : E \quad \textit{is} \quad (n,\varepsilon)\textit{-separated}\}.$$

The number

$$\mathrm{H}(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s_n(\varepsilon), \tag{38}$$

is called the topological entropy *of the map f.*

The number of periodic orbits is closely related to the topological entropy. For axiom A diffeomorphisms we have

$$\mathrm{H}(f) = \limsup_{n\to\infty} \frac{\log C(f^n)}{n},$$

where $C(f^n)$ denotes the number of fixed points of f^n.

It is also possible to use the number of periodic orbits for the estimation of topological entropy when there exists a symbolic dynamics for the map. The method is based on observation that for each admissible symbolic sequence there exist a trajectory realizing this sequence. The trajectory of length n can serve as an (n, ε)–separated set for every ε smaller that the minimum distance between the sets N_i. The approximation of topological entropy can be easily obtained by inspecting the transition matrix A corresponding to the given symbolic dynamics. The existence of symbolic dynamics means that f is semiconjugate with a subshift of a finite type with the transition matrix A. Topological entropy of a subshift of finite type with transition matrix A equals to the logarithm of an eigenvalue λ_1 of A such that $\lambda_1 \geq |\lambda_j|$ for all eigenvalues of A (see [11][Theorem 1.9, p. 340]). Topological entropy of a map semiconjugate to the subshift is not less than the topological entropy of the subshift.

Hence if all the coverings are for the same iteration p the procedure is straightforward. We produce the transition matrix directly from the coverings, compute its leading eigenvalue and we use its logarithm as a bound for the topological entropy of the pth iteration of the map. The estimation for the topological entropy for the map itself is obtained by dividing the result by p.

For example using the existence of symbolic dynamics for h^7 one can prove that (compare [15, 2]):

$$\mathrm{H}(h) \geq \frac{1}{7}\log 2 > 0.099. \tag{39}$$

Similarly one can obtain the estimation of topological entropy based on the existence of symbolic dynamics for h^2 (compare [2]):

$$\mathrm{H}(h) \geq \frac{1}{2}\log\frac{\sqrt{5}+1}{2} > 0.2406. \tag{40}$$

If coverings involve different iterates we cannot proceed in this way. We would like to obtain estimation of topological entropy based on the results from section 5. The main problem is that the iterations for which we have proven the coverings are not the same. Can we say anything about topological entropy in this case? The answer is sometimes yes. We need some additional assumptions concerning iterations of the sets N_i which will be explained now.

The method starts with the construction of auxiliary sets. They are introduced to obtain a transition matrix for the symbolic dynamics for the first iteration of the considered map. Then we give conditions under which the topological entropy associated with the shift defined by the transition matrix is not larger than the topological entropy of the map.

Let us assume that we have m disjoint sets $N_1, \ldots, N_m$ and some coverings between these sets for different iterates of the function f. The very natural idea is to reduce all the coverings to the first iterate by introducing auxiliary sets.

Consider a set N_i. Assume that we have p_i covering relations starting at N_i, i.e $N_i \stackrel{f^{n_k}}{\Rightarrow} N_{j_k}$ for $k = 1, \ldots, p_i$ and $n_i = \max\{n_1, \ldots, n_{p_i}\}$. We define the auxiliary sets by $N_i = N_i^1, N_i^2 = f(N_i), \ldots, N_i^{n_i} = f^{n_i-1}(N_i)$.

