Topological horseshoes and delay differential equations.

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Abstract

We show that if an ordinary differential equation $x' = f(x)$, where $x \in \mathbb{R}^n$ and $f \in \mathcal{C}^1$, has a topological horseshoe, then the corresponding delay equation $x'(t) = f(x(t-h))$ for small $h > 0$ also has a topological horseshoe, i.e. symbolic dynamics and an infinite number of periodic orbits. A method of computation of h is given in terms of topological properties of solutions of differential inclusion $x'(t) \in f(x(t)) + \overline{B}(0, \delta)$.

Keywords: delay differential equations, differential inclusions, fixed point index, periodic points, chaos

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1 Introduction.

The aim of this paper is to investigate the dynamical properties of delay equations with small delay using topological methods based on the notion of topological horseshoe ([GiZ]). The retarded functional differential equations (RFDEs) with small retardations were investigated by several authors (see [AP, C, D1, D2, JK, K, R1, R2]). It is known ([D1, D2, R1, R2]) that in the case of a globally Lipschitz RFDE with a sufficiently small delay there exists a finite-dimensional manifold of "Ryabov's special solutions" that attracts all solutions in the infinite-dimensional phase space of the RFDE. Thus, Ryabov's special solutions form a Lipschitz inertial manifold in the phase space of the RFDE. Recently, C. Chicone ([C]) proved that these manifolds are smooth and expansion in the small delay can be used to obtain the dynamical system on the inertial manifold. In particular, the long term dynamics of the RFDE is determined by a (smooth) ODE on the inertial manifold. In this paper we would like to present a different, topological approach to delay equations with small delay.

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We believe that our viewpoint sheds some new light on dynamical properties of such systems.

We wish to investigate the following problem:
Consider an ordinary differential equation

$$x' = f(x), \qquad x \in \mathbb{R}^n, \qquad f \in \mathcal{C}^1. \tag{1}$$

Assume that equation (1) does have a Smale's horseshoe [GH, KH] or a topological horseshoe ([GiZ] and references given there) for a suitable Poincaré map.

Does the delay differential equation

$$x'(t) = f(x(t-h)), \qquad h > 0 \tag{2}$$

have a topological horseshoe for sufficiently small delays?

Observe that the existence of a topological horseshoe implies the existence of symbolic dynamics and an infinite number of periodic orbits (see [GiZ] and references given there and Sections 6 and 8).

In this paper we show that the answer to the above question is affirmative - see Theorem 26 in Section 6. The proof is purely topological and is based on the continuation of fixed point indices of *all* periodic orbits from the topological horseshoe.

We also deal with the question of how one can obtain effectively the value of h (see Theorem 29 in Section 7). We show that this is related to some topological properties of the Poincaré map for the differential inclusion

$$x'(t) \in f(x(t)) + \overline{B}(0, \delta). \tag{3}$$

The obtained criterium can be checked rigorously on the computer just as in computer proofs of existence of symbolic dynamics for the Lorenz equations [GaZ] and other systems [Ga, W, ZNon]. For an algorithm for rigorous integration of differential inclusions see [ZKSPer].

While in this paper we focus on topological horseshoes, its continuation properties and their dynamical implications, our approach can be easily adapted to treat phenomena detected by the discrete Conley index (see [MMH] and the literature cited there).

It is quite obvious that our results can be extended to a more general class of delay equations, for example to state dependent delays. To do this one needs to find a way of switching off delay continuously by a compact homotopy.

The content of paper can be described as follows. In Section 2 we derive some basic estimates for (2), which imply the continuity and compactness for the semiflow induced by (2). In Section 3 we consider the notion of a Poincaré map for (2). The material in Sections 2 and 3 is standard and in a slightly different form can be found in textbooks [DGLW, H]. In Section 4 we recall the notion of covering relation from [GiZ] (see also [Zbif, ZSh]). This is one of the crucial tools used in this work, because closed loops of covering relations imply the existence of periodic points. In Section 5 we prove the basic continuation theorem - Theorem 18. In Section 6 apply it to prove Theorem 26, which provides an affirmative answer to the question possed at the beginning of this paper. In Section 7 we investigate and explore the connection between equations (2) and (3). As a result another main theorem in this paper - Theorem 29 - is proved. In Section 8 we apply our result to the Lorenz equations. Finally in the appendix for the convenience of the reader we have listed basic properties of the fixed point index for ANRs.

2 Delay differential equations

We start with fixing some basic notation. Let (X, ρ) be a metric space. If $Z \subset X$ and $x \in X$ then by $\mathrm{dist}\,(Z, x) = \inf_{z \in Z} \rho(z, x)$ we will denote the distance of x from set Z. For sets $Z, N \subset X$ and $\epsilon > 0$ we set $B_N(Z, \epsilon) = \{x \in N | \mathrm{dist}\,(Z, x) < \epsilon\}$. By $\overline{Z}$ we denote the closure of Z.

2.1 Basic estimates

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be smooth such that

$$\forall x,\, y \in \mathbb{R}^n \quad \|f(x) - f(y)\| \leq L\|x - y\|, \tag{4}$$

$$\forall x \in \mathbb{R}^n \quad \|f(x)\| \leq M. \tag{5}$$

Let $\bar{h} > 0$ and $v \in \mathcal{C} = C([-\bar{h}, 0], \mathbb{R}^n)$. For $h \in [0, \bar{h}]$ we consider the problem

$$x' = f(x(t - h)),\ t \geq 0 \tag{6}$$

$$x|_{[-\bar{h},0]} = v. \tag{7}$$

There is a unique solution $\bar{x} : [-\bar{h}, +\infty) \to \mathbb{R}^n$ of (6)-(7) and for $t \in [0, +\infty)$

$$\bar{x}(t) = \bar{x}(0) + \int_0^t f(\bar{x}(s - h))\mathrm{ds}. \tag{8}$$

In the sequel we will frequently use the following fundamental (see [HK])

Lemma 1 (Gronwall's Inequality) *Let C, K be nonnegative constants and $u : [a, b] \to [0, +\infty)$ be a continuous function which satisfy*

$$u(t) \leq C + \int_a^t Ku(s)\mathrm{ds}, \quad a \leq t \leq b. \tag{9}$$

Then

$$u(t) \leq Ce^{K(t-a)}, \quad a \leq t \leq b. \tag{10}$$

The next three lemmas show that the solution of the delay equation depends continuously on time, initial conditions and delay.

Lemma 2 *Assume that $h \in [0, \bar{h}]$ and $v \in \mathcal{C}$. Let $\bar{x} : [-\bar{h}, +\infty) \to \mathbb{R}^n$ be a solution of (6)-(7). Then for t_1, $t_2 \in [0, +\infty)$*

$$\|\bar{x}(t_1) - \bar{x}(t_2)\| \leq M|t_1 - t_2|. \tag{11}$$

Proof:

$$\|\bar{x}(t_1)-\bar{x}(t_2)\| = \|\int_0^{t_1} f(\bar{x}(s-h))\mathrm{ds} - \int_0^{t_2} f(\bar{x}(s-h))\mathrm{ds}\| =$$
$$\|\int_{t_2}^{t_1} f(\bar{x}(s-h))\mathrm{ds}\| \le |\int_{t_2}^{t_1} \|f(\bar{x}(s-h))\|\mathrm{ds}| \le M|t_1-t_2|.$$

∎

Lemma 3 *Assume that* $h\in[0,\bar{h}]$ *and* $x_1,\ x_2\in\mathcal{C}$. *Let* $\bar{x}_i:[-\bar{h},+\infty)\to\mathbb{R}^n$ *(*$i=1,2$*) be solution of* (6) *such that* $\bar{x}_i|_{[-\bar{h},0]}=x_i$. *Then for* $t\ge 0$

$$\|\bar{x}_1(t)-\bar{x}_2(t)\| \le \left(\|x_1(0)-x_2(0)\| + \int_{-h}^{0} L\|x_1(s)-x_2(s)\|\mathrm{ds}\right)e^{Lt}. \tag{12}$$

Proof: For $t\ge 0$

$$\|\bar{x}_1(t)-\bar{x}_2(t)\| \le \|\bar{x}_1(0)-\bar{x}_2(0)\| + \int_0^t L\|\bar{x}_1(\tau-h)-\bar{x}_2(\tau-h)\|\mathrm{d}\tau =$$
$$\|\bar{x}_1(0)-\bar{x}_2(0)\| + \int_{-h}^{t-h} L\|\bar{x}_1(s)-\bar{x}_2(s)\|\mathrm{ds} \le$$
$$\|\bar{x}_1(0)-\bar{x}_2(0)\| + \int_{-h}^{0} L\|\bar{x}_1(s)-\bar{x}_2(s)\|\mathrm{ds} + \int_0^t L\|\bar{x}_1(s)-\bar{x}_2(s)\|\mathrm{ds},$$

so the result follows by Gronwall's inequality. ∎

Lemma 4 *Assume that* $h_1,\ h_2\in[0,\bar{h}]$, $h_1<h_2$ *and* $x\in\mathcal{C}$.
Let $\bar{x}_i:[-\bar{h},+\infty)\to\mathbb{R}^n$ *(*$i=1,2$*) be a solution of*

$$x' = f(x(t-h_i)), \tag{13}$$

such that $\bar{x}_i|_{[-\bar{h},0]}=x$. *Then for* $t\ge 0$ *we have*

$$\|\bar{x}_1(t)-\bar{x}_2(t)\| \le 2(h_2-h_1)Me^{Lt}. \tag{14}$$

Proof: For $t\ge 0$

$$\|\bar{x}_1(t)-\bar{x}_2(t)\| = \|\int_0^t f(\bar{x}_1(s-h_1))\mathrm{ds} - \int_0^t f(\bar{x}_2(s-h_2))\mathrm{ds}\| =$$
$$\|\int_{-h_1}^{t-h_1} f(\bar{x}_1(s))\mathrm{ds} - \int_{-h_2}^{t-h_2} f(\bar{x}_2(s))\mathrm{ds}\| \le$$
$$\|\int_{-h_1}^{0} f(\bar{x}_1(s))\mathrm{ds} - \int_{-h_2}^{0} f(\bar{x}_2(s))\mathrm{ds}\| +$$
$$\|\int_{0}^{t-h_1} f(\bar{x}_1(s))\mathrm{ds} - \int_{0}^{t-h_2} f(\bar{x}_2(s))\mathrm{ds}\| \le$$
$$\int_{-h_2}^{-h_1} \|f(\bar{x}_2(s))\|\mathrm{ds} + \int_{-h_1}^{0} \|f(\bar{x}_1(s))-f(\bar{x}_2(s))\|\mathrm{ds} +$$
$$\int_{0}^{t-h_2} \|f(\bar{x}_1(s))-f(\bar{x}_2(s))\|\mathrm{ds} + \int_{t-h_2}^{t-h_1} \|f(\bar{x}_1(s))\|\mathrm{ds}.$$

