New lower bound estimates for quadratures of bounded analytic functions

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Abstract

We give an improved lower bound for the error of any quadrature formula for calculating $\int_{-1}^{1} f(x)\, d\alpha(x)$, where the functions f are bounded and analytic in the neighborhood of [-1,1] and α is a finite absolutely continuous Borel measure.

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1 Introduction

Let us denote by $I(f,\alpha)$ the integral

$$I(f,\alpha) = \int_{-1}^{1} f(x)\, d\alpha(x), \tag{1}$$

where α is a finite Borel measure on $[-1,1]$ that is absolutely continuous with respect to the Lebesgue measure. Sometimes we drop α and write $I(f)$, when α is known from the context.

In this paper we study the lower bound for the error in the numerical computation of $I(f,\alpha)$ for bounded analytic functions. Such investigation is motivated by the search for the optimal algorithm for the computation of $I(f,\alpha)$.

The works of Bakhvalov [1] and Petras [14] give convincing arguments for the near-optimality of Gaussian quadrature in the case when the domain of analyticity of the integrand is an ellipse; for other regions, it will be the quadrature obtained from the Gaussian quadrature transported from the unique ellipse via the Riemann mapping theorem. In Petras' article [14] one can find a demonstration of how the Gaussian quadrature fails to be nearly optimal, when the analyticity region is not an ellipse.

In their investigations Bakhvalov [1] and Petras [14] also give the lower bounds for the quadrature error for (1). However, we find these bounds to be overly pessimistic and to have some bad qualitative behavior when the domain of the analyticity of f shrinks to $[-1,1]$. To explain this in precise terms and to present our results, we need to introduce the basic notation and definitions.

Definition 1 *Let $c > 1$. Then $\mathcal{E}_c$ denotes the interior of an ellipse on the complex plane, such that the foci of $\mathcal{E}_c$ are located at points ± 1 and the sum of semi-axes is equal to c.*

Definition 2 *Let $D \subset \mathbb{C}$. We call D a* nice domain *if it is an open, connected and simply connected set which is symmetric with respect to the real axis (i.e. if $z \in D$, then $\bar{z} \in D$).*

Definition 3 *Let $D \subset \mathbb{C}$ be a nice domain. Let $M \geq 0$.*
We will write:

- *$\mathcal{A}(D)$ for the set of analytic functions on D such that $\|f\| = \sup_{z \in D} |f(z)| < \infty$,*
- *$\mathcal{A}(D, M)$ for the set of analytic functions on D such that $|f(z)| \leq M$ for all $z \in D$,*
- *$\mathcal{A}_0(D, M)$ for a subset of $\mathcal{A}(D, M)$ consisting of functions which are real on the real line.*

Let $\mathcal{Q}(n, \mathcal{R})$, where $n \in \mathbb{N}$ and $\mathcal{R} = (r_1, \ldots, r_n)$, $r_1, \ldots, r_n \in \mathbb{N} \backslash \{0\}$, denote the class of all possible (even non-linear) quadratures that use n nodes $z_1, \ldots, z_n \in [-1, 1]$ and derivatives of the integrand up to the order $r_j - 1$ for each z_j. By $\overline{\mathcal{Q}}(n, \mathcal{R})$ we denote a subclass of $\mathcal{Q}(n, \mathcal{R})$ containing quadratures of the form

$$\sum_{j=1}^{n} \sum_{k=0}^{r_j - 1} b_{kj} f^{(k)}(z_j). \tag{2}$$

Additionally, $|\mathcal{R}|$ denotes the sum $r_1 + \ldots + r_n$ and $\mathcal{R}_2 = (2, \ldots, 2)$. Note that $\mathcal{R}_2$ has a special role here, as this is the case considered by Bakhvalov [1] and Petras [14]. Most of our results, however, are for any $\mathcal{R}$.

To describe briefly the results of Bakhvalov and Petras we assume that α is the Lebesgue measure and $\mathcal{R} = \mathcal{R}_2$ (results in [1] and [14] have been established for more general weight functions and for nodes outside the real line; these are discussed in Section 2).

Let G_n denote the Gauss-Legendre quadrature with n nodes on $[-1, 1]$. The claim that the Gauss-Legendre quadrature is near-optimal for ellipses is based on the following estimates:

- There exists a bounded and positive function $\kappa_L : (1, \infty) \to \mathbb{R}_+$ such that for any $c > 1$ and for any quadrature $Q_n \in \overline{\mathcal{Q}}(n, \mathcal{R}_2)$, there is an $f_0 \in A_0(\mathcal{E}_c, M)$ such that
$$|I(f_0) - Q_n(f_0)| \geq M \kappa_L(c) c^{-2n}.$$
- There exists a bounded and positive function $\kappa_G : (1, \infty) \to \mathbb{R}_+$ such that for any $c > 1$, we have
$$|I(f) - G_n(f)| \leq M \kappa_G(c) c^{-2n} \qquad \forall f \in A_0(\mathcal{E}_c, M).$$

Observe that these estimates lead to the same asymptotic bounds on n that are needed to get the quadrature error less than ε. We obtain

$$N_L\left(M, \varepsilon, c\right) \leq n \leq N_G\left(M, \varepsilon, c\right) \tag{3}$$

where

$$N_L\left(M, \varepsilon, c\right) = \frac{\ln \frac{M}{\varepsilon} + \ln \kappa_L(c)}{2 \ln c}, \qquad N_G\left(M, \varepsilon, c\right) = \frac{\ln \frac{M}{\varepsilon} + \ln \kappa_G(c)}{2 \ln c}. \tag{4}$$

For $\varepsilon \to 0^+$ we have $N_L \approx N_G \approx (\ln(M/\varepsilon)) \Big/ (2 \ln c)$, so these lower and upper bounds are tight.

Notice that the crucial parameters of N_L and N_G are actually c and the quotient M/ε, as M and ε do not occur independently.

The motivation for our work comes from the following observation. From the estimates for $\kappa_L(c)$ given in [1, 14], it follows that

$$\lim_{c\to 1^+} \kappa_L(c) = 0. \tag{5}$$

Thus if $c-1$ is small, $N_L < 0$ in (4) unless ε is very small, so in fact the lower bound given by (4) does not have any predictive power with respect to the number of nodes required to get the error less than ε for a substantial range of the parameters c and ε.

The main technical result of our paper is a new lower bound for errors of arbitrary quadratures of bounded analytic function using N function or derivative values at some nodes, which does not suffer from the bad qualitative behavior exemplified by equation (5). This allows us to obtain more meaningful lower bounds on the cost of quadratures in the sprit of the Information Based Complexity [18] approach to the complexity of integration of bounded analytic functions (see [11] and references given there).

Our approach is based on the conformal distance on the domain of analyticity D (see Section 3 for the definition). The theorem below is an example of our lower bound for the case of the Lebesgue measure.

Theorem 1 *Let $D \subset \mathbb{C}$, $D \neq \mathbb{C}$ and let D be a nice domain such that $[-1,1] \subset D$. For any $Q \in \mathcal{Q}(n,\mathcal{R})$, where $|\mathcal{R}| = N$, and for any $M > 0$ there exists a function $f_0 \in \mathcal{A}_0(D,M)$ such that*

$$|I(f_0) - Q(f_0)| \geq \gamma M, \tag{6}$$

where

$$\gamma = \frac{2}{\left(\left(1+\frac{1}{2\delta_D}\right)^{2\delta_D}(2\delta_D+1)\right)^{2N}} \tag{7}$$

and

$$\delta_D := \sup\{\delta_D(x) : x \in [-1,1]\}, \quad \delta_D(x) := \inf\{|x-z| : z \in \mathbb{C}\setminus D\}.$$

This theorem is proved in Section 3. Corollary 2 therein contains the version of this result for the ellipse $\mathcal{E}_c$.

This result improves the results of Bakhvalov [1] and Petras [14] as it allows higher derivatives in the quadrature formula, as well as more general measures α. On the other hand, in these works the nodes used in the quadrature are not restricted to the segment $[-1,1]$. However, the most important qualitative improvement is that our bound does not tend to 0 for $c \to 1^+$.

To the best of our knowledge, the only similar result (i.e., the fact that the lower bound does not go to 0 when the ellipse shrinks to $[-1,1]$) was established by Osipenko [12] for a very particular weight function, namely the Chebyshev weight function.