Let us order the sets N_i^j in the following way

$$(N_1^1, N_1^2, \ldots, N_1^{n_1}, N_2^1, \ldots, N_2^{n_2}, \ldots, N_m^1, \ldots, N_1^{n_m}).$$

From Theorem 1 and the above definition it follows that for each sequence of covering relations

$$N_{i_1} \stackrel{f^{n_1}}{\Rightarrow} N_{i_2} \stackrel{f^{n_2}}{\Rightarrow} N_{i_3} \ldots \stackrel{f^{n_{k-1}}}{\Rightarrow} N_{i_k} \stackrel{f^{n_k}}{\Rightarrow} N_{i_{k+1}} \tag{41}$$

there exists a point x such that

$$f^{l_m}(x) \in N_{i_m}^1, \quad \text{where } l_m = \textstyle\sum_{r=1}^{m-1} n_r \text{ for } m = 1, \ldots, k+1 \tag{42}$$

$$f^{l_m+s-1}(x) \in N_{i_m}^s \quad \text{for } m = 1, \ldots, k \text{ and } s = 1, \ldots, n_m. \tag{43}$$

Moreover if $N_{i_1} = N_{i_k}$ then x can be chosen to be periodic point for f with the period $\sum_{r=1}^{k-1} n_r$.

We create the transition matrix A for the sets N_i^j as follows. The matrix has a block structure. Connections between the set $N_i = N_i^1$ and its auxiliary sets $N_i^2, \ldots, N_i^{n_i}$ produces blocks with the following structure where the upper diagonal is filled with ones:

	N_i^1	N_i^2	N_i^2	$\cdots$	$N_i^{n_i-1}$	$N_i^{n_i}$
N_i^1	0	1	0	$\cdots$	0	0
N_i^2	0	0	1	$\cdots$	0	0
$\vdots$	$\vdots$	$\vdots$		$\ddots$	$\vdots$	$\vdots$
$N_i^{n_i-1}$	0	0	0	$\cdots$	0	1
$N_i^{n_i}$	0	0	0	$\cdots$	0	0

(44)

These blocks are positioned on a diagonal within the transition matrix. The block at i-th row and j-th column corresponds to the covering between N_i and N_j. Namely, if $N_i \stackrel{f^n}{\Rightarrow} N_j$, then we put 1 in the entry corresponding to the auxiliary set N_i^{n-1} in the i-th block row and the set N_j^1 in the j-th column block, while all other entries are set to 0, otherwise the whole block is filled with 0's.

We denote the set of symbolic sequences admissible by A by Σ_A. It is easy to see that all symbolic sequences from Σ_A are realizable. Our plan is to use the transition matrix A for the estimation of the topological entropy of f. We cannot however use directly entropy of the subshift associated with A as we do not control the positions of the auxiliary sets. It may happen that these sets overlap and different symbolic sequences lead to the same orbits of f. Below we develop tools for checking if this possibility can be excluded. Let us start with some useful definitions.

Definition 3. *Let us consider a sequence of coverings:*

$$N_{i_1} \overset{\mathrm{f}^{\mathrm{k}_1}}{\Rightarrow} N_{i_2} \overset{\mathrm{f}^{\mathrm{k}_2}}{\Rightarrow} \cdots \overset{\mathrm{f}^{\mathrm{k}_r}}{\Rightarrow} N_{i_{r+1}} \tag{45}$$

The number $k_1 + k_2 + \ldots + k_r$ will be called length of the sequence (45). *We will say that the sequence* (45) *is periodic if $i_1 = i_{r+1}$.*

Definition 4. *Let us consider two sequences of coverings:*

$$N_{i'_1} \overset{\mathrm{f}^{\mathrm{k}'_1}}{\Rightarrow} N_{i'_2} \overset{\mathrm{f}^{\mathrm{k}'_2}}{\Rightarrow} \cdots \overset{\mathrm{f}^{\mathrm{k}'_{r'}}}{\Rightarrow} N_{i'_{r'+1}}, \tag{46}$$

$$N_{i''_1} \overset{\mathrm{f}^{\mathrm{k}''_1}}{\Rightarrow} N_{i''_2} \overset{\mathrm{f}^{\mathrm{k}''_2}}{\Rightarrow} \cdots \overset{\mathrm{f}^{\mathrm{k}''_{r''}}}{\Rightarrow} N_{i''_{r''+1}},. \tag{47}$$

We say that the two sequences of coverings are ε–separated if the distance between any two trajectories realizing these two sequences is larger than ε.

