Rigorous numerics for dissipative PDEs

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A Model Problem - Kuramoto-Sivashinsky PDE

Consider the Kuramoto-Sivashinsky (KS) eq.

$$u_t = -\nu u_{xxxx} - u_{xx} + 2uu_x, \qquad \nu > 0$$

where $(t, x) \in [0, \infty) \times \mathbf{R}$ subject to periodic and odd boundary conditions

$$\begin{aligned} u(t, 0) &= u(t, 2\pi) \\ u(t, -x) &= -u(t, x) \end{aligned}$$

For various values of ν a variety of dynamics,

fixed points,
periodic orbits,
heteroclinic orbits,
chaotic dynamics,

have been observed numerically.

Goal: A rigorous means of proving these numerical results.

A Model Problem - Kuramoto-Sivashinsky PDE, Fourier expansion

Fourier expansion is: $u(t,x) = \sum_{k=-\infty}^{\infty} b_k(t) e^{ikx}$

Substituting in KS and applying boundary conditions gives:

$$\dot{a}_k = k^2(1-\nu k^2)a_k - k\sum_{n=1}^{k-1} a_n a_{k-n} + 2k\sum_{n=1}^{\infty} a_n a_{n+k}$$

where $b_k = i a_k$ and $k = 1, 2, 3, \ldots$.

Linearization: $\dot{a}_k = k^2(1-\nu k^2)a_k$

- k-th mode is unstable for $k < \frac{1}{\sqrt{\nu}}$
- k-th mode is stable for $k > \frac{1}{\sqrt{\nu}}$
- the modes with $k >> \frac{1}{\sqrt{\nu}}$ should be irrelevant for the dynamics

A Model Problem - Kuramoto-Sivashinsky PDE, known results

Known results:

- the existence of global attractor, the functions from attractor are analytic - Fourier series converge at geometric rate (Foias, Temam)

- the existence of finite dimensional inertial manifold (Foias, Nicolaenko, Sell, Temam, Rossa, Jolly) (not of much use in rigorous numerics)

No analytical results dynamics more complicated than fixed points bifurcating from zero solution

Our rigorous results for Kuramoto-Sivashinsky PDE

- the existence of multiple periodic orbits for various parameter values $\nu \approx 0.1215$, 0.1212, 0.125, 0.032, 0.02991, both stable and unstable orbits

- the existence of multiple fixed points for various values o f ν and their bifurcations (joint with K. Mischaikow)

- the existence of attractive fixed points for various values of ν

Periodic point for KS-equation

$\mu = 0.127$

Symmetric attracting orbit

Theorem: Let $u_0(x) = \sum_{k=1}^{10} -2a_k \sin(kx)$, where a_k are given in table below. There exists a function $u^*(t,x)$, the classical solution of KS for $\nu = 0.127$, such that

$$\|u_0 - u^*(0,\cdot)\|_{L_2} < 8.1 \cdot 10^{-4},$$
$$\|u_0 - u^*(0,\cdot)\|_{C^0} < 6.5 \cdot 10^{-4}$$

such that u^* is periodic with respect to t.

$a_1 = 2.012088e-01$	$a_2 = 1.289978$
$a_3 = 2.012152e-01$	$a_4 = -3.778654e-01$
$a_5 = -4.231056e-02$	$a_6 = 4.316137e-02$
$a_7 = 6.940373e-03$	$a_8 = -4.156441e-03$
$a_9 = -7.945097e-04$	$a_{10} = 3.315994e-04$

Proof uses Brouwer Thm. and rigorous integration of KS-PDE

Periodic point for KS-equation

$\mu = 0.1215$

non-symmetric attracting orbit past period doubling

Theorem: Let $u_0(x) = \sum_{k=1}^{13} -2a_k \sin(kx)$, where a_k are given in table below. There exists a function $u^*(t,x)$, the classical solution of KS for $\nu = 0.1215$, such that

$$\|u_0 - u^*(0,\cdot)\|_{L_2} < 9.9 \cdot 10^{-5},$$
$$\|u_0 - u^*(0,\cdot)\|_{C^0} < 6.2 \cdot 10^{-5}$$

such that u^* is periodic with respect to t.