Let us describe briefly the content of the paper. In Section 2 we discuss in detail the results of Bakhvalov and Petras concerning the lower bounds for the integration error for arbitrary quadrature and the upper bounds for the error of the Gauss-Legendre quadrature, and which we compare. In Section 3 we develop a new lower bound for the error of an arbitrary quadrature.

2 Existing error bounds for quadratures of analytic functions

Following [14] we introduce the following definition.

Definition 4 *Let $D \subset \mathbb{C}$ be an open set such that $[-1,1] \subset D$. For a given quadrature $Q \in \mathcal{Q}(n,\mathcal{R})$ the remainder term is defined as*

$$R(f,\alpha) = I(f,\alpha) - Q(f). \tag{8}$$

The error constant of Q with respect to $\mathcal{A}(D)$ is given by

$$\rho(Q,\mathcal{A}(D),\alpha) = \sup_{f\in\mathcal{A}(D)\setminus\{0\}} \frac{|R(f,\alpha)|}{\|f\|} \tag{9}$$

and the optimal error constant by

$$\rho_n(\mathcal{A}(D),\alpha) = \inf_{Q\in\mathcal{Q}(n,\mathcal{R}_2)} \rho(Q,\mathcal{A}(D),\alpha). \tag{10}$$

A quadrature formula is called optimal *if its error constant attains* $\rho_n(\mathcal{A}(D),\alpha)$.

Notice that $\mathcal{R}_2$ is used in the definition of $\rho_n(\mathcal{A}(D),\alpha)$, as this is case investigated by Bakhvalov and Petras. We need this notion the describe theirs results, only.

2.1 Bakhvalov's lower bound for quadratures of analytic functions

The following theorem has been proven in [1, Thm. 1] (as an improvement of a previous result from [17]).

Theorem 2 *Assume that* $d\alpha = p(x)dx$ *and there exists a polynomial* $t(x)$ *such that* $p(x)/t(x) \geq \eta > 0$ *for* $x \in [-1,1]$.

Let $z_1, z_2, \ldots, z_n \in \mathbb{R}$ *(*$n \leq N$*) and let* $z_{n+1}, \ldots, z_N \in \mathbb{C}$ *be points contained in upper half-plane (*$\operatorname{Im} z > 0$*). Let* $\mathcal{E}_c$ *be an ellipse which encloses all of these points.*

For any quadrature formula of the form

$$\begin{aligned} Q_N(f) = &\sum_{j=1}^{n} \left(b_{1j}\operatorname{Re} f(z_j) + b_{2j}\operatorname{Re} f'(z_j) + b_{3j}\operatorname{Im} f(z_j) + b_{4j}\operatorname{Im} f'(z_j)\right) + \\ &\sum_{j=n+1}^{N} \left(b_{1j}\operatorname{Re} f(z_j) + b_{2j}\operatorname{Re} f'(z_j) + b_{3j}\operatorname{Im} f(z_j) + b_{4j}\operatorname{Im} f'(z_j)\right), \end{aligned} \tag{11}$$

any $c > 1$ *and* $M > 0$, *there exists a function* $f_0 \in \mathcal{A}_0(\mathcal{E}_c, M)$ *such that*

$$I(f_0) - Q_N(f_0) \geq \kappa_0 M c^{-2N},$$

where κ_0 *depends on* c *and the weight function* p *only.*

Comments:

- In terms of the notions introduced earlier, for $N = n$ we have

$$\rho_n(\mathcal{A}(D), d\alpha) \geq \kappa_0 c^{-2n}. \tag{12}$$

- In [1] the following formula for κ_0 is given (see page 67)

$$\kappa_0 = \pi P_0 (1 - c^{-1}) c^{-2m} (\sinh h)^m, \tag{13}$$

  where $h = \ln c$ (hence $\sinh h = \frac{1}{2}(c - c^{-1})$) and constants $P_0 \in \mathbb{R}_+$, $m \in \mathbb{N}$ depend on the weight function only (P_0 appears as Q_0 in [1]). In fact, [1] misprints the formula for κ_0 as $(1 - c^{-1})^{-1}$ instead of $(1 - c^{-1})$.

  The constants m and P_0 are determined as follows: after the substitution $x = \cos u$ we have

$$I(f) = \int_0^\pi f(\cos u) q(u)\, du, \qquad q(u) = p(\cos u)\sin u. \tag{14}$$

  Under the assumptions of the theorem the following holds true

$$q(u) = P(u) l(\cos u), \tag{15}$$

where l is a polynomial of degree m and $P(u) = q(u)/(l(\cos u)) \geq P_0 > 0$ for $u \in [0, \pi]$ (P appears as Q in [1]). Therefore, m is the number of zeros in $q(u)$ counted with multiplicities. It is related to the number of zeros in the weight function $p(x)$: it is the number of zeros $p(x)$ counted with multiplicities plus two if the zeros at 0 and π introduced in $q(u)$ by the factor $\sin u$ are not canceled by the singular behavior of $p(x)$, when $x \to \pm 1$. Such cancelations happen for the Chebyshev weight (see below).

– For $p(x) \equiv 1$ we have $q(u) = \sin u$. Therefore $l(z) = 1 - z^2$,

$$\frac{q(u)}{l(\cos u)} = \frac{\sin u}{1 - \cos^2 u} = \frac{1}{\sin u} \geq P_0 = 1$$

for $x \in [0, \pi]$. Hence $m = 2$.
Easy computations show that for $m = 2$ and $P_0 = 1$ we obtain

$$\begin{aligned} \kappa_0 &= \frac{\pi}{4}(1 - c^{-1})c^{-4}(c - c^{-1})^2 = \frac{\pi}{4}(c-1)^3 \frac{(c+1)^2}{c^7} = \\ &= \pi(c-1)^3 + O((c-1)^4), \quad \text{for } c \to 1^+. \end{aligned} \tag{16}$$

– For the Chebyshev weight $p(x) = 1/\sqrt{1 - x^2}$ we have

$$q(u) = p(\cos u) \sin u = 1.$$

Hence $m = 0$ and $P_0 = 1$, and consequently

$$\kappa_0 = \pi(1 - c^{-1}) = \frac{\pi(c-1)}{c}. \tag{17}$$

We obtain a counter-intuitive statement that when $c - 1$ is small (i.e. the integrated function is difficult to calculate due to the possible presence of singularities nearby), the lower bound for the error is also small. Hence the quality of the bound is rather poor and can be considerably improved.

- The reason for this overly pessimistic estimate of $\rho_n(\mathcal{A}(D), d\alpha)$ is as follows. For any n nodes, define a polynomial $f_0 \in \mathcal{A}(\mathcal{E}_c, 1)$ of degree $2n+m$ whose quadrature error is bounded from below by $\kappa_0 c^{-2n}$. For a fixed set of nodes this polynomial is the same for all $c > 1$ up to a multiplicative constant depending on c. Therefore, the functions considered have no singularities outside the ellipse. This in principle might not be a problem, since due to the Mergelyan's Theorem [16] any such function can be approximated by a polynomial, however in the proof of Theorem 2 the degree of polynomials is bounded from above by $2n + m$.

2.2 Petras' lower bounds

Petras in [14] considers the quadrature of the same type as in Theorem 2, ellipses as analyticity regions and the Szegő class of weights (measures), which are defined as follows: $d\alpha(x) = w(x)dx$, where w is a function for which $\int_0^\pi \ln w(\cos x)\, dx$ exists. It contains the class of weights considered by Bakhvalov.

The reasoning in [14] goes as follows. Petras proves the following theorem for even more general class of weight measures.