Observe that

$$\begin{aligned}\int_0^{t-h_2} \|f(\bar{x}_1(s)) - f(\bar{x}_2(s))\|\mathrm{ds} &\le \int_0^t L\|\bar{x}_1(s) - \bar{x}_2(s))\|\mathrm{ds} \\ \int_{-h_1}^0 \|f(\bar{x}_1(s)) - f(\bar{x}_2(s))\|\mathrm{ds} &= 0.\end{aligned}$$

Hence

$$\|\bar{x}_1(t) - \bar{x}_2(t)\| \le 2M|h_1 - h_2| + L\int_0^t \|\bar{x}_1(t) - \bar{x}_2(t)\|\mathrm{ds} \tag{15}$$

so the result follows by the Gronwall's inequality. ∎

2.2 Delay equations as a semidynamical system.

Let $x : [-\bar{h}, +\infty) \to \mathbb{R}^n$. For $t \ge 0$ we define $x_t \in \mathcal{C}$ by

$$x_t(\theta) = x(t+\theta),\ \theta \in [-\bar{h}, 0]. \tag{16}$$

We define $\varphi : [0, \bar{h}] \times \mathcal{C} \times [0, +\infty) \to \mathcal{C}$ by the following rule. Let $(h, x, t) \in [0, \bar{h}] \times \mathcal{C} \times [0, +\infty)$ and $x^h : [-\bar{h}, +\infty) \to \mathbb{R}^n$ be the solution of (6) such that $x^h|_{[-\bar{h},0]} = x$. We put

$$\varphi(h, x, t) = x_t^h \in \mathcal{C}. \tag{17}$$

From Lemmas 2, 3 and 4 we obtain immediately

Corollary 5 *If $t_1,\ t_2 \ge \bar{h}$ then*

$$\|\varphi(h, x, t_1) - \varphi(h, x, t_2)\| \le M|t_1 - t_2|. \tag{18}$$

Corollary 6 *If $t \ge \bar{h}$ then*

$$\|\varphi(h, x_1, t) - \varphi(h, x_2, t)\| \le \left(\|x_1(0) - x_2(0)\| + \int_{-h}^0 L\|x_1(s) - x_2(s)\|\mathrm{ds}\right) e^{Lt}. \tag{19}$$

Corollary 7 *If $0 \le h_1, h_2 \le \bar{h}$ and $t \ge 0$, then*

$$\|\varphi(h_1, x, t) - \varphi(h_2, x, t)\| \le 2|h_2 - h_1|Me^{Lt}. \tag{20}$$

Corollary 8 *The function $\varphi : [0, \bar{h}] \times \mathcal{C} \times [\bar{h}, +\infty) \to \mathcal{C}$ satisfies locally the Lipschitz condition.*

Definition 1 *For any $L \subset \mathbb{R}^n$, $\epsilon \ge 0$ we define*

$$\begin{aligned} L_{\mathcal{C}} &= \{x \in \mathcal{C} : x(0) \in L\}, \\ L(\epsilon) &= \{x \in \mathcal{C} : \quad x(0) \in L, \qquad \|x - x(0)\| \le \epsilon\}\end{aligned}$$

∎

Corollary 9 *If $K \subset \mathbb{R}^n$ and $I \subset [\bar{h}, +\infty)$ are compact, then*

$$\varphi|_{[0,\bar{h}]\times K_{\mathcal{C}}\times I} : [0,\bar{h}] \times K_{\mathcal{C}} \times I \to \mathcal{C}$$

is compact.

Proof: By the Arzelà -Ascoli lemma it suffices to prove that $\varphi([0,\bar{h}] \times K_{\mathcal{C}} \times I)$ is equicontinuous and equibounded.

Let $\epsilon > 0$ be fixed. If $(h, x, t) \in [0, \bar{h}] \times K_{\mathcal{C}} \times I$, then for θ_1, $\theta_2 \in [-\bar{h}, 0]$ we get by Lemma 2

$$\|\varphi(h, x, t)(\theta_1) - \varphi(h, x, t)(\theta_2)\| \leq M|\theta_1 - \theta_2|,$$

hence the equicontinuity follows.

Let $T_1 = \sup I$. Then by Corollary 7 we have

$$\|\varphi(h, x, t) - \varphi(0, x, t)\| \leq 2\bar{h}Me^{LT_1} = C.$$

Let $W = \{\varphi(0, x, t) : \quad x \in K_{\mathcal{C}}, \quad t \in I\}$. It follows by the compactness of K and the continuity of Φ that there exists $R > 0$ such that $W \subset \overline{B}(0, R)$, hence $\varphi(h, x, t) \in \overline{B}(0, R + C)$. ∎

3 Poincaré section.

Consider an ordinary differential equation

$$x' = f(x), \tag{21}$$

where f is as in previous section.

The goal of this section is to show, that if we have a Poincaré map, P, for equation (21), then we can prove that for small delays, h, we can define a Poincaré map P_h for a delay differential equation corresponding to (21). Then we investigate compactness properties of P_h.

Let $\Phi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be the flow generated by (21). Since

$$\varphi(h, x, t)(0) = x^h_t(0) = x^h(t),$$

we have

$$\Phi_t(x(0)) = \varphi(0, x, t)(0).$$

Let $S \subset \mathbb{R}^n$ be $n-1$-dimensional, smooth submanifold homeomorphic to $\sum V_i$, where each $V_i \approx \mathbb{R}^{n-1}$. We call S a section for f if there are $\delta > 0$, a smooth function

$$\alpha : U \to \mathbb{R}$$

such that $\Phi|_{S\times(-\delta,\delta)}$ is a homeomorphism with an open range $U = \Phi(S\times(-\delta, \delta))$ and

$$\nabla_x\alpha \cdot f(x) \geq 1, \quad x \in U, \tag{22}$$

$$S = \alpha^{-1}(0) \cap U, \tag{23}$$

$$\alpha(\Phi(S \times (-\delta, 0))) < 0,\ \alpha(\Phi(S \times (0, \delta))) > 0. \tag{24}$$

Let $\Sigma \subset S$ be open and such that each point of Σ returns to S. We define a map $P : \Sigma \to S$ as follows:

$$P(x) = \Phi_{\tau(x)}(x) \tag{25}$$

where $\tau(x) > 0$ is the time of the first return to S. The map P is called a Poincaré map and $\tau : \mathrm{dom}\,(P) = \Sigma \to \mathbb{R}$ is called the return time map. Using the fact that the solution of a differential equation depends smoothly on its initial value and the implicit function theorem, it can be proved that both P and τ are smooth functions on $\mathrm{dom}\,(P)$.

From condition (22) it follows that

$$\forall x \in \mathrm{dom}\,(P) \quad \tau(x) > \delta. \tag{26}$$

Lemma 10 *For any $0 < \eta < \delta$ and compact set $K \subset dom(P)$ there exists $\bar{h} > 0$ such that for any $(h, x, t) \in [0, \bar{h}] \times K_{\mathcal{C}} \times [\tau(x(0)) - \eta, \tau(x(0)) + \eta]$ we have*

$$\begin{aligned}
\varphi(h, x, t)(\theta) &\in U, \quad \theta \in [-\bar{h}, 0], && (27)\\
\alpha(\varphi(h, x, \tau(x(0)) - \eta)(0)) &< 0, && (28)\\
\alpha(\varphi(h, x, \tau(x(0)) + \eta)(0)) &> 0, && (29)\\
\frac{d}{dt}\alpha(\varphi(h, x, t)(0)) &> 0. && (30)
\end{aligned}$$

In particular, for any $(h, x) \in [0, \bar{h}] \times K_{\mathcal{C}}$ there exists a unique time $t_{x^h} \in [\tau(x(0)) - \eta, \tau(x(0)) + \eta]$ such that

$$\varphi(h, x, t_{x^h})(0) \in S. \tag{31}$$

Proof: We define

$$K_{[-\eta,\eta]} = \{\Phi_t(x(0)) : \, x \in K_{\mathcal{C}}, \, t \in [\tau(x(0)) - \eta, \tau(x(0)) + \eta]\}$$

From the above definition it follows that $K_{[-\eta,\eta]}$ is compact and contained in U.

There exists $0 < r < \eta$ such that

$$\overline{B}(K_{[-\eta,\eta]}, 2r) \subset U. \tag{32}$$

Let

$$T = \max_{p \in K} \tau(p) > \delta. \tag{33}$$

Let $\bar{h} > 0$ be such that $2\bar{h}Me^{L(T+\eta)} \leq r$, $M\bar{h} \leq r$. Then for $t \in [\tau(x(0)) - \eta, \tau(x(0)) + \eta]$ we have by Lemma 4

$$\|\varphi(0, x, t) - \varphi(h, x, t)\| \leq 2\bar{h}Me^{Lt} \leq 2\bar{h}Me^{L(T+\eta)} \leq r. \tag{34}$$

Observe that for $\theta \in [-\bar{h}, 0]$ and $t \in [\tau(x(0)) - \eta, \tau(x(0)) + \eta]$ we have

$$\|\varphi(0, x, t)(\theta) - \varphi(0, x, t)(0)\| = \|\Phi_{t-\theta}(x) - \Phi_t(x)\| \leq M\bar{h} \leq r.$$

Hence

$$\varphi(0,x,t)(\theta) \in \overline{B}(K_{[-\eta,\eta]}, r),$$

and from (34) we obtain

$$\varphi(h,x,t)(\theta) \in \overline{B}(K_{[-\eta,\eta]}, 2r) \subset U,$$

so (27) follows.

Conditions (28) and (29) are obviously satisfied.

A direct calculation shows that

$$\frac{d}{dt}\alpha(\varphi(h,x,t)(0)) = \nabla_{x^h(t)}\alpha \cdot (f(x^h(t-h))). \tag{35}$$

Since

$$\nabla_{x^h(t)\alpha} \cdot f(x^h(t)) > 0, \tag{36}$$

$$\|f(x^h(t)) - f(x^h(t-h))\| \le L\|x^h(t) - x^h(t-h)\| \le LM\bar{h}, \tag{37}$$

we have for $\bar{h}$ sufficiently small

$$\nabla_{x^h(t)}\alpha \cdot f(x^h(t-h)) > 0$$

and the result follows. ∎

Corollary 11 *For any $K \subset dom P$ there exists $\bar{h} > 0$ such that the functions*

$$[0,\bar{h}] \times K_{\mathcal{C}} \ni (h,x) \to t_{x^h} \in \mathbb{R}, \tag{38}$$

$$F : [0,\bar{h}] \times K_{\mathcal{C}} \ni (h,x) \to \varphi(h,x,t_{x^h}) \in S_{\mathcal{C}} \subset \mathcal{C} \tag{39}$$

are continuous and compact.

Proof: Let η and $\bar{h}$ be such that Lemma 10 holds. We eventually further decrease $\bar{h}$ to have $\tau(x) - \eta > \bar{h}$ for $x \in K$.