$a_1 = 2.559310e-01$	$a_2 = 1.096696$
$a_3 = 2.559302e-01$	$a_4 = -3.079615e-01$
$a_5 = -4.780276e-02$	$a_6 = 3.002052e-02$
$a_7 = 7.352633e-03$	$a_8 = -2.530197e-03$
$a_9 = -7.561938e-04$	$a_{10} = 1.624861e-04$
$a_{11} = 6.833008e-05$	$a_{12} = -8.789182e-06$
$a_{13} = -5.429523e-06$	

Proof uses Brouwer Thm. and rigorous integration of KS-PDE

Periodic point for KS-equation

$\mu = 0.1215$

Symmetric unstable orbit, past period doubling

Theorem: Let $u_0(x) = \sum_{k=1}^{11} -2a_k \sin(kx)$, where a_k are given in table below. There exists a function $u^*(t,x)$, the classical solution of KS for $\nu = 0.1215$, such that

$$\|u_0 - u^*(0,\cdot)\|_{L_2} < 1.27 \cdot 10^{-3},$$
$$\|u_0 - u^*(0,\cdot)\|_{C^0} < 8.26 \cdot 10^{-4}$$

such that u^* is periodic with respect to t.

$a_1 = 2.450027e-01$	$a_2 = 1.041500e+00$
$a_3 = 2.449985e-01$	$a_4 = -2.760754e-01$
$a_5 = -4.371320e-02$	$a_6 = 2.531380e-02$
$a_7 = 6.345919e-03$	$a_8 = -1.996779e-03$
$a_9 = -6.177148e-04$	$a_{10} = 1.184863e-04$
$a_{11} = 5.269771e-05$	

Proof uses Miranda Thm. and rigorous integration of KS-PDE, the orbit is apparently unstable

Periodic point for KS-equation

$\mu = 0.032$

symmetric attracting orbit, close to chaotic region

Theorem: Let $u_0(x) = \sum_{k=1}^{23} -2a_k \sin(kx)$, where a_k are given in table below. There exists a function $u^*(t,x)$, the classical solution of KS for $\nu = 0.032$, such that

$$\|u_0 - u^*(0,\cdot)\|_{L_2} < 8.9 \cdot 10^{-4},$$
$$\|u_0 - u^*(0,\cdot)\|_{C^0} < 9.5 \cdot 10^{-4}$$

such that u^* is periodic with respect to t.

$a_1 = 3.506682e-01$	$a_2 = 2.522889e-02$
$a_3 = 3.506665e-01$	$a_4 = -2.276745e+00$
$a_5 = -1.115325e+00$	$a_6 = -3.693057e-01$
$a_7 = 4.603873e-01$	$a_8 = -4.604564e-01$
$a_9 = -3.115024e-01$	$a_{10} = -1.449674e-01$
$a_{11} = 5.104894e-02$	$a_{12} = -2.165916e-02$
$a_{13} = -3.413293e-02$	$a_{14} = -2.613508e-02$
$a_{15} = 1.307623e-03$	$a_{16} = 8.752424e-05$
$a_{17} = -2.115586e-03$	$a_{18} = -2.891477e-03$
$a_{19} = -5.007345e-04$	$a_{20} = 3.374289e-05$
$a_{21} = -4.423567e-05$	$a_{22} = -2.280484e-04$
$a_{23} = -9.029570e-05$	

The main idea

Our equation

$$u_t = Lu + N(u, Du, \dots, D^r u), \qquad (1)$$

$u \in \mathbf{R}^n$, $x \in \mathbf{T}^d$, ($\mathbf{T}^d = (\mathbf{R}/2\pi)^d$ is an d-dimensional torus), L is a linear operator, N - a polynomial and by $D^s u$ we denote s-th order derivative of u,

L is diagonal in the Fourier basis $\{e^{kx}\}_{k \in \mathbf{Z}^d}$

$$Le^{ikx} = \lambda_k e^{ikx}, \qquad (2)$$

and the eigenvalues λ_k satisfy

$$\begin{aligned} \lambda_k &= -v(|k|)|k|^p && (3) \\ 0 &< v_0 \le v(|k|) \le v_1, \quad \text{for } |k| > K_- && (4) \\ p &> r. && (5) \end{aligned}$$

The fact that we are considering functions on the torus means that we impose periodic boundary conditions. We may eventually seek odd or even solutions or impose some other conditions.