Lemma 3. *For the disjoint sets $N_1, \ldots, N_m$ and coverings between them let us define auxiliary sets and create the transition matrix A in the way described above. If for some $\varepsilon_0 > 0$ all different sequences of coverings of equal length are ε_0–separated, then the topological entropy of f is not less than the logarithm of the leading eigenvalue of A.*

Proof. From the construction of the transition matrix A it follows that for any periodic sequence $\alpha \in \Sigma_A$ of period l, there exists a period–l point x_α for f, such that x_α realize the sequence α.

Let R_n be the number of periodic sequences of coverings of length n and S_n be the number of periodic symbolic sequences of period n associated with the subshift with the transition matrix A. It is clear that $R_n \leq S_n$. Now let us choose a symbolic sequence. If the sequence starts in one of the basic sets N_i we immediately obtain a sequence of coverings. If the symbolic sequence starts in an auxiliary set then after less then n_{max} shifts we obtain a symbolic sequence starting in a basic set, where n_{max} is the maximum iteration number present in the set of coverings considered. Thus $S_n \leq n_{max} R_n$. Since $S_n = \mathrm{Tr}(A^n)$, we obtain

$$R_n \geq \frac{\mathrm{Tr} A^n}{n_{max}}. \tag{48}$$

For each periodic sequence of covering relations of length n we choose one orbit realizing this sequence, we will call this set $P(n)$. From our assumptions it follows that $P(n)$ is (n, ε)-separated for $\varepsilon \leq \varepsilon_0$. Obviously

$$s_n(\varepsilon) \geq \mathrm{Card} P(n) = R_n \geq \frac{\mathrm{Tr} A^n}{n_{max}}. \tag{49}$$

Let λ_i be eigenvalues of A. From the Perron–Frobenius theorem [4] it follows that, A has a positive real eigenvalue λ_1, such that $\lambda_1 \geq |\lambda|$, where λ is any eigenvalue of A. Let us order λ_i as follows

$$\lambda_1 \geq |\lambda_2| \geq \ldots \geq |\lambda_{\dim(A)}|. \tag{50}$$

For each $n > 0$ we have

$$\mathrm{Tr} A^n = \sum_{i=1}^{\dim(A)} \lambda_i^n = \lambda_1^n \sum_{i=1}^{\dim(A)} \left(\frac{\lambda_i}{\lambda_1}\right)^n . \tag{51}$$

One can prove that there exists an increasing sequence (n_k) and a real number $a > 0$ such that

$$\mathrm{Tr} A^{n_k} \geq a\lambda_1^{n_k}. \tag{52}$$

From the above inequality and equation (49) it follows that

$$s_{n_k}(\varepsilon) \geq \frac{a\lambda_1^{n_k}}{n_{max}}, \quad \text{for } \varepsilon \leq \varepsilon_0. \tag{53}$$

Hence

$$\mathrm{H}(f) \geq \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s_n(\varepsilon) \geq \lim_{k \to \infty} \frac{1}{n_k} \log \frac{a\lambda_1^{n_k}}{n_{max}} = \log \lambda_1. \tag{54}$$

□

In general there may exist different sequences realized by a single trajectory (sequences may be not ε–separated for any positive ε). Below we describe a method how to ensure that all different sequences are ε–separated for some uniform $\varepsilon > 0$. In this case we are able to make a conclusion on topological entropy of f based on the transition matrix. For checking if the two sequences are separated we use the following observation.

Lemma 4. *Let us choose a pair of sequences (46), (47). Let us assume that there exist* $0 \leq s' \leq r'$ *and* $0 \leq s'' \leq r''$ *and* $0 \leq n = \sum_{p=1}^{s''} k''_p - \sum_{p=1}^{s'} k'_p$ *such that* $f^n(N_{i_{s'}}) \cap N_{i_{s''}} = \emptyset$. *Then the sequences (46), (47) are* ε*–separated with* $\varepsilon = \mathrm{dist}(f^n(N_{i_{s'}}), N_{i_{s''}})$.