Theorem 3 *[14, Thm. 2.1] Assume that the measure α is supported on at least $n+1$ points. Let D be a symmetric domain. Let $p_0, p_1, \ldots, p_n$ be the orthonormal polynomials with respect to the measure α, such that the degree of p_i is equal to i for $i = 0, \ldots, n$. Then*

$$\rho_n(\mathcal{A}(D), d\alpha) \geq k_n(\mathcal{A}(D), d\alpha) := \left(\sum_{\nu=0}^{n} (\sup_{z \in D} |p_\nu(z)|)^2 \right)^{-1}. \tag{18}$$

For weights in the Szegő class and $D = \mathcal{E}_c$, Petras [14, Corollary 3.1] obtains that

$$\lim_{n\to\infty} c^{2n} k_n(\mathcal{A}(\mathcal{E}_c), d\alpha) = 2\pi(1 - c^{-2}) \cdot \min_{|z|=c} |K(z^{-1})|^2 > 0, \tag{19}$$

where the so-called Szegő function is given by

$$K(z) := \exp\left(\frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} \ln(w(\cos t)|\sin t|)\, dt\right). \tag{20}$$

Observe that from the formula (20) one obtains $\rho_n c^{2n} \to 0$ for $c \to 1^+$, provided the term function $\min_{|z|=c} |K(z^{-1})|^2$ is bounded for $c \to 1^+$.

For several particular weights Petras computes an explicit lower bound for $k_n(\mathcal{A}(\mathcal{E}_c), d\alpha)$, but these bounds also show an undesirable behavior for $c \to 1^+$.

Below we list only the results for the Lebesgue measure and the Chebyshev weight.

- From Corollary 3.6 in [14] it follows that for the weight $w(x) \equiv 1$, we have

$$\rho_n(\mathcal{A}(\mathcal{E}_c), dx) \geq \pi(1 - c^{-2})^2 c^{-2n} \cdot (1 + \varepsilon_n)^{-1}, \tag{21}$$

  where

$$0 \leq \varepsilon_n \leq \frac{c^4 + 4c^2 + 18}{4nc^2(c^2 - 1)} + \frac{(n + 2)^{3/2}}{c^{n+2}}.$$

  It is clear that for a fixed n, this bound is $O((c - 1)^3)$ for $c \to 1^+$. To be more precise we have (for fixed n)

$$\rho_n(\mathcal{A}(\mathcal{E}_c), dx) \geq \frac{\pi}{c^{2n}}\left(\frac{32}{23}(c - 1)^3 nc^2 + O\left((c - 1)^4\right)\right). \tag{22}$$

- From Corollary 3.5 in [14] it follows that for the Chebyshev weight

$$d\alpha(x) = w(x)dx = dx/\sqrt{1 - x^2},$$

  we have

$$\rho_n(\mathcal{A}(\mathcal{E}_c), d\alpha) \geq \frac{\pi(1 - c^{-2})^3}{2c^{2n}}\left(1 - \frac{(2n + 3)(c^2 - 1) + c^{-2n-2}}{c^{2n+4}}\right)^{-1} \geq \frac{\pi(1 - c^{-2})^3}{2c^{2n}}. \tag{23}$$

  For $c \to 1^+$ we obtain the following estimate

$$\rho_n(\mathcal{A}(\mathcal{E}_c), d\alpha) \geq \frac{\pi}{c^{2n}}\left(\frac{3}{2n^3 + 9n^2 + 13n + 6} + \frac{6(c - 1)n}{2n^3 + 9n^2 + 13n + 6} + O\left((c - 1)^2\right)\right). \tag{24}$$

  In this case for fixed n the lower bound for $c \to 1^+$ does not go to zero, however it does go to zero when $n \to \infty$, which turns out to be unsatisfactory.

Apparently the reason for the overly pessimistic estimate for $c \to 1^+$ is that the bound in Theorem 3 obtained by considering polynomials of degree $2n$. Hence the functions producing this bound nigher have no singularities outside the ellipse nor are even close to a function with singularities.

2.3 Osipenko estimates

Osipenko in [12, Thm. 6] obtained the following explicit estimate for the Chebyshev weight $d\alpha(x) = dx/\sqrt{1 - x^2}$

$$\rho_n(\mathcal{A}(\mathcal{E}_c), d\alpha) = \frac{2\pi}{c^{2n}} + O(c^{-6n}) \tag{25}$$

and the limit behavior

$$\lim_{c\to 1^+} \rho_n(\mathcal{A}(\mathcal{E}_c), d\alpha) = 2\pi. \tag{26}$$

Osipenko uses a transformation of an ellipse to an infinite strip, which transforms the problem of integration of bounded analytic functions defined on the ellipse with the Chebyshev weight to the problem of integration of analytic periodic functions with Lebesgue measure. He uses Blaschke products to find lower estimate for the error, which is natural for this kind of problem. This should be contrasted with the polynomials used to derive lower bounds in [1, 14].

2.4 Final comments on Bakhvalov's and Petras' lower bounds

Both Bakhvalov and Petras mention that the Riemann mapping theorem allows us to transport the results for an ellipse to other domains. However, no quantitative statements related to the geometry of the domain D are given.

As it was mentioned in the introduction we have found the behavior of $\kappa_L(c)$ for $c \to 1^+$ obtained by Bakhvalov and by Petras overly pessimistic. In the argument below we will show how bad this bound is qualitatively. Namely, if $\kappa_G(c)$ were of the same order as $\kappa_L(c)$, i.e. $\lim_{c\to 1^+} \kappa_G(c) = 0$, the quadrature would be exact even for $n = 1$. This is formalized in the following remark.

Remark 4 *Let $Q \in \mathcal{Q}(n, \mathcal{R})$ and a positive bounded function $\kappa : (1, \infty) \times \mathbb{N} \to \mathbb{R}_+$ be such that*

$$|I(f) - Q(f)| \le M\kappa(c, n)c^{-2n}, \qquad \forall f \in \mathcal{A}_0(\mathcal{E}_c, M). \tag{27}$$

Assume that we have

$$\lim_{c\to 1^+} \kappa(c, n) = 0 \qquad \forall\, n \in \mathbb{N}. \tag{28}$$

Then for any $M > 0$, $c > 1$, $n \in \mathbb{N}$ and $f \in \mathcal{A}_0(\mathcal{E}_c, M)$ holds

$$I(f) = Q(f).$$

Proof. Since $\mathcal{E}_c \subset \mathcal{E}_{c_1}$ for $c < c_1$, we have

$$\mathcal{A}_0(\mathcal{E}_{c_1}, M) \subset \mathcal{A}_0(\mathcal{E}_c, M), \quad c < c_1. \tag{29}$$

The above inclusion holds in the following sense: for a function $f \in \mathcal{A}_0(\mathcal{E}_{c_1}, M)$ we consider its restriction to $\mathcal{E}_c$. It is immediate to see that $f_{|\mathcal{E}_c} \in \mathcal{A}_0(\mathcal{E}_c, M)$.

Let us fix n and take a function $f \in \mathcal{A}_0(\mathcal{E}_{c_1}, M)$. By (27) and (29)

$$|I(f) - Q(f)| \le M\kappa(c, n)c^{-2n}, \quad 1 < c \le c_1.$$

Passing to the limit $c \to 1$ we obtain

$$|I(f) - Q(f)| = 0.$$

∎

2.5 Upper bounds for Gauss-Legendre quadratures

We assume that $d\alpha(x) = dx$ and G_n denotes the Gauss-Legendre quadrature with n nodes on $[-1, 1]$. Let us define

$$r_n(c) = \rho(G_n, \mathcal{A}_0(\mathcal{E}_c, 1), dx) = \sup_{f\in\mathcal{A}_0(\mathcal{E}_c,1)} |I(f) - G_n(f)|. \tag{30}$$

Obviously

$$|I(f) - G_n(f)| \le Mr_n(c), \quad f \in \mathcal{A}_0(\mathcal{E}_c, M). \tag{31}$$

Let us list two estimates for the error of Gauss quadrature known in the literature.

Theorem 5 *The estimate for the error of the Gauss quadrature due to Rabinowitz [15, eq. (18)], see also [5, Thm. 90] and [19, Thm. 4.5], is*

$$r_n(c) \le \min\left(4, \frac{64}{15(1-c^{-2})}c^{-2n}\right). \tag{32}$$

The non-constant part of this estimate has an undesirable property. For $c \to 1$ it explodes, which may lead to non-uniform estimates in some contexts.

Bounds that are much more uniform in c for $c \to 1$ are given by Petras in [13].

Theorem 6 *The estimate for the error of the Gauss quadrature due to Petras [13, Thm. 4] is*

$$r_n(c) \le \frac{4}{c^{2n}}\left(1 + \frac{3}{2nc^2} + \frac{4}{c^{n+1}}\right).$$

In fact, [13, Thm. 4] contains four estimates for $r_n(c)$ whose mutual ratios are bounded. We have chosen the one that appears to be easiest to handle.