Replace (1) by an infinite ladder of ODEs for Fourier coefficients of $u(t,x) = \sum_k u_k(t)e^{ikx}$:

$$\frac{du_k}{dt} = \lambda_k u_k + N_k(u), \qquad \text{for all } k \in \mathbf{Z}^d. \quad (6)$$

Split the phase space for (6) into two parts: the finite dimensional part, X, containing the Fourier modes most relevant for the dynamics of (1) and the tail $T \subset X^{\perp}$. Now (6) is replaced by two problems (7) and (8). The first part consist of a finite dimensional differential inclusion for $p \in X$, given by

$$\frac{dp}{dt} \in P(Lp + N(p+T)), \qquad p \in X \qquad (7)$$

where P is a projection onto X.

The second part: the evolution of T, which is governed by an infinite set of inequalities of the form

$$\lambda_k u_{k,j} + N^-_{k,j} < \frac{du_{k,j}}{dt} < \lambda_k u_{k,j} + N^+_{k,j}, \qquad (8)$$
$$j = 1, \ldots, n \text{ and for } k \text{ not in } X$$

where $N^{\pm}_{k,j}$ are suitably chosen constants.

Obviously, to infer from (7) and (8) any information on the behavior of solutions of the full system (6) one needs some consistency conditions. A systematic treatment of this issue is at the heart of our *method of self-consistent bounds*

Important: Integrating (7) and (8) we obtain uniform bounds for all Galerkin projections of (6). We apply to Galerkin projections topological tools, to obtain periodic orbits (hopefully also the symbolic dynamics) and then we pass to the limit to get those for full PDE.

The method of self-consistent bounds

H - Hilbert space,
$e_1, e_2, \ldots$ - an orthogonal basis in H
The corresponding projections are

$$\begin{aligned} p_m = P_m a &:= (a_1, a_2, \ldots, a_m) \\ q_m = Q_m a &:= (a_{m+1}, a_{m+2}, \ldots) \end{aligned}$$

The problem:

$$\dot{a} = F(a) \tag{9}$$

F is not continuous, with dense domain in H.

$F_k \circ P_n$ is a C^1-function for $n, k \in \mathbf{N}$

Later $F(a) = L(a) + N(a)$, L - linear, N- non-linear
$e_1, e_2, \ldots$ - eigenvectors of L - very helpful

The method:

Def. Fix m, M ($m \leq M$). *A compact set $W \subset P_m(H)$ and a sequence of pairs $\{a_k^{\pm} \in \mathbf{R} \mid a_k^- < a_k^+,\ k \in \mathbf{Z}^+\}$ are self-consistent a-priori bounds for F if:*

C1 *For $k > M$, $a_k^- < 0 < a_k^+$.*

C2 *Let $\hat{a}_k := \max |a_k^{\pm}|$ and set $\hat{u} = \sum_{k=0}^{\infty} \hat{a}_k e_k$. Then, $\hat{u} \in H$, ($\{\hat{a}_k\} \in l_2$)*

C3 *The function $u \mapsto F(u)$ is continuous on*

$$W \oplus \prod_{k=m+1}^{\infty} [a_k^-, a_k^+] \subset H.$$

Moreover, if we define $\hat{f}_k = \max_{u \in W \oplus \prod_{k=m+1}^{\infty} [a_k^-, a_k^+]} |F_k(u)|$ and set $\hat{f} = \sum \hat{f}_k e_k$, then $\hat{f} \in H$. ($\{\hat{f}_k\} \in l_2$)

Notation: $T = \prod_{k=m+1}^{\infty} [a_k^-, a_k^+]$ - Tail

ISOLATION for $n > m$

For $a \in W \oplus T$ and $k > m$ holds

$$\begin{aligned} a_k = a_k^+ \quad &\Rightarrow \quad \dot{a}_k < 0 \\ a_k = a_k^- \quad &\Rightarrow \quad \dot{a}_k > 0 \end{aligned}$$