Proof. Let x' and x'' be points realizing the two sequences. Let us denote $n' = \sum_{p=1}^{s'} k'_p$ and $n'' = \sum_{p=1}^{s''} k''_p$. We have $f^{n'}(x') \in N_{i_{s'}}$, $f^{n'+n}(x') = f^{n''}(x') \in f^n(N_{i_{s'}})$ and $f^{n''}(x'') \in N_{i_{s''}}$. As $f^n(N_{i_{s'}}) \cap N_{i_{s''}} \neq \emptyset$ it is clear that the points x' and x'' are different and $\mathrm{dist}(f^{n''}(x'), f^{n''}(x'')) \geq \mathrm{dist}(N_{i_{s'}}), N_{i_{s''}})$. □

The next lemma states that it is possible to reduce the number of covering sequences that need to be checked. Let us consider a pair of sequences with equal length, starting at the same set N_i, ending both at the same set N_j such that no proper subsequences starting at the beginning have equal length:

$$\begin{aligned} i'_1 &= i''_1, \\ i'_{r'+1} &= i''_{r''+1}, \\ \sum_{p=1}^{r'} k'_p &= \sum_{p=1}^{r''} k''_p, \\ \sum_{p=1}^{s'} k'_p &\neq \sum_{p=1}^{s''} k''_p \text{ for all } 1 \leq s' < r', 1 \leq s'' < r''. \end{aligned} \tag{55}$$

Lemma 5. *If all sequences (46), (47) satisfying conditions (55) are ε–separated, then all different sequences of equal length are ε–separated.*

Proof. Let us consider two different periodic sequences of coverings with length n. If there exist $0 \leq s' \leq r'$ and $0 \leq s'' \leq r''$ such that $\sum_{p=1}^{s'} k'_p = \sum_{p=1}^{s''} k''_p$ and $i_{s'} \neq i_{s''}$ then the sequences are ε–separated with $\varepsilon = \mathrm{dist}(N_{i_{s'}}, N_{i_{s''}})$. In the opposite case we can choose subsequences satisfying conditions (55) which are different. As the subsequences are ε–separated then also the original sequences are. $\square$

Lemma 5 says that it is enough to check sequences of coverings of equal length starting both at one set and ending both at another set (which may be the same as the starting one) such that for each intermediate iteration at least one of the sequences is at the auxiliary set. It follows from this lemma that in the case when for all N_i all coverings starting at this set are for the same iteration of f all sequences are separated (there is no different sequences satisfying conditions (55)). In other cases the number of pairs of sequences which need to be checked may be infinite (we have to check symbolic sequences of all lengths). It many cases however it is possible to reduce the assumptions to finite number of cases. Several examples are shown below.

As a first example let us consider the symbolic dynamics on N_3 and N_4. One can easily prove that in addition to the coverings $N_{3,4} \overset{\mathrm{h}^7}{\Rightarrow} N_{3,4}$ there exist the following two coverings: $N_3 \overset{\mathrm{h}^2}{\Rightarrow} N_3$ and $N_3 \overset{\mathrm{h}}{\Rightarrow} N_4$. Using these coverings we produce the following covering table:

	N_3	N_4
N_3	2	1
N_4	7	7

. (56)