Let $(h,x) \in [0,\bar{h}] \times K_{\mathcal{C}}$ and $\epsilon > 0$ be fixed. It follows from Lemma 10 that if ϵ is sufficiently small then

$$\alpha(\varphi(h,x,t_{x^h} - \epsilon)(0)) < 0, \quad \alpha(\varphi(h,x,(t_{x^h} + \epsilon)(0)) > 0. \tag{40}$$

By Lemmas 3, 4 if (h_1,x_1) is sufficiently close to (h,x) then we have that

$$\alpha(\varphi(h_1,x_1,t_{x^h} - \epsilon)(0)) < 0, \quad \alpha(\varphi(h_1,x_1,t_{x^h} + \epsilon)(0)) > 0, \tag{41}$$

so

$$t_{x^h} - \epsilon < t_{x_1^{h_1}} < t_{x^h} + \epsilon. \tag{42}$$

The compactness of F follows easily because τ is bounded on the compact set K and $t_{x^h} \in [\tau(x(0)) - \eta, \tau(x(0)) + \eta]$. ∎

For $h \in [0,\bar{h}]$ we set

$$P_h = F(h,\cdot) : K_{\mathcal{C}} \ni x \to \varphi(h,x,t_{x^h}) \in S_{\mathcal{C}}.$$

This defines a map $P : [0,\bar{h}] \times K_{\mathcal{C}} \to S_{\mathcal{C}}$.

Lemma 12 *Let $K \subset dom(P)$ be a compact set. Let $r > 0$ be fixed. If $\bar{h} > 0$ is sufficiently small then for $(h, x) \in [0, \bar{h}] \times K_{\mathcal{C}}$*

$$\|P_h(x) - P_0(x)\| < r \tag{43}$$

$$\|P_h(x) - P_h(x)(0)\| < r. \tag{44}$$

Proof: Let δ, η and $\bar{h}$ be such that Lemma 10 holds. Let

$$T = \max_{p \in K} \tau(p) > \delta. \tag{45}$$

We choose η and $\bar{h}$ to have $M\eta < \frac{r}{2}$ $2\bar{h}Me^{LT} < \frac{r}{2}$ and $M\bar{h} < \frac{r}{2}$.

Let $x \in K_{\mathcal{C}}$. Then from Corollaries 5 and 7

$$\begin{aligned} \|P_h(x) - P_0(x)\| = \|\varphi(h, x, t_{x^h}) - \varphi(0, x, \tau(x(0)))\| \leq \\ \|\varphi(h, x, t_{x^h}) - \varphi(h, x, \tau(x(0)))\| + \|\varphi(h, x, \tau(x(0))) - \varphi(0, x, \tau(x(0)))\| \leq \\ M|t_{x^h} - \tau(x(0))| + 2\bar{h}Me^{L\tau(x(0))} \leq \\ M\eta + 2\bar{h}Me^{LT} < \frac{r}{2} + \frac{r}{2} = r. \end{aligned}$$

For $\theta \in [-\bar{h}, 0]$ we have

$$\begin{aligned} \|P_h(x)(\theta) - P_h(x)(0)\| = \|\varphi(h, x, t_{x^h})(\theta) - \varphi(h, x, t_{x^h})(0)\| = \\ \|x^h_{t_{x^h}}(\theta) - x^h_{t_{x^h}}(0)\| = \|x^h(t_{x^h} + \theta) - x^h(t_{x^h})\| \leq \\ M\theta \leq M\bar{h} < \frac{r}{2}, \end{aligned}$$

so

$$\|P_h(x) - P_h(x)(0)\| < r. \tag{46}$$

∎

Later we will need the following lemma about the extendability of the Poincaré map for delay equations.

Lemma 13 *Assume that $N \subset dom(P)$ is compact. There exists $\bar{h} > 0$, a compact set M, $N \subset M$ and a continuous and compact map*

$$\widetilde{P} : [0, \bar{h}] \times M_{\mathcal{C}} \to M_{\mathcal{C}},$$

such that for all $h \in [0, \bar{h}]$ and $x \in N_{\mathcal{C}}$ holds

$$\widetilde{P}(h, x) = P_h(x). \tag{47}$$

Proof: Without loss of generality we may assume that $S = \{x_n = 0\}$. We will use the coordinates $(x_1, \ldots, x_{n-1})$ on S. From the properties of Poincaré maps for ODEs it follows immediately that there exists $\beta > 0$, such that $K = \overline{B_S(N, \beta)}$ is contained in the domain of P.

Let $\bar{h}$ be chosen as in Corollary 11 for K. Let $\gamma : S \to \mathbb{R}_+$ be a continuous function, such that $\gamma(x) = 1$ for $x \in N$ and $\gamma(x) = 0$ for $x \notin K$.

Let us define a deformation retraction, $R : [0,1] \times S_{\mathcal{C}} \to S_{\mathcal{C}}$, of $S_{\mathcal{C}}$ onto the set of constant functions by

$$R(\lambda, x)(t) = \begin{cases} x(t + \lambda\bar{h}), & \text{for } t \leq -\lambda\bar{h}, \\ x(0), & \text{otherwise.} \end{cases} \tag{48}$$

Consider now the map $G : [0, \bar{h}] \times K_{\mathcal{C}} \to K_{\mathcal{C}}$ given by

$$G(h, x) = P(\gamma(x(0))h, R(1 - \gamma(x(0)), x)). \tag{49}$$

This function is obviously continuous and compact. Observe that for $x \in N_{\mathcal{C}}$ and any $h \in [0, \bar{h}]$ we have

$$G(h, x) = P(h, R(0, x)) = P(h, x).$$

For $x \in \partial K_{\mathcal{C}} = \partial(K_{\mathcal{C}})$ we have

$$G(h, x) = P(0, x(0)).$$

Observe that $P(0, x(0))$ represents in $\mathcal{C}$ a constant function equal to $P(x(0))$.

Let $A > 0$ be such that the set M given by $M = [-A, A]^{n-1}$ contains $K \cup G([0, \bar{h}] \times K)$. We will extend now the function G to $M_{\mathcal{C}}$.

To this end, we first extend continuously the Poincaré map $P : K \to S$ to $G_{const} : M \to M$, using Tietze theorem.

Now we define $\widetilde{P}$ by

$$\widetilde{P}(h, x) = \begin{cases} G(h, x), & \text{for } x \in K_{\mathcal{C}}, \\ G_{const}(x(0)) & \text{otherwise.} \end{cases} \tag{50}$$

∎

Remark 14 *Observe that what is said in this section about a Poincaré map is valid also for any iterate of Poincaré map (obviously the domain will be smaller) and also for the Poincaré map between different sections.*

4 Covering relations.

In this section we will recall a notion of *covering relation* from [GiZ]. This notion is of crucial importance in this work. For example the topological horseshoe will be defined later using it (see Section 6). The practical usefulness of it is illustrated by the results of papers [ZNon, GaZ], where it was used in the computer assisted proof of the existence of symbolic dynamics for the Rössler and Lorenz equations, respectively. In these papers the notion of a covering relation was not yet explicitly defined, but the reader may consult [Zbif], where these results were described using the covering relations point of view.

Notation: For a given norm in $\mathbb{R}^n$, by $B_n(c, r)$ we denote the open ball of radius r centered at $c \in \mathbb{R}^n$. When the dimension n is obvious from the context, we will drop the subscript n. Let $S^n(c,r) = \partial B_{n+1}(c,r)$, by the symbol S^n we will denote $S^n(0,1)$. We set $\mathbb{R}^0 = \{0\}$, $B_0(0,r) = \{0\}$, $\partial B_0(0,r) = \emptyset$.

For a given set Z, by $\operatorname{int} Z$, $\overline{Z}$, ∂Z we denote the interior, the closure and the boundary of Z, respectively. For a map $h : [0,1] \times Z \to \mathbb{R}^n$, we set $h_t = h(t,\cdot)$. By Id we denote the identity map.

Definition 2 *An h-set, N, is an object consisting of the following data*

- $|N|$ *- a compact subset of* $\mathbb{R}^n$
- $u(N), s(N) \in \{0,1,2,\dots\}$, *such that* $u(N) + s(N) = n$
- *a homeomorphism* $c_N : \mathbb{R}^n \to \mathbb{R}^n = \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)}$, *such that*

$$c_N(|N|) = \overline{B_{u(N)}}(0,1) \times \overline{B_{s(N)}}(0,1).$$

We set

$$\begin{aligned} N_c &= \overline{B_{u(N)}}(0,1) \times \overline{B_{s(N)}}(0,1), \\ N_c^- &= \partial\overline{B_{u(N)}}(0,1) \times \overline{B_{s(N)}}(0,1) \\ N_c^+ &= \overline{B_{u(N)}}(0,1) \times \partial\overline{B_{s(N)}}(0,1) \\ N^- = c_N^{-1}&(N_c^-), \quad N^+ = c_N^{-1}(N_c^+) \end{aligned}$$

Hence an h-set N, is a product of two closed balls with respect to some coordinate system. The numbers, $u(N)$ and $s(N)$, stand for the dimensions of nominally unstable and stable subspaces, respectively. The subscript c refers to the new coordinates given by homeomorphism c_N. We will call N^- (N_c^-) an exit set of N and N^+ (N_c^+) an entry set of N. Notice that if $u(N) = 0$, then $N^- = \emptyset$ and if $s(N) = 0$, then $N^+ = \emptyset$.

Sometimes, in order to make the formulas less cumbersome, we will drop the vertical bars $|$ from the notation for the support $|N|$ of N, for example $f(N)$ will stand for $f(|N|)$.

Definition 3 *Assume N, M are h-sets, such that $u(N) = u(M) = u$ and $s(N) = s(M) = s$. Let $f : |N| \to \mathbb{R}^n$ be a continuous map. We say that*

$$N \overset{f}{\Longrightarrow} M$$

(N f-covers M) iff there exists a continuous homotopy $h : [0,1] \times |N| \to \mathbb{R}^n$, satisfying the following conditions, where the map $h^c : [0,1] \times N_c \to \mathbb{R}^u \times \mathbb{R}^s$ in condition 2 below is given by $h^c(t,(p,q)) = c_M(h(t, c_N^{-1}(p,q))$,

1.

$$\begin{aligned} h_0 &= f, \\ h([0,1], N^-) \cap M &= \emptyset, \\ h([0,1], N) \cap M^+ &= \emptyset. \end{aligned}$$

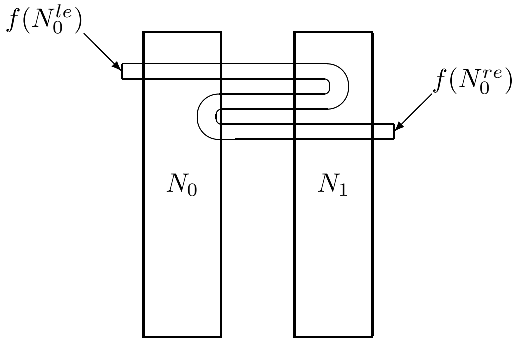

Figure 1: An example of an $f-$covering relations: $N_0 \overset{f}{\Longrightarrow} N_0$ and $N_0 \overset{f}{\Longrightarrow} N_1$.