C1,C2,C3 - give convergence

C4 - gives a priori bounds

C1,C2,C3,C4 - easy to satisfy (later)

Finite dimensional rigorous computations in m first variables

Basic Differential Inclusion:

$$\dot{p} \;\in\; P_m F(p) + \Gamma_m, \quad p \in \mathbf{R}^m, \qquad (10)$$

where $\Gamma_m = \{P_m F(p+q) - P_m F(p) \mid q \in T\}$

We say a multivalued map $p_I : [0,h] \to H$ is *upper attainable set (uas) map* for (10) if the following is true

- any C^1 function satisfying (10) and defined on the maximum interval of existence is defined on $[0,h]$

- if a C^1-function $p : [0,h] \to X_m$ satisfies (10), then $p(t) \in p_I(t)$ for $t \in [0,h]$

Theorem: Assume $W \oplus T$ are self-consistent bounds for F. If $p_I : [0, t_1] \to X_m = P_m(H)$ is uas map for (10), such that $p_I([0, t_1]) \subset W$.

Then for any $q_0 \in T$, the problem $u' = F(u)$ (and all its Galerkin projections $u' = P_n F(u)$, $n > M$) has a solution $u(t) = (p(t), q(t))$ for $t \in [0, t_1]$, such that

$$p(t) \in p_I(t), \qquad q(t) \in T, \qquad \text{for } t \in [0, t_1]$$

Why it is a easy to find a good tail = self-consistent bounds

$$u_t = Lu + N(u, Du, \ldots, D^r u)$$

$x \in \mathbf{T}^n$ (periodic boundary conditions),
L - linear, diagonal, N - polynomial

Fourier expansion $u(t) = \sum_{k \in \mathbf{Z}^n} a_k(t) e^{ik \cdot x}$

Lemma. Let $s > s_0$. If $|a_k| \leq C/|k^s|$, $|a_0| \leq C$, then there exists $D = D(C, s)$

$$|N_k| \leq \frac{D}{|k|^{s-r}}, \qquad |N_0| \leq D$$

Isolation. Assume $L(a)_k = -|k|^p a_k$, $p > r$.

Assume $|a_k| \le \frac{C}{|k^s|}$, $|a_{k_0}| = \frac{C}{|k_0|^s}$, then

$$\frac{d|a_{k_0}|}{dt} \le -|k_0|^p |a_{k_0}| + |N_{k_0}(a)| \le$$
$$-C|k_0|^{p-s} + D|k_0|^{r-s}$$
$$\frac{d|a_{k_0}|}{dt} < 0, \qquad |k_0| > M$$

Rigorous integration for dissipative PDEs

$(x, y) \in X_m \oplus Y_m \subset H$ - Hilbert space,
dim$X_m = m < \infty$, dim$Y_m \leq \infty$
$P_m : H \to H$, projection onto X_m, $Q_m = I - P_m$
F - our PDE in some basis on H

$$x' = PF(x, y) \tag{11}$$
$$y' = QF(x, y) \tag{12}$$

Idea: Replace (11 - 12) by

$$x'(t) \in P_m F(x(t), Tail(t)) \tag{13}$$
$$y(t) \in Tail(t), \tag{14}$$

where $Tail(t)$ has *finite representation* and can be computed *in finite number of operations*. $Tail_k(t)$ should decay fast enough.

We want also that: $x(t) \oplus P_n Q_m Tail(t)$, *for* $n > M$ *, is a rigorous estimate to* n*-dimensional Galerkin projection of* F*, for the initial condition* $x(0) \oplus P_n Q_m Tail(0)$

Integration of dissipative PDEs - II

$$x'(t) \in P_m F(x(t), Tail(t)) \qquad (15)$$

$$y(t) \in Tail(t), \qquad (16)$$

$x' = P_m F(x)$ - Galerkin projection, induces φ_m

One time step:
initial condition $Z \oplus Tail(0) \subset X_m \oplus Y_m$, $h > 0$
1 • find $W \oplus T[0,h]$ (rough enclosure)

$$\begin{aligned} P_m F(x,y) - P_m F(x,0) &\subset \Gamma, \qquad x \in Z \\ \varphi_{m,\Gamma}([0,h],Z) &\subset W \\ Tail([0,h]) &\subset T[0,h]. \end{aligned}$$

2 • instead of (15) consider $x' \in P_m F(x,0) + \Gamma$
- use algorithm for differential inclusions, to obtain $x(h)$ for $(x,y) \in Z \oplus Tail(0)$.