As N_3 is present in coverings involving second iteration we introduce the auxiliary sets $N_3^1 = N_3, N_3^2$ ($n_3 = 2$). For the set N_4, which is present in coverings involving 7th iteration we introduce the auxiliary sets $N_4^1 = N_4, N_4^2, \ldots, N_4^7$ ($n_4 = 7$). After ordering the auxiliary sets in the natural way $(N_3^1, N_3^2, N_4^1, N_4^2, \ldots, N_4^7)$ we obtain the following transition matrix:

$$\left(\begin{array}{cc|ccccccc} & 1 & 1 & & & & & & \\ 1 & & & & & & & & \\ \hline & & & 1 & & & & & \\ & & & & 1 & & & & \\ & & & & & 1 & & & \\ & & & & & & 1 & & \\ & & & & & & & 1 & \\ & & & & & & & & 1 \\ 1 & & 1 & & & & & & \end{array}\right). \tag{57}$$

In order to estimate the topological entropy using the above transition matrix we need to know that different sequences lead to different trajectories. To prove that different sequences of coverings are separated we need the following fact :

$$\mathrm{dist}(h(N_4), N_3) > 0, \tag{58}$$

which was proved using the interval arithmetic. Let us consider two different sequences of coverings satisfying conditions (55). It is clear that they must start in the following way:

$$N_3 \overset{\mathrm{h}}{\Rightarrow} N_4 \text{ and } N_3 \overset{\mathrm{h}^2}{\Rightarrow} N_3. \tag{59}$$

We use Lemma 4 with $s' = 1$, $s'' = 1$, $n = 1$ to show that the sequences are ε–separated with $\varepsilon = \mathrm{dist}(h(N_4), N_3) > 0$.

From Lemma 5 it follows that all different sequences of the same length are ε separated and we can use Lemma 3 for estimation of topological entropy of f. The logarithm of the leading eigenvalue of the matrix (57) is $\log \lambda_1 > 0.1859$. Hence we obtain the following estimation of the topological entropy of the Hénon map:

$$\mathrm{H}(h) \geq \log \lambda_1 > 0.1859. \tag{60}$$

Observe that using coverings for the lower iterations we have obtained much better lower bound for the topological entropy of the Hénon map than from the symbolic dynamics of h^7 (compare (39)).

In the second example we combine the coverings from symbolic dynamics on h^2, symbolic dynamics on h^7 and connection between sets N_1, N_2 and N_3, N_4. For the construction of the transition matrix we use the following coverings:

	N_1	N_2	N_3	N_4
N_1	2	2		4
N_2	2			4
N_3	6		7	7
N_4			7	7

See that we do not use all the coverings from (36). We also do not include coverings between the sets N_3 and N_4 for lower iterations. The reason is that for larger set of coverings we could not prove that all sequences are separated. The obtained bound for the topological entropy is given by

$$\mathrm{H}(h) \geq \log \lambda_1 > 0.2528, \tag{61}$$

which is slightly better than the best lower bound known so far (compare (40)).

In the previous example we have obtained some improvement in the estimation of the topological entropy in comparison with (40). We were however not able to use all the coverings and the improvement was not very large. We have found other sets with coverings for lower iterations, which allows us to obtain better lower bound for the topological entropy of the Hénon map. Let us define the following five pairwise disjoint quadrangles (see also Fig. 7):

$$
\begin{aligned}
P_1 &= ((-0.95,0.29),(-0.95,0.39),(-0.81,0.38),(-0.77,0.28)),\\
P_2 &= ((-0.75,0.29),(-0.8,0.39),(-0.31,0.34),(-0.22,0.24)),\\
P_3 &= ((0.07,0.2),(0.24,0.3),(0.44,0.25),(0.36,0.15)),\\
P_4 &= ((0.518,0.07),(0.638,0.28),(0.72,0.28),(0.62,0.07)),\\
P_5 &= ((0.7,0.12),(0.74,0.23),(0.88,0.21),(0.85,0.1)).
\end{aligned}
\tag{62}
$$