Corollary 1 *From Theorem 6 one can easily obtain*

$$r_n(c) \le \frac{26}{c^{2n}}, \tag{33}$$

$$\forall \varepsilon > 0 \quad \exists c_0(\varepsilon) \quad \forall c \ge c_0(\varepsilon) \quad r_n(c) \le \frac{4+\varepsilon}{c^{2n}}. \tag{34}$$

Remark 7 *In [13] (in part (b) of a remark just below Theorem 4 there) Petras mentions that taking f to be a suitably scaled $(2n)$-th Chebyshev polynomial of the first kind T_{2n}, i.e. $f = T_{2n}(2c^{2n})/(c^{4n}+1) \in \mathcal{A}_0(\mathcal{E}_c, 1)$ one obtains*

$$|I(f) - G_n(f)| \ge \frac{\pi(1-(4n)^{-1})}{c^{2n}(1+c^{-4n})}. \tag{35}$$

Hence, the bounds given in Theorem 6 are optimal, up to a constant independent of c and n.

Observe that from (33) it follows that if $M/\varepsilon > 26$, then to have the error less than ε for functions from $\mathcal{A}_0(\mathcal{E}_c, M)$, it is enough to use N_G nodes, where

$$N_G \ge \frac{\ln \frac{M}{\varepsilon}}{\ln c}. \tag{36}$$

2.6 Comparison of lower and upper bounds

We are now ready to compare in detail the lower bounds of Bakhalov and Petras with the bounds for the Gauss-Legendre quadrature for the ellipses with the Lebesgue measure as weight function.

Let $c > 1$ and let $\kappa_L(c)$ and $\kappa_G(c)$ be positive numbers such that

- for any $Q_n \in \overline{\mathcal{Q}}(n, \mathcal{R}_2)$ there is an $f_0 \in A_0(\mathcal{E}_c, M)$ such that

$$|I(f_0) - Q_n(f_0)| \ge M\kappa_L(c)c^{-2n}, \quad \text{and} \tag{37}$$

- for the Gauss-Legendre quadrature G_n, for any $f \in A_0(\mathcal{E}_c, M)$ we have

$$|I(f) - G_n(f)| \le M\kappa_G(c)c^{-2n}, \tag{38}$$

where κ_L is Bakhvalov's or Petras' lower bound discussed in Sections 2.1 and 2.2 and

$$\kappa_G(c) = \sup_{n\geq 1} c^{2n} r_n(c)$$

obtained from Theorem 5 or Theorem 6.

From Theorem 2 (with $\kappa_L = \kappa_0$ given by (16)) for c close to 1 we get

$$\begin{aligned}\kappa_L(c) &= (c-1)^3\pi + O((c-1)^4),\\ \kappa_G(c) &= 26.\end{aligned}$$

For large values of c (which means that we are considering very regular functions) we have

$$\kappa_L(c) = \frac{\pi}{4}c^{-2} + O(c^{-4}) \tag{39}$$

and

$$\kappa_G(c) = 4.1 + O(c^{-2}), \tag{40}$$

from (34). Note that in both cases the quotient $\kappa_G/\kappa_L \to \infty$.

Both bounds (37) and (38) are $O(c^{-2n})$ as functions of the minimal number n of nodes and they give the following estimates for n, needed to obtain the integral with error less than ε.

For the Gauss-Legendre quadrature it is enough to take $n \geq N_G$, where

$$N_G = \max\left(1, \frac{1}{2\ln c}\ln\left(\frac{M}{\varepsilon}\kappa_G(c)\right)\right),$$

while (37) implies that whatever the quadrature is we cannot take n smaller than

$$N_L = \max\left(1, \frac{1}{2\ln c}\ln\left(\frac{M}{\varepsilon}\kappa_L(c)\right)\right).$$

For $\varepsilon \to 0$ we have

$$\frac{N_L}{N_G} \approx \frac{\ln\left(\frac{M}{\varepsilon}\kappa_L(c)\right)}{\ln\left(\frac{M}{\varepsilon}\kappa_G(c)\right)} \to 1.$$

Apparently both numbers N_L and N_G are of similar magnitude up to a factor depending on c but not on n.

However, if we fix ε and let $c \to 1$, we have $\kappa_L(c) \to 0$, hence $N_L \to 1$ so the lower bound N_L loses its predictive power.

We are not concerned with the behavior of κ_L and κ_G for $c \to \infty$, because it does not necessarily make sense to increase c while keeping M constant; the functions in $\mathcal{A}_0(\mathcal{E}_c, M)$ become very flat for large c and in this limit we obtain $N_L = N_G = 1$.

Summing up, the bounds (37) and (38) might give completely different estimates N_L and N_G of the amount of information needed to bring the error below ε. For 'difficult' functions (c close to 1) we obtain the obvious bound $n \geq N_L = 1$ for a significant range of the ratio M/ε.

It appears that it makes sense to require the following condition to maintain the optimality of Gauss-Legendre quadratures on ellipses: there exists η_0 such that for all $M/\varepsilon \in \mathbb{R}_+$ and $c > 1$

$$0 < \eta_0 \leq \frac{N_L(M,\varepsilon,c)}{N_G(M,\varepsilon,c)}. \tag{41}$$

Observe that, when compared to results of Bakhvalov [1] and Petras [14], we now want the ratio to be bounded also when we change the ellipse.

3 New lower bounds

In this section, we compute lower bounds on the quadrature error in a class of analytic functions with possible singularities outside a nice domain. These results are applied to ellipses, so that they can be directly compared with known results. Since the methods may be applicable in a more general class of domains (not necessarily simply connected) we introduce distances (metrics) that could be tools for studying them in several complex variables. But we restrict our consideration to the case of simply connected domains in the complex plane, where the considered (hyperbolic) metric and distance may be described in many equivalent ways. The question which description could (and should) be applied in the case of domains being not simply connected remains open.

3.1 Definitions and description of the problem

First, let us recall that the word 'distance' is reserved for a notion of a 'distance function', i. e. the non-negative function defined for a pair of points (in our situation lying in the domain D in the complex plane) being positive for different points, symmetric and satisfying the triangle inequality. On the other hand the word 'metric' is reserved for a type of a function coming from the differential geometry that at each point of a manifold (in our case the domain D) measures the length of a vector on a tangent space (in our case $\mathbb{C}$). The metric defined on D induces in the standard way the distance on D.

By $\lambda_1(A)$ we denote the Lebesgue measure of the set $A \subset \mathbb{R}$.

We recall that *the Poincaré distance* p on the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ is given by the formula

$$p(z,w) := \frac{1}{2} \ln \frac{1+m(w,z)}{1-m(w,z)} =: \operatorname{arctanh}(m(w,z)),\ w,z \in \mathbb{D}, \tag{42}$$

where $m(w,z) = |(w-z)/(1-\bar{w}z)|$.

The Poincaré distance induces the pseudodistance c_D on any domain (i.e., a connected open set) $D \subset \mathbb{C}$ by the formula

$$c_D(w,z) := \sup\{p(F(w),F(z)) : F \in \mathcal{O}(D,\mathbb{D})\}, w,z \in D, \tag{43}$$

where $\mathcal{O}(D,\mathbb{D})$ denotes the set of holomorphic (analytic) functions from D to $\mathbb{D}$. We also put

$$c_D^*(w,z) := \tanh c_D(w,z). \tag{44}$$

We remind the following property of c_D (called *the holomorphic contractibility of* c):

$$c_G(F(w),F(z)) \le c_D(w,z) \quad \text{for any } F \in \mathcal{O}(D,G),\ w,z \in D. \tag{45}$$

In the case of simply connected domains the function c_D coincides with the distance induced by the metric γ_D (often called *hyperbolic metric* for planar domains), defined by the formula

$$\gamma_D(z;X) := \sup\{|F'(z)X|/(1-|F(z)|^2) : F \in \mathcal{O}(D,\mathbb{D})\},\ z \in D,\ X \in \mathbb{C}. \tag{46}$$

It is well-known that $\gamma_{\mathbb{D}}(z;X) = |X|/(1-|z|^2)$, $z \in \mathbb{D}$, $X \in \mathbb{C}$ (we call the function $\gamma_{\mathbb{D}}$ *the Poincaré metric*).