2.1 *If $u > 0$, then there exists a linear map $A : \mathbb{R}^u \to \mathbb{R}^u$, such that*

$$\begin{aligned} h_1^c(p,q) &= (Ap, 0), \quad \textit{where } p \in \mathbb{R}^u \textit{ and } q \in \mathbb{R}^s, && (51) \\ A(\partial B_u(0,1)) &\subset \mathbb{R}^u \setminus \overline{B_u}(0,1). && (52) \end{aligned}$$

2.2 *If $u = 0$, then*

$$h_1^c(x) = 0, \qquad \textit{for } x \in N_c \tag{53}$$

Observe that the map h^c is just a homotopy h expressed in 'nice' coordinates on N and M. In the context of the above definition we will call the map h_1^c *a model map* for the relation $N \overset{f}{\Longrightarrow} M$.

Intuitively, $N \overset{f}{\Longrightarrow} M$, if f stretches N in the 'nominally unstable' direction, so that its projection onto 'unstable' direction in M covers in topologically nontrivial manner the projection of M. In the 'nominally stable' direction N is contracted by f. As a result N is mapped across M in the unstable direction, without touching M^+.

An example of covering relation on the plane with one unstable direction is shown on Figure 1. In the case illustrated on this figure the horizontal direction is nominally expanding and the vertical one is contracting. The set N^- does have two components N^{le} and N^{re} being left and right edges, respectively.

Remark 15 *When $u > 0$, then condition (52) is equivalent to each of the following conditions*

$$\begin{aligned} \overline{B_u}(0,1) &\subset A(B_u(0,1)) \\ \|Ap\| &> 1, \qquad \textit{for } p \in \partial B_u(0,1) \\ \|Ap\| &> \|p\|, \qquad \textit{for } p \neq 0 \end{aligned}$$

Remark 16 *If $u = 0$, then it is easy to see that $N \stackrel{f}{\Longrightarrow} M$, iff $f(|N|) \subset \operatorname{int}|M|$.*

Remark 17 *When $u = 0$, we can formally define a map $A : \mathbb{R}^u \to \mathbb{R}^u$, by $A(0) = 0$. With this definition and taking into account that $\partial B_u(0,1) = \emptyset$, we see that if $N \stackrel{f}{\Longrightarrow} M$, then the conditions (51) and (52) are satisfied. We will always make use this formalism in proving theorems on covering relations.*

5 Continuation of covering relations for Poincaré map for delay equations

In this section we prove basic continuation results for the Poincaré maps for delay equation (6) for small delays. We adopt here the notation and assumptions about the existence of Poincaré map from Section 3.

Since we will be dealing in this section with covering relations on the Poincaré section S all topological and metrical notions will be relatively to S (for finite dimensional objects) and $S_{\mathcal{C}}$ (for infinite dimensional objects related to delay equations).

We will use the fixed point index theory for arbitrary ANR's introduced in [Gr] (see Appendix).

The goal of this section is to establish the following

Theorem 18 *Let $\mathcal{H} = \{N_s\}$ be a finite collection of h-sets contained in dom (P). Assume that we have a set Γ of covering relations $N_i \stackrel{P}{\Longrightarrow} N_j$ for some $N_i, N_j \in \mathcal{H}$.*

Consider any closed loop of covering relations from Γ

$$N_0 \stackrel{P}{\Longrightarrow} N_1 \stackrel{P}{\Longrightarrow} N_2 \stackrel{P}{\Longrightarrow} \dots N_{k-1} \stackrel{P}{\Longrightarrow} N_k = N_0,$$

where $N_i \in \mathcal{H}$.

For $h \geq 0$ we set

$$N^h = \{x \in N_0(\epsilon) : \quad \forall i \in \{1, \dots, k-1\} \quad P_h^i(x) \in N_i(\epsilon)\} \subset N_{\mathcal{C}}. \tag{54}$$

There are $\epsilon > 0$ and $\bar{h} > 0$, depending only on Γ and not on k, such that for $h \in [0, \bar{h}]$ the fixed point index $\operatorname{ind}(P_h^k, \operatorname{int} N^h)$ is well defined and

$$ind\,(P_h^k, \operatorname{int} N^h) = \pm 1.$$

∎

The following lemma says that open subsets of $S_{\mathcal{C}}$ are ANRs (see Appendix for a definition). Hence it makes sense (provided the map is admissible) to consider the fixed point index in Theorem 18.

Lemma 19 *If $W \subset S_{\mathcal{C}}$ is open, then W is an ANR.*

Proof: It is enough to show that $S_{\mathcal{C}}$ is an ANR. We construct a retraction $r : U_{\mathcal{C}} \to S_{\mathcal{C}}$, where $U \subset \mathbb{R}^n$ is open set in the definition of the Poincaré section. Without any loss of generality we can assume that $S = \{x_1 = 0\}$ and $U = (-\delta, \delta) \times V$, where $V \subset \mathbb{R}^n$. For any $x \in U_{\mathcal{C}}$ we define $r(x)(t) = (x_1(t) - x_1(0), x_2(t), \ldots, x_n(t))$ for $t \in [-h, 0]$. One can check that r is an retraction. Since $U_{\mathcal{C}} \subset \mathcal{C}$ is open, we have that $S_{\mathcal{C}}$ is an ANR. ∎

The proof of Theorem 18 involves several steps:

1. A continuation theorem in infinite dimension showing that $\mathrm{ind}(P_h^k, \mathrm{int}\, N^h) = \mathrm{ind}(P_0^k, \mathrm{int}\, N^0)$

2. A reduction to finite dimension. We need to show that $\mathrm{ind}\,(P_h^k, \mathrm{int}\, N^h) = \mathrm{ind}\,(P^k, \mathrm{int}\, \hat{N})$, where $\hat{N} = \{x \in N_0 : \quad \forall i \in \{1, \ldots, k-1\}\, P^i(x) \in N_i\}$.

3. A continuation theorem in finite dimension, showing that we can replace $P : N_i \to \mathbb{R}^n$ by model maps $h_{i,i}$.

4. A computation of index for compositions of model maps.

5.1 Toward the proof - a preparatory lemma

For an h-set $N \subset \mathrm{dom}\,(P)$ and $\epsilon > 0$ we put

$$\begin{aligned} N(\epsilon) &= \{x \in \mathcal{C}, \quad x(0) \in |N|, \qquad \|x - x(0)\| \leq \epsilon\} \\ N(\epsilon)^- &= \{x \in N(\epsilon) : x(0) \in N^-\}, \\ N(\epsilon)^+ &= \{x \in N(\epsilon) : x(0) \in N^+ \text{ or } \|x - x(0)\| = \epsilon\}. \end{aligned}$$

For $\epsilon > \rho > 0$ and an h-set N we set

$$\begin{aligned} N^-(\epsilon, \rho) &= \{x \in N(\epsilon) : x(0) \in \overline{B}_N(N^-, \rho)\}, && (55) \\ N^+(\epsilon, \rho) &= \{x \in N(\epsilon) : x(0) \in \overline{B}_N(N^+, \rho) && (56) \\ &\quad \text{or} \quad \epsilon - \rho \leq \|x - x(0)\| \leq \epsilon\}. \end{aligned}$$

Lemma 20 *Let $N \subset$ dom$\,(P)$, M be h-sets such that $N \overset{P}{\Longrightarrow} M$.*
Then there exist $\epsilon, \rho, \bar{h}$, $0 < 2\rho < \epsilon$ and $\bar{h} > 0$ such that for $h \in [0, \bar{h}]$

$$\begin{aligned} P_h(N^-(\epsilon, 2\rho)) \cap M(\epsilon) &= \emptyset, && (57) \\ P_h(N(\epsilon)) \cap M^+(\epsilon, 2\rho) &= \emptyset. && (58) \end{aligned}$$

Proof: From condition *1* in the definition of covering relations it follows immediately that there exists ϵ and ρ, $0 < 2\rho < \epsilon$ such that

$$\begin{aligned} \overline{B}(P(\overline{B}_N(N^-, 2\rho)), \epsilon) \cap M &= \emptyset, && (59) \\ \overline{B}(P(N), \epsilon) \cap \overline{B}_M(M^+, 2\rho) &= \emptyset. && (60) \end{aligned}$$

We fix such ϵ and ρ. Observe that we have the freedom of further decreasing ρ without violating conditions (59) and (60).

Let $K \subset \mathrm{dom}\,(P)$ be a compact neighborhood of N. By Lemma 12 there exists $\bar{h} > 0$ such that for $(h, x) \in [0, \bar{h}] \times K_{\mathcal{C}}$

$$\|P_h(x) - P_0(x)\| < \epsilon. \tag{61}$$

If $x \in N^-(\epsilon, 2\rho)$ then

$$P_0(x)(0) = \varphi(0, x, \tau(x(0)))(0) \in P(\overline{B}_N(N^-, 2\rho)),$$

so

$$P_h(x)(0) \in \overline{B}(P(\overline{B}_N(N^-, 2\rho)), \epsilon),$$

hence by (59) $P_h(x)(0) \notin M$, and consequently $P_h(x) \notin M(\epsilon)$.

To prove (58) we show that there exists $\bar{h} > 0$ such that for any $x \in N(\epsilon)$ and $0 \leq h \leq \bar{h}$ the following two conditions are satisfied

$$\begin{aligned} \|P_h(x) - P_h(x)(0)\| &< \epsilon - \rho \qquad (62) \\ P_h(x)(0) &\notin \overline{B}_M(M^+, 2\rho) \qquad (63) \end{aligned}$$

From Lemma 12 for $(h, x) \in [0, \bar{h}] \times K_{\mathcal{C}}$ it follows that for sufficiently small $\bar{h}$ we have for all $0 \leq h \leq \bar{h}$

$$\begin{aligned} \|P_h(x) - P_h(x)(0)\| &< \epsilon - 2\rho, \qquad (64) \\ \|P_h(x)(0) - P_0(x)(0)\| &< \epsilon. \qquad (65) \end{aligned}$$

From (65) it follows that

$$P_h(x)(0) \in \overline{B}(P(N), \epsilon),$$

which combined (60) proves (63). ∎

5.2 Continuation theorems

Let $N_0, \ldots, N_k \subset \mathrm{dom}\,(P)$ be h-sets such that $N_k = N_0$. Sometimes we will use the convention $N_{-1} = N_{k-1}$.

Let $N = \bigcup_{i=1}^k N_i$ and $I \subset \mathbb{R}$ be a compact interval. Assume that for some $K \subset S$, such that $N \subset K$ the map

$$F : I \times K_{\mathcal{C}} \to K_{\mathcal{C}}$$

is compact.

The assumption that the map F_λ does have the same domain and range is a technical one. It is used to have the freedom of taking arbitrary iterates of F_λ in the proof of Theorem 21 below.