3 • compute $Tail(h)$.

Representation used for KS equation

We look for solutions in

$$W \oplus T = W \oplus \Pi_{k=m+1}^{k\leq M}[a_k^-, a_k^+] \oplus \Pi_{k>M}\left[\frac{-C}{k^s}, \frac{C}{k^s}\right] \tag{17}$$

where $W \subset X_m$.

$$N_k(W \oplus T) \subset [N_k^-, N_k^+], \qquad k = m+1, \dots, M$$
$$N_k(W \oplus T) \subset \left[\frac{-D(W \oplus T)}{k^{s-2}}, \frac{D(W \oplus T)}{k^{s-2}}\right], \qquad k > M$$

We solve (estimate rigorously) the solutions of the following system of differential inclusions

$$\begin{aligned} x' &\in P_m F(x) + \Gamma, & x \in W \subset X_m \\ x_k' &\in \lambda_k x_k + [N_k^-, N_k^+], & k = m+1, \dots, \end{aligned}$$

x_k for $k > M$ are given by a single formula.

Rigorous integration for ODEs and differential inclusions - basic principles

$x \in \mathbf{R}^n$, $f : \mathbf{R}^n \to \mathbf{R}^n$ - C^1.,

$$x' = f(x), \quad x(0) = x_0 \qquad \text{(ODE)}$$

induces $\varphi(t, x_0) \in \mathbf{R}^n$

One time step:
initial condition: $X_0 \subset \mathbf{R}^n$, $h > 0$ is a time step
1• find $W \subset \mathbf{R}^n$ (rough enclosure), such that $\varphi([0, h], X_0) \subset W$

2• apply the Taylor method to (ODE), evaluate the error term on W to obtain $X_1 \subset \mathbf{R}^n$, such that

$$\varphi(h, X_0) \subset X_1$$

Rigorous integration for ODEs - comments

- all computations are performed in **interval arithmetic**

- one should be very careful in the way how step **2** is executed, straightforward interval evaluation leads to *the wrapping effect*.

- we use *the Lohner algorithm*

Rigorous integration of differential inclusion

Differential inclusion : $\Gamma \subset \mathbf{R}^n$

$$x' \in f(x) + \Gamma, \quad x(0) = x_0$$

induces $\varphi_\Gamma(t, x_0) \subset \mathbf{R}^n$

One time step:
initial condition: $X_0 \subset \mathbf{R}^n$, $h > 0$ is a time step
1• compute X_1, such that $\varphi(h, X_0) \subset X_1$

2• find $W_2 \subset \mathbf{R}^n$ (rough enclosure), such that

$$\varphi_\Gamma([0,h], X_0) \subset W_2, \quad x \in X_0$$

3• use Gronwall type lemma to find $\Delta \subset \mathbf{R}^n$,

$$\varphi_\Gamma(h, x) - \varphi(h, x) \in \Delta$$

This step requires $\frac{\partial f}{\partial x}(W_2)$

4•

$$\varphi_\Gamma(h, X_0) \subset X_1 + \Delta$$

Differential inclusions - Fundamental Lemma

For a fixed $y_c \in \mathbf{R}^{n_2}$ we compare the solutions of two ODEs

$$\begin{aligned} x_1' &= f(x_1, y_c), \\ x_2' &= f(x_2, y_c) + (f(x_2, y(t)) - f(x_2, y_c)) \\ & x_1(t_0) = x_2(t_0) = x_0 \end{aligned}$$

where $y(t)$ is given (but unknown) function.

Lemma: Let:
$[W_y] \subset \mathbf{R}^{n_2}$, convex, $y([t_0, t_0 + h]) \subset [W_y]$.
$[W_1] \subset [W_2] \subset \mathbf{R}^{n_1}$ - convex and compact.
$x_1([t_0, t_0+h]) \subset [W_1]$, $x_2([t_0, t_0+h]) \subset [W_2]$ for any continuous function $y : [t_0, t_0 + h] \to [W_y]$.