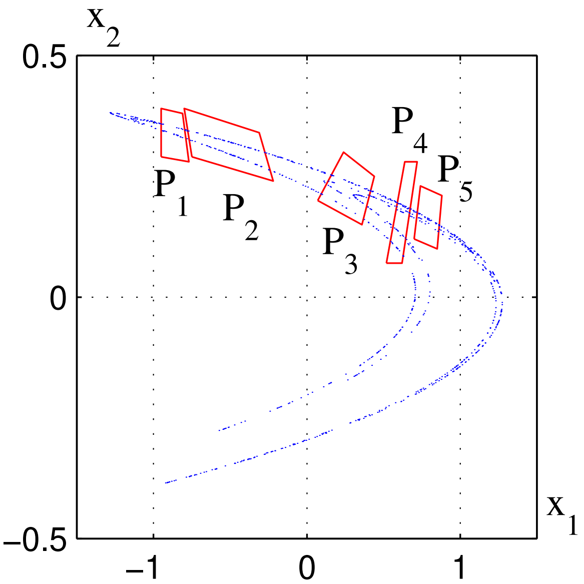

Figure 7. The sets $P_1, P_2, \ldots, P_5$.

We have proved that the following coverings take place:

- $P_1 \overset{h^2}{\Longrightarrow} P_4$,
- $P_2 \overset{h^2}{\Longrightarrow} P_1, P_2, P_3$,
- $P_3 \overset{h^2}{\Longrightarrow} P_1, P_2$,
- $P_4 \overset{h}{\Longrightarrow} P_4$, $P_4 \overset{h^3}{\Longrightarrow} P_5$,
- $P_5 \overset{h}{\Longrightarrow} P_3$.

The above coverings correspond to the following covering table:

	P_1	P_2	P_3	P_4	P_5
P_1				2	
P_2	2	2	2		
P_3	2	2			
P_4				1	3
P_5			1		

(63)

This is turn is equivalent to the following transition matrix:

$$\left(\begin{array}{cc|cc|cc|ccc|c}
 & 1 & & & & & & & & \\
 & & & & & & 1 & & & \\
\hline
 & & & 1 & & & & & & \\
1 & & 1 & & 1 & & & & & \\
\hline
 & & & & & 1 & & & & \\
1 & & 1 & & & & & & & \\
\hline
 & & & & & & 1 & 1 & & \\
 & & & & & & & & 1 & \\
 & & & & & & & & & 1 \\
\hline
 & & & & 1 & & & & &
\end{array}\right). \tag{64}$$

The logarithm of the leading eigenvalue of the above matrix is $\log\lambda_1 > 0.3381$. In order to show that the topological entropy of the Hénon map is larger than this number we prove that different sequences of symbols lead to different points. For the proof of separation of different covering sequences we need the following two facts:

$$\operatorname{dist}(h(P_4), P_5) > 0, \tag{65}$$

$$\operatorname{dist}(h(P_5), P_5) > 0, \tag{66}$$

which were checked by means of interval arithmetic.

Let us consider two covering sequences satisfying conditions (55). They must start from P_4 as for the other sets all coverings involve the same iteration number. We have to show that coverings starting with

- $P_4 \overset{\mathrm{h}}{\Rightarrow} P_4$ and $P_4 \overset{\mathrm{h}^3}{\Rightarrow} P_5$

are separated. This cannot be done directly as the condition $h^2(P_4) \cap P_5 = \emptyset$ does not hold. We consider two subcases:

- $P_4 \overset{\mathrm{h}}{\Rightarrow} P_4 \overset{\mathrm{h}}{\Rightarrow} P_4$ and $P_4 \overset{\mathrm{h}^3}{\Rightarrow} P_5$ — separated as $\operatorname{dist}(h(P_4), P_5) > 0$,
- $P_4 \overset{\mathrm{h}}{\Rightarrow} P_4 \overset{\mathrm{h}^3}{\Rightarrow} P_5$ and $P_4 \overset{\mathrm{h}^3}{\Rightarrow} P_5$ — separated as $\operatorname{dist}(h(P_5), P_5) > 0$.