For $\lambda \in I$ we set

$$N^\lambda = \{x \in N_0(\epsilon) : \quad \forall i \in \{1, \ldots, k-1\} \quad F^i_\lambda(x) \in N_i(\epsilon)\} \subset N_{\mathcal{C}}. \tag{66}$$

For any $\lambda \in I$ the set N^λ is closed. For $\lambda \in I$ we put

$$\mathrm{Fix}_\lambda(N^\lambda) = \{x \in N^\lambda : \quad F^k_\lambda(x) = x\}. \tag{67}$$

Theorem 21 *Assume that there are $0 < 2\rho < \epsilon$ such that*

$$
\begin{aligned}
F_\lambda(N_{i-1}^-(\epsilon, 2\rho)) \cap N_i(\epsilon) &= \emptyset, && (68)\\
F_\lambda(N_{i-1}(\epsilon)) \cap N_i^+(\epsilon, 2\rho) &= \emptyset, && (69)\\
\|F_\lambda(x) - F_\lambda(x)(0)\| &< \epsilon - 2\rho, \qquad \forall x \in N(\epsilon) && (70)
\end{aligned}
$$

Then the fixed point index $\mathrm{ind}(F_\lambda^k, \mathrm{int}\, N^\lambda)$ *is well defined and constant (i.e. does not depend on* λ*). In particular for* $\lambda \in I$

$$\mathrm{ind}(F_\lambda^k, \mathrm{int}\, N^\lambda) = \mathrm{ind}(F_0^k, \mathrm{int}\, N^0).$$

A theorem similar in spirit was proved in [ZSh2, Thm. 3] in a slightly different context of compact maps on a Banach space with one unstable direction. The proof below is just an adaptation of this proof.

Proof: For $\lambda \in I$

$$\mathrm{Fix}_\lambda(N^\lambda) \subset \overline{\mathrm{Im}\, F}, \tag{71}$$

where $\mathrm{Im}\, F = F(I \times N_{\mathcal{C}})$. Obviously $\mathrm{Fix}_\lambda(N^\lambda)$ is a compact set.

For $i \in \{1, \dots, k\}$ we define

$$
\begin{aligned}
C_i = \{x \in N_i(\epsilon) : x(0) \in \mathrm{int}\, N_i \setminus \left(\overline{B_{N_i}}(N_i^-, \rho) \cup \overline{B_{N_i}}(N_i^+, \rho)\right), \\
\|x - x(0)\| < \epsilon - \rho\},
\end{aligned}
$$

$$
\begin{aligned}
D_i = \{x \in N_i(\epsilon) : x(0) \in \mathrm{int}\, N_i \setminus \left(\overline{B_{N_i}}(N_i^-, 2\rho) \cup \overline{B_{N_i}}(N_i^+, 2\rho)\right), \\
\|x - x(0)\| < \epsilon - 2\rho\}.
\end{aligned}
$$

Obviously C_i, D_i are open (relatively to $S_{\mathcal{C}}$) and

$$B(D_i, \rho/2) \subset C_i, \quad B(C_i, \rho/2) \subset \mathrm{int}\, N_i(\epsilon).$$

For $\lambda \in I$ we put

$$
\begin{aligned}
C^\lambda &= \{x \in C_0 : \forall i \in \{1, \dots, k-1\} \quad F_\lambda^i(x) \in C_i\}, \\
D^\lambda &= \{x \in D_0 : \forall i \in \{1, \dots, k-1\} \quad F_\lambda^i(x) \in D_i\}.
\end{aligned}
$$

Obviously C^λ, D^λ are open and $D^\lambda \subset C^\lambda \subset \mathrm{int}\, N^\lambda$.

We will show that, for $\lambda \in I$

$$F_\lambda^k(x) \neq x, \quad x \in N^\lambda \setminus D^\lambda. \tag{72}$$

Let us take any $\lambda \in I$ and $x \in N^\lambda \setminus D^\lambda$. Since $x \in N^\lambda \setminus D^\lambda$, we have that there exists $0 \leq i \leq k-1$ such that (we choose a smallest such i)

$$x_i = F_\lambda^i(x) \in N_i(\epsilon) \setminus D_i. \tag{73}$$

We will show that $x_i \notin F_\lambda(N_{i-1})$ or $F_\lambda(x_i) \notin N_{i+1}$, which means that $x \notin N^\lambda$ or $F_\lambda^k(x) \neq x$.

From (73) it follows that one of the following conditions is satisfied

$$\begin{aligned} x_i(0) &\in \overline{B_{N_i}}(N_i^-, 2\rho) && (74)\\ x_i(0) &\in \overline{B_{N_i}}(N_i^+, 2\rho) && (75)\\ \|x_i - x_i(0)\| &\geq \epsilon - 2\rho && (76) \end{aligned}$$

Observe that (76) contradicts (70). If (74) holds, then from (68) it follows that

$$F_\lambda(x_i) \notin N_{i+1}(\epsilon), \tag{77}$$

hence if $i < k-1$, then $x \notin N^\lambda$. If $i = k-1$ then $F_\lambda(x_i) = F_\lambda^k(x) \neq x$.

If (75) holds then from (69) it follows that

$$x_i \notin F_\lambda(N_{i-1}(\epsilon)), \tag{78}$$

hence if $i > 0$ then we have the contradiction, because $x_i = F_\lambda^i(x) \in F_\lambda(N_{i-1}(\epsilon))$. If $i = 0$, then $F_\lambda^k(x) \neq x$. This concludes the proof of (72).

From (72) it follows that for $\lambda \in I$

$$\mathrm{Fix}_\lambda(D^\lambda) = \mathrm{Fix}_\lambda(C^\lambda) = \mathrm{Fix}_\lambda(\mathrm{int}\, N^\lambda) = \mathrm{Fix}_\lambda(N^\lambda). \tag{79}$$

Let $\lambda_0 \in I$ be fixed. It follows from the uniform continuity of continuous maps on compact sets that there exists an interval Λ, open in I, $\lambda_0 \in \overline{\Lambda}$, such that for every λ_1, $\lambda_2 \in \overline{\Lambda}$, $i \in \{1,\dots,k-1\}$ and $x \in \mathrm{Im}F$ one has

$$\|F_{\lambda_1}^i(x) - F_{\lambda_2}^i(x)\| < \frac{\rho}{2}. \tag{80}$$

We will show now the following inclusions for $\lambda \in \overline{\Lambda}$

$$\mathrm{Im}F \cap D^\lambda \subset \mathrm{Im}F \cap C^{\lambda_0} \subset \mathrm{Im}F \cap \mathrm{int}\, N^\lambda. \tag{81}$$

Let $x \in \mathrm{Im}F \cap D^\lambda$. Then for $i \in \{1,\dots,k-1\}$ $F_\lambda^i(x) \in D_i$. Since $\|F_\lambda^i(x) - F_{\lambda_0}^i(x)\| < \frac{\rho}{2}$, we have for $i \in \{1,\dots,k-1\}$ $F_{\lambda_0}^i(x) \in B(D_i, \rho/2) \subset C_i$, hence $x \in \mathrm{Im}F \cap C^{\lambda_0}$.

Let $x \in \mathrm{Im}F \cap C^{\lambda_0}$ and $\lambda \in \overline{\Lambda}$. Then for all $i \in \{1,\dots,k-1\}$ $F_{\lambda_0}^i(x) \in C_i$ and $\|F_\lambda^i(x) - F_{\lambda_0}^i\| < \frac{\rho}{2}$, so $F_\lambda^i(x) \in B(C_i, \rho/2) \subset \mathrm{int}\, N_i(\epsilon)$, hence $x \in \mathrm{int}\, N^\lambda$.

It follows that for $\lambda \in \overline{\Lambda}$

$$\mathrm{Fix}_\lambda(C^{\lambda_0}) = \mathrm{Fix}_\lambda(N^\lambda). \tag{82}$$

Since $\mathrm{Fix}_\lambda(N^\lambda)$ is a compact set, we have from Lemma 19 it follows that the fixed point indices of F_λ^k relatively to the sets $\mathrm{int}\, N^\lambda$, C^{λ_0}, D^λ are well defined. From the excision property of the fixed point index we conclude that for $\lambda \in \overline{\Lambda}$

$$\mathrm{ind}(F_\lambda^k, D^\lambda) = \mathrm{ind}(F_\lambda^k, C^{\lambda_0}) = \mathrm{ind}(F_\lambda^k, \mathrm{int}\, N^\lambda). \tag{83}$$

We put

$$\mathrm{Fix}_{\overline{\Lambda}}(C^{\lambda_0}) = \bigcup_{\lambda \in \overline{\Lambda}} \mathrm{Fix}_\lambda(C^{\lambda_0}) \subset C^{\lambda_0}.$$

It follows that

$$\mathrm{Fix}_{\overline{\Lambda}}(C^{\lambda_0}) = \bigcup_{\lambda\in\overline{\Lambda}} \mathrm{Fix}_\lambda(N^\lambda),$$

so by the compactness of F we get that $\mathrm{Fix}_{\overline{\Lambda}}(C^{\lambda_0})$ is compact. Now from the homotopy property of the fixed point index, we obtain for $\lambda \in \overline{\Lambda}$

$$\mathrm{ind}(F^k_\lambda, C^{\lambda_0}) = \mathrm{ind}(F^k_{\lambda_0}, C^{\lambda_0}), \tag{84}$$

so

$$\mathrm{ind}(F^k_\lambda, D^\lambda) = \mathrm{ind}(F^k_{\lambda_0}, D^{\lambda_0}),$$

hence

$$\mathrm{ind}(F^k_\lambda, \mathrm{int}\, N^\lambda) = \mathrm{ind}(F^k_{\lambda_0}, \mathrm{int}\, N^{\lambda_0}).$$

By the connectedness of I we get that for $\lambda \in I$

$$\mathrm{ind}(F^k_\lambda, \mathrm{int}\, N^\lambda) = \mathrm{ind}(F^k_0, \mathrm{int}\, N^0).$$

∎

Let $N_0, \ldots, N_k$ be h-sets such that $N_k = N_0$. Let $N = \bigcup_{i=1}^k N_i$. Assume that for $i \in \{1, \ldots, k\}$

$$F_i : [0,1] \times N_{i-1} \to \mathbb{R}^n$$

are continuous. For $\lambda \in [0,1]$ we set

$$\hat{N}^\lambda = \{x \in N_0 : \quad \forall i \in \{1, \ldots, k-1\} \quad F_{i,\lambda} \circ \ldots \circ F_{1,\lambda}(x) \in N_i\}, \tag{85}$$

The proof of Theorem 21 contains also the proof of the following (see also [Ztmna, Thm. 2.2])

Theorem 22 *Assume that there is $\rho > 0$ such that*

$$F_{i,\lambda}(\overline{B}_N(N^-_{i-1}, \rho)) \cap N_i = \emptyset, \tag{86}$$

$$F_{i,\lambda}(N_{i-1}) \cap \overline{B}(N^+_i, \rho) = \emptyset. \tag{87}$$

Then the fixed point index $\mathrm{ind}(F_{k,\lambda} \circ \ldots \circ F_{1,\lambda}, \mathrm{int}\, \hat{N}^\lambda)$ *is well defined and does not depend on* λ.