Analogously, we get a version of holomorphic contractibility of γ, namely the inequality

$$\gamma_G(F(w);F'(w)X) \le \gamma_D(w;X),\ w \in D; X \in \mathbb{C}, \tag{47}$$

for any $F \in \mathcal{O}(D,G)$. For domains $D \subset G$ in $\mathbb{C}$ we may use the holomorphic contractibility for the inclusion function $\iota : D \mapsto G$, where $D \subset G \subset \mathbb{C}$, which gives, among others, the inequality $\gamma_D(z;1) \ge \gamma_G(z;1)$ for all $z \in D$.

Note that although we defined the functions c_D and γ_D in a general situation we shall consider them in the special case of D being a simply connected domain.

The geometry induced by the Poincaré distance is an example of a non-Euclidean geometry. Recall that the lines (geodesics) in this geometry are diameters and the arcs of circles lying in $\mathbb{D}$ and being orthogonal to the unit circle $\partial\mathbb{D}$. In particular, for three consecutive points x, y, z on such geodesics one has the equality $p(x,z) = p(x,y) + p(y,z)$. Note also that the biholomorphic mappings transform geodesics into geodesics, and the geodesics in the domain D satisfy the equality $c_D(x,z) = c_D(x,y) + c_D(y,z)$ for three consecutive points lying in the geodesic. The distance of two points w, z from the simply connected domain D lying in a geodesic may be given with the help of the function γ_D as follows. Assume that $\alpha : [0,1] \to D$ is a parametrization of the part of the geodesic passing through w and z that lies between w and z, then

$$c_D(w,z) = \int_0^1 |\alpha'(t)|\gamma_D(\alpha(t);1)\,dt.$$

We should also keep in mind that the Poincaré distance on $\mathbb{D}$ and the Poincaré metric are invariant under holomorphic automorphisms of the unit disk ($\mathrm{Aut}(\mathbb{D})$). Recall that

$$\mathrm{Aut}(\mathbb{D}) = \left\{e^{i\theta}m_\eta : \theta \in \mathbb{R}, \eta \in \mathbb{D}\right\},$$

where $m_\eta(z) := (\eta - z)/(1 - \bar{\eta}z)$, $z \in \mathbb{D}$.

A special role in our considerations will be played by *the finite Blaschke products.*

Definition 5 *Functions of the form $B(z) := e^{i\theta}\prod_{j=1}^n m_{\eta_j}(z)$, $z \in \mathbb{D}$, where $\eta_j \in \mathbb{D}$, $j = 1, \ldots, n$, $\theta \in \mathbb{R}$, $n \in \mathbb{N} \setminus \{0\}$ are called finite Blaschke products.*

Some of basic properties of the finite Blaschke products are that they extend holomorphically to a neighborhood of $\overline{\mathbb{D}}$ (they are rational with poles lying outside of the closed unit disk). The finite Blaschke product B is a proper holomorphic mapping of $\mathbb{D}$ onto $\mathbb{D}$. Moreover, $|B(z)| = 1$ for $|z| = 1$.

We refer the reader to any of the textbooks [16], [6], [7] and [10]. In the last reference the theory of holomorphically invariant metrics and distances in several complex variables is presented.

In higher-dimensional case the metric γ_D depends on points $z \in D$ and vectors X from the tangent space to D; that is the reason why the value of the differential at vector $X \in \mathbb{C}$ (generally $\mathbb{C}^n$) is studied. However, the facts that we use are standard in the theory of one complex variable and may be found in many textbooks on the theory of complex variable. As to the theory of (bounded) holomorphic functions, in addition to the textbooks mentioned above, we refer the reader to [8] and [9] (where one may also see how the Blaschke products appear naturally when considering some extremal problems in the theory of analytic functions). Out of many possible references for the properties of the Carathéodory distance (induced by the hyperbolic metric) we recommend the paper [2] and the references therein concerning estimates for the hyperbolic metric in the ellipses. Note that the hyperbolic density σ_D considered in [2] is related to γ_D by the relation $\gamma_D(z;X) = |X|\sigma_D(z)$. The paper [2] could also possibly be applied to sharpen some of the results presented in the paper in the case of ellipses.

In this section, unless otherwise stated, the domain $D \subset \mathbb{C}$ containing $[-1,1]$ is simply connected, $D \neq \mathbb{C}$ and D is symmetric with respect to the x-axis, i.e. $z \in D$ iff $\bar{z} \in D$. Let α be a finite, positive, Borel measure on $[-1,1]$ absolutely continuous with respect to Lebesgue measure.

Let $f_D : D \to \mathbb{D}$ be a conformal mapping (i.e. biholomorphic) such that $f_D(0) = 0$, $f_D([0,1]) \subset [0,1)$ (the latter is possible because of the symmetry of D). Note also that the function f_D is actually unique (it follows from the uniqueness part of the Riemann mapping theorem). The set $\mathbb{R} \cap D$ is a geodesic. We shall often make use of the identity

$$c_D^*(w,z) = m(f_D(w), f_D(z)),\ w, z \in D.$$

Given an integer k let $r(k)$ be the least even integer bigger than or equal to k. Certainly, $r(k)$ is either k or $k+1$.

For the sequence of n distinct points $X := (x_1, \ldots, x_n)$, where $-1 \leq x_1 < \ldots < x_n \leq 1$ and the sequence of n positive integers $\mathcal{K} = (k_1, \ldots, k_n)$ we define

$$\mathcal{F}(D;X;\mathcal{K}) := \{f \in \mathcal{O}(D,\mathbb{D}) : f^{(l)}(x_j) = 0 : l = 0, \ldots, k_j - 1;\ j = 1, \ldots, n\},$$

$$\mathcal{F}_r(D;X;\mathcal{K}) := \{f \in \mathcal{F}(D;X;\mathcal{K}) : f(D\cap\mathbb{R}) \subset \mathbb{R}\},$$
$$\mathcal{F}_+(D;X;\mathcal{K}) := \{f \in \mathcal{F}_r(D;X;\mathcal{K}) : f \geq 0 \text{ on } D\cap\mathbb{R}\}$$

and

$$J_a(D;X;\mathcal{K}) := \sup\left\{\left|\int_{-1}^1 g(x)\,d\alpha(x)\right| : g \in \mathcal{F}_a(D;X;\mathcal{K})\right\},$$

where a is $+$, r or empty sign.

We are now in a position to prove the following lemma.

Lemma 8 *Let D, f_D, α, X and $\mathcal{K}$ be defined as above. Then there is exactly one $f \in \mathcal{F}_+(D;X;\mathcal{K})$ such that*

$$\int_{-1}^1 f(x)\,d\alpha(x) = J_+(D;X;\mathcal{K}).$$

Moreover, f is given by the formula

$$f(z) = \prod_{j=1}^n \left(\frac{f_D(z) - f_D(x_j)}{1 - f_D(x_j)f_D(z)}\right)^{r(k_j)},\ z \in D \tag{48}$$

and

$$J_+(D;X;\mathcal{K}) = \int_{-1}^1 \left(\prod_{j=1}^n (c_D^*(x,x_j))^{r(k_j)}\right) d\alpha = \int_{-1}^1 \left(\prod_{j=1}^n m(f_D(x), f_D(x_j))^{r(k_j)}\right) d\alpha.$$

Proof. Let $g \in \mathcal{F}_+(D;X;\mathcal{K})$. The non-negativity of g together with the vanishing of derivatives at x_j implies that the multiplicity of g at x_j is at least $r(k_j)$. Let f be the function given by the formula (48). Then the function $h := g/f$ extends to a well-defined holomorphic function on D. Moreover, the function f is the composition of the finite Blaschke product with the conformal function f_D so that $\lim_{z\to\partial D}|f(z)| = 1$ and thus $\limsup_{z\to\partial D}|h(z)| \leq 1$. This together with the maximum principle for holomorphic functions implies that $|h(z)| \leq 1$, $z \in D$. Additionally, the maximum principle gives that the equality at one point $z \in D$ holds iff h is constant. And the non-negativity of f and g on $[-1,1]$ implies that this constant is one. Consequently, either $g(z) = f(z)$, $z \in D$ or $|g(z)| < |f(z)|$, $z \in D\setminus\{x_1,\ldots,x_n\}$, which completes the proof. ∎

Remark 9 *It is obvious that*

$$J_+(D;X;\mathcal{K}) \leq J_r(D;X;\mathcal{K}) \leq J(D;X;\mathcal{K}). \tag{49}$$

Moreover, the second inequality above is actually an equality. To see this take any $g \in \mathcal{F}(D;X;\mathcal{K})$. Let $|\omega| = 1$ be such that $\omega\int_{-1}^1 g(x)\,d\alpha(x) = \left|\int_{-1}^1 g(x)\,d\alpha(x)\right|$. Define $h(\lambda) := (\omega g(\lambda) + \overline{\omega g(\bar{\lambda})})/2$, $\lambda \in D$. Then $h \in \mathcal{F}_r(D;X;\mathcal{K})$ and $h(x) = \mathrm{Re}\,(\omega g(x))$, $x \in [-1,1]$. Consequently,

$$\left|\int_{-1}^1 g(x)\,d\alpha(x)\right| = \mathrm{Re}\left(\omega\int_{-1}^1 g(x)\,d\alpha(x)\right) = \int_{-1}^1 h(x)\,d\alpha(x), \tag{50}$$

which implies the inequality $J(D;X;\mathcal{K}) \leq J_r(D;X;\mathcal{K})$.