From Lemma 5 and Lemma 3 it follows that the topological entropy of the Hénon map is

$$\mathrm{H}(h) \geq \log\lambda > 0.3381. \tag{67}$$

The entropy computations presented in this section can be summarized in the following

Theorem 5. $$0.3381 < \mathrm{H}(h) \leq \log 2 \approx 0.693.$$

Proof. The upper bound is obtained from [9][Theorem 4]. □

We have obtained a rigorous estimation for the topological entropy of the Hénon map which is significantly better than the one known before. We believe that it is possible to find other covering relations which would allow to obtain even better bounds. This hypotheses results from the estimation of topological entropy based on the number of low–period cycles, $\mathrm{H}(h) \approx 0.465$ (compare [3, 10]).

Acknowledgments

This research has been supported by Polish Committee of Scientific Research KBN, grants no. 2/P03A/021/15 and 2/P03A/011/18. This work was prepared during the one of the authors (GZ) stay at the Institute for Nonlinear Science, University of California, San Diego sponsored by the Fulbright fellowship and other author (PZ) stay at CDSNS, Georgia Tech, Atlanta.

References

[1] R. Bowen. Periodic points and measures for axiom A diffeomorphisms. *Trans. Amer. Math. Soc.*, 154:377–397, 1971.

[2] Z. Galias. Rigorous numerical studies of the existence of periodic orbits for the Hénon map. *J. of Universal Computer Science, Springer*, 4(2):114–124, 1998. (http://www.iicm.edu/jucs_4_2).

[3] Z. Galias. All periodic orbits with period $n \leq 26$ for the Hénon map. In *Proc. European Conference on Circuit Theory and Design, ECCTD'99*, volume 1, pages 361–364, Stresa, 1999.

[4] F. R. Gantmacher. *Applications of the Theory of Matrices.* Interscience, New York, 1959.

[5] M. Hénon. A two dimensional map with a strange attractor. *Commun. Math. Phys.*, 50:69–77, 1976.

[6] O. Knüppel. Profil — programmer's runtime optimized fast interval library. Technical Report 93.4, Technical University Hamburg – Harburg, July 1993.

[7] M. Misiurewicz and B. Szewc. Existence of a homoclinic point for the Hénon map. *Comm. Math. Phys.*, 75:285–291, 1980.

[8] A. Neumaier. *Interval methods for systems of equations.* Cambrigde University Press, 1990.

[9] S. Newhouse. Entropy and volume. *Ergodic Theory and Dynamical Systems*, 8:283–299, 1988.

[10] S. Newhouse and T. Pignataro. On the estimation to topological entropy. *Journal of Statistical Physics*, 72:1331–1351, 1993.

[11] C. Robinson. *Dynamical Systems: Stability, Symbolic Dynamics, and Chaos.* CRC Press, USA, 1995.

[12] D. Stoffer and K. Palmer. Rigorous verification of chaotic behavior of maps using validated shadowing. *Nonlinearity*, 12:1683–1698, 1999.

[13] A. Szymczak. A combinatorial procedure for finding isolating neighbourhoods and index pairs. *Proc. Roy. Soc. Edingurgh*, A127:1075–1088, 1997.

[14] M. Viana. Dynamics: A probabilistic and geomtric perspective. *Proc. of ICM'98 Berlin.*

[15] P. Zgliczyński. Computer assisted proof of chaos in the Rössler equations and the Hénon map. *Nonlinearity*, 10(1):243–252, 1997.

[16] P. Zgliczyński. Multidimensional perturbations of one-dimensional maps and stability of Sharkovskii ordering. *Int. J. of Bifurcation and Chaos*, 9(9):1867–1876, 1999.

[17] P. Zgliczyński. Sharkovskii's theorem for multidimensional perturbations of one–dimensional maps. *Ergodic Theory and Dynamical Systems*, 19:1655–1684, 1999.

[18] P. Zgliczyński. On periodic points for systems of weakly coupled 1-dim maps. *Nonlinear Analysis. TMA*, 2000.