For the proof of this theorem we have to drop from the proof of Theorem 21 any condition involving an estimate for $\|x - x(0)\|$ and in each argument involving the map F_λ on N_{i-1} we need to substitute it by $F_{i-1,\lambda}$. For example we define

$$\mathrm{Fix}(\hat{N}^\lambda) = \{x \in \hat{N} \;:\; F_{k,\lambda} \circ F_{k-1,\lambda} \circ \cdots \circ F_{1,\lambda}(x) = x\}$$

5.3 Reduction to finite dimension

Theorem 23 *We make the same assumptions as in Theorem 18. Then there exists $\epsilon > 0$ and $\bar{h} > 0$, depending only on Γ (the set of covering relations) and not on k, such that for $h \in [0, \bar{h}]$ the fixed point index $\mathrm{ind}(P_h^k, \mathrm{int}\, N^h)$ is well defined and constant (i.e. does not depend on h). In particular for $h \in [0, \bar{h}]$*

$$\mathrm{ind}(P_h^k, \mathrm{int}\, N^h) = \mathrm{ind}(P_0^k, \mathrm{int}\, N^0).$$

Proof: By Lemma 13 we can extend the domain of P_h for $h \in [0, \bar{h}]$, so that $P_h : M_{\mathcal{C}} \to M_{\mathcal{C}}$. Now from Lemma 20 applied to all the covering relations in Γ and Lemma 12 we know there exist $0 < 2\rho < \epsilon$ and $\bar{h} > 0$ such that for any $h \in [0, \bar{h}]$ and $i \in \{1, \ldots, k\}$

$$\begin{aligned}
P_h(N_{i-1}^-(\epsilon, 2\rho)) \cap N_i(\epsilon) &= \emptyset, && (88)\\
P_h(N_{i-1}(\epsilon)) \cap N_i^+(\epsilon, 2\rho) &= \emptyset, && (89)\\
\|P_h(x) - P_h(x)(0)\| &< \epsilon - 2\rho, \qquad \forall x \in N(\epsilon), && (90)
\end{aligned}$$

so the result follows by Theorem 21. ∎

Our next goal is to reduce the computation of $\mathrm{ind}\,(P_h^k, \mathrm{int}\, N^h)$ to the computation of the index of the Poincaré map on the subspace of constant functions in $S_{\mathcal{C}}$.

For $\lambda \in [0, 1]$ we define

$$\beta_\lambda : [-\bar{h}, 0] \to [-\bar{h}, 0]$$

by

$$\beta_\lambda(s) = \begin{cases} 0 & , s \in [-\lambda\bar{h}, 0] \\ s + \lambda\bar{h} & s \in [-\bar{h}, -\lambda\bar{h}]. \end{cases} \tag{91}$$

Observe that

$$\beta_1(s) = 0, \quad \beta_0(s) = s, \quad s \in [-\bar{h}, 0].$$

We define a homotopy

$$H : [0, 1] \times N_{\mathcal{C}} \to S_{\mathcal{C}}$$

by the formula

$$H(\lambda, x)(s) = P_0(x)(\beta_\lambda(s)). \tag{92}$$

It follows that $H_0 = P_0$ and $H_1(x) = P_0(x)(0)$ is a constant map for any $x \in N_{\mathcal{C}}$.

We put

$$\tilde{N}^\lambda = \{x \in N_0(\epsilon) : \quad \forall i \in \{1, \ldots, k-1\} \quad H_\lambda^i(x) \in N_i(\epsilon)\}. \tag{93}$$

Let $\mathcal{C}_0 \subset \mathcal{C}$ be the subspace of constant functions. We put

$$\hat{N} = \{x \in N_0 : \quad \forall i \in \{1, \ldots, k-1\} \quad P^i(x) \in N_i\}. \tag{94}$$

Theorem 24 *Under the above assumptions*

$$\operatorname{ind}(P_0^k, \operatorname{int} N^0) = \operatorname{ind}(P^k, \operatorname{int} \hat{N}). \tag{95}$$

Proof: By the compactness of $H_0 = P_0$ on $N_{\mathcal{C}}$, one can easily check that H is compact. Obviously $\tilde{N}^0 = N^0$. Since P_0 fulfils assumptions (88)-(90), we have that the homotopy H fulfils the assumptions (68)-(70), hence by Theorem 21 we get

$$\operatorname{ind}(P_0^k, \operatorname{int} N^0) = \operatorname{ind}(H_1^k, \operatorname{int} \tilde{N}^1). \tag{96}$$

Since

$$H_1(N_{\mathcal{C}}) \subset \mathcal{C}_0, \tag{97}$$

$$H_1|_{\mathcal{C}_0 \cap N_{\mathcal{C}}} = P|_N, \tag{98}$$

$$\mathcal{C}_0 \cap \tilde{N}^1 = \hat{N}, \tag{99}$$

we have by the commutativity property of the fixed point index ([Gr]) we get

$$\operatorname{ind}(H_1^k, \operatorname{int} \tilde{N}^1) = \operatorname{ind}(P^k, \operatorname{int} \hat{N}). \tag{100}$$

∎

5.4 Continuation in finite dimension - the completion of the proof of Theorem 18.

In view of the results in the preceding subsections it remains to compute $\operatorname{ind}(P^k, \operatorname{int} \hat{N})$.

From Definition 2 it follows that for $i \in \{1, \ldots, k\}$ there exists a homotopy

$$h_i : [0,1] \times N_{i-1} \to \mathbb{R}^n$$

such that

$$h_{i,0} = P,$$

$$h_i([0,1], N_{i-1}^-) \cap N_i = \emptyset,$$

$$h_i([0,1], N_{i-1}) \cap N_i^+ = \emptyset,$$

and conditions 2.1-2.2 hold.

By Theorem 22 the fixed point index

$$\operatorname{ind}(h_{k,\lambda} \circ \ldots \circ h_{1,\lambda}, \operatorname{int} \hat{N}^\lambda)$$

is well defined and does not depend on λ.

Since $h_{i,0} = P$, we have $\hat{N}^\lambda = \hat{N}$ and we obtain

$$\operatorname{ind}(P^k, \operatorname{int} \hat{N}) = \operatorname{ind}(h_{k,1} \circ \ldots \circ h_{1,1}, \operatorname{int} \hat{N}^1).$$

Direct calculations based on the conditions 2.1-2.2 show (see for example [ZNon] for the case of one unstable direction) that

$$\operatorname{ind}(h_{k,1} \circ \ldots \circ h_{1,1}, \operatorname{int} \hat{N}^1) = \pm 1.$$

This finishes the proof of Theorem 18.

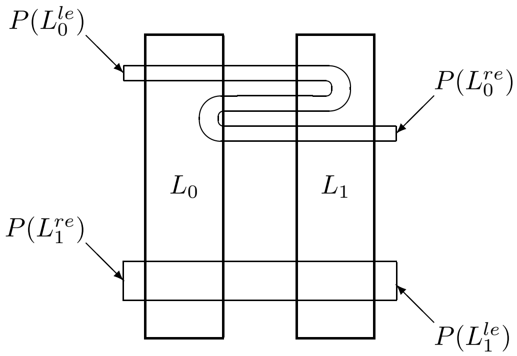

Figure 2: Topological horseshoe on the plane.

6 Continuation of topological horseshoes

Consider now an ordinary differential equation

$$x' = f(x), \qquad x \in \mathbb{R}^n, \quad f \in \mathcal{C}^1 \tag{101}$$

Let P be a Poincaré map for (101).

Definition 4 *Let L_0, $L_1 \subset S$ be disjoint h-sets and $L = L_0 \cup L_1$. We say that the Poincaré map P has a topological horseshoe (on L_0 and L_1) if*

$$L_i \overset{P}{\Longrightarrow} L_j \quad \text{for} \quad i, j = 0, 1. \tag{102}$$

An example of topological horseshoe for a planar map with one unstable direction is shown on Figure 2. In the case illustrated on this figure the horizontal direction is nominally expanding and the vertical one is contracting. The sets L_i^- do have two components L_i^{le} and L_i^{re} being left and right edges, respectively.

From Theorem 18 we obtain

Corollary 25 *Assume that a suitable Poincaré map, P, for (101) has a topological horseshoe. Then there are $\epsilon > 0$ and $\bar{h} > 0$ such that for any $c = (c_0, \ldots, c_{k-1}) \in \{0,1\}^k$ and $h \in [0, \bar{h}]$ there exists $x \in L_{c_0}(\epsilon)$*

$$P_h^i(x) \in L_{c_i}(\epsilon), \tag{103}$$

$$P_h^k(x) = x. \tag{104}$$

Proof: First we modify f outside a sufficiently large ball (a ball containing the horseshoe) , so that f satisfies (6) and (7). Let $c = (c_0, \ldots, c_n)$ be fixed. For $i \in \{0, \ldots, k\}$ we define h-sets N_i by

$$N_i = L_{c_i}, \quad N_0 = N_k.$$

Obviously $N_{i-1} \overset{P}{\Longrightarrow} N_i$ for $i \in \{1, \ldots, k\}$. Let $N = \bigcup_{i=1}^k N_i$. By Theorem 18 there are $\epsilon > 0$ and $\bar{h} > 0$, depending only on $P|_{L_0}$, $P|_{L_1}$, and not on c, such that for $h \in [0, \bar{h}]$

$$\text{ind}(P_h^k, \text{int} N^h) = \pm 1,$$

so the result follows. ∎

Let $\Sigma_2 = \{0,1\}^{\mathbb{N}}$ and $\sigma : \Sigma_2 \to \Sigma_2$ be a shift map.

Theorem 26 *Assume that a suitable Poincaré map, P, for (101) has a topological horseshoe. Let*

$$I_h = \{x \in L(\epsilon) : \quad \forall i \in \mathbb{N} \quad P_h^i(x) \in L(\epsilon)\}.$$

Then there are $\epsilon > 0$ and $\bar{h} > 0$ such that for any $h \in [0, \bar{h}]$ there exists a continuous, surjective map $g : I_h \to \Sigma_2$ such that P_h is semi-conjugated to σ in the set I_h i.e.

$$g \circ P_h = \sigma \circ g. \tag{105}$$

Moreover, for any n-periodic sequence $c \in \Sigma_2$ there exists $x \in g^{-1}(c)$ such that $P_h^n(x) = x$.

Proof: Let $\epsilon > 0$ and $\bar{h} > 0$ be as in Corollary 25. We define $g : I_h \to \Sigma_2$ by

$$g(x)_i = \begin{cases} 1 & , P_h^i(x) \in L_1(\epsilon) \\ 0 & , P_h^i(x) \in L_0(\epsilon). \end{cases} \tag{106}$$

Obviously, g is continuous and (105) holds. Since P_h is compact, we have that the image of g is compact. The subset of periodic sequences is dense in Σ_2, and by Corollary 25 is contained in the image of g, so g is surjective. ∎

As the last corollary we state the result about a continuation of symbolic dynamics in case of a transversal homoclinic periodic orbit.