On the other hand $J_+(D;X;\mathcal{K})$ is, in general, less than $J_r(D;X;\mathcal{K})$. It can already be seen when considering $n = 1$, $k_1 = 1$, $d\alpha(x) = dx$ and x_1 close to -1. In fact, first note that for $x_1 = -1$ we get the inequalities

$$1 > \frac{f_D(x) - f_D(x_1)}{1 - f_D(x_1)f_D(x)} > \left(\frac{f_D(x) - f_D(x_1)}{1 - f_D(x_1)f_D(x)}\right)^2 > 0,\ x \in (-1,1] \tag{51}$$

so the inequality

$$\int_{-1}^1 \left(\frac{f_D(x) - f_D(x_1)}{1 - f_D(x_1)f_D(x)}\right) dx > \int_{-1}^1 \left(\frac{f_D(x) - f_D(x_1)}{1 - f_D(x_1)f_D(x)}\right)^2 dx \tag{52}$$

holds for $x_1 \geq -1$ sufficiently close to -1.

Remark 10 *Recall that finite Blaschke products are extremal in many problems that involve bounded holomorphic functions on the unit disk. In the context of the optimal quadrature formula the Blaschke products have been used by Osipenko [12] and Bojanov [3, 4] for analytic functions on the unit circle. Therefore, it is natural that the function for which the supremum in Lemma 8 is attained is, up to a conformal mapping f_D, a finite Blaschke product.*

3.2 Lower estimate

We now compute lower bounds on

$$\begin{aligned} J_+(D;N) \quad := \quad & \inf\{J_+(D;(x_1,\ldots,x_n);(k_1,\ldots,k_n)) : \\ & \qquad n\in\mathbb{N}, -1\le x_1<\ldots<x_n\le 1, k_1+\ldots+k_n=N\}. \end{aligned}$$

First we recall the classical Koebe one-quarter theorem.

Theorem 11 (see e.g. [7, Thm. 14. 7. 8]) The image of an injective holomorphic function $f:\mathbb{D}\to\mathbb{C}$ contains the disk centered at $f(0)$ with radius $\frac{1}{4}|f'(0)|$.

Before we proceed further with estimates for nice domains we present a result on a more general class of domains. First we remind that for any domain $D\subset\mathbb{C}$, $D\neq\mathbb{C}$ we define $\delta_D(x):=\inf\{|x-z| : z\in\mathbb{C}\setminus D\}$, $x\in D$.

Lemma 12 *Let D be a simply connected domain in $\mathbb{C}$, with $D\neq\mathbb{C}$ (we do not assume the symmetry of D!). Let $z_0\in D$. Then $\gamma_D(z_0;1)\ge L/\delta_D(z_0)$ where $L=\frac{1}{4}$. If D is additionally convex then we may take in the inequality $L=\frac{1}{2}$.*

Proof. Let $g:\mathbb{D}\to D$ be the conformal mapping such that $g(0)=z_0$. Applying Theorem 11 to g we get that $\delta_D(z_0)\ge|g'(0)|/4$. But then $\gamma_D(z_0;1)\ge\left|\left(g^{-1}\right)'(z_0)\right|=1/|g'(0)|$ which finishes the proof in the general case.

Assume additionally that D is convex. Then after translating and rotating the set D, we can assume that $D\subset H:=\{\operatorname{Re} z>0\}$ and $z_0=\delta_D(z_0)$. Define the biholomorphism $F:H\to\mathbb{D}$ by $F(z)=(z-1)/(1+z)$ for $z\in H$. From (46) and (47) it follows that

$$\gamma_D(z_0;1)\ge\gamma_H(z_0;1)=\frac{|F'(z_0)|}{1-|F(z_0)|^2}.$$

Taking into account that $z_0=\delta_D(z_0)>0$ we obtain the following estimate

$$\gamma_D(z_0;1)\ge\frac{1}{2z_0}=\frac{1}{2\delta_D(z_0)}.$$

∎

Recall now that we assume that D is a simply connected domain, symmetric with respect to the real axis and such that $[-1,1]\subset D\subset\mathbb{C}$, $D\neq\mathbb{C}$. In such a case, we have

$$\delta_D=\sup\{\delta_D(x) : x\in[-1,1]\}.$$

Observe that δ_D is the radius of the largest disk with the center in $[-1,1]$ that is contained in D.

Lemma 13 *For all $w,z\in[-1,1]$ the following inequality holds $c_D(w,z)\ge L|w-z|/\delta_D$, where $L=\frac{1}{4}$. Moreover, in the case D is additionally convex we may take $L=\frac{1}{2}$. Consequently,*

$$m(f_D(w),f_D(z))=c_D^*(w,z)=\tanh c_D(w,z)\ge\frac{\exp\left(\frac{2L|w-z|}{\delta_D}\right)-1}{\exp\left(\frac{2L|w-z|}{\delta_D}\right)+1},\ w,z\in[-1,1]. \qquad (53)$$

Proof.
Using the simple fact that $\mathbb{R} \cap D$ is a geodesic and applying Lemma 12, we get

$$c_D(w,z) = \int_0^1 |w-z|\gamma_D(tw+(1-t)z;1)\,dt \geq \frac{L|w-z|}{\delta_D}.$$

As to the last inequality in (53), recall that tanh is an increasing function and so we obtain

$$c_D^*(w,z) = \tanh c_D(w,z) \geq \tanh \frac{L}{\delta_D}|w-z| = \frac{\exp\left(\frac{2L|w-z|}{\delta_D}\right)-1}{\exp\left(\frac{2L|w-z|}{\delta_D}\right)+1},\ w,z \in [-1,1].$$

∎

Let us prove the general estimate for J_+.

Theorem 14 *Given a positive number $N \in \mathbb{N}$ we have*

$$J_+(D;N) \geq \sup_{\varepsilon>0}\left\{\left(\frac{\exp\left(\frac{2L\varepsilon}{\delta_D}\right)-1}{\exp\left(\frac{2L\varepsilon}{\delta_D}\right)+1}\right)^{2N}(\alpha([-1,1])-\omega(2N\varepsilon,\alpha))\right\}, \tag{54}$$

where $\omega(\delta,\alpha) := \sup\{\alpha(A) : A \subset [-1,1]$ is a Borel subset, $\lambda_1(A) \leq \delta\}$.
Moreover,

$$\lim_{\delta_D\to 0} J_+(D;N) = \alpha([-1,1]). \tag{55}$$

Proof. Fix $\varepsilon > 0$. For any compact set K denote $K^\varepsilon := \{z \in \mathbb{C} : |z-x| < \varepsilon$ for some $x \in K\}$. Let $r := r(k_1) + \ldots + r(k_n)$. By decreasing the set of integration, applying Lemma 8 and the estimate (53), and keeping in mind that the integrands take the values in the interval $[0,1)$ we get

$$J_+(D;(x_1,\ldots,x_n),(k_1,\ldots,k_n)) \geq \int_{[-1,1]\setminus\{x_1,\ldots,x_n\}^\varepsilon}\left(\frac{\exp\left(\frac{2L\varepsilon}{\delta_D}\right)-1}{\exp\left(\frac{2L\varepsilon}{\delta_D}\right)+1}\right)^r d\alpha.$$

Since $n \leq N$, we get that $r \leq 2N$ so

$$J_+(D;(x_1,\ldots,x_n),(k_1,\ldots,k_n)) \geq \left(\frac{\exp\left(\frac{2L\varepsilon}{\delta_D}\right)-1}{\exp\left(\frac{2L\varepsilon}{\delta_D}\right)+1}\right)^{2N}\int_{[-1,1]\setminus\{x_1,\ldots,x_n\}^\varepsilon} d\alpha.$$

Since $\lambda_1(\{x_1,\ldots,x_n\}^\varepsilon) \leq 2n\varepsilon \leq 2N\varepsilon$ the desired result follows. ∎

Note that Theorem 14 gives a substantial improvement of the estimates in [1], [14], since $J(D;N)$ is estimated from below by a function tending to $\alpha([-1,1])$ as $\delta_D \to 0$. Moreover, the estimates in [1], [14] are studied in detail for ellipses only.