Then the following inequality holds for $t \in [t_0, t_0+h]$ and for $i = 1, \ldots, n_1$

$$|x_{1,i}(t) - x_{2,i}(t)| \leq \left(\int_{t_0}^{t} e^{J(t-s)} C \, ds \right)_i , \qquad (18)$$

where

$$\begin{aligned} [\delta] &= \{ f(x, y_c) - f(x, y) \mid x \in [W_1], y \in [W_y] \}, \\ C_i &\geq \sup \left| [\delta_i] \right| , \quad i = 1, \ldots, n_1 \\ J_{ij} &\geq \sup \frac{\partial f_i}{\partial x_j}([W_2], [W_y]) \text{if } i = j, \\ J_{ij} &\geq \sup \left| \frac{\partial f_i}{\partial x_j}([W_2], [W_y]) \right| \text{if } i \neq j. \end{aligned}$$

Tail evolution

Our problem $a_k' = \lambda_k a_k + N_k(a)$,
$\lambda_k \to -\infty$, for $|k| \to \infty$

W, $T([0,h])$ - the rough enclosure for $Z \oplus T(0)$ for $t \in [0,h]$

For $k > m$ we have

$$N_k^{\pm} = N_k^{\pm}(W, T([0,h]))$$
$$\lambda_k a_k + N_k^{-} < \frac{da_k}{dt} < \lambda_k a_k + N_k^{+},$$

hence

$$b_k^{\pm} = \frac{N_k^{\pm}}{-\lambda_k}, \tag{19}$$
$$T(h)_k^{\pm} = \left(T(0)_k^{\pm} - b_k^{\pm}\right) e^{\lambda_k h} + b_k^{\pm} \tag{20}$$

It remains to put $T(h)$ for $k > M$ in the form

$$T(h)_k^{\pm} = \frac{\pm C(T(h))}{k^{s(T(h))}}$$

For $k > M$ we have

$$0 < b_k^+ \leq \frac{C(b)}{k^{s(b)}}$$
$$T(0)_k^+ = \frac{C(T(0))}{k^{s(T(0))}}$$
$$T(h)_k^+ \leq T(0)_k^\pm e^{\lambda_k h} + b_k^\pm.$$

$$T(h)_k^+ \leq \frac{C(T(0))}{k^{s(T(0))}} e^{\lambda_k h} + \frac{C(b)}{k^{s(b)}}.$$

Let

$$E = e^{h\lambda_{M+1}}(M+1)^{s(b)-s(T(0))}.$$

then (modulo some conditions on M, h)

$$e^{\lambda_k h} \leq \frac{E}{k^{s(b)-s(T(0))}}, \quad k > M$$

and finally we can set

$$T_k^\pm(h) = \pm \frac{C(T(0))E + C(b)}{k^{s(b)}}.$$

About the computations

- gnu C++

- interval arithmetic - from CAPD package devoloped in Krakow, Poland

- we use the Lohner algorithm to integrate differential inclusions

Some computation data

On 3GHz machine, Linux, gnu C++

- $\nu = 0.127$, $m = 10$, $M = 3 * m$, $h = 1e - 3$, $order = 4$, $T/2 \approx 1.12$, computation time around 10 sec

- $\nu = 0.1215$, $m = 13$, $M = 3*m$, $h = 4e-4$, $order = 6$, $T \approx 3.07$, computation time around 240 sec

- $\nu = 0.032$, $m = 23$, $M = 3*m$, $h = 1.5e-4$, $order = 5$, $T/2 \approx 0.41$, computation time around 300 sec

Conclusions

- rigorous numerics for dissipative PDEs is possible

- global existence and uniqueness theorems are not required, interesting solutions are constructed

- could be applied to (I hope): Ginzburg-Landau, Navier-Stokes in 2D and 3D

Future work

• prove chaos (symbolic dynamics) for KS $\nu \approx 0.029$ or $\nu \approx 0.1212$

• Construct an *rigorous* C^1*-algorithm* for dissipative PDE.

This will make possible to **rigorously** apply a lot of dynamical system theory to dissipative PDEs.