Theorem 27 *Assume that (2) has a transversal homoclinic periodic orbit, then the delay equation $x'(t) = f(x(t-\tau))$ for sufficiently small delays $\tau > 0$ has a symbolic dynamics on two symbols for some Poincaré map.*

Proof: Consider a section S transversal to our periodic orbit, γ. Let $x_0 = \gamma \cap S$. Let P be a Poincaré map on S. Obviously $P(x_0) = x_0$. From our assumptions it follows that stable and unstable manifolds of x_0 intersect transversally. The existence of covering relations defining a topological horseshoe for P^k for some k is an easy consequence of the construction of Smale's horseshoe in the proof of the Birkhoff-Smale homoclinic point theorem (see for example [GH, KH]). The result now follows from Theorem 26. ∎

7 Differential inclusions, small delays and symbolic dynamics

Continuation theorems from previous sections say nothing about the size of $\bar{h}$. Obviously one could trace bounds on $\bar{h}$ going through the proof of Theorem 18, but we believe that this will usually lead to overpessimistic estimates. As the remedy we propose study the differential inclusion

$$x'(t) \in f(x(t)) + \overline{B}(0, \delta). \tag{107}$$

We will argue in this section that its solutions give rigorous enclosure for solutions of delay equation $x'(t) = f(x(t-h))$. This will allow to state Theorem 29 about the existence of symbolic dynamics, which gives an explicit expression for $\bar{h}$.

Some notation: for the set X by 2^X we denote the set of all subsets of X. The multivalued map from X to Y is a map $g : X \to 2^Y$. To denote multivalued maps we will use the caligraphic font, double arrows and the range will be denoted by Y instead of 2^Y, for example $\mathcal{P} : X \rightrightarrows Y$ means that for $x \in X$ the image $\mathcal{P}(x) \subset Y$.

7.1 Definitions

Definition 5 *Let $\mathcal{P} : X \rightrightarrows Y$ be a multivalued map.*

The map $f : X \to Y$ is called a selector of $\mathcal{P}$ *iff $f(x) \in \mathcal{P}(x)$ for all $x \in X$.*

Definition 6 *For $x_0 \in \mathbb{R}^n$ by the solution set of (107) through x_0, denoted by $Sol(x_0, f, \delta)$, we understand the set of differentiable functions $x : [0, t_m) \to \mathbb{R}^n$, such that*

$$\begin{aligned} x'(t) &\in f(x(t)) + \overline{B}(0, \delta), \qquad t \in [0, t_m) &(108)\\ x(0) &= x_0 &(109)\\ t_m &= \infty \qquad \text{or} \qquad \lim_{t \to t_m} \|x(t)\| = \infty. &(110)\end{aligned}$$

Definition 7 *Let S be a section for $x' = f(x)$ (see Section 3). Let α, U, S, Φ satisfy conditions (22–24). We say that S is a section for (107) if it is a section for $x' = f(x)$ and the following modification of (22) holds for some $\delta > 0$*

$$\nabla_x \alpha \cdot \big(f(x) + \overline{B}(0, \delta)\big) \geq 1, \quad x \in U. \tag{111}$$

Definition 8 *Assume that S is a section for (107). We define a multivalued Poincaré map $\mathcal{P} : S \supset dom(\mathcal{P}) \rightrightarrows S$ as follows*

- *$x_0 \in S$, $x_0 \in dom(\mathcal{P})$ iff for any $x : [0, t_m) \to \mathbb{R}^n$, $x \in Sol(x_0, f, \delta)$ there exists $0 < \tau_r = \tau_r(x, S) < t_m$, such that $x(\tau_r) \in S$ and for any $0 < t < \tau$ one has $x(t) \notin S$.*
- *for $x_0 \in dom(\mathcal{P})$ we set*

$$\mathcal{P}(x_0) = \{x(\tau_r(x, S)) \mid x \in Sol(x_0, f, \delta)\} \tag{112}$$

- *for $x_0 \in dom(\mathcal{P})$ we define the set of return times to S*

$$\tau_r(x_0) = \{\tau_r(x, S) \mid x \in Sol(x_0, f, \delta)\} \tag{113}$$

Definition 9 *Assume N, M are h-sets, such that $u(N) = u(M) = u$ and $s(N) = s(M) = s$. Let $\mathcal{G} : |N| \rightrightarrows \mathbb{R}^n$ be a multivalued map. We say that*

$$N \stackrel{\mathcal{G}}{\Longrightarrow} M$$

(N $\mathcal{G}$-covers M) iff there exists a continuous selector $g : |N| \to \mathbb{R}^n$ of $\mathcal{G}$, such that the following conditions are satisfied

$$
\begin{aligned}
N &\overset{g}{\Longrightarrow} M, &(114)\\
c_M^{-1}\left(conv(c_M(\mathcal{G}(N^-))\right) \cap M &= \emptyset, &(115)\\
c_M^{-1}\left(conv(c_M(\mathcal{G}(N))\right) \cap M^+ &= \emptyset, &(116)
\end{aligned}
$$

where $conv(Z)$ denotes the closure of the convex hull of Z.

We have the following

Lemma 28 *Let N, M be h-sets. Assume that $\mathcal{G} : |N| \rightrightarrows \mathbb{R}^n$ be a multivalued map.*

If $N \overset{\mathcal{G}}{\Longrightarrow} M$, then for any continuous selector, r, of $\mathcal{G}$

$$N \overset{r}{\Longrightarrow} M.$$

Proof: Let g be the selector of $\mathcal{G}$ from Definition 9.

It is easy to see that the homotopy $R : [0,1] \times |N| \to \mathbb{R}^n$ defined by

$$R(\lambda, x) = c_M^{-1}\left((1-\lambda)c_M(r(x)) + \lambda c_M(g(x))\right) \tag{117}$$

satisfies conditions from the point 1 in Definition 3. It easy to see that we can glue this homotopy with the homotopy h from the covering relation for $N \overset{g}{\Longrightarrow} M$, to obtain a new homotopy proving that $N \overset{r}{\Longrightarrow} M$. ∎

7.2 Basic theorem

Definition 10 *Let $\Gamma = (\mathcal{N}, Cov\}$ be a set of abstract covering relations on $\mathbb{R}^n$. By this we understand that $\mathcal{N} = \{N_i \mid i \in I\}$ (is a collection of abstract h-sets) and $Cov \subset \mathcal{N} \times \mathcal{N}$.*

Let $g : X \supset dom(g) \to X$ be continuous and assume that $|N_i| \subset dom(P)$ for all $N_i \in \mathcal{N}$. We will say that g has the symbolic dynamics induced by Γ if for any infinite chain of covering relations from Γ,

$$N_{i_0} \Longrightarrow N_{i_1} \Longrightarrow N_{i_2} \ldots, \qquad (N_{i_k}, N_{i_{k+1}}) \in Cov \quad k = 0, 1, \ldots$$

there exists $x \in X$ such that for every $k = 0, 1, \ldots$

$$g^k(x) \in N_{i_k}. \tag{118}$$

Moreover, if the sequence N_{i_k} is periodic, then x in (118) can be chosen to be a periodic point for g with the same period.

Let $\Gamma = ((N_i)_{i\in I}, Cov)$ be a set covering relations, such that N_i are some h-sets contained in $\mathbb{R}^n$. To deal with delay differential equations we define for any $\epsilon > 0$ a derived abstract set of covering relations $\Gamma(\epsilon) = ((N_i(\epsilon)_{i\in I}, Cov(\epsilon))$ by

$$(N_i, N_j) \in Cov \qquad \text{iff} \qquad (N_i(\epsilon), N_j(\epsilon)) \in Cov(\epsilon)$$

Theorem 29 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be bounded and Lipschitz. Let M and L be such that*

$$M = \sup_{x \in \mathbb{R}^n} \|f(x)\|, \qquad \|f(x) - f(y)\| \leq L\|x - y\|, \quad x, y \in \mathbb{R}^n. \tag{119}$$

Consider a differential inclusion

$$x' \in f(x) + \overline{B}(0, \delta). \tag{120}$$

Assume that we have a finite set, Γ, of covering relations for multivalued Poincaré map $\mathcal{P} : S \supset dom(\mathcal{P}) \rightrightarrows S$ induced by (120).

Let $\tau > 0$ smaller than the infimum return time to section S for $\mathcal{P}$ on h-sets appearing in Γ, i.e. for any covering relation $N \overset{\mathcal{P}}{\Longrightarrow} M$ in Γ holds

$$\tau_r(x) > \tau, \qquad \forall x \in |N|. \tag{121}$$

Then for all h with $0 \leq h \leq \bar{h} = \min\left(\frac{c\delta}{LM}, \tau\right)$, where c is any number in $(0, 1)$, the delay differential equation

$$x'(t) = f(x(t - h)) \tag{122}$$

has the symbolic dynamics induced by $\Gamma(\epsilon)$ for some $\epsilon > 0$.

Proof: Let $\mathcal{P}_\kappa$ denote the Poincaré map for differential inclusion (120) with $\delta = \kappa$. Obviously $\mathcal{P} = \mathcal{P}_\delta$.

Let us chose c, c_1, such that $1 > c_1 > c > 0$. Let us set $\epsilon = \frac{c_1 \delta}{L} > M\bar{h}$. Let $x \in C([-\bar{h}, T), \mathbb{R}^n)$ be a solution of the delay differential equation (122) such that $\|x(t) - x(0)\| \leq \epsilon$ for $t \in [-\bar{h}, 0]$, then

$$x'(t) = f(x(t - h)) \in f(x(t)) + \overline{B}(0, \epsilon L) \subset f(x(t)) + \overline{B}(0, c_1 \delta). \tag{123}$$

This shows that for any covering relation $N_i \overset{\mathcal{P}}{\Longrightarrow} N_j$ from Γ the Poincaré map P_h is well defined for $x \in N_i(\epsilon)$ for all $h \in [0, \bar{h}]$. Moreover we have

$$P_h(x)(0) \in \mathcal{P}_{c_1 \delta}(x(0)) \subset \mathcal{P}(x(0)) \tag{124}$$

Now we will show the result analogous to Lemma 20.

Lemma 30 *Let $\mathcal{P} : S \supset dom(\mathcal{P}) \rightrightarrows S$, ϵ and $\bar{h}$ be as defined above.*

Let $N \subset dom(\mathcal{P})$, M be h-sets such that $N \overset{\mathcal{P}}{\Longrightarrow} M$.

Then there exists ρ with $0 < 2\rho < \epsilon$ such that for $h \in [0, \bar{h}]$

$$P_h(N^-(\epsilon, 2\rho)) \cap M(\epsilon) = \emptyset, \tag{125}$$

$$P_h(N(\epsilon)) \cap M^+(\epsilon, 2\rho) = \emptyset. \tag{126}$$

After Lemma 30 is established the remainder of the proof is the same as proofs of Theorem 18 and Col. 25 (to obtain periodic orbits). The case of nonperiodic chains of covering relations is left to the reader. ∎

Proof of Lemma 30: Take any $0 < 2\rho < \epsilon$. Assume that $\|x_0 - x_1\| < 2\rho$. We will show that there exists $\rho_0 > 0$ such that for $0 \leq \rho \leq \rho_0$ one has

$$\mathcal{P}_{c_1\delta}(x_1) \subset \mathcal{P}(x_0) \tag{127}$$

Take any $y_1 \in \mathcal{P}_{c_1\delta}(x_1)$. Then there exists a function $y(t)$ and $T > 0$ (the return time to S), such that $y_1 = y(T)$ and

$$y'(t) \in f(y(t)) + \overline{B}(0, c_1\delta). \tag{128}$$

Let $d : [0, \infty] \to [0, 1]$ be any smooth function, such that $d(0) = 1$ and $d(t) = 0$ for $t \geq \tau$.