Theorem 15 *Let $D \subset \mathbb{C}$ be a domain as above (i.e. simply connected, symmetric with respect to the x-axis, $[-1,1] \subset D$, $D \neq \mathbb{C}$) and let $\alpha = \lambda_1$. Then for any positive integer N, we have*

$$J_+(D;N) \geq 2L^{2N}\frac{\delta_D^{(2N\delta_D)/L}}{(\delta_D+L)^{(2N/L)(\delta_D+L)}}.$$

(Recall that $L = \frac{1}{4}$ in general case, and $L = \frac{1}{2}$ for D convex.) When D is convex, this inequality yields

$$J_+(D;N) \geq 2\left((1+1/(2\delta_D))^{2\delta_D}(2\delta_D+1)\right)^{-2N} \geq 2\exp(-2N)(2\delta_D+1)^{-2N}.$$

Proof.

Since for $t \geq 0$

$$\frac{\exp(t)-1}{\exp(t)+1} \geq \frac{t}{2+t}$$

by (53) and Lemma 8 we get

$$J_+(D;N) \geq \inf\left\{ \int_{-1}^{1} \prod_{j=1}^{n} \left(\frac{L|x-x_j|}{\delta_D + L|x-x_j|} \right)^{r(k_j)} dx : n \in \mathbb{N},\ -1 \leq x_1 < \ldots < x_n \leq 1,\ k_1 + \ldots + k_n = N \right\}.$$

The Jensen inequality now implies that $J_+(D;N)$ is not less than the infimum of

$$2\exp\left(1/2 \sum_{j=1}^{n} r(k_j) \int_{-1}^{1} \left(\ln(L|x-x_j|) - \ln(\delta_D + L|x-x_j|)\right) dx \right). \tag{56}$$

taken over all sequences $-1 \leq x_1 < \ldots < x_n \leq 1$, $k_1 + \ldots + k_n = N$.

The integral in (56) equals

$$\begin{aligned} I_j = 2\ln L + (1-x_j)\ln(1-x_j) + (1+x_j)\ln(1+x_j) + \\ - (1/L)\left(L(1-x_j)+\delta_D\right)\ln(\delta_D + L(1-x_j)) + \\ - (1/L)\left(L(1+x_j)+\delta_D\right)\ln(\delta_D + L(1+x_j)) + (2/L)\delta_D \ln\delta_D. \end{aligned}$$

We now rewrite it in the form

$$I_j = g(x_j) + g(-x_j) + 2\ln L + (2/L)\delta_D \ln \delta_D,$$

where

$$g(t) = (1+t)\ln(1+t) - \frac{1}{L}(L(1+t)+\delta_D)\ln(L(1+t)+\delta_D),\ t \in [-1,1].$$

By setting $h(t) := g(t) + g(-t)$, $t \in [-1,1]$ we get

$$\begin{aligned} h'(t) &= g'(t) - g'(-t) \\ &= \ln\frac{1+t}{1-t} - \ln\frac{L(1+t)+\delta_D}{L(1-t)+\delta_D} \\ &= \ln\frac{(1+t)(L(1-t)+\delta_D)}{(1-t)(L(1+t)+\delta_D)}. \end{aligned}$$

It is clear that h is even and $h'(0) = 0$. Moreover, we claim that $h'(t) > 0$ for $t \in (0,1)$.
Indeed $h'(t) > 0$ iff $(1+t)(L(1-t)+\delta_D) > (1-t)(L(1+t)+\delta_D)$. This condition is equivalent to

$$L(1-t^2) + (1+t)\delta_D > L(1-t^2) + (1-t)\delta_D,$$

which is satisfied for $t > 0$.

The calculations above show that the function defined by the formula

$$\begin{aligned} (1+t)\ln(1+t) + (1-t)\ln(1-t) - (1/L)\left(L(1+t)+\delta_D\right)\ln(\delta_D + L(1+t)) + \\ - (1/L)\left(L(1-t)+\delta_D\right)\ln(\delta_D + L(1-t)) \end{aligned}$$

attains its minimum on the interval $[-1,1]$ at $t = 0$. Since $r = \sum r(k_j) \leq 2N$, we get

$$\ln\left(J_+(D;N)/2\right) \geq 2N\left(\ln L - (1/L)\left(L+\delta_D\right)\ln(L+\delta_D) + (1/L)\,\delta_D \ln\delta_D\right)$$

and consequently

$$J_+(D;N) \geq 2L^{2N} \frac{\delta_D^{(2N\delta_D)/L}}{(\delta_D + L)^{(2N/L)(\delta_D+L)}}.$$

Note that the last expression tends to 2 as $\delta_D \to 0$ (compare Theorem 14).

On the other hand, in the case when D is convex, we have

$$\begin{aligned} J_+(D;N) &\geq 2L^{2N} \frac{\delta_D^{(2N\delta_D)/L}}{(\delta_D + L)^{(2N/L)(\delta_D+L)}} \\ &= 2\left((1+1/(2\delta_D))^{2\delta_D}(2\delta_D+1)\right)^{-2N} \\ &> 2\exp(-2N)(2\delta_D+1)^{-2N}, \end{aligned}$$

in view of the inequality $(1+1/x)^x < e$ for $x > 0$. ∎

Proof of Theorem 1: Without loss of generality we may assume that $M = 1$. Let us take any fixed quadrature Q, i.e., we know the nodes x_j and integers $k_1, \ldots, k_n$. Based on these nodes let us construct f, the unique function for which the supremum in the definition of $J_+(D; X; \mathcal{K})$ is attained (cf. Lemma 8). Since $f \in \mathcal{F}(D; X; \mathcal{K})$, we have $f^{(l)}(x_j) = 0$ for $l = 0, \ldots, k_j - 1$; $j = 1, \ldots, n$ and consequently f (and $-f$) provides the quadrature Q with the same information. Therefore,

$$Q(f) = Q(-f).$$

From Theorem 15 it follows that $I(f) \geq \gamma$, where γ is defined by (7). Clearly, $I(f) = -I(f)$.

Thus

$$\begin{aligned} 2\gamma &\leq |I(f) - I(-f)| \\ &\leq |I(f) - Q(f)| + |Q(-f) - I(-f)|, \end{aligned}$$

and $|I(f) - Q(f)| \geq \gamma$ or $|Q(-f) - I(-f)| \geq \gamma$. Hence the required function f_0 is f or $-f$.

∎

Notice that for "reasonable quadratures", i.e., such that $Q(g) = 0$ for $g \equiv 0$, the proof is even simpler: $Q(f) = Q(g) = 0$ and $|I(f) - Q(f)| = |I(f)| \geq \gamma$.

3.3 The case of ellipses

In the case when D is an ellipse

$$\mathcal{E}_c := \left\{ (x,y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1 \right\},$$

with $a, b > 0$ such that $a + b = c$ and $a^2 - b^2 = 1$, simple computations lead to the relations $a = (c^2+1)/(2c)$, $b = (c^2-1)/(2c)$ and the formula

$$\delta_{\mathcal{E}_c}(x) = \begin{cases} \sqrt{a^2-1}\sqrt{1-x^2}, & x \in [-1/a, 1/a], \\ \\ \min\{|x \pm a|\}, & x \in [-1,1] \setminus (-1/a, 1/a). \end{cases}$$

Consequently, $\delta_{\mathcal{E}_c} = \sqrt{a^2-1} = (c^2-1)/(2c)$.