Consider now a function $x(t) = y(t) + d(t)(x_1 - x_0)$. We have

$$\begin{aligned} x'(t) = y'(t) + d'(t)(x_1 - x_0) \in f(y(t)) + \overline{B}(0, c_1\delta) + d'(t)(x_1 - x_0) \subset \\ \subset f(x(t)) + \overline{B}(0, (L + |d'|)\|x_1 - x_0\| + c_1\delta) \end{aligned}$$

hence for $\rho \leq \frac{(1-c_1)\delta}{2(L+|d'|)}$ we obtain

$$x'(t) \in f(x(t)) + \overline{B}(0, \delta).$$

Obviously $x(T) = y(T)$. Observe that this must be the first return to S, because for $t \leq \tau$ $x(t) \notin S$ (by the definition of τ) and for $t > \tau$ $x(t) = y(t)$. This proves (127).

By combining (124) and (127) we obtain for $x_1, x_2 \in N(\epsilon)$, such that $\|x_1 - x_2\| \leq 2\rho \leq 2\rho_0$ the following condition

$$P_h(x_1)(0) \in \mathcal{P}_{c_1\delta}(x_1(0)) \subset \mathcal{P}(x_2(0)). \tag{129}$$

Now we are ready for the proof of conditions (125) and (126). From condition (115) in Definition 9 we have

$$\mathcal{P}(N^-) \cap M = \emptyset. \tag{130}$$

From the above condition and from (129) it follows that for $\rho \leq \rho_0$ holds

$$\left\{P_h(x)(0) \mid x \in N^-(\epsilon, 2\rho)\right\} \cap M \subset \mathcal{P}(N^-) \cap M = \emptyset. \tag{131}$$

This proves (125).

To establish (126) we will show that for sufficiently small ρ and for any $x \in N(\epsilon)$ the following two conditions are satisfied

$$P_h(x)(0) \notin \overline{B}_M(M^+, 2\rho) \tag{132}$$

$$\|P_h(x) - P_h(x)(0)\| < \epsilon - 2\rho. \tag{133}$$

From condition (116) in Definition 9 we have a constant $\eta > 0$, such that

$$\operatorname{dist}(\mathcal{P}(N), M^+) > \eta. \tag{134}$$

Hence from (124) it follows that (132) is satisfied for $\rho \leq \rho_1 = \min(\rho_0, \eta/2)$.

To prove (133) observe that since the return time is larger than the delay we have

$$\|P_h(x) - P_h(x)(0)\| \leq M\bar{h} = \epsilon - (\epsilon - M\bar{h}) < \epsilon - 2\rho_2, \tag{135}$$

where $0 < \rho_2 < \frac{\epsilon - M\bar{h}}{2}$. Hence (133) holds for $\rho \leq \min(\rho_2, \rho_1)$. ∎

8 Application to the Lorenz equations.

In this section we combine Theorem 18 and results from [GaZ] to show that the Lorenz equations [Lo] with small delay possess symbolic dynamics.

Let $F : X \to X$ be any continuous map and $N \subset X$. By $F_{|N}$ we will denote the map obtained by restricting the domain of F to the set N. The maximal invariant part of N (with respect to F) is defined by

$$\mathrm{Inv}(N,F) = \bigcap_{i\in\mathbb{Z}} F_{|N}^{-i}(N).$$

The Lorenz equations [Lo] are given by

$$\begin{aligned} \dot{x} &= s(y-x), \\ \dot{y} &= rx - y - xz, \\ \dot{z} &= xy - qz. \end{aligned} \tag{136}$$

We set $(s, r, q) = (10, 28, 8/3)$. These are the most popular parameter values in the literature. The flow generated by Eq. (136) exhibits a so-called strange attractor (see [Tu] for the computer assisted proof of this fact).

We focus on the Poincaré map P generated by (136) on the section $\Theta := \{(x, y, z) \in \mathbb{R}^3, \quad z = r - 1\}$.

The following result was proved in [GaZ]

Theorem 31 *For all parameter values in a sufficiently small neighborhood of $(s, r, q) = (10, 28, 8/3)$ there exists a Poincaré section $N \subset \Theta$ such that the Poincaré map P induced by Eq. (136) is well defined and continuous.*

There exists a continuous surjective map $\pi : Inv(N, P^2) \to \Sigma_2$, such that

$$\pi \circ P^2 = \sigma \circ \pi.$$

The preimage of any periodic sequence from Σ_2 contains a periodic point of P^2 with the same principal period.

The above theorem is a consequence of Theorem 18 and the following Lemma, which was established in [GaZ] with computer assistance (see Lemma 4 and Fig. 3 and 4 in [GaZ])

Lemma 32 *There are h-sets $N_{00}, N_{01}, N_{10}, N_{11} \subset \Theta$ such that for all parameter values in sufficiently small neighborhood of $(s, r, q) = (10, 28, 8/3)$ $N_{ij} \subset Dom(P)$ for $i, j = 0, 1$ and the following conditions hold*

$$N_{00} \overset{P}{\Longrightarrow} N_{11}, \quad N_{01} \overset{P}{\Longrightarrow} N_{10} \quad N_{10} \overset{P}{\Longrightarrow} N_{00}, N_{01}, \quad N_{11} \overset{P}{\Longrightarrow} N_{00}, N_{01}$$

Let us recall from [GaZ] that the sets N_{ij} for $i, j = 0, 1$ are not disjoint. We have two disjoint pairs $\{N_{00}, N_{01}\}$ and $\{N_{10}, N_{11}\}$. For each pair we obtain a full symbolic dynamics on two symbols for P^2.

Let us denote by $L_{s,r,q} : \mathbb{R}^3 \to \mathbb{R}^3$ the vector field on the right-hand side of (136). From Theorem 18 and Lemma 32 we immediately obtain the following

Theorem 33 *Let us fix* $(s, r, q) = (10, 28, 8/3)$*. Consider the Lorenz system (136) with small delay*

$$x'(t) = L_{s,r,q}(x(t-h)). \tag{137}$$

Let $N = N_{00} \cup N_{01}$ *and*

$$I_h = \{x \in N(\epsilon) : \quad \forall i \in \mathbb{N} \quad P_h^{2i}(x) \in N(\epsilon)\}$$

Then there are $\epsilon > 0$ *and* $\bar{h} > 0$ *such that for any* $h \in [0, \bar{h}]$ *there exists a continuous, surjective map* $g : I_h \to \Sigma_2$ *such that* P_h^2 *is semi-conjugated to* σ *in the set* I_h *i.e.*

$$g \circ P_h^2 = \sigma \circ g. \tag{138}$$

Moreover, for any n*-periodic sequence* $c \in \Sigma_2$ *there exists* $x \in g^{-1}(c)$ *such that* $P_h^{2n}(x) = x$*.*

Proof: To obtain periodic orbits we apply Theorem 18 to the following chains of covering relations build from the following elementary loops

$$N_{00} \stackrel{P}{\Longrightarrow} N_{11} \stackrel{P}{\Longrightarrow} N_{00}, N_{01}$$
$$N_{01} \stackrel{P}{\Longrightarrow} N_{10} \stackrel{P}{\Longrightarrow} N_{00}, N_{01}$$

The trajectories corresponding to nonperiodic symbolic sequences are obtained via the compactness argument as in the proof of Theorem 26. ∎

The actual value of $\bar{h}$ can be established using Theorem 29. One can do it either analytically or, which will be probably much more efficient, using rigorous numerics. In order to do that one can perform an rigorous integration of $x' = L_{r,s,q}(x) + \overline{B}(0, \delta)$ for various values of δ looking for the largest possible value of δ for which the covering relations listed in Lemma 32 hold for the induced multivalued Poincaré map. For an algorithm for rigorous integration of differential inclusions see [ZKSPer].

8.1 Other examples.

Other computer assisted proofs of the existence of nontrivial symbolic dynamics based on covering relations give rise to theorems analogous to Theorem 33. This applies to the following systems

- Rössler equations [ZNon],
- Chua circuit [Ga],
- Kuramoto-Shivashinsky ODE [W].

Similarly one can extend the results about the Lorenz equations for various sets of parameter values from [MM, MMS] to small delays. These results have been obtained using the discrete Conley index[MMH], but one can easily see that, if N is an isolating neighborhood for some Poincaré map for $x' = f(x)$, then the set $N(\epsilon)$ with ϵ small enough is an isolating neighborhood for the Poincaré map induced by $x'(t) = f(x(t-h))$ for small delays. It turns out that the same homotopy of turning-off the delay, which we are using throughout the paper, works and we obtain exactly the same bounds for h and ϵ.

9 Appendix. Fixed point index for ANRs.

In this section we recall the fixed point index theory for metric ANR's introduced in [Gr].

A metric space X is called an ANR if it is an retract of an open set in a normed space. Let $U \subset X$ be open and $f : U \to X$ be continuous. We say that f is admissible if the fixed point set of f

$$\mathrm{Fix}(f) = \{x \in U : \quad f(x) = x\}$$

is compact. Let X, Y be topological spaces. The map $f : X \to Y$ is compact if it is continuous and $\overline{f(X)}$ is compact.

A compact homotopy $h_t : U \to X$ will be called admissible provided the set $\bigcup_{t=0}^{1} \mathrm{Fix}(h_t)$ is compact.

Assume that $f : U \to X$ is an admissible, compact map. Then the fixed point index of f in U is defined and has the following properties:

Integer $\mathrm{ind}(f, U) \in \mathbb{Z}$,

Excision If $U^{'} \subset U$ and $\mathrm{Fix}(f) \subset U^{'}$, then

$$\mathrm{ind}(f, U) = \mathrm{ind}(f, U^{'}). \tag{139}$$

Fixed points If $\mathrm{ind}(f, U) \neq 0$, then $\mathrm{Fix}(f) \neq \emptyset$.

Homotopy If $h_t : U \to X$ is an admissible homotopy, then $\mathrm{ind}(h_0, U) = \mathrm{ind}(h_1, U)$.

Multiplicativity If $f_1 : U_1 \to X_1$ and $f_2 : U_2 \to X_2$ are admissible, then the product $f_1 \times f_2 : U_1 \times U_2 \to X_1 \times X_2$ is admissible and

$$\mathrm{ind}(f_1 \times f_2, U_1 \times U_2) = \mathrm{ind}(f_1, U_1)\,\mathrm{ind}(f_2, U_2). \tag{140}$$

Commutativity Let $U \subset X$ and $U' \subset X'$ be open and assume that $f : U \to X'$, $g : U' \to X$ are continuous. If one of the composites

$$gf : V = f^{-1}(U') \to X,\ fg : V' = g^{-1}(U) \to X',$$

is admissible, then so is the other and, in that case,

$$\mathrm{ind}(gf, V) = \mathrm{ind}(fg, V'). \tag{141}$$

Note that by the commutativity of the fixed point index if $f : U \to X$ is admissible and $h : X \to X'$ is homeomorphism, then $hfh^{-1} : h(U) \to X'$ is admissible and

$$\mathrm{ind}(f, U) = \mathrm{ind}(hfh^{-1}, h(U)).$$

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