Therefore, as an immediate consequence of Theorem 15, we get the following lower bound in the case of the ellipse and α being the Lebesgue measure.

Corollary 2 *Let $c > 1$. Then*

$$J_+(\mathcal{E}_c; N) \geq 2 \left(\left(\frac{c^2 - 1 + c}{c^2 - 1} \right)^{(c^2-1)/c} \left(\frac{c^2 - 1 + c}{c} \right) \right)^{-2N} . \tag{57}$$

Theorem 16 *Let $Q \in \overline{\mathcal{Q}}(n, \mathcal{R})$, such that $|\mathcal{R}| = N$ be a quadrature on $\mathcal{E}_c$. Let c be close to 1. For the error constant of Q to be less than ε (i.e., $\rho(Q, \mathcal{A}(\mathcal{E}_c), 1) < \varepsilon$), we must have*

$$N > N_L(M, \varepsilon, c) = \frac{-\ln \frac{M}{\varepsilon}}{4(c-1)\ln(c-1)} \left(1 + O\left(\left| \frac{1}{\ln(c-1)} \right| \right) \right). \tag{58}$$

Proof. Let us remind the reader that for all functions appearing in the definition of $J_+(\mathcal{E}_c; N)$ we have had a bound $|f(z)| \leq 1$ for $z \in \mathcal{E}_c$.
Therefore from (57) (see also the proof of Thm. 1) it follows there exists a function $f_0 \in \mathcal{A}_0(\mathcal{E}_c, M)$ such that

$$|I(f_0) - Q(f_0)| \geq M \left(\left(\frac{c^2 - 1 + c}{c^2 - 1} \right)^{(c^2-1)/c} \left(\frac{c^2 - 1 + c}{c} \right) \right)^{-2N} . \tag{59}$$

Therefore, to have error less than ε we need to take $N \geq N_L$, where

$$N_L = \frac{1}{2} \left(\ln \frac{M}{\varepsilon} \right) \left(\ln \left(\left(\frac{c^2 - 1 + c}{c^2 - 1} \right)^{(c^2-1)/c} \left(\frac{c^2 - 1 + c}{c} \right) \right) \right)^{-1} . \tag{60}$$

Let $\Delta = c - 1$. Then for $c \to 1$ we obtain

$$\begin{aligned} D \quad &:= \quad \ln \left(\left(\frac{c^2 - 1 + c}{c^2 - 1} \right)^{(c^2-1)/c} \left(\frac{c^2 - 1 + c}{c} \right) \right) \\ &= \quad \frac{c^2 - 1}{c} \left(\ln(c^2 - 1 + c) - \ln(c - 1) - \ln(c + 1) \right) + \ln \left(1 + \frac{c^2 - 1}{c} \right) \\ &= \quad (2\Delta + O(\Delta^2)) \left(\ln(1 + O(\Delta)) - \ln \Delta - \ln(2 + \Delta) \right) + \ln(1 + O(\Delta)) \\ &= \quad (2\Delta + O(\Delta^2)) \left(O(\Delta) - \ln \Delta + O(1) \right) + O(\Delta) \\ &= \quad -2\Delta \ln \Delta + O(\Delta). \end{aligned}$$

Therefore

$$D^{-1} \quad = \quad \frac{-1}{2\Delta \ln \Delta} \left(1 + O\left(\left| \frac{1}{\ln \Delta} \right| \right) \right),$$

and from (60) we obtain

$$N_L \quad = \quad \frac{-1}{4} \left(\ln \frac{M}{\varepsilon} \right) \frac{1}{\Delta \ln \Delta} \left(1 + O\left(\left| \frac{1}{\ln \Delta} \right| \right) \right).$$

∎

4 Conclusions

For an ellipse $\mathcal{E}_c$ and α being the Lebesgue measure, let us compare N_L, the lower bound for the pieces of information required, with N_G, the estimate of the number of nodes in the Gauss-Legendre quadrature needed to obtain an error less than ε, for $f \in \mathcal{A}_0(\mathcal{E}_c, M)$. From (36) we obtain for $c \to 1^+$

$$\begin{aligned}
\frac{N_L}{N_G} &= \frac{-\ln \frac{M}{\varepsilon}}{4(c-1)\ln(c-1)} \left(1 + O\left(\left|\frac{1}{\ln(c-1)}\right|\right)\right) \cdot \left(\frac{\ln \frac{M}{\varepsilon}}{\ln c}\right)^{-1} \\
&= -\frac{\ln c}{4(c-1)\ln(c-1)} \left(1 + O\left(\left|\frac{1}{\ln(c-1)}\right|\right)\right) \\
&= -\frac{1 + O(c-1)}{4\ln(c-1)} \left(1 + O\left(\left|\frac{1}{\ln(c-1)}\right|\right)\right) \\
&\approx -\frac{1}{4\ln(c-1)}.
\end{aligned}$$

We see that $N_L/N_G \to 0$ for $c \to 1^+$, hence we have not obtained (41). It will be interesting to see whether the lower bound can be improved to obtain a positive lower bound for this ratio not dependent on c. By Remark 7 the estimate for error for the Gauss-Legendre quadrature is optimal and the improvement should be sought through better estimation of $J_+(\mathcal{E}_c, N)$.

References

[1] N.S. Bakhvalov. On the optimal speed of integrating analytic functions (in russian). *Journal of Computational Mathematics and Mathematical Physics*, 7:63–75, 1967.

[2] R. Banuelos and T. Carroll. Stretching convex domains and the hyperbolic metric. *Quarterly J. Math.*, 61(3):265–273, 2010.

[3] B.D. Bojanov. Optimal rate of integration and ε–entropy of a class of analytic functions (in russian). *Mat. Zamietki*, 14:551–556, 1973.

[4] B.D. Bojanov. The best quadrature formula for a certain class of analytic functions. *Applicationes Mathematicae*, 14:441–446, 1974.

[5] H. Brass. *Quadraturverfahren.* Vandenhoeck and Ruprecht, Göttingen, 1997.

[6] J. Conway. *Functions of One Complex Variable I*, volume 11 of *Graduate Texts in Mathematics.* Springer Verlag, 1978.

[7] J. Conway. *Functions of One Complex Variable II*, volume 159 of *Graduate Texts in Mathematics.* Springer Verlag, 1995.

[8] P.L. Duren. *Theory of H^p spaces*, volume 38 of *Pure and Applied Mathematics.* Academic Press, New York-London, 1970.

[9] J.B. Garnett. *Bounded analytic functions*, volume 96 of *Pure and Applied Mathematics.* Academic Press, New York-London, 1981.

[10] M. Jarnicki and P. Pflug. Invariant distances and metrics in complex analysis. *De Gruyter Expositions in Mathematics*, 9, 1993.

[11] M. Kowalski, A. Werschulz, and H. Woźniakowski. Is Gauss quadrature optimal for analytic functions? *Num. Math.*, 47:89–98, 1985.

[12] K.Yu. Osipenko. Exact values of n-widths and optimal quadratures on classes of bounded analytic and harmonic functions. *Journal of Approximation Theory*, 82:156–175, 1995.

[13] K. Petras. Gaussian integration of Chebyshev polynomials and analytic functions. *Numerical Algorithms*, 10:187–202, 1995.

[14] K. Petras. Gaussian Versus Optimal Integration of Analytic Functions. *Constructive Approximation*, 14:231–245, 1998.

[15] P. Rabinowitz. Rough and ready error estimates in Gaussian integration of analytic functions. *Comm. ACM*, 12:268–270, 1969.

[16] W. Rudin. *Real and Complex Analysis.* McGraw Hill Company, 1966.

[17] I.F. Sharygin. Lower bounds for the errors of quadrature formulae in classes of functions. *Journal of Computational Mathematics and Mathematical Physics*, 3:370–376, 1963.

[18] J.F. Traub, G.W. Wasilkowski, and H. Woźniakowski. *Information-Based Complexity.* Academic Press, 1988.

[19] L.N. Trefethen. Is Gauss quadrature better than Clenshaw-Curtis? *SIAM Review*, 50:67–87, 2008.